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A Note on Positively Invariant Cones*

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Abstract.
Given a closed convex pointed cone $\mathcal{C} \subset R^n$ which is positively invariant with respect to motions of the differential equation $\dot{x} = Ax$ (A being a real $(n \times n)$ matrix), it is proven that a necessary and sufficient condition for asymptotic stability of $\mathcal{C}$ (and therefore of the linear span of $\mathcal{C}$) is

$$\left.\begin{array}{r} Ax \in \mathcal{C} \\ x \in \mathcal{C} \end{array}\right\} \Rightarrow x = 0.$$

In case $\mathcal{C} = R^n_+$, this result yields a known equivalence from the theory of M-matrices.

1. Introduction

Consider the linear autonomous differential equation

$$\dot{x}(t) = Ax(t); \qquad t \geqslant 0, \tag{1.1}$$

where $x \in R^n$ and A is a real constant $(n \times n)$ matrix. With regard to (1.1) we require the following.

Definition 1.2. Let $\mathcal{G}$ be a nonempty subset of R^n.

(1.2.1) $\mathcal{G}$ is said to be *positively invariant* (with respect to motions of (1.1)) if for each initial state $x_0 \in \mathcal{G}$, the motion emanating from x_0 remains in $\mathcal{G}$; that is, $e^{At}\mathcal{G} \in \mathcal{G}\ \forall t \geqslant 0$.

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(1.2.2) $\mathcal{G}$ is said to be *asymptotically stable* provided that for every $x_0 \in \mathcal{G}$ we have $e^{At}x_0 \to 0$ as $t \to \infty$; in case span $(\mathcal{G}) = R^n$, this means $\operatorname{Re}\lambda < 0 \ \forall \lambda \in$ spectrum (A).

The characterization of asymptotic stability of a positively invariant set $\mathcal{G}$ containing the origin is a problem of general interest, since in certain dynamical models it may be useful to determine whether or not motions constrained to remain in $\mathcal{G}$ deteriorate to the origin. The main purposes of the present work are to obtain such a characterization (in algebraic terms) in case $\mathcal{G}$ is closed convex pointed cone, and to point out connections with results from the theory of M-matrices.

Prior to stating the main result of this note, some further terminology is needed.

Definition 1.3. Let $\mathcal{G}$ be a nonempty subset of R^n.

(1.3.1) $\mathcal{G}$ is a *cone* if $\alpha\mathcal{G} \subset \mathcal{G} \ \forall \alpha \geq 0$; that is, $\alpha g \in \mathcal{G} \ \forall \alpha \geq 0, \forall g \in \mathcal{G}$.

(1.3.2) A cone $\mathcal{G}$ is *pointed* if $\mathcal{G} \cap \{-\mathcal{G}\} = \{0\}$; that is, if $g \in \mathcal{G}$, $-g \in \mathcal{G}$, then $g = 0$.

The main result of the present work gives a characterization of asymptotic stability of closed, convex, pointed, positively invariant cones:

Theorem 1.4. *Let $\mathcal{C} \subset R^n$ be a closed convex pointed cone which is positively invariant. Then a necessary and sufficient condition for asymptotic stability of $\mathcal{C}$ (and therefore of the linear span of $\mathcal{C}$) is*

$$\left.\begin{matrix} Ax \in \mathcal{C} \\ x \in \mathcal{C} \end{matrix}\right\} \Rightarrow x = 0. \tag{1.5}$$

Theorem 1.4, and the following corollary, (in which we take $\mathcal{C} = R^n_+$, the nonnegative orthant), are proven in section 2.

Corollary 1.6. *Let A be a real $(n \times n)$ matrix whose off-diagonal entries are nonnegative. Then the following are equivalent*:

$$\operatorname{Re}\lambda < 0 \ \forall \lambda \in \text{spectrum}(A) \tag{1.7}$$

$$\left.\begin{matrix} Ax \geq 0 \\ x \geq 0 \end{matrix}\right\} \Rightarrow x = 0 \tag{1.8}$$

There exists $u \geq 0$ such that $-A^T u > 0$. (Here "$\geq$" and "$>$" hold componentwise.) (1.9)

Corollary 1.6 represents a new proof of a known result in the theory of M-matrices. In 1937 Ostrowski [5] proved that a real $(n \times n)$ matrix A with nonnegative off-diagonal entries has an inverse with all nonpositive entries if and only if $\operatorname{Re}\lambda < 0 \ \forall \lambda \in$ spectrum(A), while in 1966 Robert [6] showed that this is equivalent to (1.8) and (1.9). (For a thorough treatment of M-matrices as well as bibliographic background, the reader is referred to Berman and Plemmons [1].)

2. Proof of Main Result.

Our first task is to obtain a geometric characterization of positive invariance. To this end, we require the following specialization of a result due to Nagumo [4]; see also Yorke [8]. (The euclidean distance from a point $x \in R^n$ to a set $\mathcal{G} \subset R^n$ is denoted $d(x, \mathcal{G})$. Let $\partial\mathcal{G}$ denote the boundary of $\mathcal{G}$.)

Lemma 2.1. *Let $\mathcal{G}$ be a closed subset of R^n. Then a sufficient condition for positive invariance of $\mathcal{G}$ is*

$$\liminf_{t \to 0^+} \frac{d(x + tAx, \mathcal{G})}{t} = 0 \quad \forall x \in \partial\mathcal{G} \tag{2.2}$$

For $\mathcal{G} \subset R^n$ closed, convex, and $x \in \partial\mathcal{G}$, we denote by $\mathfrak{N}_{\mathcal{G}}(x)$ the *cone of outward normal vectors to $\mathcal{G}$ at x*; that is,

$$\mathfrak{N}_{\mathcal{G}}(x) = \{v \in R^n : \langle v, y - x \rangle \leq 0 \ \forall y \in \mathcal{G}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n. Note that $v \in \mathfrak{N}_{\mathcal{G}}(x)/\{0\}$ means that v is an (outward pointing) normal to a hyperplane which supports $\mathcal{G}$ at x. The *support cone of $\mathcal{G}$ at $x \in \partial\mathcal{G}$* is

$$\mathcal{S}_{\mathcal{G}}(x) = \{w \in R^n : \langle w, v \rangle \leq 0 \ \forall v \in \mathfrak{N}_{\mathcal{G}}(x)\}.$$

Hence $\{x + \mathcal{S}_{\mathcal{G}}(x)\}$ is the intersection of all halfspaces supporting $\mathcal{G}$ at x.

Definition 2.3. Let $\mathcal{G} \subset R^n$ be closed and convex. For $x \in \partial\mathcal{G}$, we say that a vector *$h \in R^n$ is subtangential to $\mathcal{G}$ at x* provided that $h \in \mathcal{S}_{\mathcal{G}}(x)$.

Lemma 2.4. *Let $\mathcal{G}$ be a closed convex subset of R^n. Then a necessary and sufficient condition for $\mathcal{G}$ to be positively invariant is that Ax be subtangential to $\mathcal{G}$ at each $x \in \partial\mathcal{G}$.*

Proof. (i) *Necessity.* Let $x_0 = x(0) \in \partial\mathcal{G}$. Since $x(t)$ remains in $\mathcal{G}$, it readily follows that

$$\frac{x(t) - x_0}{t} \in \mathcal{S}_{\mathcal{G}}(x_0) \quad \forall t > 0.$$

Letting $t \to 0^+$, we obtain $\dot{x}(0) = Ax_0 \in \mathcal{S}_{\mathcal{G}}(x_0)$.

(ii) *Sufficiency.* In view of Lemma 2.1, it is enough to show that subtangentiality of Ax to $\mathcal{G}$ at x for each $x \in \partial\mathcal{G}$ implies (2.2). (Actually, the reverse implication holds as well, but is not required.) Suppose (2.2) did not hold. Then for some $x \in \partial\mathcal{G}$ there exist $\varepsilon > 0$ and a sequence $\{t_i\}_{i=1}^{\infty}$, $t_i \to 0^+$, such that $d(x + t_i Ax, \mathcal{G}) \geq \varepsilon t_i$. Denote the closed ball of radius ρ centered at $z \in R^n$ by $B(z, \rho)$. We then have

$$\left\{x + \bigcup_{i=1}^{\infty} B(t_i Ax, t_i \varepsilon)\right\} \cap \mathcal{G} = \phi \tag{2.5}$$

We claim that

$$\left\{x+\bigcup_{t>0} B(tAx,t\varepsilon)\right\}\cap\mathcal{G}=\phi \tag{2.6}$$

Indeed, if $\tilde{t}>0$ was such that $\{x+B(\tilde{t}Ax,\tilde{t}\varepsilon)\}\cap\mathcal{G}\neq\phi$, then convexity of $\mathcal{G}$ implies $\{x+B(tAx,t\varepsilon)\}\cap\mathcal{G}\neq\phi$ $\forall t\in[0,\tilde{t}]$, which violates (2.5). Now, in view of (2.6), $Ax\notin\mathcal{S}_{\mathcal{G}}(x)$, contradicting subtangentiality. □

We shall require the following elementary results on subtangentiality to cones:

Lemma 2.7. *Let $\mathcal{C}\subset R^n$ be a closed convex cone, and let $x\in\partial\mathcal{C}$. Then $\langle v,x\rangle=0$ $\forall v\in\mathfrak{N}_{\mathcal{C}}(x)$.*

Proof. Let $v\in\mathfrak{N}_{\mathcal{C}}(x)$ and $\alpha>0$. Then $\alpha x\in\mathcal{C}$, $2\alpha x\in\mathcal{C}$, and therefore

$$\langle v,\alpha x-x\rangle\leqslant 0;\ \langle v,2\alpha x-x\rangle\leqslant 0.$$

Taking $\alpha=2/3$, we obtain $-1/3\langle v,x\rangle\leqslant 0$ and $1/3\langle v,x\rangle\leqslant 0$, whence $\langle v,x\rangle=0$. □

Lemma 2.8. *A vector y is subtangential to a closed convex cone $\mathcal{C}$ at $x\in\partial\mathcal{C}$ if and only if $y+z$ is subtangential to $\mathcal{C}$ at x $\forall z\in\mathcal{C}$.*

Proof. The "if" is immediate, upon taking $z=0$. To verify the "only if", let $v\in\mathfrak{N}_{\mathcal{C}}(x)$ and $z\in\mathcal{C}$, whence $\langle v,z-x\rangle\leqslant 0$. Since by Lemma 2.7, $\langle v,x\rangle=0$, $\langle v,y-x\rangle\leqslant 0$ implies $\langle v,y+z\rangle\leqslant 0$. □

We shall also make use of the following lemma due to 0. Hájek [2] and Schneider and Vidyasagar [7], which generalizes the well-known result that a nonnegative matrix possesses a nonnegative eigenvector:

Lemma 2.9. *Let $\mathcal{C}\subset R^n$ be a closed, convex, pointed and positively invariant cone. Then $\mathcal{C}$ contains an eigenvector of A.*

We are now in position to prove our main result.

Proof of Theorem 1.4. (i) *Necessity.* Assume that $\mathcal{C}$ is asymptotically stable and that there exists $\hat{x}\in\mathcal{C}/\{0\}$ such that $A\hat{x}\in\mathcal{C}$. Let $w\in\partial\{\hat{x}+\mathcal{C}\}$. Then $w=\hat{x}+x$, $x\in\partial\mathcal{C}$. Lemma 2.4 implies that Ax is subtangential to $\mathcal{C}$ at x, and since $A\hat{x}\in\mathcal{C}$, Lemma 2.8 implies $Aw=A\hat{x}+Ax$ is subtangential to $\mathcal{C}$ at x. Hence Aw is subtangential to $\{\hat{x}+\mathcal{C}\}$ at w (since the normal cone does not change under a shift). Lemma 2.4 now implies that $\{\hat{x}+\mathcal{C}\}$ is positively invariant. Since $\mathcal{C}$ is pointed it follows that $0\notin\{\hat{x}+\mathcal{C}\}$. Closedness of $\{\hat{x}+\mathcal{C}\}$ therefore implies $e^{At}\hat{x}\nrightarrow 0$ as $t\to\infty$, contradicting asymptotic stability of $\mathcal{C}$.

(ii) *Sufficiency.* Suppose (1.5) holds, but that $\mathcal{C}$ was not asymptotically stable. Consider the nonempty set

$$\mathcal{Z}=\{x\in\mathcal{C}: e^{At}x\nrightarrow 0 \text{ as } t\to\infty\}.$$

In view of the structure of solutions to (1.1) ([3,p.135]), $x \in \mathfrak{Z}$ implies that there exists $\varepsilon > 0$ such that $e^{At}x \notin B(0, \varepsilon)$ $\forall t \geq 0$. Note that $\overline{\mathfrak{Z}}$ (the closure of $\mathfrak{Z}$) is a closed subcone of the pointed cone $\mathcal{C}$, and is therefore itself pointed. To see that $\overline{\mathfrak{Z}}$ is convex, let x^1 and x^2 be any points in $\mathfrak{Z}$. Then there exists $\hat{\varepsilon} > 0$ such that $e^{At}x_i \notin B(0, \hat{\varepsilon})$ $\forall t \geq 0$, $i = 1,2$. Since $\mathcal{C}$ is pointed, there exists a hyperplane H with associated open halfspaces H^+, H^- such that $\{0\} \neq \{H^- \cap \mathcal{C}\} \subset B(0, \varepsilon)$. From positive invariance of $\mathcal{C}$ it then follows that for $i = 1,2$ we have $e^{At}x_i \in \overline{H^+ \cap \mathcal{C}}$ $\forall t \geq 0$. If $x = \lambda x^1 + (1 - \lambda)x^2$ for $\lambda \in [0, 1]$, then convexity of $\overline{H^+ \cap \mathcal{C}}$ implies $e^{At}x \in \overline{H^+ \cap \mathcal{C}}$ $\forall t \geq 0$, whence $\mathfrak{Z}$ and therefore $\overline{\mathfrak{Z}}$ are convex. Note that $\overline{\mathfrak{Z}}$ is invariant with respect to motions of (1.1), since $\mathfrak{Z}$ clearly is.

Upon applying Lemma 2.9 to $\mathcal{C} = \overline{\mathfrak{Z}}$, we conclude that there exists $\bar{x} \in \overline{\mathfrak{Z}}/\{0\}$ such that $A\bar{x} = \lambda \bar{x}$ for some real λ. The definition of $\mathfrak{Z}$ implies $\lambda \geq 0$. But then $A\bar{x} \in \mathcal{C}$, violating (1.5). □

Remark 2.10. With regard to Theorem 1.4, it is worthwhile to investigate the roles played by the hypotheses of positive invariance and pointedness.

(2.10.1) If $\mathcal{C}$ is a closed convex pointed cone *not positively invariant*, then neither the necessity nor the sufficiency parts of the theorem need be true. For a breakdown of the necessity part, consider

$$A = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}; \qquad \mathcal{C} = \text{conic hull}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} \right\}.$$

Then $\mathcal{C}$ is asymptotically stable and (1.5) is violated at $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. A sufficiency breakdown is illustrated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \qquad \mathcal{C} = \text{conic hull}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Here (1.5) holds, but $\mathcal{C}$ is not asymptotically stable.

(2.10.2) In Theorem 1.4 the role of closedness is readily seen to be vital by considering the example with $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$\mathcal{C} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 > 0 \right\} \cup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

(2.10.3) Pointedness of $\mathcal{C}$ is vital to the necessity part of the theorem; consider the case $A = -I$ and

$$\mathcal{C} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \geq 0 \right\}.$$

In fact, we have

(2.10.4) *If* $\mathcal{C} \subset R^n$ *is a nonpointed closed convex cone which is positively invariant, then* (1.5) *is impossible; that is, there exists* $x \in \mathcal{C}/\{0\}$ such that $Ax \in \mathcal{C}$.

Outline of proof. Nonpointedness implies that $\mathcal{C}$ contains a nonzero subspace; denote by $\mathcal{M}$ the maximal subspace contained in $\mathcal{C}$. Let $x \in \mathcal{M}/\{0\}$. Then for Ax to be subtangential to $\mathcal{C}$ at x, we must have $Ax = a^1 + a^2$ where $a^1 \in \mathcal{M}$ and a^2 is contained in the (pointed) cone $P\mathcal{C}$, P being the projection of $\mathcal{C}$ along $\mathcal{M}$ onto some direct sum complement of $\mathcal{M}$. Hence $Ax \in \mathcal{C}$. □

Finally, we will prove the corollary of Theorem 1.4 which was given in section 1.

Proof of Corollary 1.6. The assumptions on A imply that Ax is subtangential to R^n_+ $\forall x \in \partial R^n_+$, whence R^n_+ is positively invariant. In view of Theorem 1.4, asymptotic stability of R^n_+, and therefore of R^n, is equivalent to (1.5). Now recall that asymptotic stability of R^n is equivalent to negativity of the real part of the spectrum of A, and note that for $\mathcal{C} = R^n_+$, (1.5) becomes (1.8). It remains to show that (1.8) is equivalent to (1.9). To this end, consider the following linear programming problem, for given $y \in R^n$:

$$\begin{aligned}&\text{maximize } \langle y, x \rangle \\ &\text{subject to } Ax \geqslant 0,\ x \geqslant 0\end{aligned} \qquad (P_y)$$

Note that (1.8) holds if and only if (P_y) has a bounded maximum (necessarily zero) for every $y \geqslant 0$. From the duality theory of linear programming we then conclude that (1.8) is equivalent to "dual feasibility" holding for each $y \geqslant 0$; that is, for each $y \geqslant 0$ there exists $u_y \geqslant 0$ such that

$$-y - A^T u_y \geqslant 0$$

(where "T" denotes transpose). This is equivalent to the existence of $u \geqslant 0$ such that $-A^T u > 0$. □

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