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Zero-sum Markov Games with Stopping and Impulsive Strategies

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Abstract. Three kinds of zero-sum Markov games with stopping and impulsive strategies are considered. For these games we find the saddle point strategies and prove that, the value of the game depends continuously on the initial state.

In the paper the following three kinds of zero-sum, two-person Markov games are considered: games with stopping, games with impulsive controls and games with impulsive control and stopping. These games do not exhaust the variety of the zero-sum games with stopping and impulsive strategies but are typical in this theory.

The form of the associated cost functional depends on the kind of game. For instance, for the game of the first type the functional is of the form:

$$J_x(\tau,\delta) = E_x\left\{\int_0^{\tau\wedge\delta} e^{-\alpha s} f(x_s)\,ds + \chi_{\tau<\delta} e^{-\alpha\tau}\Psi_1(x_\tau) + \chi_{\delta\leqslant\tau} e^{-\alpha\delta}\Psi_2(x_\delta)\right\}$$

If τ and δ are stopping times chosen by the first and second player respectively, then the first player pays to the second one in average the total amount equal to $J_x(\tau,\delta)$.

A similar, but a more complicated functional is in the case of games with impulsive controls, and games with impulsive control and stopping.

Under assumptions introduced by M. Robin in [8], we prove the existence of the saddle point strategies for these games. We show also that the values of the games depend continuously on the initial state. This way we generalize Bismut results contained in [2] and solve a problem posed by M. Robin in [8].

An analogous game to the one with impulsive controls was considered independently and under stronger assumptions by J. P. Lepeltier in [6].

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1. Stochastic Game with Optimal Stopping

1.1. Introduction

Let (E, B) be a locally compact, with countable base state space endowed with the Borel σ-field B, and $X=(x_t, F_t, P_x)$ a right continuous, homogeneous Markov process on the space E.

By C we will denote the space of all continuous, bounded functions on E, by C_0 the sub-space of C of functions vanishing at infinity and by $(\Phi(t))_{t\geqslant 0}$ the Markov semigroup associated with the process X. We will assume

(A1) $\forall f\in C, \quad t\geqslant 0 \qquad \Phi(t)f\in C$

(A2) $\forall f\in C_0, \quad t\geqslant 0 \qquad \Phi(t)f\in C_0$

(A3) if d denotes metric compatible with the topology on E, such that every ball is compact (it is well known that such metric exists), then

$$\forall T > 0 \; \gamma_T(R) \stackrel{\text{def}}{=} \sup_{x\in E} P_x\Big\{ \sup_{0\leqslant t\leqslant T} d(x_t, x) > R \Big\} \to 0 \text{ as } R \to \infty$$

Let us consider the following two person zero-sum game. The players choose as their strategies Markov times. If τ and δ are the strategies of the first and second player respectively, then the first pays to the second one the total amount equal to $J_x(\tau, \delta)$. The functional $J_x(\tau, \delta)$ has the form

$$J_x(\tau,\delta) = E_x\Big\{ \int_0^{\tau\wedge\delta} e^{-\alpha s} f(x_s)\, ds + \chi_{\tau<\delta} e^{-\alpha\tau}\Psi_1(x_\tau) + \chi_{\delta\leqslant\tau} e^{-\alpha\delta}\Psi_2(x_\delta) \Big\} \tag{1.1}$$

where α is a positive constant, and functions f, Ψ_1, Ψ_2 satisfy the following assumption

(A4) $f, \Psi_1, \Psi_2 \in C, \qquad \forall x \in E \quad \Psi_1(x) \geqslant \Psi_2(x)$

We define lower and upper value of our game, $\underline{v}$ and $\bar{v}$ respectively

$$\underline{v}(x) = \sup_\delta \inf_\tau J_x(\tau, \delta) \qquad \bar{v}(x) = \inf_\tau \sup_\delta J_x(\tau,\delta) \tag{1.2}$$

Under the above 4 assumptions we will prove that the game has value v: $v = \underline{v} = \bar{v}$ and that the value is continuous function. Moreover we will find the saddle point strategy (τ, δ), such that $v(x) = J_x(\tau, \delta)$. These results will be obtained with the aid of the penalty method, introduced for stopping games by N. V. Krylov [4,5]. This method consists in finding for each $\beta > 0$ a solution of the following penalized equation

$$v^{\beta} = R_{\alpha}\left[f-\beta\left(v^{\beta}-\Psi_1\right)^{+}+\beta\left(v^{\beta}-\Psi_2\right)^{-}\right] \tag{1.3}$$

and showing that $v^{\beta} \to v$ as $\beta \to \infty$.

Zero sum stochastic game with optimal stopping appeared first in Dynkin's paper [3] for stochastic sequences. Continuous time analog of this game was considered first by Krylov in [4, 5].

From the point of view of variational inequalities and for diffusion processes such games were investigated by Bensoussan and Friedman in [1]. Bismut in [2] studied similar games using results from convex analysis. In the Part 1 we present similar results to those obtained in [1, 2, 4, 5], but proved under very general conditions (A.1), (A.2), (A.3), (A.4) introduced by Robin in [8].

1.2. The Main Result

First we have to establish solvability of the penalized equation (1.3).

Proposition 1. *If the assumption* (A1) *and* (A4) *are satisfied then exists unique solution* v^{β} *of the equation* (1.3). *Moreover,* $v^{\beta} \in C \cap D_{\tilde{A}}$, *where* $D_{\tilde{A}}$ *denotes the domain of weak infinitesimal operator* $\tilde{A}$.

Proof. Proof is almost identical with the proof of Corollary 1 [10] where the general case of Dynkin's game is considered. Namely from the Lemma 1 [9] we know that (1.3) is equivalent to

$$v^{\beta} = R_{\alpha+\beta}\left[f-\beta\left(v^{\beta}-\Psi_1\right)^{+}+\beta\left(v^{\beta}-\Psi_2\right)^{-}+\beta v^{\beta}\right] \tag{1.4}$$

Since the function $F(x) = a-\beta(x-b)^{+}+\beta(x-c)^{-}+\beta x$ for $b \geqslant c$ is lipschitzian with the parameter β, the equation (1.4) can be solved with the aid of Banach principle. □

Now, our aim is to obtain the existence of the value v of the game. The next two lemmas will play an important role in the proof of this fact.

Lemma 1. *The solution* v^{β} *of the penalized equation* (1.3) *has the following form*

$$\begin{aligned} v^{\beta}(x) &= \inf_{u^1 \in M_{\beta}} \sup_{u^2 \in M_{\beta}} I_x(u^1, u^2) = \sup_{u^2 \in M_{\beta}} \inf_{u^1 \in M_{\beta}} I_x(u^1, u^2) \\ &= I_x(\hat{u}^{1\beta}, \hat{u}^{2\beta}) \end{aligned} \tag{1.5}$$

where M_{β} *is a family of progressively measurable processes* $(u_s)_{s \geqslant 0}$ *with values from the interval* $[0, \beta]$ *and*

$$\begin{aligned} I_x(u^1, u^2) = E_x \int_0^{\infty} &\exp\left(-\int_0^t \left(\alpha+u_s^1+u_s^2\right) ds\right) \\ &\times \left[f(x_t)+u_t^1\Psi_1(x_t)+u_t^2\Psi_2(x_t)\right] dt \end{aligned}$$

and

$$\hat{u}_s^{1\beta} = \begin{cases} \beta & \text{if } v^\beta(x_s) > \Psi_1(x_s) \\ 0 & \text{if } v^\beta(x_s) \leq \Psi_1(x_s) \end{cases} \qquad \hat{u}_s^{2\beta} = \begin{cases} \beta & \text{if } v^\beta(x_s) < \Psi_2(x_s) \\ 0 & \text{if } v^\beta(x_s) \geq \Psi_2(x_s) \end{cases}. \tag{1.6}$$

Lemma 2. *The following two identities are true*

$$\begin{aligned} v^\beta(x) &= \inf_\tau \sup_\delta E_x\Big\{ \int_0^{\tau \wedge \delta} e^{-\alpha t} f(x_t)\,dt + \chi_{\tau<\delta} e^{-\alpha\tau}\Psi_1(x_\tau) + \chi_{\delta \leq \tau} e^{-\alpha\delta}\Psi_2(x_\delta) \\ &\qquad + \chi_{\tau<\delta} e^{-\alpha\tau}\big(v^\beta - \Psi_1\big)^+(x_\tau) - \chi_{\delta\leq\tau} e^{-\alpha\delta} \\ &\qquad \times \big(v^\beta - \Psi_2\big)^-(x_\delta)\Big\} \\ &= \sup_\delta \inf_\tau E_x\Big\{ \int_0^{\tau \wedge \delta} e^{-\alpha t} f(x_t)\,dt + \chi_{\tau<\delta} e^{-\alpha\tau}\Psi_1(x_\tau) + \chi_{\delta \leq \tau} e^{-\alpha\delta}\Psi_2(x_\delta) \\ &\qquad + \chi_{\tau<\delta} e^{-\alpha\tau}\big(v^\beta - \Psi_1\big)^+(x_\tau) - \chi_{\delta\leq\tau} e^{-\alpha\delta}\big(v^\beta - \Psi_2\big)^-(x_\delta)\Big\} \end{aligned} \tag{1.7}$$

First lemma is a Markov analog of Corollary 2 [10] and the second one is proved in the second part of proof of Theorem 3 [10]. Using these lemmas we show the main theorem.

Theorem 1. *Suppose the assumptions* (A1)–(A4) *are satisfied. Then there exists the value of the game v, which is the continuous function and $v^\beta \to v$ as $\beta \to \infty$ uniformly on each compact. Moreover the times $\hat{\tau}, \hat{\delta}$*

$$\hat{\tau} = \inf\{s \geq 0 \colon v(x_s) = \Psi_1(x_s)\} \qquad \hat{\delta} = \inf\{s \geq 0 \colon v(x_s) = \Psi_2(x_s)\} \tag{1.8}$$

are the saddle stopping times, that is,

$$v(x) = J_x(\hat{\tau}, \hat{\delta})$$

(We suppose that in (1.8) infimum over empty set is equal $+\infty$).

Proof. We will prove only the first part of the theorem since the second one is almost identical with the proof of existence of the optimal stopping time in [8]. So we need to establish the convergence of v^β to the value of the game v. Similarly as M. Robin in the proof of Theorem I 2 [8] we will base on the fact of density in C_0 of the domain D_A of infinitesimal operator A in C_0, and we consider three cases:

$$1)\ \Psi_1, \Psi_2 \in D_A, \quad 2)\ \Psi_1, \Psi_2 \in C_0, \quad \text{and} \quad 3)\ \Psi_1, \Psi_2 \in C.$$

Step 1. Suppose $\Psi_1, \Psi_2 \in D_A$. Then since

$$\alpha v^{\beta} - A v^{\beta} = f - \beta\left(v^{\beta} - \Psi_1\right)^{+} + \beta\left(v^{\beta} - \Psi_2\right)^{-}$$

and

$$\alpha\left(v^{\beta} - \Psi_2\right) - A\left(v^{\beta} - \Psi_2\right) = f - \alpha\Psi_2 + A\Psi_2 - \beta\left(v^{\beta} - \Psi_1\right)^{+} + \beta\left(v^{\beta} - \Psi_2\right)^{-}$$

we have

$$\hat{v}^{\beta} \stackrel{\text{def}}{=} v^{\beta} - \Psi_2 = R_{\alpha}\left(g - \beta\left(\hat{v}^{\beta} + \Psi_2 - \Psi_1\right)^{+} + \beta\left(\hat{v}^{\beta}\right)^{-}\right) \tag{1.9}$$

where $g \stackrel{\text{def}}{=} f - \alpha\Psi_2 + A\Psi_2$.

So from Lemma 2 we obtain

$$\hat{v}^{\beta}(x) = \sup_{u^2 \in M_{\beta}} \inf_{u^1 \in M_{\beta}} E_x\left\{\int_0^{\infty} \exp\left(-\int_0^s \left(\alpha + u_r^1 + u_r^2\right) dr\right)\left(g(x_s) + u_s^1\left(\Psi_1(x_s) - \Psi_2(x_s)\right)\right) ds\right\} \tag{1.10}$$

and therefore

$$\hat{v}^{\beta}(x) \geqslant \frac{-c}{\alpha + \beta} \tag{1.11}$$

where c is constant such that $\|g\| \leqslant c$.

On the other hand, analogously

$$\bar{v}^{\beta}(x) \stackrel{\text{def}}{=} v^{\beta}(x) - \Psi_1(x) = \inf_{u^1 \in M_{\beta}} \sup_{u^2 \in M_{\beta}} E_x\left\{\int_0^{\infty} \exp\left(-\int_0^s \left(\alpha + u_r^1 + u_r^2\right) dr\right)\left(g'(x_s) + u_s^2\left(\Psi_2(x_s) - \Psi_1(x_s)\right)\right) ds\right\} \leqslant \frac{c_1}{\alpha + \beta} \tag{1.12}$$

where $g' \stackrel{\text{def}}{=} f - \alpha\Psi_1 + A\Psi_1$ and $\|g'\| \leqslant c_1$.

Thus summarizing (1.7), (1.11), and (1.12) the following two approximations are true

$$\left\|v^{\beta} - \inf_{\tau} \sup_{\delta} J(\tau, \delta)\right\| \leqslant \frac{\max\{c, c_1\}}{\alpha + \beta} \tag{1.13}$$

and

$$\left\| v^{\beta} - \sup_{\delta} \inf_{\tau} J(\tau, \delta) \right\| \leqslant \frac{\max\{c, c_1\}}{\alpha + \beta} \tag{1.14}$$

If β tends to $+\infty$ we obtain the assertion of the theorem in the case 1.

Step 2. Let Ψ_1 and Ψ_2 belong to C_0. Since D_A is dense in C_0, then there exist two sequences $(\Psi_1^n)_{n=1,2\dots}$ and $(\Psi_2^n)_{n=1,2,\dots}$ of functions from C_0 such that

$$\Psi_1^n \to \Psi_1, \quad \Psi_2^n \to \Psi_1 \qquad \text{in } C \text{ as } n \to \infty$$

and

$$\forall n \ \Psi_1^n, \Psi_2^n \in D_A, \qquad \Psi_1^n \geqslant \Psi_2^n$$

Let us define

$$J_x^n(\tau, \delta) = E_x\left\{ \int_0^{\tau \wedge \delta} e^{-\alpha s} f(x_s)\, ds + \chi_{\tau < \delta} e^{-\alpha\tau} \Psi_1^n(x_\tau) + \chi_{\delta \leqslant \tau} e^{-\alpha\delta} \Psi_2^n(x_\delta) \right\} \tag{1.15}$$

Since

$$\|J_x(\tau, \delta) - J_x^n(\tau, \delta)\| \leqslant \|\Psi_1 - \Psi_1^n\| + \|\Psi_2 - \Psi_2^n\|$$

then

$$\left\| \inf_{\tau} \sup_{\delta} J_x(\tau, \delta) - \inf_{\tau} \sup_{\delta} J_x^n(\tau, \delta) \right\| \leqslant \|\Psi_1 - \Psi_1^n\| + \|\Psi_2 - \Psi_2^n\| \tag{1.16}$$

From the Step 1 we know that for each $n = 1, 2, \dots$

$$\left\| v_n^{\beta} - \inf_{\tau} \sup_{\delta} J^n(\tau, \delta) \right\| \to 0 \quad \text{as} \quad \beta \to \infty \tag{1.17}$$

where

$$v_n^{\beta}(x) = \inf_{u^1 \in M_\beta} \sup_{u^2 \in M_\beta} E_x\left\{ \int_0^{\infty} \exp\left(-\int_0^s (\alpha + u_r^1 + u_r^2)\, dr \right) \left(f(x_s) + u_s^1 \Psi_1^n(x_s) + u_s^2 \Psi_2^n(x_s) \right) ds \right\}$$

Moreover,

$$\begin{aligned}|v^\beta(x)-v_n^\beta(x)| &\leq \sup_{u^1\in M_\beta}\sup_{u^2\in M_\beta} E_x\left\{\int_0^\infty \exp\left(-\int_0^s\left(\alpha+u_r^1+u_r^2\right)dr\right)\right.\\ &\qquad\left.\times\left(u_s^1|\Psi_1(x_s)-\Psi_1^n(x_s)|+u_s^2|\Psi_2(x_s)-\Psi_2^n(x_s)|\right)ds\right\} \\ &\leq \|\Psi_1-\Psi_1^n\|+\|\Psi_2-\Psi_2^n\|\end{aligned} \tag{1.18}$$

Summarizing (1.16), (1.17), and (1.18) we obtain

$$\begin{aligned}\left\|\inf_\tau\sup_\delta J_x(\tau,\delta)-v^\beta\right\| &\leq \left\|\inf_\tau\sup_\delta J_x(\tau,\delta)-\inf_\tau\sup_\delta J_x^n(\tau,\delta)\right\| \\ &\quad+\left\|\inf_\tau\sup_\delta J_x^n(\tau,\delta)-v_n^\beta\right\|+\|v_n^\beta-v^\beta\| \\ &\leq 2(\|\Psi_1-\Psi_1^n\|+\|\Psi_2-\Psi_2^n\|) \\ &\quad+\left\|v_n^\beta-\inf_\tau\sup_\delta J_x^n(\tau,\delta)\right\|\end{aligned}$$

Passing with $\beta\to+\infty$ and then with $n\to+\infty$ we have the convergence

$$v^\beta\to\inf_\tau\sup_\delta J(\tau,\delta) \qquad \text{in } C \text{ as } \beta\to\infty$$

The analogous considerations lead to the convergence

$$v^\beta\to\sup_\delta\inf_\tau J(\tau,\delta) \qquad \text{in } C \text{ as } \beta\to\infty$$

Step 3. Suppose now that $\Psi_1,\Psi_2\in C$. As we know, there exists an increasing family of compacts K_n such that $\bigcup_{n=1}^{\infty} K_n=E$.

Let for each $n=1,2,\ldots,$ $h_n\colon E\to[0,1]$ be a continuous function with compact support, equal 1 on the set K_n. We put

$$\Psi_1^n(x)\stackrel{\text{def}}{=}h_n(x)\cdot\Psi_1(x) \qquad \Psi_2^n(x)\stackrel{\text{def}}{=}h_n(x)\cdot\Psi_2(x)$$

Obviously

$$\forall n\ \|\Psi_1^n\|\leq\|\Psi_1\| \quad \|\Psi_2^n\|\leq\|\Psi_2\|, \qquad \Psi_1^n\geq\Psi_2^n,\ \Psi_1^n \text{ and } \Psi_2^n\in C_0$$

and $\Psi_1^n\to\Psi_1$, $\Psi_2^n\to\Psi_2$ uniformly on each compact as $n\to\infty$. In the second step we proved that

$$\forall n \;\; v_n^\beta \to v_n = \inf_\tau \sup_\delta J^n(\tau,\delta) \text{ uniformly as } \beta \to \infty \tag{1.19}$$

Now we will show that $v_n \to v = \inf_\tau \sup_\delta J(\tau,\delta)$ uniformly on each compact.

Let $K = K(x_0, R_1) = \{x\colon d(x_0, x) \leqslant R_1\}$ be a closed ball with center in the point x_0 and the radius equal R_1. For $T > 0$, $R > 0$ we define the sets:

$$D = \left\{\omega\colon \sup_{0 \leqslant t \leqslant T} d(x_t(\omega), x) \leqslant R\right\}$$

$$\tilde{K} = \{y\colon d(x_0, y) \leqslant R_1 + R\} = K(x_0, R_1 + R)$$

We have

$$\begin{aligned}|J_x(\tau,\delta) - J_x^n(\tau,\delta)| \leqslant &\big[E_x\chi_{\tau\leqslant T}\chi_{\tau<\delta}e^{-\alpha\tau}|\Psi_1(x_\tau) - \Psi_1^n(x_\tau)| \\ &+ E_x\chi_{\delta\leqslant T}\chi_{\delta\leqslant\tau}e^{-\alpha\delta}|\Psi_2(x_\delta) - \Psi_2^n(x_\delta)|\big] \\ &+ 2e^{-\alpha T}(\|\Psi_1\| + \|\Psi_2\|) = \mathrm{I} + \mathrm{II}\end{aligned} \tag{1.20}$$

Further,

$$\begin{aligned}\mathrm{I} = &E_x\chi_D\big[\chi_{\tau\leqslant T}\chi_{\tau<\delta}e^{-\alpha\tau}|\Psi_1(x_\tau) - \Psi_1^n(x_\tau)| \\ &+ \chi_{\delta\leqslant T}\chi_{\delta\leqslant\tau}e^{-\alpha\delta}|\Psi_2(x_\delta) - \Psi_2^n(x_\delta)|\big] \\ &+ E_x\chi_{\Omega\setminus D}\big[\chi_{\tau\leqslant T}\chi_{\tau<\delta}e^{-\alpha\tau}|\Psi_1(x_\tau) - \Psi_1^n(x_\tau)| + \chi_{\delta\leqslant T}\chi_{\delta\leqslant\tau}e^{-\alpha\delta} \\ &\times |\Psi_2(x_\delta) - \Psi_2^n(x_\delta)|\big] = \mathrm{III} + \mathrm{IV}\end{aligned} \tag{1.21}$$

From the definition of $\gamma_T(R)$ (see the assumption (A3)) we obtain

$$\mathrm{IV} \leqslant 2\gamma_T(R)(\|\Psi_1\| + \|\Psi_2\|) \tag{1.22}$$

Moreover,

$$\mathrm{III} \leqslant \sup_{y\in\tilde{K}} |\Psi_1(y) - \Psi_1^n(y)| + \sup_{y\in\tilde{K}} |\Psi_2(y) - \Psi_2^n(y)| \tag{1.23}$$

uniformly with respect to $x \in K$. Thus (1.20), (1.21), (1.22) imply

$$\begin{aligned}\sup_{x\in K} |J_x(\tau,\delta) - J_x^n(\tau,\delta)| \leqslant \mathrm{II} + \mathrm{III} + \mathrm{IV} \leqslant &2\big(e^{-\alpha T} + \gamma_T(R)\big) \\ &\times (\|\Psi_1\| + \|\Psi_2\|) + \sup_{y\in\tilde{K}} |\Psi_1(y) - \Psi_1^n(y)| \\ &+ \sup_{y\in\tilde{K}} |(\Psi_2(y) - \Psi_2^n(y)|\end{aligned}$$

and since the right hand side is independent on τ and δ we have

$$\sup_{x\in K}\left|\inf_{\tau}\sup_{\delta} J_x(\tau,\delta)-\inf_{\tau}\sup_{\delta} J_x^n(\tau,\delta)\right|$$
$$\leqslant 2\left(e^{-\alpha T}+\gamma_T(R)\right)\left(\|\Psi_1\|+\|\Psi_2\|\right) \tag{1.24}$$
$$+\sup_{y\in\tilde{K}}\left|\Psi_1(y)-\Psi_1^n(y)\right|+\sup_{y\in\tilde{K}}\left|\Psi_2(y)-\Psi_2^n(y)\right|$$

Now we see that since

$$\sup_{x\in K}\left|\inf_{\tau}\sup_{\delta} J_x(\tau,\delta)-v^\beta(x)\right|$$
$$\leqslant \sup_{x\in K}\left|\inf_{\tau}\sup_{\delta} J_x(\tau,\delta)-\inf_{\tau}\sup_{\delta} J_x^n(\tau,\delta)\right| \tag{1.25}$$
$$+\left\|\inf_{\tau}\sup_{\delta} J^n(\tau,\delta)-v_n^\beta\right\|+\sup_{x\in K}\left|v_n^\beta(x)-v^\beta(x)\right|$$

we have to estimate the last term of this inequality. Using the similar considerations as in the proof of inequalities (1.8), (1.20)–(1.23) we obtain for $x\in K$

$$\left|v^\beta(x)-v_n^\beta(x)\right|\leqslant \sup_{u^1\in M_\beta}\sup_{u^2\in M_\beta} E_x\left\{\int_0^\infty \exp\left(-\int_0^s\left(\alpha+u_r^1+u_r^2\right)dr\right)\right.$$
$$\left.\times\left[u_s^1\left|\Psi_1(x_s)-\Psi_1^n(x_s)\right|+u_s^2\left|\Psi_2(x_s)-\Psi_2^n(x_s)\right|\right]ds\right\}$$
$$=\sup_{u^1\in M_\beta}\sup_{u^2\in M_\beta}\left[E_x\left\{\int_0^T\chi_D(\cdots)\,ds\right\}+E_x\left\{\int_0^T\chi_{\Omega\setminus D}(\cdots)\,ds\right\}\right.$$
$$\left.+E_x\left\{\int_T^\infty(\cdots)\,ds\right\}\right]$$
$$\leqslant \sup_{y\in\tilde{K}}\left|\Psi_1(y)-\Psi_1^n(y)\right|+\sup_{y\in\tilde{K}}\left|\Psi_2(y)-\Psi_2^n(y)\right|$$
$$+2\left(e^{-\alpha T}+\gamma_T(R)\right)\left(\|\Psi_1\|+\|\Psi_2\|\right) \tag{1.26}$$

The last approximation is uniform with respect to $x\in K$. Thus summarizing (1.24), (1.25), and (1.26) we have

$$\sup_{x\in K}\left|\inf_{\tau}\sup_{\delta} J_x(\tau,\delta)-v^\beta(x)\right|\leqslant 4\left(e^{-\alpha T}+\gamma_T(R)\right)\left(\|\Psi_1\|+\|\Psi_2\|\right)$$
$$+2\sup_{y\in\tilde{K}}\left|\Psi_1(y)-\Psi_1^n(y)\right|$$
$$+2\sup_{y\in\tilde{K}}\left|\Psi_2(y)-\Psi_2^n(y)\right|$$
$$+\left\|\inf_{\tau}\sup_{\delta} J^n(\tau,\delta)-v_n^\beta\right\|$$

Going to the limits with $\beta \to \infty$, $n \to \infty$, $R \to \infty$ and finally with $T \to \infty$ we obtain

$$\lim_{\beta \uparrow \infty} \sup_{x \in K} \left| \inf_{\tau} \sup_{\delta} J_x(\tau, \delta) - v^{\beta}(x) \right| = 0$$

Changing the role of operators inf and sup we have the same convergence for the lower value of the game.

Thus v^{β} tends simultaneously to the lower and upper value of the game. So v^{β} tends uniformly on each compact to the value of the game.

1.3. Some Additional Results

The aim of this point is to introduce some new characterizations associated with Dynkin's problem. These results will be useful for the game with impulsive controls. The first theorem is an analog of Theorem I 32. [8], which put the interpretation on optimal reward in stopping as the maximum element in certain class of functions.

Theorem 2. *Suppose the assumptions* (A1)–(A4) *are satisfied. Then the value of the game v is the greatest, bounded, continuous function w satisfying the inequalities*

$$\begin{cases} w(x) \leqslant \Psi_1(x) \\ w(x) \leqslant E_x\left\{ \int_0^{\hat{\delta} \wedge t} e^{-\alpha s} f(x_s)\, ds + e^{-\alpha \hat{\delta} \wedge t} w(x_{\hat{\delta} \wedge t}) \right\} \\ \text{for } t \geqslant 0 \end{cases} \tag{1.27}$$

where $\hat{\delta} = \hat{\delta}(w) = \inf\{s \geqslant 0 \colon w(x_s) \leqslant \Psi_2(x_s)\}$

On the other hand, v is the least bounded, continuous function w satisfying the inequalities

$$\begin{cases} w(x) \geqslant \Psi_2(x) \\ w(x) \geqslant E_x\left\{ \int_0^{\hat{\tau} \wedge t} e^{-\alpha s} f(x_s)\, ds + e^{-\alpha \hat{\tau} \wedge t} w(x_{\hat{\tau} \wedge t}) \right\} \\ \text{for } t \geqslant 0 \end{cases} \tag{1.28}$$

where $\hat{\tau} = \hat{\tau}(w) = \inf\{s \geqslant 0 \colon w(x_s) \geqslant \Psi_1(x_s)\}$

The proof of this theorem is similar to the one of Theorem I 3.2 [8] and can be found in [11].

Sometimes, particularly in the case of games with impulsive control, inequality $\Psi_1 \geqslant \Psi_2$ is not satisfied. This situation consider the following

Theorem 3. *Let $f, \Psi_1, \Psi_2 \in C$ and the assumptions* (A1)–(A3) *are satisfied. Then zero-sum game with the functional* (1.1) *has the same value as the analogous game with the functions* $\Psi_1 \vee \Psi_2 = \max\{\Psi_1, \Psi_2\}$, Ψ_2 *instead of the functions* Ψ_1 *and* Ψ_2 *in* (1.1).

Proof. Let's denote

$$v(x) = \inf_\tau \sup_\delta J_x(\tau,\delta) \quad \tilde{v}(x) = \inf_\tau \sup_\delta \tilde{J}_x(\tau,\delta)$$

where

$$\tilde{J}_x(\tau,\delta) = E_x\left\{ \int_0^{\tau\wedge\delta} e^{-\alpha s} f(x_s)\,ds + \chi_{\tau<\delta} e^{-\alpha\tau}\Psi_1 \vee \Psi_2(x_\tau) + \chi_{\delta\leqslant\tau} e^{-\alpha\delta}\Psi_2(x_\delta)\right\}$$

First note that

$$v(x) \geqslant \inf_\tau J_x(\tau,0) = \Psi_2(x), \quad v(x) \leqslant \sup_\delta J_x(0,\delta) = \Psi_1 \vee \Psi_2(x) \qquad (1.29)$$

From Theorem 1, for arbitrary Markov times τ, δ we have

$$\tilde{J}_x(\tilde{\tau},\delta) \leqslant \tilde{J}_x(\tilde{\tau},\tilde{\delta}) = \tilde{v}(x) \leqslant \tilde{J}(\tau,\tilde{\delta})$$

where

$$\tilde{\tau} = \inf\{s\geqslant 0:\ \tilde{v}(x_s) = \Psi_1 \vee \Psi_2(x_s)\}$$
$$\tilde{\delta} = \inf\{s\geqslant 0:\ \tilde{v}(x_s) = \Psi_2(x_s)\}$$

We will show that $(\tilde{\tau},\tilde{\delta})$ is the saddle point for the functional J_x too. Since

$$\tilde{J}_x(\tau,\tilde{\delta}) = E_x\left\{ \int_0^{\tau\wedge\tilde{\delta}} e^{-\alpha s} f(x_s)\,ds + \chi_{\tau<\tilde{\delta}}\chi_{\{\Psi_1(x_\tau)\leqslant\Psi_2(x_\tau)\}} e^{-\alpha\tau}\Psi_2(x_\tau) + \chi_{\tau<\tilde{\delta}}\chi_{\{\Psi_1(x_\tau)>\Psi_2(x_\tau)\}} e^{-\alpha\tau}\Psi_1(x_\tau) + \chi_{\tilde{\delta}\leqslant\tau} e^{-\alpha\tilde{\delta}}\Psi_2(x_{\tilde{\delta}})\right\}$$

on the set $\{\Psi_1(x_\tau)\leqslant\Psi_2(x_\tau)\}$, we have $\tilde{v}(x_\tau)=\Psi_2(x_\tau)$ and then $\tau\geqslant\tilde{\delta}$, thus $\{\Psi_1(x_\tau)\leqslant\Psi_2(x_\tau)\}\subset\{\tau\geqslant\tilde{\delta}\}$. Hence

$$\tilde{J}_x(\tau,\tilde{\delta}) = J_x(\tau,\tilde{\delta}) \qquad \text{for each Markov time } \tau \qquad (1.30)$$

Particularly

$$\tilde{J}_x(\tilde{\tau},\tilde{\delta}) = J_x(\tilde{\tau},\tilde{\delta}) \qquad (1.31)$$

On the other hand for each time δ we have trivially

$$\tilde{J}_x(\tilde{\tau},\delta) \geqslant J_x(\tilde{\tau},\delta) \qquad (1.32)$$

Summarizing (1.30)–(1.32) for Markov times τ and δ we finally have

$$J_x(\tilde{\tau},\delta) \leqslant J_x(\tilde{\tau},\tilde{\delta}) = \tilde{J}_x(\tilde{\tau},\tilde{\delta}) \leqslant J_x(\tau,\tilde{\delta}) \tag{1.33}$$

which finishes the proof of theorem. □

Remark. One can consider the zero-sum game with the functional J_x^1

$$J_x^1(\tau,\delta) = E_x\left\{\int_0^{\tau\wedge\delta} e^{-\alpha s} f(x_s)\,ds + \chi_{\tau\leqslant\delta} e^{-\alpha\tau}\Psi_1(x_\tau) + \chi_{\delta<\tau} e^{-\alpha\delta}\Psi_2(x_\delta)\right\} \tag{1.34}$$

We can easily notice, that the value of this game is identical with the value of the game with functional J_x. For new game we have the theorem analogous to the previous one, with the functions Ψ_1, and $\Psi_1 \wedge \Psi_2 = \min\{\Psi_1, \Psi_2\}$ instead of Ψ_1 and Ψ_2 in the functional (1.34)

2. Stochastic Game with Impulsive Controls and a Constant Time Delay

2.1. Formulation of the Problem

In this part we consider games with impulses and we will frequently use results from the second Chapter of Robin's dissertation [8].

Let $\Omega = D(0,\infty; E)$ be the space of right continuous with left hand limits functions from R^+ in E,

$$x_s(\omega) = \omega(s), \quad \Theta_t(\omega)(s) = \omega(s+t) \qquad \text{for } s,t \geqslant 0,$$

Θ_t is a translation operator, and

$$F_t^0 = \sigma\{x_s : s \leqslant t\}, \qquad F^0 = F_\infty^0$$

We denote by F_t and F the universal completions of F^0.

Next assume that $X = (\Omega, F_t, \Theta_t, x_t, P_x)$ is the homogeneous Markov process. Moreover we introduce function f defined on E and functions c, d on $E \times E$, compact sets $U_1, U_2 \subset E$ and a constant number $h > 0$. Besides of the assumptions (A1), (A2), (A3) we introduce the following assumption (A5):

(A5) $f \in C$, and $c, d \in C(E \times E)$—the set of bounded, continuous, real functions on $E \times E$

Let us assume now that two players choose strategies W and Z respectively of the form:

$$W = \{\tau^1, \xi^1; \tau^2, \xi^2; \ldots, \tau^i, \xi^i; \ldots\}$$
$$Z = \{\delta^1, \eta^1; \delta^2, \eta^2; \ldots, \delta^i, \eta^i; \ldots\} \tag{2.1}$$

where $(\tau^i)_{i=1,2,\ldots}$ and $(\delta^i)_{i=1,2,\ldots}$ denotes increasing sequences of $(F_t)_{t\geq 0}$ stopping times, and ξ^i, η^i are F_{τ^i} or F_{δ^i} respectively measurable random variables with the values ξ^i in U_1 and η^i in U_2.

The first player shifts process at times $\tau^i + h$ to the states ξ^i, provided that in the interval $[\tau^i, \tau^i + h]$ no other shift of the process is done. This means that first player comes to decision of shift at time τ^i, and unless no other shifts in the interval $[\tau^i, \tau^i, +h]$ have appeared, this one is realized with the constant time delay equal h. The second player has the analogous possibilities of control. Moreover, if both players decided to shift process at the same moment, then it is passed according to the desire of the second player (obviously one can consider the analogous game in which such advantage will have the first player). If the player decided to shift process at time τ^i, then provided that in time interval $[\tau^i - h; \tau^i]$ no one came to decision of shift, no influence on the evolution of the process in the time interval $(\tau^i, \tau^i + h)$ is possible—all decisions in the interval $(\tau^i, \tau^i + h)$ are cancelled.

The decisive role in the run of the controlled process will play the following stopping times

$$\begin{cases} \rho^1(W,Z) = \tau^1 \wedge \delta^1 = \tau^{r^1} \wedge \delta^{s^1} \\ \rho^2(W,Z) = \tau^{r^2} \wedge \delta^{s^2} \\ \dots\dots\dots\dots\dots\dots \\ \rho^n(W,Z) = \tau^{r^n} \wedge \delta^{s^n} \end{cases} \tag{2.2}$$

where

$$\begin{array}{ll} r^1 = 1 & s^1 = 1 \\ r^2 = \min\{i: \tau^i \geq \rho^1 + h\} & s^2 = \min\{i: \delta^i \geq \rho^1 + h\} \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ r^n = \min\{i: \tau^i \geq \rho^{n-1} + h\} & s^n = \min\{i: \delta^i \geq \rho^{n-1} + h\} \end{array}$$

If a minimum is taken over the empty set, then we put suitable r^i or s^i equal plus infinity. We assume also that

$$\tau^\infty \overset{\text{def}}{=} +\infty \quad \text{and} \quad \delta^\infty \overset{\text{def}}{=} +\infty.$$

Let us denote first, that τ^{r^i} and δ^{s^i} for $i=1,2,\ldots$ are $(F_t)_{t\geq 0}$ Markov times. So in fact the following impulsive control is active on the process X

$$V(W,Z) = \{\rho^1, \zeta^1; \rho^2, \zeta^2; \ldots \rho^i, \zeta^i, \ldots\} \tag{2.3}$$

where

$$\zeta^1 = \begin{cases} \xi^1 & \text{if } \tau^1 < \delta^1 \\ \eta^1 & \text{if } \tau^1 \geq \delta^1 \end{cases}$$

$$\dots\dots\dots\dots\dots\dots$$

$$\zeta^n = \begin{cases} \xi^{r^n} & \text{if } \tau^{r^n} < \delta^{s^n} \\ \eta^{s^n} & \text{if } \tau^{r^n} \geq \delta^{s^n} \end{cases}$$

Obviously we have $\rho^i + h \leq \rho^{i+1}$ for $i=1,2,\ldots$, and ζ^i is F_{ρ^i} measurable.

Under the influence of impulsive control V the new measure P^V on the space of trajectories $D(0,\infty: E)$ is constructed. One can find this construction in second chapter of [8]. We will rely on the following properties of such measure:

$$P_x^V = P_x^{n-1} \quad \text{on} \quad F_{(\rho^n+h)^-} \tag{2.4}$$

where

$$P_x^0 = P_x$$
$$P_x^1 = P_x^0 \quad \text{on} \quad F_{(\rho^1+h)^-}$$
$$P_x^1\left(A\cap\Theta_{\rho_h^1}^{-1}B\right) = E_x^0\left[\chi_A P_{\xi^1}(B)\right] \qquad \text{for each } A \in F_{\rho_h^1-}$$
$$B\in F, \text{ where } \rho_h^1 = \rho^1 + h$$

. . . .

$$P_x^n = P_x^{n-1} \quad \text{on} \quad F_{(\rho^n+h)^-}$$
$$P_x^n(A\cap\Theta_{\rho_h^n}^{-1}B) = E_x^{n+1}[\chi_A P_{\xi^n}\cap(B)] \qquad \text{for each } A\in F_{\rho_h^n-},$$
$$B\in F \text{ where } \rho_h^n \overset{\text{def}}{=} \rho^n + h \tag{2.5}$$

Now we can consider the functional of the game:

$$J_x(W,Z) = E^V\left\{\int_0^\infty e^{-\alpha s} f(x_s)\,ds + \sum_{i=1}^\infty e^{-\alpha\tau^{r^i}\wedge\delta^{s^i}}\left[\chi_{\tau^{r^i}<\delta^{s^i}} c\left(x_{\tau^{r^i}},\xi^{r^i}\right)\right.\right.$$
$$\left.\left. + \chi_{\delta^{s^i}\leqslant\tau^{r^i}} d\left(x_{\delta^{s^i}},\eta^{s^i}\right)\right]\right\} \tag{2.6}$$

The first player using the strategy W wants to minimize the functional, while the second one with the aid of strategy Z is interested in maximizing it. Let us denote the upper value of the game by v

$$v(x) = \inf_W \sup_Z \quad J_x(W,Z) \tag{2.7}$$

We will show later that v presents in fact the value of the game.

2.2. *Characterizations of the saddle strategies*

Let us define first two operators M_1 and M_2 by

$$M_1 w(x) = \inf_{\xi\in U_1}\left[c(x,\xi) + e^{-\alpha h} w(\xi)\right]$$
$$M_2 w(x) = \sup_{\eta\in U_2}\left[d(x,\eta) + e^{-\alpha h} w(\eta)\right] \tag{2.8}$$

and the function Ψ

$$\Psi(x) \overset{\text{def}}{=} E_x\left\{\int_0^h e^{-\alpha s} f(x_s)\,ds\right\} \tag{2.9}$$

Since sets U_1 and U_2 are compact, M_1 and M_2 transform C into C, and there exist measurable functions $\hat{\xi}(x)$ and $\hat{\eta}(x)$ realizing infimum and supremum in the definition of M_1 and M_2, respectively. Moreover, from the assumptions (A1) and (A5) we have that $\Psi \in C$.

The following lemma will play an important role in this section

Lemma 1. *Let the assumptions* (A1)–(A3), (A5) *be satisfied. Then the equation*

$$\bar{v}(x) = \inf_{\tau} \sup_{\delta} E_x\left\{\int_0^{\tau \wedge \delta} e^{-\alpha s} f(x_s)\,ds + e^{-\alpha(\tau \wedge \delta)} \times \left[\chi_{\tau<\delta} M_1 \bar{v}(x_\tau) + \chi_{\delta \leqslant \tau} M_2 \bar{v}(x_\delta) + \Psi(x_{\tau \wedge \delta})\right]\right\} \tag{2.10}$$

has unique bounded, continuous solution $\bar{v}$.

Proof. We define on C the transformation Γ

$$\Gamma: C \ni g \to \Gamma(g) = \inf_{\tau} \sup_{\delta} E_x\left\{\int_0^{\tau \wedge \delta} e^{-\alpha s} f(x_s)\,ds + e^{-\alpha(\tau \wedge \delta)} \times \left[\chi_{\tau<\delta} M_1 g(x_\tau) + \chi_{\delta \leqslant \tau} M_2 g(x_\delta) + \Psi(x_{\tau \wedge \delta})\right]\right\} \tag{2.11}$$

We easy notice (Theorems 1 and 3, Part I), that Γ acts from C into C. Since for $g_1, g_2 \in C$

$$\| M_i g_1 - M_i g_2 \| \leqslant e^{-\alpha h} \| g_1 - g_2 \| \qquad i = 1,2$$

then

$$\|\Gamma(g_1) - \Gamma(g_2)\| \leqslant e^{-\alpha h} \| g_1 - g_2 \| \tag{2.12}$$

and Γ is the contraction on C.

Further we will proceed to show the identity $\bar{v} = v$. With this aim we will need the following lemma

Lemma 2. *Assume the assumptions* (A1), (A2), (A3), (A4) *are satisfied. Let*

$$w(x) = \inf_{\tau} \sup_{\delta} E_x\left\{\int_0^{\tau \wedge \delta} e^{-\alpha s} f(x_s)\,ds + e^{-\alpha(\tau \wedge \delta)}\left[\chi_{\tau<\delta} \Psi_1(x_\tau) + \chi_{\delta \leqslant \tau} \Psi_2(x_\delta)\right]\right\} \tag{2.13}$$

$$\hat{\tau} = \inf\{s \geqslant 0: w(x_s) = \Psi_1(x_s)\}$$

$$\hat{\delta} = \inf\{s \geqslant 0: w(x_s) = \Psi_2(x_s)\} \tag{2.14}$$

ρ is arbitrary Markov time, and ξ is F_ρ measurable random variable. We put

$$\rho_h = \rho + h \qquad \tilde{\tau} = \rho_h + \hat{\tau} \circ \Theta_{\rho_h} \qquad \tilde{\delta} = \rho_h + \hat{\delta} \circ \Theta_{\rho_h} \tag{2.15}$$

On the space (Ω, F, P) we define a new probabilistic measure $\tilde{P}$ satisfying the properties

$$\tilde{P} = P \quad \text{on} \quad F_{\rho_h^-}$$
$$\tilde{P}\left(A \cap \Theta_{\rho_h}^{-1} B\right) = E\left(\chi_A P_\xi(B)\right) \qquad \text{for each } A \in F_{\rho_h^-}$$
$$A \subset \{\rho < +\infty\} \quad \text{and} \quad B \in F$$

(*such measure there exists from Lemma* II 1.1 [8]). *Then*

$$e^{-\alpha\rho_h} w(\xi) = \tilde{E}\left\{ \int_{\rho_h}^{\tilde{\tau} \wedge \tilde{\delta}} e^{-\alpha s} f(x_s)\, ds + e^{-\alpha(\tilde{\tau} \wedge \tilde{\delta})} \left[\chi_{\tilde{\tau} < \tilde{\delta}} \Psi_1(x_{\tilde{\tau}}) + \chi_{\tilde{\delta} \leqslant \tilde{\tau}} \Psi_2(x_{\tilde{\delta}}) \right] \,\middle|\, F_{\rho_h^-} \right\} \text{ } P \text{ almost surely} \tag{2.16}$$

Moreover for a Markov time τ, $\tau \geqslant \rho_h$ we have P almost surely

$$e^{-\alpha\rho_h} w(\xi) \leqslant E\left\{ \int_{\rho_h}^{\tau \wedge \tilde{\delta}} e^{-\alpha s} f(x_s)\, ds + e^{-\alpha(\tau \wedge \tilde{\delta})} \left[\chi_{\tau < \tilde{\delta}} \Psi_1(x_\tau) + \chi_{\tilde{\delta} \leqslant \tau} \Psi_2(x_{\tilde{\delta}}) \right] \,\middle|\, F_{\rho_h^-} \right\} \tag{2.17}$$

and for a Markov time $\delta \geqslant \rho_h$

$$e^{-\alpha\rho_h} w(\xi) \geqslant \tilde{E}\left\{ \int_{\rho_h}^{\tilde{\tau} \wedge \delta} e^{-\alpha s} f(x_s)\, ds + e^{-\alpha(\tilde{\tau} \wedge \delta)} \left[\chi_{\tilde{\tau} < \delta} \Psi_1(x_{\tilde{\tau}}) + \chi_{\delta \leqslant \tilde{\tau}} \Psi_2(x_\delta) \right] \,\middle|\, F_{\rho_h^-} \right\} \tag{2.18}$$

P almost surely.

The proof of this lemma is similar to the one in [8] and is in [11].

The following theorem contains the main result of part 2.

Theorem 1. *If the assumptions* (A1), (A2), (A3), (A5) *are satisfied then*

$$v(x) = \bar{v}(x) \tag{2.19}$$

where $v(x)$ is defined in (2.7) *and $\bar{v}(x)$ is the solution of the equation* (2.10).

Let us introduce Markov times

$$\hat{\tau} = \inf\{s \geq 0: M_1 v(x_s) \vee M_2 v(x_s) + \Psi(x_s) = v(x_s)\}$$
$$\hat{\delta} = \inf\{s \geq 0: M_2 v(x_s) + \Psi(x_s) = v(x_s)\} \tag{2.20}$$

and assume that $\hat{\xi}(x)$ ($\hat{\eta}(x)$) is a measurable function realizing the infimum in $\xi \to c(x,\xi) + e^{-\alpha h} v(\xi)$ on U_1 (supremum in $\eta \to d(x,\eta) + e^{-\alpha h} v(\eta)$ on U_2).

Then the saddle strategies $(\hat{W}, \hat{Z})$ are defined in the following way

$$\hat{\tau}^1 = \begin{cases} \hat{\tau} & \text{if } \hat{\tau} \leq \hat{\delta} \\ \hat{\delta} + \frac{h}{2} & \text{if } \hat{\delta} < \hat{\tau} \end{cases}$$

. . . .

$$\hat{\tau}^{i+1} = \begin{cases} \hat{\rho}^i + h + \hat{\tau} \circ \Theta_{\hat{\rho}^i + h} & \text{on } \Theta^{-1}_{\hat{\rho}^i + h}(\{\hat{\tau} < \infty\} \cap \{\hat{\tau} \leq \hat{\delta}\}) \\ \hat{\rho}^i + h + \hat{\delta} \circ \Theta_{\hat{\rho}^i + h} + \frac{h}{2} & \text{on } \Theta^{-1}_{\hat{\rho}^i + h}(\{\hat{\delta} < \hat{\tau}\}) \\ \infty & \text{on } \Theta^{-1}_{\hat{\rho}^i + h}(\{\hat{\tau} = +\infty, \hat{\delta} = +\infty\}) \end{cases} \tag{2.21}$$

$$\hat{\xi}^i = \begin{cases} \hat{\xi}(x_{\hat{\tau}^i}) \text{ on } \{\hat{\tau}^i < \infty\} \\ \text{arbitrary } \xi_0 \in U_1 \text{ on } \{\hat{\tau}^i = +\infty\} \end{cases}$$

$$\hat{\delta}^1 = \begin{cases} \hat{\delta} & \text{if } \hat{\delta} \leq \hat{\tau} \\ \hat{\tau} + \frac{h}{2} & \text{if } \hat{\tau} < \hat{\delta} \end{cases} \tag{2.22}$$

$$\hat{\delta}^{i+1} = \begin{cases} \hat{\rho}^i + h + \hat{\delta} \circ \Theta_{\rho^i + h} & \text{on } \Theta^{-1}_{\hat{\rho}^i + h}(\{\hat{\delta} < \infty\} \cap \{\hat{\delta} \leq \hat{\tau}\}) \\ \hat{\rho}^i + h + \hat{\tau} \circ \Theta_{\hat{\rho}^i + h} + \frac{h}{2} & \text{on } \Theta^{-1}_{\hat{\rho}^i + h}(\{\hat{\tau} < \hat{\delta}\}) \\ \infty & \text{on } \Theta^{-1}_{\hat{\rho}^i + h}(\{\hat{\delta} = +\infty\} \cap \{\hat{\tau} = +\infty\}) \end{cases} \tag{2.23}$$

$$\hat{\eta}^i = \begin{cases} \hat{\eta}(x_{\hat{\delta}^i}) & \text{on } \{\hat{\delta}^i < \infty\} \\ \text{arbitrary } \eta_0 \in U_2 & \text{on } \{\hat{\delta}^i = +\infty\} \end{cases} \tag{2.24}$$

Proof. First we will show that for the strategies defined with the aid of (2.21)–(2.24), where v is replaced by $\bar{v}$, we have the identity

$$\bar{v}(x) = J_x(\hat{W}, \hat{Z}) \tag{2.25}$$

In fact, from the Lemma 1

$$\bar{v}(x) = E_x^0\left\{\int_0^{\hat{\tau}^1\wedge\hat{\delta}^1} e^{-\alpha s} f(x_s)\,ds + e^{-\alpha(\hat{\tau}^1\wedge\hat{\delta}^1)} \times\left[\chi_{\hat{\tau}^1\leqslant\hat{\delta}^1} M_1\bar{v}(x_{\hat{\tau}^1}) + \chi_{\hat{\delta}^1\leqslant\hat{\tau}^1} M_2\bar{v}(x_{\hat{\delta}^1}) + \Psi(x_{\hat{\tau}^1\wedge\hat{\delta}^1})\right]\right\} \tag{2.26}$$

Using the definition of $\hat{\xi}^1$ and $\hat{\eta}^1$ we have

$$e^{-\alpha\hat{\tau}^1} M_1\bar{v}(x_{\hat{\tau}^1}) = \left[c(x_{\hat{\tau}^1},\hat{\xi}^1) + e^{-\alpha h}\bar{v}(\hat{\xi}^1)\right]e^{-\alpha\hat{\tau}^1}$$

$$e^{-\alpha\hat{\delta}^1} M_2\bar{v}(x_{\hat{\delta}^1}) = \left[d(x_{\hat{\delta}^1},\hat{\eta}^1) + e^{-\alpha h}\bar{v}(\hat{\eta}^1)\right]e^{-\alpha\hat{\delta}^1}$$

and hence

$$\bar{v}(x) = E_x^0\left\{\int_0^{\hat{\tau}^1\wedge\hat{\delta}^1+h} e^{-\alpha s} f(x_s)\,ds + e^{-\alpha(\hat{\tau}^1\wedge\hat{\delta}^1)} \times\left[\chi_{\hat{\tau}^1<\hat{\delta}^1} c(x_{\hat{\tau}^1},\hat{\xi}^1) + \chi_{\hat{\delta}^1\leqslant\hat{\tau}^1} d(x_{\hat{\delta}^1},\hat{\eta}^1)\right] + e^{-\alpha(\hat{\tau}^1\wedge\hat{\delta}^1+h)}\bar{v}(\hat{\zeta}^1)\right\} \tag{2.27}$$

where we set

$$\hat{\zeta}^1 = \begin{cases}\hat{\xi}^1 & \text{if } \hat{\tau}^1<\hat{\delta}^1\\ \hat{\eta}^1 & \text{if } \hat{\tau}^1\geqslant\hat{\delta}^1\end{cases}$$

$$\hat{\zeta}^n = \begin{cases}\hat{\xi}^n & \text{if } \hat{\tau}^n\geqslant\hat{\delta}^n\\ \hat{\eta}^n & \text{if } \hat{\tau}^n\geqslant\hat{\delta}^n\end{cases} \tag{2.28}$$

Now we exert (2.16) from Lemma 2. We obtain

$$e^{-\alpha(\hat{\tau}^1\wedge\hat{\delta}^1+h)}\bar{v}(\hat{\zeta}^1) = E_x^1\left\{\int_{\hat{\rho}^1+h}^{\hat{\tau}^2\wedge\hat{\delta}^2+h} e^{-\alpha s} f(x_s)\,ds + e^{-\alpha(\hat{\tau}^2\wedge\hat{\delta}^2)}\left[\chi_{\hat{\tau}^2<\hat{\delta}^2} M_1\bar{v}(x_{\hat{\tau}^2}) + \chi_{\hat{\delta}^2\leqslant\hat{\tau}^2} M_2\bar{v}(x_{\hat{\delta}^2})\right] \Big| F(\hat{\rho}^1+h)^-\right\} \tag{2.29}$$

So from (2.27) we have inductively for $i=1,2,\ldots$

$$\bar{v}(x) = E_x^{i-1}\left\{\int_0^{\hat{\tau}^1\wedge\hat{\delta}^1+h} e^{-\alpha s} f(x_s)\,ds + \sum_{j=1}^{i} e^{-\alpha(\hat{\tau}^j\wedge\hat{\delta}^j)} \times\left[\chi_{\hat{\tau}^j<\hat{\delta}^j} c(x_{\hat{\tau}^j},\hat{\xi}^j) + \chi_{\hat{\delta}^j\leqslant\hat{\tau}^j} d(x_{\hat{\delta}^j},\hat{\eta}^j)\right] + e^{-\alpha(\hat{\rho}^i+h)}\bar{v}(\hat{\zeta}^i)\right\} \tag{2.30}$$

Since $\hat{\rho}^i=\hat{\tau}^i\wedge\hat{\delta}^i\to\infty$ as $i\to\infty$, the proof of the identity (2.25) is established.

The proof of theorem with be finished if we assure that $\hat{W}$ and $\hat{Z}$ are the saddle strategies indeed. Suppose the strategy Z is arbitrary

$$Z = \{\delta^1, \eta^1; \delta^2, \eta^2 \cdots\}$$

Then we take the strategy $\hat{W}$ of the form

$$\hat{\tau}^1 = \begin{cases} \hat{\tau} & \text{if } \hat{\tau} \leqslant \delta^1 \\ \delta^1 + \dfrac{h}{2} & \text{if } \delta^1 < \hat{\tau} \end{cases}$$

$$\rho^1 = \hat{\tau}^1 \wedge \delta^1$$

$$\hat{\tau}^2 = \begin{cases} \rho^1 + h + \hat{\tau} \circ \Theta_{\rho^1+h} & \text{on } \Theta^{-1}_{\rho^1+h}(\{\hat{\tau} < \infty\}) \cap \{\rho^1 + h + \hat{\tau} \circ \Theta_{\rho^1+h} \leqslant \delta^{s^2}\} \\ \delta^{s^2} + \dfrac{h}{2} & \text{on } \{\rho^1 + h + \hat{\tau} \circ \Theta_{\rho^1+h} > \delta^{s^2}\} \\ \infty & \text{on } \Theta^{-1}_{\rho^1+h}(\{\hat{\tau} = \infty\}) \cap \{\delta^{s^2} = \infty\} \end{cases}$$

. . . .

$$\rho^i = \hat{\rho}^i \wedge \delta^{s^i}$$

$$\hat{\tau}^{i+1} = \begin{cases} \rho^i + h + \hat{\tau} \circ \Theta_{\rho^i+h} & \text{on } \Theta^{-1}_{\rho^i+h}(\{\hat{\tau} < \infty\}) \cap \{\rho^i + h + \hat{\tau} \circ \Theta_{\rho^i+h} \leqslant \delta^{s^{i+1}}\} \\ \delta^{s^{i+1}} + \dfrac{h}{2} & \text{on } \{\rho^i + h + \hat{\tau} \circ \Theta_{\rho^i+h} > \delta^{s^{i+1}}\} \\ \infty & \text{on } \Theta^{-1}_{\rho^i+h}(\{\hat{\tau} = \infty\}) \cap \{\delta^{s^{i+1}} = \infty\} \end{cases} \tag{2.31}$$

where s^i is defined in (2.2) and $\hat{\xi}^i$ in (2.22). Let us note that on the game influences in fact not strategy Z but $\bar{Z}$

$$\bar{Z} = \{\delta^{s_1}, \eta^{s_1}, \delta^{s_2}, \eta^{s_2}, \dots\} \tag{2.32}$$

We will prove the inequality

$$\bar{v}(x) = J_x(\hat{W}, \hat{Z}) \geqslant J_x(\hat{W}, \bar{Z}) \tag{2.33}$$

First from (2.10) we have

$$\bar{v}(x) \geqslant E_x\left\{\int_0^{\hat{\tau} \wedge \delta^{s^1}} e^{-\alpha s} f(x_s)\, ds + e^{-\alpha(\hat{\tau} \wedge \delta^{s^1})}\left[\chi_{\hat{\tau} < \delta^{s^1}} M_1 \bar{v}(x_{\hat{\tau}}) \right.\right.$$
$$\left.\left. + \chi_{\delta^{s^1} \leqslant \hat{\tau}} M_2 \bar{v}(x_{\delta^{s^1}}) + \Psi(x_{\hat{\tau} \wedge \delta^{s^1}})\right]\right\}$$

Next to simplify the notations we put $\delta^{s^1}=\delta^i$ and $\eta^{s^i}=\eta^i$. Since

$$M_2\bar{v}(x_{\delta^1}) \geqslant d(x_{\delta^1},\eta^1)+e^{-\alpha h}\bar{v}(\eta^1)$$

then

$$\bar{v}(x) \geqslant E_x^0\Big\{\int_0^{\hat{\tau}^1\wedge\delta^1+h} e^{-\alpha s}f(x_s)\,ds+e^{-\alpha(\hat{\tau}^1\wedge\delta^1)} \times\big[\chi_{\hat{\tau}^1<\delta^1}c(x_{\hat{\tau}^1},\hat{\xi}^1)+\chi_{\delta^1\leqslant\hat{\tau}^1}d(x_{\delta^1},\eta^1)\big]+e^{-\alpha(\hat{\tau}^1\wedge\delta^1+h)}\bar{v}(\zeta^1)\Big\} \quad (2.35)$$

where this time ζ^i denotes

$$\zeta^1=\begin{cases}\hat{\xi}^1 & \text{if } \hat{\tau}^1<\delta^1\\ \eta^1 & \text{if } \delta^1\leqslant\hat{\tau}^1\end{cases}\cdots$$

$$\zeta^n=\begin{cases}\hat{\xi}^n & \text{if } \hat{\tau}^n<\delta^n\\ \eta^n & \text{if } \delta^n\leqslant\hat{\tau}^n\end{cases} \quad (2.36)$$

From (2.18) we have

$$e^{-\alpha(\hat{\tau}^1\wedge\delta^1+h)}\bar{v}(\zeta^1) \geqslant E_x^1\Big\{\int_{\rho^1+h}^{\hat{\tau}^2\wedge\delta^2+h} e^{-\alpha s}f(x_s)\,ds +e^{-\alpha(\hat{\tau}^2\wedge\delta^2)}\big[\chi_{\hat{\tau}^2<\delta^2}M_1\bar{v}(x_{\hat{\tau}^2})+\chi_{\delta^2\leqslant\hat{\tau}^2}M_2\bar{v}(x_{\delta^2})\big]\,\big|\, F_{(\rho^1+h)^-}\Big\} \quad (2.37)$$

Thus inductively we obtain

$$\bar{v}(x) \geqslant E^{i-1}\Big\{\int_0^{\hat{\tau}^i\wedge\delta^i+h} e^{-\alpha s}f(x_s)\,ds+\sum_{j=1}^{i} e^{-\alpha(\hat{\tau}^j\wedge\delta^j)} \times\big[\chi_{\hat{\tau}^j<\delta^j}c(x_{\hat{\tau}^j},\hat{\xi}^j)+\chi_{\delta^j\leqslant\hat{\tau}^j}d(x_{\delta^j},\eta^j)\big]+e^{-\alpha(\rho^i+h)}\bar{v}(\zeta^i)\Big\} \quad (2.38)$$

Since $\rho^i=\hat{\tau}^i\wedge\delta^i\to\infty$ as $i\to\infty$ the inequality (2.33) is satisfied.

It remains to prove for each strategy W the inequality

$$J_x(\hat{W}, \hat{Z}) \leqslant J_x(W, \hat{Z}) \tag{2.39}$$

The proof of this fact is based on the Lemmas 1 and 2, and is analogous as the one of inequality (2.33). So the proof of theorem is established. □

3. Stochastic Game with Impulsive Control and Stopping

3.1. Introduction

In the third part we consider more complicated situation. There are two players. The first one controls a Markov process by means of the instantaneous impulses and the second one only stops the game. Shifts are realized immediately, without any delay. Similarly as in Part 2 we consider Markov process $X = (\Omega, F_t, x_t, P_x)$, where $\Omega = D(0, \infty; E)$. But to describe the evolution of the process, controlled by instantaneous shifting, we need to take new probability space $\tilde{\Omega} = \Omega^N$, $\tilde{F} = F^{\otimes N}$, $\tilde{P}$.

Let us introduce the projection subspaces of $\tilde{\Omega}$

$$\Omega_1 = \Omega \times \Omega, \ldots \qquad \Omega_n = (\Omega)^{n+1} \tag{3.1}$$

and their σ algebras

$$F_t^1 = F_t \otimes F_t \ldots \qquad F_t^n = (F_t)^{\otimes (n+1)} \tag{3.2}$$

We also define a translation operator $\Theta_{n,t}$ for

$$\begin{aligned} [\omega]_n &= (\omega_1, \omega_2, \ldots \omega_{n+1}) \in \Omega_n \\ (\Theta_{n,t}[\omega]_n)(s) &= (\Theta_t \omega_1(s), \ldots \Theta_t \omega_{n+1}(s)) \end{aligned} \tag{3.3}$$

an impulsive control W consists of a sequence of pairs $W = (\tau^i, \xi^i)_{i=1,2,\ldots,}$ of F_t^{i-1} stopping times τ^i and $F_{\tau^{i-1}}$ measurable random variables ξ^i with the values in U-compact subset of E.

Let us denote by $G_{\tau^1}, \ldots G_{\tau^n}$ the following σ-fields

$$\begin{aligned} G_{\tau^1} &= \sigma\{F_{\tau^1-}^1, F_{\tau^1}^0 \otimes \{\varnothing, \Omega\}\} \\ G_{\tau^n} &= \sigma\{F_{\tau^n-}^n, F_{\tau^n}^{n-1} \otimes \{\varnothing, \Omega\}\} \end{aligned} \tag{3.4}$$

Then from the Lemma A2 of chapter 5 in [8] the projections P^n of the measure $\tilde{P} = P^W$ of the controlled with the aid of strategy W process X on the spaces Ω^n

for $n=1,2\ldots$ have the following properties

$$\begin{cases} P_x^0 = P_x \\ \cdots\cdots \\ P_x^n = P_x^{n-1}\otimes\varepsilon_{\varphi_y} \quad \text{on} \quad G_{\tau^n} \\ P_x^n\left(\Theta_{n,\tau^n}^{-1}B \mid G_{\tau^n}\right) = \varepsilon_{\varphi_{x^1_{\tau^n}(\omega_1)}}\otimes\cdots\otimes\varepsilon_{\varphi_{y_{x^n_{\tau^n}(\omega_n)}}}\otimes P_{\xi^n([\omega]_{n-1})}(B) \\ P_x^{n-1}\otimes\varepsilon_{\varphi_y} \qquad \text{a.e. on } \{\tau^n<\infty\} \text{ where } B\in F. \\ \varphi_y\in\Omega,\ \varphi_y(t)\equiv y \qquad \text{for each } t\geqslant 0 \end{cases} \tag{3.5}$$

The process controlled by W has the following form $(y_t)_{t\geqslant 0}$

$$y_t(\omega) = x_t^{k+1}(\omega_{k+1}) \text{ for } t\in[\tau^k,\tau^{k+1}] \text{ (we put } \tau^0=0) \tag{3.6}$$

Suppose the assumptions (A1)–(A3) (Section 1) are satisfied and

$$\text{(A6)} \quad \begin{cases} 0\leqslant f\in C \\ \Psi\in C \\ c(x,\xi)\geqslant k>0 \quad \text{is a continuous bounded function on } E\times U \end{cases}$$

The second player chooses as his strategy stopping time δ, which is a positive random variable, F_t^i Markov time on the interval $[\tau^i,\tau^{i+1})$.

We define for our game the functional

$$J_x(W,\delta) = \lim_{N\to\infty} J_x^N(W,\delta) \tag{3.7}$$

where

$$J_x^N(W,\delta) = E_x^N\left\{\int_0^{\tau^{N+1}\wedge\delta} e^{-\alpha s}f(y_s)\,ds + \sum_{n=1}^{N+1}\left[\chi_{\tau^n<\delta}c(x^n_{\tau^n},\xi^n)\right.\right.$$
$$\left.\left.\times e^{-\alpha\tau^n} + \chi_{\delta\leqslant\tau^n}e^{-\alpha\delta}\Psi(y_\delta)\right]\right\} \tag{3.8}$$

and the lower value of the game v

$$v(x) = \sup_\delta \inf_W \quad J_x(W,\delta) \tag{3.9}$$

So we have a zero-sum game, where the first player proceeds to minimizing (3.7), using the impulsive control W, while the second one with the aid of Markov time δ wants to maximize (3.7).

3.2. *Main theorem*

Similarly as in Sections 1 and 2 we are interested in the saddle strategies for the game.

The following theorem holds:

Theorem 1. *Under the assumptions* (A1)–(A3), (A6), *function* $v(x)$ *is continuous and presents the value of the game. Moreover we have the following saddle strategies*

$$\begin{cases} \hat\tau^1 = \inf\{s \geqslant 0 \colon v(x_s) = Mv \vee \Psi(x_s)\} \\ \hat\xi^1 = \hat\xi(x_{\hat\tau}1) \text{ where } \hat\xi(x) \text{ is the measurable function realizing the infimum in} \\ \qquad Mv(x) = \inf_{\xi \in U} \{c(x,\xi) + v(\xi)\} \end{cases}$$

$$\begin{cases} \hat\tau^2(\omega_1, \omega_2) = \hat\tau^1(\omega_1) + \hat\tau^1(\omega_2)\Theta_{1, \hat\tau^1(\omega_1)} \\ \hat\xi^2 = \hat\xi(x^2_{\hat\tau^2}) \end{cases}$$

. . . .

$$\begin{cases} \hat\tau^n(\omega_1, \dots \omega_n) = \hat\tau^{n-1}(\omega_1, \dots \omega_{n-1}) + \hat\tau^1(\omega_n)\Theta_{1, \hat\tau^{n-1}(\omega_1, \dots \omega_{n-1})} \\ \hat\xi^n = \hat\xi(x^n_{\hat\tau^n}) \end{cases} \tag{3.10}$$

$$\hat\delta^1 = \inf\{s \geqslant 0 \colon v(x_s) = \Psi(x_s)\} \tag{3.11}$$

Proof. The proof consists of three steps. First we will show the continuity of $v(x)$ using the uniform approximation of v.

Step 1. Let us consider the same game, but with impulsive control consisting only in n shifts. We denote by v_n the lower value of such game. It turns out, we have the following identities

Lemma 1. $$v_0(x) = \sup_\delta E_x\left\{\int_0^\delta e^{-\alpha s} f(x_s)\,ds + e^{-\alpha\delta}\Psi(x_\delta)\right\}$$

. . . .

$$v_n(x) = \sup_\delta \inf_\tau E_x\left\{\int_0^{\tau\wedge\delta} e^{-\alpha s} f(x_s)\,ds + \chi_{\tau<\delta} e^{-\alpha\tau} Mv_{n-1} \vee \Psi(x_\tau) + \chi_{\delta\leqslant\tau} e^{-\alpha\delta}\Psi(x_\delta)\right\}$$

The lemma can be proved by induction basing on the suitable (in our situation) version of Lemma 2 2.

Step 2. Using the technics due to Menaldi [7] we prove similarly as Robin [8] uniform convergence v_n to v, as $n \to \infty$. For the details see [11]. Since from the Lemma 1 for each n, the function v_n is continuous, then v is continuous too.

Step 3. It remains to check that strategies (3.10) and (3.11) are saddle indeed. We prove this similarly as in Theorem 2 1 using the version of Lemma 2 2. □

Final Remarks

This paper is based on the part of the author's thesis which was written under the guidance of Prof. J. Zabczyk. In his dissertation the author considers in addition the approximations associated with the game with stopping. He introduces similarly as Robin in [8] for stopping, two kinds of approximations. Namely: approximation of the process and approximations of the functions Ψ_1 and Ψ_2 in the functional 1 (1.1). Moreover, the error of the penalty method for this game is obtained.

Last time the game with impulses has appeared in J. P. Lepeltier paper [6]. She considers for diffusion the same game as the author in part 2, under the very strong assumption that there exists positive $\beta > 0$, such that

$$\frac{1}{\alpha}(\sup f - \inf f) + \beta \leqslant \inf_{y \in U_1} c(x, y) - \sup_{y \in U_2} d(x, y)$$

for each $x \in E$.

The case of Markov nonzero-sum games with stopping or impulses seemed so far to be unsolved.

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