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Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field

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1 Introduction

In this paper we study an inverse boundary value problem for the Schrödinger equation in the presence of a magnetic potential. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqq 3$, with smooth boundary. The Schrödinger equation in a magnetic potential is given by

$$(1.1) \qquad H_{\vec{A},q} = \sum_{j=1}^{n} \left(\frac{1}{i} \frac{\partial}{\partial x_j} + A_j(x) \right)^2 + q(x), \ i = \sqrt{-1},$$

where $\vec{A} = (A_1, A_2, \ldots, A_n) \in C^1(\bar{\Omega})$ is the magnetic potential and $q \in L^\infty(\Omega)$ is the electric potential. The magnetic field is the rotation of the magnetic potential, $\operatorname{rot}(\vec{A})$.

We assume that $\vec{A}$ and q are real-valued function and thus (1.1) is self-adjoint. We also assume that zero is not a Dirichlet eigenvalue of (1.1) on Ω, so that the boundary value problem

$$(1.2) \qquad \begin{cases} H_{\vec{A},q} u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f \in H^{\frac{1}{2}}(\partial\Omega) \end{cases}$$

has a unique solution $u \in H^1(\Omega)$. The Dirichlet-to-Neumann map $\Lambda_{\vec{A},q}$ which maps $H^{\frac{1}{2}}(\partial\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$, is defined by

$$(1.3) \qquad \Lambda_{\vec{A},q} : f \to \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} + i(\vec{A}\cdot\nu) f, \ f \in H^{\frac{1}{2}}(\partial\Omega)$$

where u is the unique solution to (1.2), and ν is the unit outer normal on $\partial\Omega$.

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The inverse boundary value problem for (1.1) is to recover information of $\vec{A}$ and q from knowledge of $\Lambda_{\vec{A},q}$. This problem is closely related to the inverse scattering problem for (1.1) with a fixed energy. This problem was considered in [H–N] where it is assumed that the magnetic field is small. The inverse problem at a fixed energy in the case that the magnetic field is 0 was solved by Novikov ([N]).

As it was noted in [Su], the Dirichlet to Neumann map $\Lambda_{\vec{A},q}$ is invariant under a gauge transformation in the magnetic potential: $\vec{A} \to \vec{A} + \nabla g$, where $g \in C^1_\Omega$, where we denote

$$C^s_\Omega = \{f \in C^s(\mathbb{R}^n), \operatorname{supp} f \subset \Omega\} \tag{1.4}$$

Thus, $\Lambda_{\vec{A},q}$ carries information about the magnetic field instead of information about $\vec{A}$. The natural question is whether $\Lambda_{\vec{A},q}$ determines uniquely $\operatorname{rot}(\vec{A})$ and q. In [Su], this question was answered affirmatively for $\vec{A}$ in the C^2_Ω class and q in the $L^\infty(\Omega)$ class, under the assumption that $\operatorname{rot}(\vec{A})$ is small in the L^∞ topology. Namely, we have

Theorem A. *Let $\vec{A}_j \in C^2_\Omega, q_j \in L^\infty(\Omega)$, $j = 1,2$. Assume that zero is not a Dirichlet eigenvalue for $H_{\vec{A}_j,q_j}$, $j = 1,2$. Then there exists a constant $\varepsilon = \varepsilon(\Omega) > 0$ such that if $\|\operatorname{rot}(\vec{A}_j)\|_{L^\infty(\Omega)} < \varepsilon$, $j = 1,2$, and*

$$\Lambda_{\vec{A}_1,q_1} = \Lambda_{\vec{A}_2,q_2}\,,$$

then

$$\operatorname{rot}(\vec{A}_1) = \operatorname{rot}(\vec{A}_2) \text{ and } q_1 = q_2 \text{ in } \Omega\,.$$

The main purpose of this paper is to consider the above problem for C^∞ class of potentials $\vec{A}$ and q. In this case we are able to remove the smallness assumption on $\operatorname{rot}(\vec{A})$. We have

Theorem B. *Let $\vec{A}_j \in C^\infty(\bar{\Omega})$, $q_j \in C^\infty(\bar{\Omega})$, $j = 1,2$. Assume that zero is not a Dirichlet eigenvalue of $H_{\vec{A}_j,q_j}$, $j = 1,2$. If*

$$\Lambda_{\vec{A}_1,q_1} = \Lambda_{\vec{A}_2,q_2}\,,$$

then

$$\operatorname{rot}(\vec{A}_1) = \operatorname{rot}(\vec{A}_2) \text{ and } q_1 = q_2 \text{ in } \Omega\,.$$

If we assume $\vec{A}_j \in C^\infty_\Omega$, then Theorem B holds even for $q_j \in L^\infty(\Omega)$. We have

Theorem C. *Let $\vec{A}_j \in C^\infty_\Omega$, $q_j \in L^\infty(\Omega)$, $j = 1,2$. Assume that zero is not a Dirichlet eigenvalue of $H_{\vec{A}_j,q_j}$, $j = 1,2$. If*

$$\Lambda_{\vec{A}_1,q_1} = \Lambda_{\vec{A}_2,q_2}\,,$$

then

$$\mathrm{rot}(\vec{A}_1) = \mathrm{rot}(\vec{A}_2) \text{ and } q_1 = q_2 \text{ in } \Omega .$$

In the next three sections we shall give proofs for Theorems B and C. Our proofs are a combination of methods developed in [L-U], [N-U] and [Su]. We remark that in the case $\vec{A} = 0$, global uniqueness was established in [S-U].

In the next section we shall show that $\Lambda_{\vec{A},q}$ determines the boundary values of $\mathrm{rot}(\vec{A})$ and its normal derivatives of all orders at the boundary. Namely, we have

Theorem D. *Let $\vec{A}_j \in C^\infty(\bar{\Omega})$ and $q_j \in C^\infty(\bar{\Omega})$, $j = 1, 2$. Assume that zero is not a Dirichlet eigenvalue of $H_{\vec{A}_j, q_j}$, $j = 1, 2$. If*

$$\Lambda_{\vec{A}_1, q_1} = \Lambda_{\vec{A}_2, q_2} ,$$

then we can find $\phi \in C^\infty(\Omega)$ vanishing to first order at $\partial\Omega$ such that

$$\vec{A}_1 = \vec{A}_2 + \nabla\phi$$

to infinite order at $\partial\Omega$.

As one can see from the proof of Theorem D one can prove also that $\Lambda_{\vec{A},q}$ determines also the boundary values of q and its normal derivatives of all orders at the boundary (This was proven in [K-V] in the cases $\vec{A} = 0$.) However, this fact will not be used to prove Theorem B and C. Theorem D reduces the proof of Theorem B to the proof of Theorem C by an extension argument.

In Sect. 3 we shall construct a family of exponentially growing functions (in certain directions) of the form

$$u(x,\xi) = e^{\xi\cdot x + \phi(x,\hat{\xi})}(1 + \omega(x,\xi)) \tag{1.5}$$

satisfying the equation $H_{\vec{A},q}u = 0$, with the property that $\omega(x,\xi)$ decays as $|\xi| \to \infty$, where $\xi \in \mathbb{C}^n$, $\xi\cdot\xi = 0$, $\hat{\xi} = \xi/|\xi|$. It is the construction of ω which presents some difficulties. In [Su], ω was constructed for $\vec{A} \in C^2_\Omega$, $q \in L^\infty(\Omega)$ under the assumption that $\|\mathrm{rot}(\vec{A})\|_{L^\infty(\Omega)}$ is small. In Sect. 3 we shall construct ω for $\vec{A} \in C^\infty(\bar{\Omega})$ and $q \in L^\infty(\Omega)$ without the smallness assumption on $\mathrm{rot}(\vec{A})$. This will be done by intertwining the operator $e^{-x\cdot\xi}H_{\vec{A},q}(e^{x\cdot\xi})$ with the operator $e^{-x\cdot\xi}\Delta(e^{x\cdot\xi})$ and then using the solutions for the latter constructed in [S-U]. A similar approach was developed in [N-U] for the elasticity system.

Section 4 is devoted to the proof of Theorem B and C. We shall first reduce Theorem B to Theorem C. The proof of C follows from results in Sect. 3 and modification of arguments in [Su].

2 Boundary determination

In this section we prove Theorem D in the introduction. To do this we proceed as in [L-U]. We write the operator in boundary normal coordinates and then we factorize the operator. This leads to a Riccati type equation for the Dirichlet to Neumann map similar to the one derived in [L-U] for the case that the operator is the Laplace-Beltrami operator of a Riemannian metric. In that case similar Riccati type equations were derived by different methods by Cheney, Isaacson and Somersalo [S-I-C] and, in the isotropic case, by Sylvester [S].

The Schrödinger operator (1.1) can be rewritten as

$$P(x,D) = -\Delta + \sum_{j=1}^{n} 2A_j(x)D_{x_j} + G(x) \tag{2.1}$$

where

$$G = \vec{A}^2 - 2i\nabla\cdot\vec{A} + q, \quad D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j} \tag{2.2}$$

The Dirichlet to Neumann map $\Lambda_{\vec{A},q}$ is a pseudodifferential operator of order 1. In boundary normal coordinates, $(x^1,\dots,x^{n-1},x^n) = (x',x_n)$ with $\partial\Omega$, (resp. Ω) is given locally by $x^n = 0$, (resp. $x_n > 0$) (see [L-U]). We have that

$$-\Delta = D_{x^n}^2 + iE(x)D_{x^n} + Q(x,D_{x'}) \tag{2.3}$$

where $Q(x,D_{x'})$ a second order operator in $D_{x'}$, depending smoothly on x^n with positive principal symbol. We remark that $E(x)$ and $Q(x,D_{x'})$ are known in this case. Using (2.3), the operator (2.1) can be written as

$$P(x,D) = D_{x^n}^2 + iE(x)D_{x^n} + Q(x,D_{x'}) + \sum_{j=1}^{n-1} 2C_j(x)D_{x^j} + 2\tilde{A}_nD_{x^n} + G \tag{2.4}$$

where $\tilde{A}_n, C_j = 1,\dots,n-1$ are the components of $\vec{A}$ in boundary normal coordinates. Now we define

$$M(x,D) = e^{if}P(x,D)(e^{-if}) = -\Delta + \sum_{j=1}^{n-1}\tilde{A}_jD_{x^j} + \tilde{G} \tag{2.5}$$

with

$$f(x',x^n) = \int_0^{x^n}\tilde{A}_n(x',s)ds\,.$$

The point is that we can factorize (2.5) as a product of two first order operators. Namely, we have

Proposition 2.6. *There exists a pseudodifferential operator $B(x,D_{x'})$ of order 1 in x' depending smoothly on x^n such that*

$$M(x,D) = (D_{x^n} + iE(x) - iB(x,D_{x'}))(D_{x^n} + iB(x,D_{x'})) \tag{2.7}$$

modulo a smoothing operator.

Proof. In order for (2.7) to be valid we need that B satisfies the following Ricatti equation

$$i[D_{x^n}, B] - E(x)B + B^2 - Q - \sum_{j=1}^{n-1} \tilde{A}_j D_{x^j} - \tilde{G} = 0 \tag{2.8}$$

modulo smoothing. Therefore the full symbol $b(x, \eta')$ of $B(x, D_{x'})$ must satisfy

$$\sum_\alpha \frac{1}{\alpha!} \partial^\alpha_{\eta'} b D^\alpha_{x'} b - h + \partial_{x^n} b - Eb = 0 \tag{2.9}$$

where

$$h(x, \eta') = g_2(x, \eta') + g_1(x, \eta') + \sum_{j=1}^{n-1} \tilde{A}_j \eta_j + \tilde{G}$$

with $g_2(x, \eta') + g_1(x, \eta')$ the full-symbol of $Q(x, D_{x'})$ and $g_2(x, \eta')$ the principal symbol.

We write

$$b(x, \eta') \sim \sum_{j \leqq 1} b_j(x, \eta') \tag{2.10}$$

with b_j homogeneous of degree j in η'. Grouping the homogeneous terms of degree two in (2.9) we obtain

$$b_1^2 - g_2 = 0 .$$

We then choose

$$b_1 = -\sqrt{g_2} . \tag{2.11}$$

The homogeneous terms of degree one in (2.9) is

$$2b_0 b_1 + \sum_{|\alpha|=1} \partial^\alpha_{\eta'}(b_1) D^\alpha_{x'}(b_1) - g_1 + \partial_{x^n} b_1 - \sum_{j=1}^{n-1} \tilde{A}_j \eta_j - Eb_1 = 0 .$$

We choose

(2.12)

$$b_0 = \frac{1}{2\sqrt{g_2}} \left(\sum_{|\alpha|=1} \partial^{\alpha'}_{\eta'}(\sqrt{g_2}) D^{\alpha'}_{x'}(\sqrt{g_2}) - g_1 - \partial_{x^n} \sqrt{g_2} + E\sqrt{g_2} + \sum_{j=1}^{n-1} \tilde{A}_j \eta_j \right)$$

The homogeneous term of degree zero in (2.9) is

$$2b_{-1} b_1 + \sum_{\substack{j,k,K \\ 0 \leqq j,k \leqq 1 \\ |K| = j+k}} \frac{1}{K!} \partial^K_{\eta'}(b_j) D^K_{x'}(b_k) + \frac{\partial}{\partial x^n} b_0 - \tilde{G} - Eb_0 . \tag{2.13}$$

We then choose

$$b_{-1} = \frac{1}{2\sqrt{g_2}} \left(\sum_{\substack{j,k,K \\ 0 \leqq j,k \leqq 1 \\ |K| = j+k}} \frac{1}{K!} \partial^K_{\eta'}(b_j) D^K_{x'}(b_k) + \frac{\partial b_0}{\partial x^n} - Eb_0 - \tilde{G} \right) . \tag{2.14}$$

The terms $b_j, j \leqq -2$ are chosen in a similar fashion, (see (1.9) in [L-U]) completing the proof of the proposition.

The point of the construction above is that

Proposition 2.15.

$$\Lambda_{\vec{A},q} = B(x, D_{x'})|_{x^n=0} - 2i\tilde{A}_n|_{x^n=0} \text{ mod. smoothing} . \tag{2.16}$$

Proof. Using Proposition (2.6) and the same arguments as in the proof of Proposition 1.2 in [L-U] we conclude that

$$\frac{\partial u}{\partial x^n}|_{x^n=0} = B(x, D_{x'})u|_{x^n=0} \text{ mod. smoothing}$$

where u solves the Dirichlet problem

$$M(x, D)u = 0 \text{ in } \Omega; \quad u|_{\partial\Omega} = f .$$

Proof of Theorem D. We first look at the principal symbol of (2.16). Using (2.16) we have that the terms homogeneous of degree one in the full symbol of $\Lambda_{\vec{A},q}$ is

$$b_1(x, \eta')|_{x^n=0} = \sqrt{g_2}|_{x^n=0} . \tag{2.17}$$

The term homogeneous of degree zero in the full symbol of $\Lambda_{\vec{A},q}$ is

$$(b_0 - 2i\tilde{A}_n)|_{x^n=0} \tag{2.18}$$

From (2.12) then we observe that we can determine knowing the full symbol of the Dirichlet to Neumann map

$$\frac{1}{2\sqrt{g_2}}\sum_{j=1}^{n-1}\tilde{A}_j\eta^j - 2i\tilde{A}_n \tag{2.19}$$

at $x^n = 0$. Therefore, since $\tilde{A}_j$ are real-valued, we conclude that we can recover, $\tilde{A}_j, \tilde{A}_n$ at $x^n = 0$ at this stage.

The term homogeneous of degree -1 in the full symbol of $\Lambda_{\vec{A},q}$ is

$$b_{-1}|_{x^n=0} .$$

Therefore using (2.14), (2.17) and (2.18) we conclude that we can recover from $b_{-1}|_{x^n=0}$

$$\frac{\partial b_0}{\partial x^n} - \tilde{G}$$

at $x^n = 0$. Then using (2.12) we can determine from the full symbol of the DN map

$$\frac{1}{2\sqrt{g_2}}\frac{\partial}{\partial x^n}\left(\sum_{j=1}^{n-1}\tilde{A}_j\eta^j\right) - \tilde{G} \tag{2.20}$$

at $x^n = 0$. Since $\tilde{A}_j$, $j = 1, \ldots, n-1$, is real-valued we determine from (2.20)

$$\frac{\partial}{\partial x^n}\tilde{A}_j,\ j = 1, \ldots, n-1$$

at $x^n = 0$. Proceeding inductively we can then prove that we can recover from the homogeneous terms of degree $-k$ in the full symbol of $\Lambda_{\vec{A},q}$.

$$\left(\frac{\partial}{\partial x^n}\right)^k \tilde{A}_j|_{x^n=0},\ j = 1, \ldots, n-1 .$$

The result now follows from (2.5) since we know $\tilde{A}_n|_{x^n=0}$ and the Taylor series of $\tilde{A}_j$, $j = 1, \ldots, n-1$, at $x^n = 0$ □

3 Construction of solutions

Let $\vec{A} \in C^\infty_\Omega$, $q \in L^\infty(\Omega)$. We extend $q = 0$ outside Ω. We look for solution of the form

$$U(x, \xi) = e^{\xi \cdot x + \phi(x, \hat{\xi})}(1 + \omega(x, \xi)),\ x \in \Omega , \tag{3.1}$$

in the null space of $H_{\vec{A},q}$, with the property that $\omega(x, \xi)$ behaves like $|\xi|^{-1}$ as $|\xi|$ tends to ∞, where $\xi \in \mathbb{C}^n$, $\xi \cdot \xi = 0$ and $\hat{\xi} = \xi/|\xi|$. This is reduced to construct ϕ and ω from the following two equations.

$$\xi \cdot \nabla\phi = -i\xi \cdot \vec{A} \tag{3.2}$$

$$\Delta\omega + 2(\xi + \nabla\phi + i\vec{A}) \cdot \nabla\omega - g\omega = g \tag{3.3}$$

where

$$g = \vec{A}^2 - i\nabla \cdot \vec{A} + q - 2i\vec{A} \cdot \nabla\phi - \nabla\phi \cdot \nabla\phi - \Delta\phi . \tag{3.4}$$

As in [Su], we construct ϕ by Fourier transforming (3.2). This leads to

$$\phi(x, \hat{\xi}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix \cdot \eta} \left(\frac{\xi \cdot \vec{A}^\wedge(\eta)}{\xi \cdot \eta}\right) d\eta , \tag{3.5}$$

where $\wedge$ denotes the Fourier transform. One can easily verify that $\phi(\cdot, \hat{\xi}) \in C^\infty(\mathbb{R}^n)$ is a solution to (3.2), which satisfies

$$\|\phi(\cdot, \hat{\xi})\|_{C^m(\bar{\mathcal{O}})} \leqq C(\mathcal{O}, m)\|\vec{A}\|_{C^m_\Omega} \tag{3.6}$$

for any bounded domain $\mathcal{O} \subset \mathbb{R}^n$. Moreover, $\phi(x, \hat{\xi})$ is differentiable in $\hat{\xi}$ on the manifold $\{\xi \in \mathbb{C}^n, \xi \cdot \xi = 0, |\xi| = 1\}$.

In the rest of this section we shall construct ω. Since we are looking for solution u on Ω, we may rewrite the equation (3.3) as

$$\Delta_\xi \omega + \vec{B}\cdot\nabla\omega - h\omega = h \tag{3.8}$$

where

$$\Delta_\xi = \Delta + 2\xi\cdot\nabla, \vec{B} = 2(\nabla\phi + i\vec{A})\psi,\ h = g\psi \tag{3.9}$$

and $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi|_\Omega \equiv 1$. We denote

$$M_\xi = \Delta_\xi + \vec{B}\cdot\nabla,\ L_\xi = M_\xi - h\,. \tag{3.10}$$

We shall construct ω by inverting L_ξ. To do this, we need to invert M_ξ first. Assume that $\xi = s\xi_0 + O(\frac{1}{s})$, where $\xi_0 \in \mathbb{C}^n, |\xi_0| = 1$ and $\xi\cdot\xi = 0$.

Lemma 3.11. *The operator M_ξ has a bounded inverse $M_\xi^{-1} : L^2(\Omega) \to H^1(\Omega)$ for $|\xi|$ sufficiently large. Moreover, for any $f \in L^2(\Omega)$,*

$$\|M_\xi^{-1}(f)\|_{L^2(\Omega)} \leqq C|\xi|^{-1}\|f\|_{L^2(\Omega)} \tag{3.12}$$

$$\|M_\xi^{-1}(f)\|_{H^1(\Omega)} \leqq C\|f\|_{L^2(\Omega)} \tag{3.13}$$

where C is a constant independent of ξ.

Lemma 3.11 is the key step in constructing the solution ω. Once Lemma 3.11 is proven, analogous arguments to the ones in [Su] (lemma 2.4) can be used to show that Lemma 3.11) is valid with M_ξ replaced by L_ξ. This gives us

Proposition 3.14. *For $|\xi|$ large enough, there is a solution ω satisfying* (3.3). *Moreover,*

$$\|\omega\|_{L^2(\Omega)} \leqq \frac{C}{|\xi|},\ \|\nabla\omega\|_{L^2(\Omega)} \leqq C\,, \tag{3.15}$$

where C is a constant independent of $|\xi|$.

To prove Lemma 3.11, we first introduce some notations. We denote by $L^m(\mathbb{R}^n, Z)$ the space of pseudodifferential operators of order m in the Shubin class [Sh] where $Z = \{\xi \in \mathbb{C}^n, \xi\cdot\xi = 0, |\xi| \geqq 1\}$ (see [N-U]). We define $\Lambda_\xi \in L^s(\mathbb{R}^n, Z)$ as the properly supported pseudodifferential operator with principal symbol

$$\sigma(\Lambda_\xi^s) = (|\eta|^2 + |\xi|^2)^{\frac{s}{2}}\,,$$

for the definition of properly supported see [N-U].

We denote

$$\tilde{M}_\xi = M_\xi\Lambda_\xi^{-1},\ \tilde{\Delta}_\xi = \Delta_\xi\Lambda_\xi^{-1}\,.$$

Lemma 3.16. *Let $N \in \mathbb{Z}^+$. There exist A_ξ and $B_\xi \in L^0(\mathbb{R}^n, Z)$ properly supported satisfying the following: For any $\phi_1 \in C_0^\infty(\mathbb{R}^n)$ there exist $\phi_2, \phi_3, \phi_4 \in C_0^\infty(\mathbb{R}^n)$ and $r > 0$ such that*

$$\phi_1\tilde{M}_\xi A_\xi = \phi_1 B_\xi \phi_2 \Lambda_\xi^{-1}(\Delta_\xi + \phi_3 R_\xi^{(-N)}\phi_4) \tag{3.17}$$

where

$$R_\xi^{(-N)} : H^\alpha(\mathbb{R}^n) \to H^{\alpha+N}(\mathbb{R}^n)$$

is a bounded linear operator with

$$\|R_\xi^{(-N)}\|_{H^\alpha(\mathbb{R}^n), H^{\alpha+N}(\mathbb{R}^n)} \leqq C_\alpha |\xi|^{-N} \tag{3.18}$$

for $\xi \in Z, |\xi| \geqq r$ *and any* $\alpha \in \mathbb{R}$. *Morever* $\phi_j, j = 2,3,4$ *are taken to satisfy*

$$\phi_1\phi_2 = \phi_1,\ \phi_2\phi_3 = \phi_2,\ \phi_1\phi_4 = \phi_1 \tag{3.19}$$

and

$$\phi_1 B_\xi \phi_2 = \phi_1 B_\xi,\ \phi_2 \Lambda_\xi^{-1} \phi_3 = \phi_2 \Lambda_\xi^{-1} . \tag{3.20}$$

We refer the reader to [N-U] for the proof of Lemma 3.16.

Proof of lemma 3.11. By defining

$$C_\xi = B_\xi \Lambda_\xi^{-1} \tag{3.21}$$

we conclude using (3.17) and (3.20)

$$\phi_1 \tilde{M}_\xi A_\xi = \phi_1 C_\xi \left(\Delta_\xi + \phi_3 R_\xi^{(-N)} \phi_4\right) . \tag{3.22}$$

Take now $\phi_1' \in C_0^\infty(\mathbb{R}^n)$ such that $\phi_1' = 1$ on supp ϕ_1. It is easy to see that there exists a linear operator $\tilde{C}_\xi^{-1}$ such that

$$\phi_1' C_\xi \tilde{C}_\xi^{-1} = \phi_1' \tag{3.23}$$

and

$$\|\tilde{C}_\xi^{-1}\|_{H^\alpha \cdot H^{\alpha-1}} \leqq C_\alpha |\xi| \tag{3.24}$$

for any $\alpha \in \mathbb{R}$. We are taking in the rest of the proof $\xi \in Z$, $|\xi| \geqq r$, for an appropriate $r > 0$. Let us choose now $\phi_2' \in C_0^\infty(\mathbb{R}^n)$ such that

$$\phi_1' C_\xi \phi_2' = \phi_1 C_\xi . \tag{3.25}$$

Let us consider the equation

$$\left(\Delta_\xi + \phi_3 R_\xi^{(-N)} \phi_4\right) v = \phi_2' \tilde{C}_\xi^{-1} \phi_1 f \tag{3.26}$$

From (3.22), (3.23) and (3.25) we have

$$\phi_1 \tilde{M}_\xi A_\xi v = \phi_1^3 \phi_1' C_\xi \tilde{C}_\xi^{-1} \phi_1 f = \phi_1^2 \phi_1' f = \phi_1^2 f .$$

Hence, since $\phi_1 = 1$ on Ω, if v is a solution of (3.26), then

$$w = \Lambda_\xi^{-1} A_\xi \phi_1'' v \tag{3.27}$$

satisfies

$$M_\xi w = f \quad \text{in } \Omega . \tag{3.28}$$

Here we have taken $\phi_1'' \in C_0^\infty(\mathbb{R}^n)$ such that $\phi_1 \tilde{M}_\xi A_\xi \phi_1'' = \phi_1 \tilde{M}_\xi A_\xi$. Using (3.18), (3.24) and the representation theorem for compactly supported distributions (see for instance [M] Thm. 2.13, page 92), it is easy to prove that for any $f \in L^2(\mathbb{R}^n)$,

$$v = \left(I + \Delta_\xi^{-1}\phi_3 R_\xi^{(-N)}\phi_4\right)^{-1}\Delta_\xi^{-1}\phi_2'\tilde{C}_\xi^{-1}\phi_1 f \in H^1_{loc}(\mathbb{R}^n) \tag{3.29}$$

satisfies (3.26). Furthermore, using (3.24) and the estimates for Δ_ξ^{-1} ([S-U]) we have

$$\|\phi_1'' v\|_{H^\alpha(\mathbb{R}^n)} \leqq C_\alpha |\xi|^\alpha \|f\|_{L^2(\mathbb{R}^n)} \tag{3.30}$$

for $\alpha = 0, 1$. Now we take $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^n)$ such that $\psi_1 \phi_1'' = \phi_1''$, $\Lambda_\xi^{-1} A_\xi \psi_1 = \psi_2 \Lambda_\xi^{-1} A_\xi \psi_1$. Then using the estimate (1.12) in [N-U] we conclude that $w \in H^2(\mathbb{R}^n)$ and

$$\|w\|_{H^\alpha(\mathbb{R}^n)} \leqq C_\alpha |\xi|^{\alpha-1} \|f\|_{H^1(\mathbb{R}^n)}, \alpha = 0, 1 . \tag{3.31}$$

This concludes the proof. □

4 Proofs of theorems

In this section we prove Theorems B and C. We first reduce Theorem B to Theorem C.

Let $\vec{A}_j \in C^\infty(\bar{\Omega})$, $q_j \in C^\infty(\bar{\Omega})$, $j = 1, 2$ and $\Lambda_{\vec{A}_1, q_1} = \Lambda_{\vec{A}_2, q_2}$. Let B_R be a ball with center at the origin and radius R so that $\bar{\Omega} \subset B_R$. Let ϕ be as in Theorem D. Using this result and the invariance of the Dirichlet to Neumann map, we can change $\vec{A}_2$ by $\vec{A}_2 + \nabla\phi$ and assume that $\vec{A}_1 = \vec{A}_2$ to infinite order at $\partial\Omega$. We choose a vector function $\vec{A}_1'$ on B_R so that $\vec{A}_1' \in C_0^\infty(B_R)$ and $\vec{A}_1'|_\Omega \equiv \vec{A}_1$, and define

$$\vec{A}_2'(x) = \begin{cases} \vec{A}_2(x), & x \in \Omega \\ \vec{A}_1'(x), & x \in \bar{B}_R \backslash \Omega . \end{cases}$$

Then $\vec{A}_2' \in C_0^\infty(B_R)$. We simply extend q_j, to $q_j' \in L^\infty(B_R), j = 1, 2$ so that

$$q_j'(x) = \begin{cases} q_j(x), & x \in \Omega \\ 0, & x \in B_R \backslash \Omega . \end{cases}$$

Consider now the operator $H_{\vec{A}_j', q_j'}$, $j = 1, 2$. We may change the radius R slightly so that zero is not a Dirichlet eigenvalue for both $H_{\vec{A}_j', q_j'}$, and thus $\Lambda_{\vec{A}_j', q_j'}$ is well-defined, $j = 1, 2$. Since $\Lambda_{\vec{A}_1, q_1} = \Lambda_{\vec{A}_2, q_2}$, and $\vec{A}_1' = \vec{A}_2'$ and $q_1' = q_2'$ on $B_R \backslash \Omega$, it follows from a well-known fact in inverse boundary value problems that $\Lambda_{\vec{A}_1', q_1'} = \Lambda_{\vec{A}_2', q_2'}$ on ∂B_R. Therefore, Theorem B is reduced to Theorem C.

The proof of Theorem C follows similar lines of argument to the ones given in [Su]. We give only an outline.

Proof of Theorem C. The proof begins with the following identity.

$$i\int_{\Omega}(\vec{A}_1 - \vec{A}_2)\cdot(u_1\nabla\bar{u}_2 - u_2\nabla\bar{u}_1)dx + \int_{\Omega}(\vec{A}_1^2 - \vec{A}_2^2 + q_1 - q_2)u_1\bar{u}_2 dx = 0 \tag{4.1}$$

where u_j is any solution of $H_{\vec{A}_j,q_j}u_j = 0$, $j = 1,2$, and we have assumed that $\Lambda_{\vec{A}_1,q_1} = \Lambda_{\vec{A}_2,q_2}$. See Proposition 3.1 in [Su] for a proof which is based on an integration by parts argument.

The next step is to replace u_j, $j = 1,2$ in (4.1) by exponentially growing solutions constructed in Sect. 3. Let k, γ_1, γ_2 be three mutually orthogonal vectors in $\mathbb{R}^n$ with $|\gamma_1| = |\gamma_2| = 1$. Let $\zeta, \xi \in \mathbb{C}^n$ be given by $\zeta = \gamma_1 + i\gamma_2$, $\xi = s\zeta + g(s,k)\gamma_1$, where s is a positive real parameter and

$$g(s,k) = |k|^2(2(|k|^2 + 4s^2)^{\frac{1}{2}} + 4s)^{-1} .$$

Define

$$\xi_1 = \frac{ik}{2} + \xi, \quad \bar{\xi}_2 = \frac{ik}{2} - \xi . \tag{4.2}$$

One can check that $\xi_1 \cdot \xi_1 = \xi_2 \cdot \xi_2 = 0$. We now construct

$$u_j(x, \xi_j) = e^{\xi_j \cdot x + \phi_j(x,\hat{\xi}_j)}(1 + \omega_j(x, \xi_j)) \tag{4.3}$$

solution of $H_{\vec{A}_j,q_j}u_j = 0$, $j = 1,2$. Substituting (4.3) into (4.1) and letting s tend to ∞, one gets

$$\int_{\Omega} e^{ikx + \phi_1^* + \bar{\phi}_2^*}\zeta\cdot(\vec{A}_1 - \vec{A}_2)dx = 0 \tag{4.4}$$

where $\phi_j^*(x,\zeta) = \phi_j(x,\widehat{\zeta})$, $j = 1,2$.

(4.4) is all we need to deduce that

$$\operatorname{rot}(\vec{A}_1) = \operatorname{rot}(\vec{A}_2) . \tag{4.5}$$

See Sect. 4.1 in [Su] for details. (4.5) implies that there exist $p \in C^1_{\Omega}$ so that

$$\vec{A}_1 - \vec{A}_2 = \nabla p \quad \text{in } \Omega . \tag{4.6}$$

To prove $q_1 = q_2$, we first recall that $H_{\vec{A},q}$ is invariant under gauge transformations. This fact together with (4.6) implies that $\Lambda_{\vec{A}_2,q_2} = \Lambda_{\vec{A}_2+\nabla p,q_2} = \Lambda_{\vec{A}_1,q_2}$. On the other hand, $\Lambda_{\vec{A}_1,q_1} = \Lambda_{\vec{A}_2,q_2}$. So we have

$$\Lambda_{\vec{A}_1,q_1} = \Lambda_{\vec{A}_1,q_2} .$$

This means that we may assume $\vec{A}_1 = \vec{A}_2$ when we prove $q_1 = q_2$. Letting $\vec{A}_1 = \vec{A}_2$ in (4.1) we get

$$\int_\Omega (q_1 - q_2) u_1 \bar{u}_2 dx = 0 . \tag{4.7}$$

Now substituting (4.3) into (4.7) and letting s tend to ∞ we get

$$\int_\Omega e^{ikx + \phi_1^* + \bar{\phi}_2^*} (q_1 - q_2) dx = 0 ,$$

from which the result follows immediately. □

Note added in proof. Esking and Ralston have recently solved the inverse scattering problem at a fixed energy for the Schrödinger equation in a magnetic field assuming that the potentials are exponentially decaying.

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