Math. Ann. 282, 463–471 (1988)

Mathematische Annalen

Remarks on the Existence Problem of Positive Kähler-Einstein Metrics

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Recently, there has been interesting progress on the problem of existence of Kähler-Einstein metrics on compact Kähler manifolds with positive first Chern class (cf. [Si, Ti, T–Y]). The approach proposed by Tian is of particular interest. In [Ti], he defined an invariant $\alpha(M)$ for compact Kähler manifolds with $c_1(M)>0$, and proved that, if $\alpha(M)>\frac{n}{n+1}$, where $n=\dim M$, then there exists a Kähler-Einstein metric on M. The definition of $\alpha(M)$ is as follows.

Let g be a Kähler metric on M. Let

$$P(M,g)=\{u\in C^2(M,R): g_{\alpha\bar\beta}+u_{,\alpha\bar\beta}>0\},$$

where $u_{,\alpha\bar\beta}=\frac{\partial^2 u}{\partial z^\alpha \partial z^{\bar\beta}}$, and $P_0(M,g)=\{u\in P(M,g): \max_M u=0\}$. It is proved in [Ti] that there exists a constant $\alpha>0$ such that

$$\int_M e^{-\alpha u}d\mu \leqq C \quad \text{for all } u\in P_0(M,g). \tag{1}$$

Define $\alpha(M,g)=\sup\{\alpha>0$: there exists a constant C such that (1) holds.$\}$ It can be shown that $\alpha(M,g)$ does not depend on the specific metric g in a given cohomological class. Therefore, we may define $\alpha(M)$ to be $\alpha(M,g)$ for any Kähler metric g such that its Kähler form $\omega_g\in c_1(M)$.

In the present notes it is our aim to introduce another invariant $\eta(M)$ which also gives a criterion for the existence of Kähler-Einstein metrics. Moreover, Tian's criterion can be easily derived from the new criterion [assuming the validity of Tian's inequality (1)]. Like $\alpha(M)$, the invariant $\eta(M)$ also arises from an inequality which is a generalization of Moser's inequality on $S^2=\mathbb{C}P^1$. This generalized Moser inequality was first conjectured and proposed to be used in the proof of existence of Kähler-Einstein metrics by Aubin ([Au]).

Recall the two functionals defined on $P(M,g)$ as follows

$$I(u)=\frac{1}{V}\int_M u(1-\mathfrak{M}(u))d\mu,$$

$$J(u)=\int_0^1 \frac{1}{t} I(tu)dt.$$

Here, V is the volume of (M,g) and $\mathfrak{M}(u)$ is the complex Monge-Ampère operator defined by

$$\mathfrak{M}(u)=\det(g_{\alpha\bar\beta}+u_{,\alpha\bar\beta})/\det(g_{\alpha\bar\beta}).$$

Recall the following inequality for I and J (cf. [Au, B–M]):

$$0\leqq\frac{n+1}{n}J(u)\leqq I(u)\leqq(n+1)J(u).$$

The generalized Moser inequality takes the following form:

$$\int_M e^{-u}d\mu\leqq C\exp\left[\eta J(u)-\frac{1}{V}\int_M u\,d\mu\right]. \tag{2}$$

It was conjectured by Aubin that there exist constants $\eta>0$ and $C>0$ such that (2) holds for all $u\in P(M,g)$. Until now, it is unknown whether this is true or not. What we can prove is a weaker result which is actually an easy consequence of a result of Bando and Mabuchi [B–M].

Let $Q^\varepsilon(M,g)=\{u\in P(M,g):R^u_{\alpha\bar\beta}\geqq\varepsilon g^u_{\alpha\bar\beta}\}$, where $g^u_{\alpha\bar\beta}=g_{\alpha\bar\beta}+u_{,\alpha\bar\beta}$ and $R^u_{\alpha\bar\beta}$ is the Ricci curvature of the metric g^u.

Proposition 1. *For $\varepsilon>0$ there exists $C_\varepsilon=C_\varepsilon(M,g)$ such that*

$$\int_M e^{-u}d\mu\leqq C_\varepsilon\exp\left[(n+1)J(u)-\frac{1}{V}\int_M u\,d\mu\right] \tag{3}$$

for all $u\in Q^\varepsilon(M,g)$.

Proof. By Bando and Mabuchi ([B–M], Prop. 3.6), there exists $A_\varepsilon=A_\varepsilon(M,g)>0$ such that for any $u\in Q^\varepsilon(M,g)$

$$\begin{aligned}-\min\nolimits_M u&\leqq-\frac{1}{V}\int_M u\mathfrak{M}(u)d\mu+A_\varepsilon\\&=I(u)-\frac{1}{V}\int_M u\,d\mu+A_\varepsilon.\end{aligned}$$

Using the inequality $I(u)\leqq(n+1)J(u)$ we have

$$-u(x)\leqq(n+1)J(u)-\frac{1}{V}\int_M u\,d\mu+A_\varepsilon.$$

Taking the exponential of the above inequality and integrating over M we get (3).

Now, we define $\eta(M,g)$ to be the infimum of all positive numbers η such that (2) holds for all $u\in Q^\varepsilon(M,g)$ with some $C=C(\varepsilon,\eta,M,g)$. Then define

$$\eta(M)=\inf\{\eta(M,g):\omega_g\in c_1(M)\}.$$

By Proposition 1, one has $\eta(M)\leqq n+1$ for any compact Kähler manifold with positive first Chern class.

Theorem 2. *Let M be a compact Kähler manifold with $c_1(M)>0$. If $\eta(M)<1$, then there exists a Kähler-Einstein metric on M.*

Proof. Since $\eta(M)<1$, we may take a Kähler metric g on M with $\omega_g\in c_1(M)$ and $\eta(M,g)<1$. There exists a function $f\in C^\infty(M)$ such that $R_{\alpha\bar\beta}=g_{\alpha\bar\beta}+f_{,\alpha\bar\beta}$. It will be convenient to normalize f by requiring that

$$\frac{1}{V}\int_M e^f d\mu=1\,. \tag{4}$$

The problem of finding Kähler-Einstein metrics can be reduced to solving the following Monge-Ampère equation

$$\mathfrak{M}(u)=e^{f-u},\qquad u\in P(M,g)\,. \tag{5}$$

As in [Au], one attempts to solve (5) by continuity method. Consider the family of equations:

$$\mathfrak{M}(u)=e^{f-tu},\qquad u\in P(M,g)\,, \tag*{$(6)_t$}$$

where $t\in[0,1]$. It can be shown that there exists a smooth family of solutions $\{u_t\}$ of $(6)_t$ for $t\in[0,\tau)$ with some $\tau\in(0,1)$. Moreover, if one has a priori estimates

$$\|u_t\|_{C^0(M)}\leqq C \tag{7}$$

for all $t\in[0,\tau)$, then $\tau=1$ and u_t subconverges in $C^2(M)$ to a solution u_1 of $(6)_1=(5)$ (cf. [Au] or [Ti]). It is also important to notice that if we set $g^t=g^{u_t}$ and denote the Ricci curvature of g^t by $R^t{}_{\alpha\bar\beta}$, then $R^t{}_{\alpha\bar\beta}=(1-t)g_{\alpha\bar\beta}+tg^t_{\alpha\bar\beta}$. This implies $u_t\in Q^t(M,g)$.

Consider now the following family of functionals defined on $P(M,g)$:

$$F_t(u)=J(u)-\frac{1}{V}\int_M u\,d\mu-\frac{1}{t}\log\left[\frac{1}{V}\int_M e^{f-tu}d\mu\right],$$

where $t\geqq 0$, with the understanding that

$$F_0(u)=J(u)-\frac{1}{V}\int_M u\,d\mu+\frac{1}{V}\int_M e^f u\,d\mu\,.$$

If $v(s)$ is a smooth curve in $P(M,g)$ with $\frac{d}{ds}v(s)|_{s=0}=\zeta$, then one has (cf. [Au])

$$\frac{d}{ds}F_t(v)|_{s=0}=-\frac{1}{V}\int_M \mathfrak{M}(u)\zeta\,d\mu+\left(\int_M e^{f-tu}\zeta\,d\mu\right)\left(\int_M e^{f-tu}d\mu\right)^{-1}.$$

This means the Euler-Lagrange equation of F_t is

$$\frac{1}{V}\mathfrak{M}(u)=\left(\int_M e^{f-tu}d\mu\right)^{-1}e^{f-tu}\,. \tag{8}$$

Note that $F_t(u+c)=F_t(u)$ for any constant c, which implies whenever u is a critical point of F_t so is $u+c$. Note also that u is a critical point of F_t iff u is a solution to

$$\frac{1}{V}\mathfrak{M}(u)=ce^{f-tu},$$

for any positive constant c. In fact, integrating the above equation over M we get $c\int_M e^{f-tu}d\mu=1$, which means u satisfies (8). Now set

$$v_t=u_t-\frac{1}{V}\int_M u_t d\mu, \qquad t\in[0,\tau).$$

Then v_t is a critical point of F_t, and $\int_M v_t d\mu=0$. We have

$$\begin{aligned}\frac{d}{dt}F_t(v_t)&=\frac{d}{ds}F_t(v_s)|_{s=t}+\frac{1}{t^2}\log\left[\frac{1}{V}\int_M e^{f-tv_t}d\mu\right]\\&\quad+\frac{1}{t}\left(\int_M v_t e^{f-tv_t}d\mu\right)\left(\int_M e^{f-v_t}d\mu\right)^{-1}\\&=\frac{1}{t^2}\log\left[\frac{1}{V}\int_M e^{f-tv_t}d\mu\right]+\frac{1}{tV}\int_M v_t\mathfrak{M}(v_t)d\mu\\&=-\frac{1}{t}[F_t(v_t)-J(v_t)]-\frac{1}{t}I(v_t).\end{aligned}$$

It follows that

$$\frac{d}{dt}[tF_t(v_t)]=J(v_t)-I(v_t).$$

Integrating this to get

$$F_t(v_t)=\frac{1}{t}\int_0^t[J(v_s)-I(v_s)]ds\leqq 0.$$

Let $\delta\in(0,\tau)$. Then we have $v_t\in Q^\delta(M,g)$ for $t\in[\delta,\tau)$. Since $\eta(M,g)<1$, one may take $\eta<1$ so that

$$\log\int_M e^{-v_t}d\mu\leqq \eta J(v_t)+C_\delta \tag{9}$$

Therefore

$$\begin{aligned}\frac{1}{t}\log\int_M e^{f-tv_t}d\mu&\leqq\log\left(\int_M e^{-tv_t}d\mu\right)^{1/t}+C\\&\leqq\log\int_M e^{-v_t}d\mu+C'_\delta\\&\leqq\eta J(v_t)+C''_\delta.\end{aligned}$$

It follows that

$$F_t(v_t)\geqq(1-\eta)J(v_t)-C''_\delta.$$

Since $F_t(v_t)\leqq 0$, we see that

$$J(v_t)\leqq C \quad \text{for} \quad t\in[\delta,\tau).$$

By [B–M], for any $u\in Q^\delta(M,g)$ we have

$$\operatorname{Max}u-\operatorname{Min}u\leqq I(u)+C(g,\delta). \tag{10}$$

Hence,

$$\operatorname{Max} v_t - \operatorname{Min} v_t \leqq (n+1)J(v_t) + C(g,\delta) \leqq C.$$

where $t \in [\delta, \tau)$. On the other hand, from $v_t \in P(M,g)$ and $\int_M v_t d\mu = 0$ one deduces $\operatorname{Max} v_t \leqq C$ (cf. [Au] or [Ti]). It follows that

$$\|v_t\|_{C^0(M)} \leqq C. \tag{11}$$

Notice now that by integrating $(6)_t$ we have

$$V = \int_M e^{f - tu_t} d\mu,$$

which combining with $u_t = v_t + \frac{1}{V}\int_M u_t d\mu$ gives

$$\exp\left(\frac{t}{V}\int_M u_t d\mu\right) = \frac{1}{V}\int_M e^{f - tv_t} d\mu.$$

So, $\left|\frac{1}{V}\int_M u_t d\mu\right| \leqq C$ for $t \in [\delta, \tau)$. This together with (11) shows $\|u_t\|_{C^0(M)} \leqq C$, i.e. (7) holds. It follows that (5) has a solution u_1 which gives rise to a Kähler-Einstein metric on M. The proof of Theorem 2 is complete.

Proposition 3. $\eta(M) \leqq (n+1)[1 - \alpha^*(M)]$, *where* $\alpha^*(M) = \operatorname{Min}\{\alpha(M), 1\}$. *In particular,* $\alpha(M) > \frac{n}{n+1}$ *implies* $\eta(M) < 1$.

Proof. We need only to show

$$\eta(M,g) \leqq (n+1)[1 - \alpha^*(M)] \tag{12}$$

for any Kähler metric g on M with $\omega_g \in c_1(M)$. Let α be any positive number less than $\alpha^*(M)$. For $u \in Q^\varepsilon(M,g)$, since $u - \operatorname{Max} u \in P_0(M,g)$, we have

$$\int_M e^{-\alpha(u - \operatorname{Max} u)} d\mu \leqq C. \tag{13}$$

Using (10) and (13) we estimate

$$\begin{aligned}
\int_M e^{-u} d\mu &= \int_M e^{-\alpha(u - \operatorname{Max} u)} e^{-(1-\alpha)u} e^{-\alpha \operatorname{Max} u} d\mu \\
&\leqq C \exp[-(1-\alpha)\operatorname{Min} u - \alpha \operatorname{Max} u] \\
&\leqq C_\varepsilon \exp[(1-\alpha)I(u) - \operatorname{Max} u] \\
&\leqq C_\varepsilon \exp\left[(1-\alpha)(n+1)J(u) - \frac{1}{V}\int_M u d\mu\right].
\end{aligned}$$

This means $\eta(M,g) \leqq (n+1)(1-\alpha)$. Letting $\alpha \to \alpha^*(M)$, we see (12) holds.

By Theorem 2 and Proposition 3, it is easily seen that $\alpha(M) > \frac{n}{n+1}$ implies the existence of Kähler-Einstein metrics on M, which is just Tian's criterion.

Remark 1. Proposition 3 indicates that $\eta(M)<1$ may be a better criterion than $\alpha(M)>\frac{n}{n+1}$. However, we do not know how to estimate $\eta(M)$ directly. The only known estimation of $\eta(M)$ is Proposition 3 which relates the upper bound of $\eta(M)$ with the lower bound of $\alpha(M)$. So we have not obtained any new existence results. We should also mention that in the special case $M=\mathbb{C}P^1$, we have $\eta(\mathbb{C}P^1)=1$. This is a consequence of Moser's inequality [Mo]. In the improved version [On], this inequality may be written as

$$\int_M e^{-u}d\mu \leqq V\exp\left[J(u)-\frac{1}{V}\int_M u\,d\mu\right].$$

Notice that $V=4\pi$ for $M=\mathbb{C}P^1$ and $J(u)=\frac{1}{2V}\int_M |\nabla u|^2 d\mu$ for $n=1$. It can be proved that on $\mathbb{C}P^n$ equipped with a suitable multiple of the Fubini-Study metric [so that $\omega_g\in c_1(\mathbb{C}P^n)$], the above inequality holds for all $u\in P(\mathbb{C}P^n,g)$ with $u(x)=u(d(x,p))$, where $d(x,p)$ is the distance from x to a fixed point p.

Remark 2. It would be interesting to know whether Proposition 1 is still true if we replace the condition $u\in Q^\varepsilon(M,g)$ by $u\in P(M,g)$. If it remained true, then there would be a constant $\tau_n>0$ depending only on n such that on any compact Kähler manifold one can always solve Eq. $(6)_t$ for $t\in[0,\tau_n)$. Indeed, one can replace inequality (9) in the proof of Theorem 2 by

$$\log\int_M e^{-tv_t}d\mu \leqq (n+1)J(tv_t)+C.$$

On the other hand, from

$$\frac{d}{dt}J(tu)=\frac{d}{dt}\int_0^1\frac{1}{s}I(stu)ds=\frac{d}{dt}\int_0^t\frac{1}{s}I(su)ds$$
$$=\frac{1}{t}I(tu)\geqq\frac{n+1}{n}\frac{1}{t}J(tu),$$

one derives

$$J(tu)\leqq t^{\frac{n+1}{n}}J(u).$$

Therefore,

$$\frac{1}{t}\log\int_M e^{-tv_t}d\mu\leqq(n+1)t^{1/n}J(v_t)+C,$$

and

$$F_t(v_t)\geqq[1-(n+1)t^{1/n}]J(v_t)-C.$$

From here one can follow the proof of Theorem 2 to show that $(6)_t$ has a solution for $t\in[0,\tau_n)$, where $\tau_n=1/(n+1)^n$. This means that on any compact Kähler manifold M with $c_1(M)>0$, there exists a Kähler metric g with $\mathrm{Ric}(g)\geqq(\tau_n-\varepsilon)$ for any $\varepsilon>0$. As explained in [T-Y], such a result will imply $c_1(M)^n\leqq c(n)$ for all compact Kähler manifolds M with $c_1(M)>0$, where $c(n)$ is a constant depending only on the dimension.

Remark 3. As in [Ti], for every compact subgroup G of $\mathrm{Aut}(M)$, we may define $\eta_G(M)$ by requiring in the definition of $\eta(M)$ that the metrics g and the functions $u\in Q^\varepsilon(M,g)$ be G-invariant. In such cases, the proof of Theorem 2 can be easily modified to show that $\eta_G(M)<1$ implies the existence of Kähler-Einstein metrics on M. [Just notice that, since the solutions of $(6)_t$ are unique for $t\in(0,1)$, the v_t's have to be G-invariant.] Also, similar to Proposition 3, we have $\eta_G(M) \leqq (n+1)[1-\alpha_G(M)]$. The necessity of introducing $\alpha_G(M)$ and $\eta_G(M)$ will be seen in the following discussions.

In the proof of Theorem 3 we have used the fact that $(6)_t$ is the Euler-Lagrange equation of the functional F_t. Of the family of functionals $\{F_t\}$, the most important one is $F=F_1$ whose critical points correspond to Kähler-Einstein metrics on M. So we would like to discuss some basic properties of F. To emphasize the dependence of F on the metrics g, we set

$$F_g(u)=L_g(u)-G_g(u),$$

where

$$L_g(u)=J_g(u)-\frac{1}{V}\int_M u\,d\mu_g,$$

$$G_g(u)=\log\left[\frac{1}{V}\int_M e^{f_g-u}d\mu_g\right].$$

Lemma 4. *Let $v\in P(M,g)$ and $g^v_{\alpha\bar\beta}=g_{\alpha\bar\beta}+v_{,\alpha\bar\beta}$. Then for $u\in P(M,g^v)$ it holds that*

$$L_g(v)+L_{g^v}(u)=L_g(u+v), \tag{14}$$

$$G_g(v)+G_{g^v}(u)=G_g(u+v). \tag{15}$$

Proof. Let $L(\,,)$ be the functional defined in [Ma]. One checks that $L_g(v)=L(0,v)$, $L_{g^v}(u)=L(v,u+v)$, and $L_g(u+v)=L(0,u+v)$. It follows from the 1-cocycle property of $L(\,,)$ that (14) holds (cf. [Ma, Theorem (2.3)]). Next, we have

$$(f_g)_{,\alpha\bar\beta}=(R_g)_{\alpha\bar\beta}-g_{\alpha\bar\beta}=-(\log\det(g_{\alpha\bar\beta}))_{,\alpha\bar\beta}-g_{\alpha\bar\beta}.$$

Combining this with the analogous identity for f_{g^v} we get

$$(f_{g^v}-f_g)_{,\alpha\bar\beta}=-(\log\mathfrak{M}_g(v))_{,\alpha\bar\beta}-v_{,\alpha\bar\beta},$$

which implies

$$f_{g^v}=f_g-\log\mathfrak{M}_g(v)-v-C, \tag{16}$$

for some constant C. By condition (4), one determines $C=G_g(v)$. It is then easy to derive (15) from (16).

As an immediate consequence of Lemma 4, we get

$$F_g(v)+F_{g^v}(u)=F_g(u+v), \tag{17}$$

for $v\in P(M,g)$ and $u\in P(M,g^v)$.

Now, let $\mathrm{Aut}_0(M)$ denote the connected component of the automorphism group $\mathrm{Aut}(M)$ containing the identity. For any $\varphi\in\mathrm{Aut}_0(M)$ there exists $v_\varphi\in C^\infty(M)$, unique upto a constant, such that

$$(\varphi^*g)_{\alpha\bar\beta}=g_{\alpha\bar\beta}+(v_\varphi)_{,\alpha\bar\beta}=(g^{v_\varphi})_{\alpha\bar\beta}.$$

Proposition 5. *Let g be a Kähler-Einstein metric on a compact Kähler manifold M with $\omega_g \in c_1(M) > 0$. Then, for any $\varphi \in \mathrm{Aut}_0(M)$ we have*

$$F_g(u) = F_g(\varphi^* u + v_\varphi) \quad \textit{for all } u \in P(M, g). \tag{18}$$

Proof. We have

$$F_g(u) = F_{\varphi^* g}(\varphi^* u) = F_{g^{v_\varphi}}(\varphi^* u) = F_g(\varphi^* u + v_\varphi) - F_g(v_\varphi)$$

by (17). It suffices to show $F_g(v_\varphi) = 0$ for any $\varphi \in \mathrm{Aut}_0(M)$. Let φ_t be a smooth curve in $\mathrm{Aut}_0(M)$ such that $\varphi_0 = \mathrm{id}$ and $\varphi_1 = \varphi$. Then $v_t = v_{\varphi_t}$ is a smooth curve in $P(M, g)$ $\left(\text{assuming } \int_M v_t d\mu_g = 0\right)$ with $v_0 = 0$ and $v_1 = v_\varphi$. Notice that each g^{v_t} is a Kähler-Einstein metric, so v_t is a critical point of F_g for each t. It follows that $\frac{d}{dt} F_g(v_t) \equiv 0$, and hence $F_g(v_\varphi) = F_g(0) = 0$. This completes the proof.

Remark 4. Relation (18), reflecting one of the important holomorphically invariant properties of the functional F on a Kähler-Einstein manifold, was first discovered by Onofri for $M = \mathbb{C}P^1$ in [On], where he used it to improve the Moser inequality on $\mathbb{C}P^1$. Later, it was found that such a relation is very useful in solving the problem of prescribing Gaussian curvatures on $S^2 = \mathbb{C}P^1$ (cf. [C–Y, C–D]).

Proposition 6. *Let M be a compact Kähler manifold with $c_1(M) > 0$. If M admits a nonzero holomorphic vector field, then $\eta(M) \geqq 1$.*

Proof. If there is no Kähler-Einstein metric on M, then the conclusion follows from Theorem 2. So we assume M has a Kähler-Einstein metric g with $\omega_g \in c_1(M)$. We need only to show $\eta(M, g^w) \geqq 1$ for any $w \in P(M, g)$. Now, for $\varphi \in \mathrm{Aut}_0(M)$ let v_φ be the function as above. By (17) we have

$$F_{g^w}(v_\varphi - w) = F_g(v_\varphi) - F_g(w) = -F_g(w). \tag{19}$$

Notice that $(v_\varphi - w) \in Q^1(M, g^w)$. If $\eta(M, g^w) < 1$, then inequality (2) (with $\eta < 1$) implies there exist $\varepsilon > 0$ and $C > 0$ such that

$$F_{g^w}(v_\varphi - w) \geqq \varepsilon J_{g^w}(v_\varphi - w) - C. \tag{20}$$

Combining (19) and (20) gives

$$J_{g^w}(v_\varphi - w) \leqq C(\varepsilon, w) \tag{21}$$

for all $\varphi \in Aut_0(M)$. On the other hand, since each g^{v_φ} is Kähler-Einstein, $v_\varphi - w$ satisfies Eq. (5) with respect to the metric g^w. As in the proof of Theorem 2, one shows by (21) that $\{v_\varphi - w\}$ is uniformly bounded in $C^0(M)$. Estimates for higher order derivatives of solutions of (5) (cf. [Au] or [Ti]) then shows $\{v_\varphi - w\}$ is uniformly bounded in $C^k(M)$ for any $k \geqq 1$. It follows that $\{g^{v_\varphi} = \varphi^* g : \varphi \in \mathrm{Aut}_0(M)\}$ is a compact family of Kähler-Einstein metrics. However, if M admits a nonzero holomorphic vector field, one knows from the proof of Matsushima theorem (see [Ko, p. 96]) that the first nonzero eigenvalue of the complex Laplacian of a Kähler-Einstein metric equals 1, and the gradient vector field of a first eigenfunction generates one-parameter family $\{\varphi_t\}$ of holomorphic transformations in $\mathrm{Aut}_0(M)$. Since the gradient flow contracts an open set of M to a lower-dimensional subset, it is clear that the family $\{\varphi^* g\}$ can not be compact. The contradiction shows that $\eta(M) \geqq 1$.

Remark 5. The above result shows that if M admits nonzero holomorphic vector fields, then one has to use $\alpha_G(M)$ instead of $\alpha(M)$ to prove the existence of Kähler-Einstein metrics, where G is the maximal compact subgroup of $\mathrm{Aut}(M)$ (cf. [Ti, T-Y]). On the other hand, if the maximal compact subgroup G is normal in $\mathrm{Aut}(M)$ (we do not know whether this is possible), which implies the action of $\mathrm{Aut}(M)$ preserves G-invariant metrics, one can prove in a similar way that $\eta_G(M) \geqq 1$ and hence $\alpha_G(M) \leqq \frac{n}{n+1}$.

Finally, if u is a critical point of F_g, one obtains the second variation formula of F_g at u by a direct computation and making use of (17). The result is, for any $v \in C^\infty(M)$ with $\int_M v\, d\mu_{g^u} = 0$ one has

$$d^2 F_g(u)(v,v) = d^2 F_{g^u}(0)(v,v) = \frac{1}{V} \int_M (|\nabla_{g^u} v|^2 - v^2)\, d\mu_{g^u} \geqq 0 .$$

The equality holds if and only if the first eigenvalue of the complex Laplacian of g_u is 1 (hence M admits nonzero holomorphic vector fields) and v is a first eigenfunction.

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Received July 3, 1987; in revised form December 14, 1987