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Elliptic Equations and Maps of Bounded Length Distortion

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1. Introduction

Consider the second order elliptic partial differential equation

$$L(u)=\sum_{i,j=1}^{n}\partial_i(a_{ij}\partial_j u)=0 \tag{1.1}$$

whose coefficients a_{ij} are C^1-functions in a domain D of R^n. If G is another domain in R^n and if $f: G\to D$ is a C^2-map whose jacobian $J(x,f)$ is positive in G, then f induces a similar elliptic operator $f^{\#}L$ in G with the property that $u\circ f$ is a solution of $f^{\#}L(v)=0$ whenever u is a solution of $L(u)=0$. For this classical result see e.g. [VI, p. 38]. If the functions a_{ij} are merely L^∞-functions in D and if (1.1) is interpreted in the weak sense, then a proper counterpart of the locally diffeomorphic C^2-maps seems to be the class of locally bilipschitz maps. In this paper we show that there is, however, a more general class of maps $f: G\to D$ which enjoy the above invariance property and which are not necessarily locally injective. We say that these maps are of *bounded length distortion*, abbreviated BLD. More precisely, a continuous map $f: G\to R^n$ is L-BLD for some $L\geqq 1$ if f is discrete, open and sense-preserving and if for each path α in G we have

$$l(\alpha)/L\leqq l(f\alpha)\leqq Ll(\alpha).$$

Here $l(\alpha)$ denotes the length of the path α. Although a BLD map is locally lipschitz, it need not be a local homeomorphism. A typical example is the map $(r,\phi,z)\mapsto(r,2\phi,z)$ in the cylindrical coordinates of R^3. This map is 2-BLD in R^3.

We shall consider the second order elliptic differential equation

$$\nabla\cdot A(x,\nabla u(x))=0 \tag{1.2}$$

in divergence form with $|A(x,h)|\approx|h|^{p-1}$ for some $p>1$. For $p\neq 2$ the Eq. (1.2) is non-linear. The invariance property applies to the wider class of equations $\nabla\cdot A=B$. The Eq. (1.2) have been chosen for the sake of simplicity. They include the important Euler equations of the variational integrals

$$\int F(x,\nabla u(x))dm(x)$$

where $F(x,h)\approx|h|^p$.

The Laplace equation $\Delta u=0$ in the plane is invariant under analytic and antianalytic functions. This phenomenon has a proper counterpart for Eq. (1.2) with $p=n$. In all dimensions the invariance property is enjoyed by the quasiregular (QR) maps, which form a generalization of the complex analytic functions to higher dimensional euclidean spaces. This invariance property has been studied by Reshetnyak [Re_2], see also [GLM]. The BLD maps form a proper subclass of QR maps, and we shall use the theory of QR maps in several occasions. For the basic properties of QR maps, see [MRV_1].

In Sect. 2 we give several equivalent characterizations for BLD. For the study of the aforementioned invariance property the most useful characterization is the following: A continuous map $f: G\to R^n$ is L-BLD if and only if (i) f is absolutely continuous on lines (ACL), (ii) for almost every $x\in G$

$$|h|/L\leqq|f'(x)h|\leqq L|h|$$

for all $h\in R^n$ and (iii) $J(x,f)\geqq 0$ a.e. in G.

Section 3 is devoted to the study of the Eq. (1.2) and the corresponding induced equations $\nabla\cdot f^{\#}A=0$. In particular, we show that if the equations $\nabla\cdot A=0$ and $\nabla\cdot f^{\#}=0$ are both elliptic and satisfy a natural boundedness condition, then under weak assumptions on the map $f: G\to D$, f is BLD for $p\neq n$ and QR for $p=n$.

Some properties of BLD maps are studied in Sect. 4. These include distortion, normal families, boundary behavior and the multiplicity function

$$N(y,f,A)=\operatorname{card}(A\cap f^{-1}(y))$$

of a BLD map f. For example, we show that for every $y\in R^n$, $N(y,f,R^n)\leqq L^{2n}$ whenever $f: R^n\to R^n$ is L-BLD. An interesting problem associated with f is the structure of the branch set B_f of f. It turns out that B_f in the BLD case is somewhat simpler than in the QR case; we show that $R^n\setminus B_f$ is a uniform domain [MS] for a BLD map $f: R^n\to R^n$. On the other hand, for every $n\geqq 3$ and $\varepsilon>0$ there is a BLD map $f: R^n\to R^n$ such that the Hausdorff dimension of B_f exceeds $n-\varepsilon$.

As a by-product we obtain new examples of finite quasiconformal groups of R^n, $n\geqq 4$, not quasiconformally conjugate to Möbius groups.

2. Definitions for BLD Maps

2.1. In this section we give various characterizations for BLD maps and also some preliminary material needed in the later sections. The main results are given in 2.16. Throughout the paper, the notation $f: G\to R^n$ will include the assumptions that G is a domain in R^n, $n\geqq 2$, and that f is continuous. We start with the following definition:

Let $L\geqq 1$. A map $f: G\to R^n$ is said to be of *L-bounded length distortion*, abbreviated L-BLD, if (i) f is ACL,

$$\text{(ii)} \qquad |h|/L\leqq|f'(x)h|\leqq L|h|$$

for all $h\in R^n$ for almost every $x\in G$, and (iii) $J(x,f)\geqq 0$ a.e. in G.

Here we use the notation and terminology of [MRV_1]. Thus $f'(x): R^n\to R^n$ is the formal derivative and $J(x,f)$ the jacobian of f at x. Note that since f is ACL [MRV_1, 2.16], the partial derivatives of f, and hence $f'(x)$ and $J(x,f)$, exist a.e.

We shall frequently employ the theory of quasiregular (QR) maps. The basic reference of this theory is [MRV$_1$]. We recall the definition. If $T: R^n \to R^n$ is a linear map, we let $|T|$ and $l(T)$ denote the maximum and the minimum, respectively, of $|Th|$ over all unit vectors $h \in R^n$. A map $f: G \to R^n$ is K-QR, $K \geqq 1$, if f is ACL^n and if

$$|f'(x)|^n/K \leqq J(x,f) \leqq K l(f'(x))^n \tag{2.2}$$

a.e. in G. The first inequality alone guarantees that f is K^{n-1}-QR.

2.3. **Lemma.** *Suppose that $f: G \to R^n$ is L-BLD. Then:*

(1) *If the line segment $[x,y]$ lies in G,*

$$|f(x)-f(y)| \leqq L|x-y|. \tag{2.4}$$

Thus f is locally L-lipschitz.

(2) *f is K-QR with $K = L^{2(n-1)}$.*

Proof. Suppose first that $\gamma = [x,y]$ is parallel to a coordinate axis, that f is absolutely continuous on γ and that (ii) holds a.e. on γ with respect to the linear measure. Then (2.4) follows by integration. By continuity, (2.4) holds whenever γ is parallel to a coordinate axis. Hence f is locally nL-lipschitz. By the theorem of Rademacher, f is differentiable a.e. Moreover, f is absolutely continuous on every line segment in G. Hence we can repeat the argument and obtain (2.4) in the general case.

To prove (2) we first observe that f is ACL$^\infty$ and hence ACL^n. The inequalities (2.2) are true at every point $x \in G$ such that f is differentiable at x and (ii) holds. This follows by a simple calculation involving the eigenvalues of $f'(x)^* f'(x)$, cf. [Vä$_2$, p. 44].

2.5. *Remarks and Examples.* (a) The condition (ii) of 2.1 is equivalent to the inequalities

$$|f'(x)| \leqq L, \qquad l(f'(x)) \geqq 1/L.$$

These and (iii) imply that $J(x,f) > 0$ a.e. in G. This is also true but less trivial for QR maps. A QR map is BLD if and only if $|f'(x)|$ is essentially bounded away from 0 and ∞.

(b) As noted in the proof of 2.3, a BLD map is a.e. differentiable, and the formal derivative $f'(x)$ is thus a.e. the ordinary derivative of f. Moreover, f is not only ACL but absolutely continuous on every line segment, up to the boundary.

(c) Clearly every locally L-bilipschitz map $f: G \to R^n$ is L-BLD. However, a BLD map need not be a local homeomorphism. As a typical example we can consider the winding map $f: R^n \to R^n$, defined by $f(r,\phi,z) = (r,k\phi,z)$ in the cylindrical coordinates, $k = 2, 3, \ldots$. This map is k-BLD.

(d) Suppose that $n=2$ and that $f: G \to R^2$ is complex analytic. If $f'(z) = 0$ for some $z \in G$, f cannot be BLD. The function $f(z) = e^z$ is not BLD in R^2, but its restriction to the strip $\{(x,y): |x| < c\}$ is e^c-BLD.

(e) If $f: G \to R^n$ is BLD and C^1, the jacobian $J(x,f)$ never vanishes. Hence f is a local homeomorphism. For C^1 QR maps in dimensions $n \geqq 3$, this is an open question. However, this holds for C^3 QR maps and for C^2 QR maps in dimensions $n=3$ and $n \geqq 4$, respectively. This easily follows from [F, Theorem 3.4.3] and from the fact that the $(n-2)$-measure of $f B_f$ is positive whenever $B_f \neq \emptyset$.

2.6. We recall some topological terminology. A map $f: G \to R^n$ is open if it maps open sets onto open sets. It is discrete if $f^{-1}(y)$ is a discrete set of points for every $y \in R^n$. It is sense-preserving if $\mu(y, f, D) > 0$ for each $y \in fD \setminus f\partial D$ and for each domain $D \subset\subset G$, that is, $\bar{D}$ is compact in G. Here μ is the topological index. The branch set B_f of f is the set of all points $x \in G$ at which f is not a local homeomorphism. For discrete open maps, the topological dimension of B_f is at most $n-2$. For the basic facts about these properties we refer to [MRV$_1$, pp. 8–12].

2.7. **Lemma.** *An L-BLD map $f: G \to R^n$ has the following properties:*

(1) *f is discrete, open and sense-preserving.*

(2) *$f|G \setminus B_f$ is locally L-bilipschitz.*

(3) *If f is injective, $f^{-1}: fG \to G$ is L-BLD.*

Proof. By a result of Reshetnyak [Re$_1$], a QR map is either constant or discrete, open and sense-preserving. Since a BLD map is not constant, (1) follows from 2.3 (2).

To prove (2), fix an open ball $B \subset G \setminus B_f$ such that $f|B$ is injective. By 2.3, $f|B$ is L-lipschitz. Moreover, $f|B$ is quasiconformal, that is, an injective QR map. Hence $g = (f|B)^{-1}$ is also quasiconformal and thus ACL and a.e. differentiable. If f is differentiable and satisfies (ii) at $x \in B$ and if g is differentiable at $y = f(x)$, then $g'(y) = f'(x)^{-1}$, and thus

$$|g'(y)| \leq L, \qquad l(g'(y)) \geq 1/L.$$

Since a lipschitz map preserves the property of being of measure zero, these inequalities are true for almost every $y \in fB$. Hence g is L-BLD and thus locally L-lipschitz. This proves (2) and (3).

2.8. **Lemma.** *If $f: G \to R^n$ is BLD, then $m(B_f) = m(fB_f) = 0$.*

Proof. By [MRV$_1$, 2.14], $J(x, f) = 0$ whenever f is differentiable at a branch point $x \in B_f$. Hence, by 2.5 (a), $m(B_f) = 0$. Since f is locally lipschitz, this implies $m(fB_f) = 0$. We remark that the result is also true for QR maps but considerably deeper.

2.9. *Path Lifting.* We next recall some results on path lifting for discrete open maps. This is perhaps the most important tool in the theory of BLD maps. It will frequently be used together with the inequalities of Lemma 2.15.

A path in R^n is a continuous map $\alpha: \Delta \to R^n$ of an interval $\Delta \subset R^1$. In this paper Δ is of the type $[a, b)$ or $[a, b]$. We formulate the results for half open intervals; the obvious modifications for closed intervals are easy consequences.

Suppose that $f: G \to R^n$ is discrete, open and sense-preserving. Let $\beta: [a, b) \to R^n$ be a path and let $x \in f^{-1}(\beta(a))$. A maximal lift of β starting at x is a path $\alpha: [a, c) \to G$ such that $\alpha(a) = x$, $f\alpha = \beta|[a, c)$ and α is not a proper subpath of another path with these properties. There is always at least one maximal lift α of β starting at x. If $c = b$, α is called a total lift of β. If $c < b$, $\alpha(t) \to \partial G$ as $t \to c$.

We let $i(x, f)$ denote the local index of f at $x \in G$. Let $\beta: [a, b) \to R^n$ be a path, let $x_1, \ldots, x_k$ be distinct points in $f^{-1}(\beta(a))$, and set

$$m = i(x_1, f) + \ldots + i(x_k, f).$$

By [Ri$_2$, Theorem 1], f has a maximal sequence $\alpha_1, \ldots, \alpha_m$ of lifts starting at $\{x_1, \ldots, x_k\}$. This means:

(i) α_j is a maximal lift of β, $1 \leqq j \leqq m$.
(ii) $\operatorname{card}\{j : \alpha_j(a) = x_i\} = i(x_i, f)$, $1 \leqq i \leqq k$.
(iii) $\operatorname{card}\{j : \alpha_j(t) = x\} \leqq i(x_i, f)$ for all $x \in G$, $t \in [a, b)$.

A domain $D \subset\subset G$ is a normal domain of f if $f\partial D = \partial f D$. If, in addition, $x \in D$ with $D \cap f^{-1}(f(x)) = \{x\}$, D is called a normal neighborhood of x. If $x \in G$, we let $U(x, f, r)$ denote the x-component of $f^{-1}B(f(x), r)$. Here and later $B(y, r)$ denotes the open ball with center y and radius r. If $U = U(x, f, r) \subset\subset G$, U is a normal domain of f and $fU = B(f(x), r)$. There is $r_0 > 0$ such that U is a normal neighborhood of x for $r \leqq r_0$.

If $A \subset G$, we write

$$N(y, f, A) = \operatorname{card}(A \cap f^{-1}(y)),$$

$$N(f, A) = \sup\{N(y, f, A) : y \in R^n\},$$

$$N(f) = N(f, G).$$

If $A \subset\subset G$, $N(f, A)$ is finite. If U is a normal neighborhood of x, $N(V, f) = i(x, f)$ for every neighborhood $V \subset U$ of x. We give as a lemma a result of Rickman [Ri$_2$, Theorem 2], which will be useful in the sequel. We assume that $f : G \to R^n$ is discrete, open and sense-preserving.

2.10. **Lemma.** *Suppose that D is a normal domain of f and that $\beta : [a, b) \to fD$ is a path with locus $|\beta|$. Then $D \cap f^{-1}|\beta|$ can be expressed as the union of the loci of $N(f, D)$ total lifts $\alpha_j : [a, b) \to D$ of β such that* $\operatorname{card}\{j : \alpha_j(t) = x\} = i(x, f)$ *for every* $x \in D \cap f^{-1}|\beta|$.

2.11. **Corollary.** *Suppose that D is a normal neighborhood of x, that $y \in D$ and that $\beta : [a, b] \to fD$ is a path joining $f(x)$ to $f(y)$. Then there is a lift $\alpha : [a, b] \to D$ of β joining x to y.*

If f: $G \to R^n$, $x \in G$, and $r > 0$, we set as in [MRV$_1$]

$$L^*(x, f, r) = \sup\{|y - x| : y \in \partial U\},$$

$$l^*(x, f, r) = \inf\{|y - x| : y \in \partial U\},$$

where $U = U(x, f, r)$. We next estimate these numbers for normal neighborhoods U of x. A more general situation will be considered in 4.20.

2.12. **Lemma.** *Suppose that $f : G \to R^n$ is L-BLD and that $U = U(x, f, r)$ is a normal neighborhood of x. Then*

$$L^*(x, f, r) \leqq Lr, \qquad l^*(x, f, r) \geqq r/L.$$

Proof. Choose $y \in \partial U$ with $|y - x| = l^*(x, f, r)$. Since the line segment $[x, y]$ lies in G, 2.3 yields

$$r = |f(y) - f(x)| \leqq L l^*(x, f, r),$$

which is the second inequality of 2.12.

To prove the first inequality we consider a point $z \in S(f(x), r) = \partial B(f(x), r)$ and the radial path $\beta_z : [0, r) \to R^n$ from $f(x)$ to z, defined by

$$\beta_z(t) = f(x) + t(z - f(x))/r.$$

By 2.10, $U \cap f^{-1}|\beta_z|$ can be covered by m maximal lifts of β_z starting at x, $m = N(U, f) = i(x, f)$. From [MRV$_1$, 7.10] or from [Vä$_3$, 2.6] it easily follows that for almost every z, each lift α_z is absolutely continuous on every interval $[s, r]$, $s > 0$. By 2.8, almost every $|\beta_z|$ meets fB_f in a set of linear measure zero. Since $f|G \setminus B_f$ is locally L-bilipschitz, we have then $|\alpha_z'(t)| \leqq L$ for almost every $t \in [0, r)$. Hence

$$|\alpha_z(t) - x| = |\alpha_z(t) - \alpha_z(0)| \leqq Lr$$

for all $t \in [0, r)$ and for almost every $z \in S(f(x), r)$. Thus $|y - x| \leqq Lr$ for a dense set of points $y \in U$. This implies $L^*(x, f, r) \leqq Lr$.

2.13. **Corollary.** *If $f : G \to R^n$ is L-BLD and if $U(x, f, r)$ is a normal neighborhood of x, then*

$$|y - x|/L \leqq |f(y) - f(x)| \leqq L|y - x|$$

whenever $|y - x| \leqq r/L$.

2.14. **Lemma.** *If $f : G \to R^n$ is L-BLD and if f is differentiable at $x \in G$, then $x \in G \setminus B_f$, $|f'(x)| \leqq L$, $l(f'(x)) \geqq 1/L$, and $J(x, f) > 0$.*

Proof. This follows directly from 2.7 (1), 2.13 and [MRV$_1$, 2.14].

If α is a path in R^n, we let $l(\alpha)$ denote its length. We next prove the fundamental property of BLD maps:

2.15. **Lemma.** *If $f : G \to R^n$ is L-BLD and if α is a path in G, then*

$$l(\alpha)/L \leqq l(f\alpha) \leqq Ll(\alpha).$$

Proof. By 2.3, f is locally L-lipschitz. This implies the second inequality of 2.15. The proof of the first inequality is somewhat harder. Setting $\beta = f\alpha$ we may assume that $l(\beta) < \infty$ and that $\beta : [0, l(\beta)] \to R^n$ is a parametrization by arc length. Let P be a partition of $[0, l(\beta)]$. From [MRV$_1$, 2.9] it easily follows that there is a refinement $P' = \{t_0, \ldots, t_k\}$ of P and numbers $r_i > 0$ such that for each $i \in \{0, \ldots, k\}$, $U(\alpha(t_i), f, r_i)$ is a normal neighborhood of $\alpha(t_i)$ and $\beta[t_{i-1}, t_i] \subset B_{i-1} \cup B_i$, where $B_i = B(\beta(t_i), r_i)$. For each $i = 1, \ldots, k$ choose $s_i \in [t_{i-1}, t_i]$ with $\beta(s_i) \in B_{i-1} \cap B_i$. Then 2.13 implies

$$|\alpha(s_i) - \alpha(t_{i-1})| \leqq L(s_i - t_{i-1}), \qquad |\alpha(t_i) - \alpha(s_i)| \leqq L(t_i - s_i).$$

Hence

$$\sum_{i=1}^{k} |\alpha(t_i) - \alpha(t_{i-1})| \leqq \sum_{i=1}^{k} (|\alpha(t_i) - \alpha(s_i)| + |\alpha(s_i) - \alpha(t_{i-1})|)$$

$$\leqq L \sum_{i=1}^{k} ((t_i - s_i) + (s_i - t_{i-1})) = Ll(\beta).$$

This yields $l(\alpha) \leqq Ll(\beta)$ as required.

We conclude this section with a theorem which summarizes several results above. We use the standard notation

$$L(x,f)=\limsup_{y\to x}\frac{|f(y)-f(x)|}{|y-x|},\qquad l(x,f)=\liminf_{y\to x}\frac{|f(y)-f(x)|}{|y-x|}.$$

2.16. **Theorem.** *For $f: G\to R^n$ and $L\geqq 1$ the following conditions are equivalent:*

(1) *f is L-BLD.*

(2) *For each $x\in G$ there is $r>0$ such that*

$$|y-x|/L\leqq|f(y)-f(x)|\leqq L|y-x|$$

for all $y\in B(x,r)$, and $J(x,f)\geqq 0$ a.e.

(3) *f is discrete, open and sense-preserving, and*

$$l(\alpha)/L\leqq l(f\alpha)\leqq Ll(\alpha)$$

for every path α in G.

(4) *$L(x,f)\leqq L$ and $l(x,f)\geqq 1/L$ for each $x\in G$, and $J(x,f)\geqq 0$ a.e.*

(5) *f is QR and $|f'(x)|\leqq L$, $l(f'(x))\geqq 1/L$ a.e.*

Proof. We first show that (1) ⇒ (2) ⇒ (4) ⇒ (5) ⇒ (1). Suppose that f is L-BLD. By 2.7, f is discrete and open. Hence each $x\in G$ has a normal neighborhood, and (2) follows from 2.13.

The implication (2) ⇒ (4) is trivial. If (4) holds, f is locally L-lipschitz and hence ACL^n. Moreover, Rademacher's theorem implies that f is differentiable a.e. and that $|f'(x)|\leqq L$, $l(f'(x))\geqq 1/L$ a.e. Hence

$$|f'(x)|^n\leqq L^{2n}l(f'(x))^n\leqq L^{2n}J(x,f)$$

a.e., and f is thus QR. Since a QR map is ACL and has $J(x,f)\geqq 0$ a.e., the implication (5) ⇒ (1) is clear.

Since (1) ⇒ (3) follows at once from 2.7 and 2.15, it remains to show that (3) ⇒ (1). If the line segment γ from x to y lies in G, (3) implies

$$|f(y)-f(x)|\leqq l(f\gamma)\leqq Ll(\gamma)=L|x-y|.$$

Thus f is locally L-lipschitz and consequently ACL^∞. Moreover f is differentiable a.e. and $|f'(x)|\leqq L$ a.e. Since f is sense-preserving, $J(x,f)\geqq 0$ a.e., see [MRV_1, 2.14]. Fix $x\in G$ such that f is differentiable at x. It suffices to show that $|f'(x)h|\geqq|h|/L$ for all $h\in R^n$. Suppose that this is false for some $h=h_0$. Then

$$|f'(x)h_0|/|h_0|=\alpha<1/L. \tag{2.17}$$

We may assume that $x=0=f(x)$. Let $U=U(0,f,r)$ be a normal neighborhood of 0. Now f has the expansion

$$f(h)=f'(0)h+\varepsilon(h)h, \tag{2.18}$$

where $\varepsilon(h)\to 0$ as $h\to 0$. Pick $t>0$ so small that $h=th_0\in U$ and

$$2|\varepsilon(h)|\leqq 1/L-\alpha. \tag{2.19}$$

Let γ be the line segment from 0 to $f(h)$. By 2.11 there is a lift γ^* of γ from 0 to h. Then (3) implies

$$|f(h)| = l(\gamma) \geqq l(\gamma^*)/L \geqq |h|/L. \tag{2.20}$$

On the other hand, (2.17), (2.18), and (2.19) yield

$$\begin{aligned}|f(h)| &\leqq |f'(0)h| + |\varepsilon(h)|\,|h| \leqq \alpha|h| + (1/L - \alpha)\,|h|/2 \\ &= (1/L + \alpha)\,|h|/2 < |h|/L.\end{aligned}$$

This contradicts (2.19) and completes the proof of the theorem.

3. Elliptic Equations

3.1. Suppose that D is a domain in R^n and that $p > 1$. Let $A: D \times R^n \to R^n$ be an elliptic second order partial differential operator in divergence form, that is, A satisfies the following conditions:

(a) For each $\varepsilon > 0$ there is a closed set $F \subset D$ such that $m(D \setminus F) < \varepsilon$ and $A|F \times R^n$ is continuous.

(b) There are positive numbers γ_1, γ_2 such that for almost every $x \in D$,

$$|A(x,h)| \leqq \gamma_1 |h|^{p-1} \quad \text{(boundedness)}, \tag{3.2}$$

$$A(x,h) \cdot h \geqq \gamma_2 |h|^p \quad \text{(ellipticity)}, \tag{3.3}$$

for all $h \in R^n$.

An ACL^p-function $u: D \to R^1$ is a solution of the equation

$$\nabla \cdot A(x, \nabla u(x)) = 0 \tag{3.4}$$

if

$$\int_D A(x, \nabla u(x)) \cdot \nabla \phi(x)\, dm(x) = 0$$

for all $\phi \in C_0^\infty(D)$. Note that we always assume that an ACL^p-function is continuous. We often write $\nabla \cdot A = 0$ for the Eq. (3.4).

3.5. *Remarks.* (a) It is possible to consider equations $\nabla \cdot A = B$, which are more general than (3.4). Here A and B may also depend on u; for conditions on A and B see [Se, p. 247].

(b) If A also satisfies the inequality

$$(A(x,h_1) - A(x,h_2)) \cdot (h_1 - h_2) > 0 \tag{3.6}$$

whenever $h_1 \neq h_2$ for almost every $x \in D$, then the solutions of (3.4) can be used to build a potential theory similar to the classical potential theory formed by harmonic functions. In general, this theory is non-linear, see [GLM] and [HK].

(c) An important class of Eq. (3.4) is obtained as Euler equations of the variational integrals

$$\int_G F(x, \nabla u(x))\, dm(x),$$

where $F(x,h)\approx|h|^p$. In particular, the case $F(x,h)=h^p$ gives the so-called p-harmonic equation

$$\nabla\cdot(|\nabla u|^{p-2}\nabla u)=0.$$

3.7. Next let f be an ACL map of a domain $G\subset R^n$ into D. The pullback $f^{\#}A$ of A is defined by

$$f^{\#}A(x,h)=J(x,f)f'(x)^{-1}A(f(x),f'(x)^{-1*}h) \tag{3.8}$$

whenever $x\in G$ is such that $J(x,f)\neq 0$. If $J(x,f)$ does not exist or if $J(x,f)=0$, we set

$$f^{\#}A(x,h)=A(f(x),h).$$

However, this case does not play any role in the sequel. We recall that in (3.8), $f'(x)$ is the formal derivative of f at x, $f'(x)^{-1}$ is its inverse, and T^* is the transpose of the linear map $T: R^n\to R^n$. The main motivation to (3.8) is Theorem 3.17 below, which shows that under suitable conditions, solutions of $\nabla\cdot A=0$ give rise to solutions of $\nabla\cdot f^{\#}A=0$.

We next investigate the relation between the operators A and $f^{\#}A$.

3.9. **Theorem.** *Suppose that $f: G\to D$ is* ACL, *that $J(x,f)>0$ a.e. and that A and $f^{\#}A$ satisfy* (a) *and* (b) *of* 3.1 *in D and in G, respectively. Then*

(1) *f is L-BLD if $p\neq n$,*

(2) *f is K-QR if $p=n$ and if $J(x,f)$ is locally integrable.*

In both cases, L and K depend only on the constants for A and $f^{\#}$ A and on p and n.

Proof. Let γ_1', γ_2' be the constants for $f^{\#}A$ in (3.2) and (3.3). Let E be the set of all $x\in D$ which do not satisfy (3.2) and (3.3), and let $E'\subset G$ be the corresponding set for $f^{\#}A$. Then $m(E)=m(E')=0$. From [RR, Lemma 7, p. 348] it follows that $J(x,f)=0$ a.e. in $f^{-1}E$. Hence $m(f^{-1}E)=0$. Fix $x\in G\setminus(E'\cup f^{-1}E)$ such that $J(x,f)>0$. Let $h\in R^n$ and write $h'=f'(x)^*h$. Then

$$\begin{aligned}J(x,f)\gamma_2|h|^p&\leqq J(x,f)A(f(x),h)\cdot h=f^{\#}A(x,h')\cdot h'\\&\leqq\gamma_1'|h'|^{p-1}|h'|=\gamma_1'|f'(x)^*h|^p,\end{aligned}$$

and choosing h such that $|h|=1$ and $|f'(x)^*h|=l(f'(x))$ we obtain

$$\gamma_2 J(x,f)\leqq\gamma_1' l(f'(x))^p. \tag{3.10}$$

On the other hand,

$$\begin{aligned}J(x,f)\gamma_1|h|^p&\geqq J(x,f)A(f(x),h)\cdot h=f^{\#}A(x,h')\cdot h'\\&\geqq\gamma_2'|h'|^p=\gamma_2'|f'(x)^*h|^p,\end{aligned}$$

and choosing h such that $|h|=1$ and $|f'(x)^*h|=|f'(x)|$ we obtain

$$\gamma_1 J(x,f)\geqq\gamma_2'|f'(x)|^p. \tag{3.11}$$

The inequalities (3.10) and (3.11) hold a.e. in G.

If $p=n$ and if $J(x,f)$ is locally integrable, then these inequalities imply that f is K-QR with $K=\max(\gamma_1/\gamma_2', \gamma_1'/\gamma_2)$. Next assume that $p\neq n$. Now (3.10) and (3.11) yield

$$|f'(x)|\leqq Hl(f'(x)) \tag{3.12}$$

a.e. in G, where $H=(\gamma_1\gamma_1'/\gamma_2\gamma_2')^{1/p}$. Let first $p<n$. Then (3.10) gives

$$l(f'(x))^n\leqq J(x,f)\leqq(\gamma_1'/\gamma_2)l(f'(x))^p,$$

and hence

$$l(f'(x))\leqq(\gamma_1'/\gamma_2)^{1/(n-p)} \tag{3.13}$$

a.e. in G. By (3.11) we have

$$|f'(x)|^p\leqq(\gamma_1/\gamma_2')J(x,f)\leqq(\gamma_1/\gamma_2')\,|f'(x)|^n,$$

and hence

$$|f'(x)|\geqq(\gamma_2'/\gamma_1)^{1/(n-p)}>0 \tag{3.14}$$

a.e. in G; note that $J(x,f)>0$ a.e. Now (3.12), (3.13), and (3.14) yield

$$|f'(x)|\leqq L,\qquad l(f'(x))\geqq 1/L$$

a.e. in G, where $L=HK^{1/(n-p)}$. Hence f is L-BLD.

Next let $p>n$. Then (3.10) gives

$$\gamma_2 l(f'(x))^n\leqq\gamma_2 J(x,f)\leqq\gamma_1' l(f'(x))^p,$$

and thus

$$l(f'(x))\geqq(\gamma_2/\gamma_1')^{1/(p-n)}$$

a.e. in G. From (3.11) we obtain

$$\gamma_1|f'(x)|^n\geqq\gamma_1 J(x,f)\geqq\gamma_2'|f'(x)|^p,$$

and hence

$$|f'(x)|\leqq(\gamma_1/\gamma_2')^{1/(p-n)}$$

a.e. in G. Consequently,

$$|f'(x)|\leqq L',\qquad l(f'(x))\geqq 1/L'$$

a.e. in G, where $L'=K^{1/(p-n)}$. Hence f is L'-BLD.

We next consider the converse of Theorem 3.9:

3.15. **Theorem.** *Suppose that the Eq.* (3.4) *satisfies* (a) *and* (b) *of* 3.1 *and that $f: G\to D$ is L-BLD. Then the operator $f^{\#}A$ also satisfies* (a) *and* (b) *in G with constants γ_1', γ_2' depending only on the constants γ_1, γ_2 of A and on L, p and n. In the case $p=n$ it suffices to assume that f is K-QR; then γ_1', γ_2' depend on γ_1, γ_2, K, p, and n.*

Proof. Suppose that $f: G\to D$ is L-BLD. For every set $E\subset G$ we have $m(E)=0$ if and only if $m(fE)=0$. This follows from the local bilipschitz property of $f|G\setminus B_f$ and from 2.8. Hence it is not difficult to show that $f^{\#}A$ satisfies the condition (a) in G.

To prove (b) let $x \in G$ be such that f is differentiable at x and that (3.2) and (3.3) are true for A at $f(x)$. Let $h \in R^n$ and write $h^* = f'(x)^{-1*}h$. With Lemma 2.14 we obtain

$$\begin{aligned}|f^{\#}A(x,h)| &= |J(x,f)f'(x)^{-1}A(f(x),h^*)| \\ &\leqq L^{n-1}|A(f(x),h^*)| \leqq L^{n-1}\gamma_1|h^*|^{p-1} \\ &\leqq L^{n+p-2}\gamma_1|h|^{p-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}f^{\#}A(x,h)\cdot h &= J(x,f)A(f(x),h^*)\cdot h^* \\ &\geqq J(x,f)\gamma_2|h^*|^p \geqq \gamma_2 L^{2-n-p}|h|^p.\end{aligned}$$

These inequalities hold a.e. in G. Thus $f^{\#}A$ satisfies (3.2) and (3.3) with

$$\gamma_1' = L^{n+p-2}\gamma_1, \qquad \gamma_2' = L^{2-n-p}\gamma_2.$$

If $p=n$ and if $f: G \to D$ is non-constant and K-QR, the proof for (a) is similar to the above. For the property $m(E)=0 \Leftrightarrow m(fE)=0$, see [MRV$_1$, 8.4]. The inequalities in (b) follow as above from the quasiregularity conditions (2.2) with $\gamma_1' = K\gamma_1$, $\gamma_2' = \gamma_2/K$; note that $J(x,f)>0$ a.e. The case where f is a constant is trivial.

3.16. *Remark.* If A satisfies (3.6), it is easy to see that $f^{\#}A$ also satisfies (3.6) whenever f is BLD or, in the case $p=n$, QR.

The final theorem of this section states that a BLD map preserves the solutions of A and $f^{\#}A$. In the QR case this question has been studied in [Re$_2$] and in [GLM]. We shall adopt the method of [GLM].

3.17. **Theorem.** *Let $f: G \to D$ be L-BLD, and suppose that u is a solution of $\nabla \cdot A = 0$ in D. Then $v = u \circ f$ is a solution of $\nabla \cdot f^{\#}A = 0$ in G.*

Proof. Approximating u by smooth functions we see that v is ACL^p. To prove that v is a solution of $\nabla \cdot f^{\#}A = 0$ it suffices to show, by the partition of unity, that v is a solution of $\nabla \cdot f^{\#}A = 0$ in a normal neighborhood $U = U(x_0, f, r)$ of a point $x_0 \in G$. Set $B = fU = B(f(x_0), r)$.

Let $\psi \in C_0^\infty(U)$. We must show that

$$I = \int_U f^{\#}A(x, \nabla v(x)) \cdot \nabla\psi(x)\,dm(x) = 0.$$

Define $\psi^*: B \to R^1$ by

$$\psi^*(y) = \sum_{x \in f^{-1}(y) \cap U} i(x,f)\psi(x).$$

Then $\psi^* \in C_0(U)$ by [Ma, 5.4]. Setting

$$k = N(f,U), \qquad M = \sup_{x \in U} |\nabla\psi(x)|$$

we next show that ψ^* is kML-lipschitz. Fix $y, z \in B$, and let $\beta: [0, |z-y|] \to B$ be the segmental path

$$\beta(t) = y + t\frac{z-y}{|z-y|}.$$

Let $\alpha_j:[0,|z-y|]\to U$, $1\leq j\leq k$, be the lifts of β given by Lemma 2.10 (modified for closed intervals). Since f is L-BLD, each α_j is L-lipschitz, and hence $\psi\circ\alpha_j$ is ML-lipschitz. Now

$$\psi^*(\beta(t))=\sum_{j=1}^{k}\psi(\alpha_j(t))$$

for all $t\in[0,|z-y|]$, and hence

$$|\psi^*(z)-\psi^*(y)|\leq\sum_{j=1}^{k}|\psi(\alpha_j(|z-y|))-\psi(\alpha_j(0))|\leq kML|z-y|.$$

Thus ψ^* is kML-lipschitz.

Since f defines a closed map of U onto B, the set $V=B\setminus f[U\cap B_f]$ is open. If $B(y,r')$ is a ball in V, then $U\cap f^{-1}B(y,r')$ has exactly k components, each of which is mapped homeomorphically onto $B(y,r')$. We choose a sequence $B_1, B_2,\dots$ of disjoint open balls in V which almost cover V and hence B. Let $U_{i1},\dots,U_{ik}$ be the components of $U\cap f^{-1}B_i$, let $f_{ij}:U_{ij}\to B_i$ be the homeomorphism defined by f, and set $g_{ij}=f_{ij}^{-1}$. If $z\in B_i$, we have

$$\psi^*(z)=\sum_{j=1}^{k}\psi(g_{ij}(z)),$$

and hence

$$\nabla\psi^*(z)=\sum_{j=1}^{k}f'(g_{ij}(z))^{-1*}\nabla\psi(g_{ij}(z)) \tag{3.18}$$

a.e. in B_i. Since $m(B_f)=0$ and since $\nabla v(x)=f'(x)^*\nabla u(f(x))$ a.e. in U, we obtain

$$\begin{aligned}I&=\int_U J(x,f)f'(x)^{-1}A(f(x),\nabla u(f(x)))\cdot\nabla\psi(x)dm(x)\\&=\sum_{i=1}^{\infty}\sum_{j=1}^{k}\int_{U_{ij}}J(x,f)A(f(x),\nabla u(f(x)))\cdot f'(x)^{-1*}\nabla\psi(x)dm(x)\\&=\sum_{i=1}^{\infty}\sum_{j=1}^{k}\int_{B_i}A(y,\nabla u(y))\cdot f'(g_{ij}(y))^{-1*}\nabla\psi(g_{ij}(y))dm(y),\end{aligned}$$

by the transformation formula for integrals. By (3.18) this yields

$$\begin{aligned}I&=\sum_{i=1}^{\infty}\int_{B_i}A(y,\nabla u(y))\cdot\nabla\psi^*(y)dm(y)\\&=\int_B A(y,\nabla u(y))\cdot\nabla\psi^*(y)dm(y).\end{aligned}$$

Since u is a solution of $\nabla\cdot A=0$ and since ψ^* is a lipschitz function with compact support in B, the last integral vanishes. Hence $I=0$ as required.

3.19. *Remarks.* (a) The conditions (b) of 3.1 did not play any role in the proof of Theorem 3.17. Hence the theorem holds for all reasonable equations of the form $\nabla\cdot A=B$.

(b) If $p=n$, it suffices to assume that f is QR in Theorem 3.16, cf. [GLM].

(c) Suppose that A and A_1 are operators which satisfy (a) and (b) of 3.1 in the domains G and D, respectively. Let $f:G\to D$. If $u\circ f$ is a solution of $\nabla\cdot A=0$ whenever u is a solution of $\nabla\cdot A_1=0$, is f BLD or constant for $p\neq n$ and QR for $p=n$?

4. Properties of BLD Maps

4.1. In this section we study distortion, normal families, boundary behavior and multiplicity functions of BLD maps. We pay a special attention to those properties which distinguish BLD maps from QR maps.

4.2. *Immersions.* Suppose that $f: G \to R^n$ is an immersion (local homeomorphism). Then f is L-BLD if and only if f is locally L-bilipschitz, see 2.5 and 2.7. These maps have been rather extensively studied by several people under the name local quasi-isometries or quasi-isometries, especially by John, Martio-Sarvas, Gehring, and Gevirtz. The following basic result follows from [Jo_1, Theorem III, p. 93]; a somewhat sharper result is given in [Jo_2, p. 276]:

4.3. **Lemma.** *If $f: G \to R^n$ is an L-BLD immersion and if $B(x,r) \subset G$, then $f|B(x,r/L^2)$ is L-bilipschitz. Hence an L-BLD immersion $f: R^n \to R^n$ is an L-bilipschitz homeomorphism onto R^n.*

4.4. *Remark.* The analogue of 4.3 for QR maps is true for $n \geqq 3$ but false for $n=2$ [MRV_2, 2.3].

4.5. *Normal Families.* Since BLD maps form a subclass of locally lipschitzian maps, normal family properties of BLD maps are much easier than the corresponding results in the QR case. We first need a lemma.

4.6. **Lemma.** *If $f: G \to R^n$ is L-BLD and if $B(x,r) \subset G$, then $B(f(x), r/L) \subset fG$.*

Proof. If the lemma is false, then there is a boundary point b of fG with $|b-f(x)| < r/L$. Let α be a maximal lift starting at x of the segmental path β from $f(x)$ to b, see 2.9. Then α converges to ∂G and hence $l(\alpha) \geqq r$. Since

$$l(f\alpha) \leqq l(\beta) < r/L \leqq l(\alpha)/L,$$

this is impossible by 2.16(3).

4.7. **Theorem.** *Let G be a domain in R^n and let $W_L(G)$ denote the family of all L-BLD maps $f: G \to R^n$. Then every sequence in $W_L(G)$ has a subsequence converging either to ∞ or to an L-BLD map, uniformly in compact sets.*

Proof. Since each member of $W_L(G)$ is L-lipschitz in convex sets, $W_L(G)$ is equicontinuous. If $F \subset G$ is compact, there is $M < \infty$ such that each pair of points in F can be joined by a path of length at most M. This implies

$$d(fF) \leqq LM \tag{4.8}$$

for each $f \in W_L(G)$.

Let J be a sequence in $W_L(G)$. If J has a subsequence J_1 converging to ∞ at some point of G, then by (4.8), J_1 converges to ∞ uniformly in compact sets. Otherwise J is uniformly bounded in compact sets. By Ascoli's theorem J has a subsequence $f_1, f_2, \ldots$ converging to a map $f: G \to R^n$ uniformly in compact subsets. Since every f_j is L-lipschitz in convex sets, so is f. Hence f is a.e. differentiable with $|f'(x)| \leqq L$. Moreover, f is sense-preserving [$\mathrm{Vä}_1$, 4.3], and hence $J(x,f) \geqq 0$ a.e. It remains to show that $l(f'(x_0)) \geqq 1/L$ at every point x_0 of differentiability.

To this end assume that $l(f'(x_0)) = a < 1/L$. Write

$$f(x_0 + h) = f(x_0) + f'(x_0)h + |h|\varepsilon(h)$$

where $\varepsilon(h) \to 0$ as $h \to 0$. Choose $r > 0$ such that $\bar{B}(x_0, r) \subset G$ and such that $|\varepsilon(h)| \leq (L^{-1} - a)/2 = q$ for $|h| \leq r$. Then

$$f\bar{B}(x_0, r) \subset f'(x_0)\bar{B}(x_0, r) + qr\bar{B}^n = A(r).$$

By 4.6, $B(f_j(x_0), r/L) \subset f_j B(x_0, r)$ for all j. Since $l(f'(x_0)) < 1/L$, there is a point $b \in S(f(x_0), r/L)$ such that $d(b, A(r)) \geq qr$. Hence we can choose points $y_j \in \bar{B}(x_0, r)$ with $d(f_j(y_j), A(r)) \geq qr$. We may assume that $y_j \to y \in \bar{B}(x_0, r)$. Then $f_j(y_j) \to f(y)$, and thus $f(y) \notin A(r)$, a contradiction.

4.9. *Remark.* Observe that the limit map in 4.7 is never a finite constant. In this respect BLD maps differ essentially from QR maps.

4.10. *Multiplicity.* We next derive upper bounds for the multiplicity numbers of BLD maps, defined in 2.9. No corresponding results exist for QR maps. The function $f(z) = z^k$ gives a counterexample for $n = 2$ and for higher dimensions see [MRV$_2$, 4.9].

Since an L-BLD map is $L^{2(n-1)}$-QR, the following result follows directly from [Ma, 6.1]:

4.11. **Theorem.** *If $f: G \to R^n$ is L-BLD, then $i(x, f) \leq L^{2(n-1)}$ for all $x \in G$.*

4.12. **Theorem.** *If $f: G \to R^n$ is L-BLD and if $B(x, r) \subset G$, then $N(f, B(x, ar)) \leq L^{2n}(1-a)^{-n}$ for every $a < 1$.*

Proof. We show that

$$N(y, f, B(x, ar)) \leq L^{2n}(1-a)^{-n}$$

for each $y \in R^n$. We may assume that $x = 0 = y$ and that $r = 1$. Write

$$B(x, a) \cap f^{-1}(y) = \{x_1, \ldots, x_k\}, \qquad k = N(y, f, B(x, ar)).$$

Set $b = (1-a)/L$ and define for each $e \in S^{n-1}$ a segmental path $\beta_e: [0, b) \to R^n$ by $\beta_e(t) = te$. Applying 2.9 we choose a maximal sequence $(\alpha_e^1, \ldots, \alpha_e^m)$ of lifts of β_e starting at $x_1, \ldots, x_k$. Then

$$l(\alpha_e^j) \leq L l(f\alpha_e^j) \leq L l(\beta_e) = 1 - a$$

for every j. Hence α_e^j does not converge to ∂B^n, which implies that α_e^j is a total lift defined on the whole interval $[0, b)$. It follows that $N(z, f, B^n) \geq m \geq k$ for all $z \in B(b) \setminus fB_f$. Since $m(fB_f) = 0$ by 2.8, this yields

$$\int_{B^n} J(x, f)\,dm(x) = \int_{R^n} N(z, f, B^n)\,dm(z)$$
$$= \int_{R^n \setminus fB_f} N(z, f, B^n)\,dm(z) \geq k\,m(B^n)\,b^n$$

by a transformation formula for integrals, see [RR, p. 260]. Since $J(x, f) \leq L^n$ a.e., we obtain

$$k \leq L^n/b^n = L^{2n}(1-a)^{-n}.$$

4.13. **Theorem.** *If $f: R^n \to R^n$ is L-BLD, then $N(f) \leqq L^{2n}$. Moreover, f is a closed map onto R^n.*

Proof. The inequality follows at once from 4.12 and the surjectivity from 4.6. If f is not closed, then there is a sequence (x_j) converging to ∞ such that $|x_i - x_j| \geqq 2$ for $i \neq j$ and such that $f(x_j) \to y \in R^n$. By 4.6, $fB(x_j, 1)$ contains the ball $B_j = B(f(x_j), 1/L)$. Since $y \in B_j$ for large j, $f^{-1}(y)$ is infinite, a contradiction.

4.14. **Theorem.** *Suppose that $f: G \to R^n$ is L-BLD and that $A \subset G$ is compact. Then*

$$N(f, A) \leqq cL^{2n}\left(1 + \frac{d(A)}{d(A, \partial G)}\right)^n$$

where c depends only on n.

Proof. Write $\delta = d(A, \partial G)$ and assume first that $\delta \leqq 2d(A)$. There is a constant c_1 depending only on n such that A can be covered by balls $B_j = B(x_j, \delta/2)$, $1 \leqq j \leqq k$, such that $x_j \in A$ and $k \leqq c_1(d(A)/\delta)^n$. Since $B(x_j, \delta) \subset G$, 4.12 gives $N(f, B_j) \leqq 2^n L^{2n}$. Hence

$$N(f, A) \leqq 2^n L^{2n} k \leqq 2^n L^{2n} c_1 (d(A)/\delta)^n,$$

and we choose $c = 2^n c_1$.

Next assume that $\delta > 2d(A)$. Pick $x_0 \in A$. Then $A \subset B(x_0, \delta/2)$ and since $B(x_0, \delta) \subset G$, 4.12 again gives $N(f, A) \leqq 2^n L^{2n}$. Hence we can choose $c = 2^n$ in this case.

4.15. *Normal Domains.* We recall from 2.9 that a domain $D \subset\subset G$ is a normal domain of $f: G \to R^n$ if $f\partial D = \partial f D$. The x-component $U(x, f, r)$ of $f^{-1}B(f(x), r)$ is a normal domain if and only if it has a compact closure in G. If, in addition, it meets $f^{-1}(f(x))$ only at x, it is a normal neighborhood of x.

4.16. **Lemma.** *Suppose that $f: G \to R^n$ is L-BLD and that $B(x, s) \subset G$. If $1/q \geqq 6(n+1)L^{2n+1}$, then $\bar{U}(x, f, qs) \subset B(x, s)$, and hence $U(x, f, qs)$ is a normal domain of f.*

Proof. We may assume that $x = 0 = f(0)$ and that $s = 1$. Set $a = 1/(n+1)$ and let $\{x_1, \ldots, x_k\} = f^{-1}(0) \cap B(a)$ with $x_1 = 0$. Then 4.12 implies $1 \leqq k \leqq L^{2n}(1-a)^{-n}$. For $j = 1, \ldots, k$ we let V_j denote the union of the loci of all maximal lifts of all radial paths $\beta: [0, q) \to R^n$, $\beta(t) = te$, $e \in S^{n-1}$, starting at x_j. Then $V_j \subset B(x_j, Lq)$.

We first show that the set $W = B(a - Lq) \cap f^{-1}B(q)$ is covered by $V_1 \cup \ldots \cup V_k$. Let $y \in W$, let γ be the segmental path from $f(y)$ to 0 and let α be a maximal lift of γ starting at y. Since $l(\alpha) \leqq Lq$, α is a total lift with end point $\alpha(0)$ in $B(a)$. Hence $\alpha(0) = x_j$ for some $j = 1, \ldots, k$. Thus the inverse of α is one of the paths defining V_j, and consequently, $y \in V_j$.

Assume that $\bar{U}(x, f, q) \not\subset B^n$. Then it is a connected set joining 0 and S^{n-1}. Hence W meets every sphere $S(t)$, $0 < t < a - Lq$. Since $d(V_j) \leqq 2Lq$ and since $V_1 \subset B(Lq)$, this implies $(2k-1)Lq \geqq a - Lq$, and hence

$$\begin{aligned} \frac{1}{q} &\leqq \frac{2kL}{a} \leqq 2L^{2n+1}(n+1)(1+1/n)^n < 2e(n+1)L^{2n+1} \\ &< 6(n+1)L^{2n+1}, \end{aligned}$$

a contradiction.

For the next lemma we recall that the Hausdorff distance $d_H(A,B)$ of non-empty compact sets A, B in R^n is

$$d_H(A,B)=\max\left(\max_{x\in A} d(x,B),\ \max_{y\in B} d(y,A)\right).$$

4.17. **Lemma.** *Suppose that $f: G\to R^n$ is L-BLD, that $x\in G$, and that $U=U(x,f,r)$ is a normal domain of f. Then $d_H(U\cap f^{-1}(f(x)), \partial U)\leqq Lr$.*

Proof. Set $y=f(x)$. If $x_0\in U\cap f^{-1}(y)$, then 4.6 implies that $d(x_0,\partial U)\leqq Lr$. Next assume that $z\in\partial U$. Choose s, $0<s<r$, such that $U_1=U(z,f,s)$ is a normal neighborhood of z. Set $V=B(y,r)\cap B(f(z),s)$. By [MRV$_1$, 2.5], f maps every component of $U\cap f^{-1}V$ onto V; hence $f[U\cap U_1]=V$. Choose a point $y_1\in V\cap[y,f(z)]$ and then $x_1\in U\cap U_1\cap f^{-1}(y_1)$. Choosing maximal lifts of the segments $[y_1,f(z)]$ and $[y_1,y]$ starting at x_1 we find a total lift $\alpha:[0,r]\to\bar U$ of the segment β defined by $\beta(t)=f(z)+t(y-f(z))/r$. Then $\alpha(r)=x_2\in U\cap f^{-1}(y)$, and

$$|x_2-z|\leqq l(\alpha)\leqq Ll(\beta)=Lr.$$

Thus $d(z,U\cap f^{-1}(y))\leqq Lr$ and the result follows.

4.18. *Distortion.* The numbers $l^*(x,f,r)$ and $L^*(x,f,r)$ were estimated in 2.12 in the case where $U(x,f,r)$ is a normal neighborhood of x. With the aid of 4.16 we next derive more general estimates for these numbers. A simple topological lemma is needed.

4.19. **Lemma.** *Suppose that $f: G\to R^n$ is open, that $x\in G$ and that $U=U(x,f,r)$ is a normal domain of f contained in a bounded domain $D\subset G$ with connected boundary. Then ∂U is connected.*

Proof. Since ∂D is connected, D has a connected complement CD. Hence CD is contained in the unbounded component E of CU. Set $F=CU\setminus E$ and $B=fU=B(f(x),r)$. Then F is a compact subset of D. Since f is open, $\partial fF\subset f\partial F\subset f\partial U=\partial B$, which implies $fF\subset\bar B$. Hence $F=\emptyset$, since f cannot be open at the points of ∂F.

4.20. **Theorem.** *For every $L\geqq 1$ and $n\geqq 2$ there are numbers $c_1, c_2\geqq 1$ with the following properties: If $f: G\to R^n$ is L-BLD and if $B(x,c_1r)\subset G$, then $U(x,f,r)$ is a normal domain of f, and*

(1) $r/L\leqq l^*(x,f,r)\leqq Lr$,

(2) $r/L\leqq L^*(x,f,r)\leqq c_2r$.

Proof. Let $c_0=6(n+1)L^{2n+1}$, and suppose that $B(x,c_0r)\subset G$. By 4.16, $U=U(x,f,r)$ is a normal domain of f with $\bar U\subset B(x,c_0,r)$. Set $y=f(x)$ and choose $z\in\partial U$ with $|z-x|=l^*(x,f,r)$. Since the segment $[z,x]$ lies in G, we have $r=|f(z)-y|\leqq L|z-x|$, which is the first inequality of (1). From 4.17 we obtain

$$l^*(x,f,r)=d(x,\partial U)\leqq Lr,$$

which is the second inequality of (1).

The first inequality of (2) is a trivial consequence of (1). To prove the second inequality of (2) is more difficult. Setting $K=3L+2^{n+2}L^{2n+1}$ we show that it is true for $c_1=\max(c_0,2K)$ and for $c_2=K$. Suppose that $B(x,c_1r)\subset G$ and that $L^*(x,f,r)>Kr$. Since $U\subset B(x,c_1r)\subset G$, 4.19 implies that ∂U is connected. Hence, by (1), ∂U

meets the spheres $S(x,t)$ for all $t\in[Lr,Kr]$. Let m be the largest integer satisfying $(2m+1)L\leqq K$. Then $Kr/(2m+1)=s\geqq Lr$. Now the spheres $S(x,(2j+1)s)$ meet ∂U for $j=0,\ldots,m$. Thus 4.17 implies

$$N(y,f,B(x,Kr))=N\geqq m.$$

Since $(2m+3)L>K$, we obtain

$$N>(K-3L)/2L=2^{n+1}L^{2n}.$$

On the other hand, since $B(x,2Kr)\subset G$, 4.12 yields $N\leqq 2^n L^{2n}$, a contradiction.

4.21. *Hausdorff Measures.* If $A\subset R^n$ and $0\leqq p\leqq n$, we let $m_p(A)$ and $\dim_H A$ denote the p-dimensional Hausdorff measure and the Hausdorff dimension of A, respectively. We next show that $m_p(A)$ is quasi-invariant and $\dim_H A$ is invariant under BLD maps. For bilipschitz maps these results are trivial and hence the invariance of $\dim_H A$ is clear whenever A lies outside B_f because a BLD map $f\colon G\to R^n$ is locally bilipschitz in $G\setminus B_f$. Note that the QR analogues are false, see [GV].

4.22. **Theorem.** *Suppose that $f\colon G\to R^n$ is L-BLD, that $0\leqq p\leqq n$ and that $A\subset G$. Then $m_p(fA)\leqq L^p m_p(A)$. If $r_G(A)=d(A)/d(A,\partial G)<\infty$, then $m_p(fA)\geqq m_p(A)/c$ where c depends only on L, p, n, and $r_G(A)$.*

Proof. Since f is locally L-lipschitz, the first inequality is clear. To prove the second inequality we choose $t>0$ and estimate the approximating measure $m_p^t(fA)$. Set $\delta=d(A,\partial G)$, $A_1=A+\bar{B}(\delta/2)$ and

$$q=\min(\delta/2c_1,\delta/2L,t/c_2),$$

where c_1 and c_2 are given by 4.20. Let $\varepsilon>0$ and choose a countable covering of fA by balls $B_j=B(y_j,r_j)$ such that $r_j\leqq q$, $\sum r_j^p\leqq m_p(fA)+\varepsilon$ and $y_1,y_2,\ldots$ are distinct points in fA. From 4.14 we obtain an estimate $N(f,A_1)\leqq c_3$ with c_3 depending only on L, n, and $r_G(A)$. Hence the sets $Q_j=A_1\cap f^{-1}(y_j)$ have cardinalities at most c_3. Since $c_1r_j\leqq c_1q\leqq\delta/2$, 4.20 implies that $U(x)=U(x,f,r_j)$ is a normal domain of f for each $x\in Q_j$. Moreover,

$$R_x=L^*(x,f,r_j)\leqq c_2r_j.$$

We show that the sets $U(x)$, $x\in Q_j$, $j=1,2,\ldots$, cover A. If $a\in A$, then $f(a)$ belongs to some B_j. Choose a maximal lift α of the segment from $f(a)$ to y_j, starting at a. Then

$$l(\alpha)\leqq L|f(a)-y_j|<Lr_j\leqq\delta/2,$$

and hence α terminates at a point $x\in Q_j$. Thus $a\in U(x)$ as required.

Since $R_x\leqq c_2q\leqq t$, we obtain

$$m_p^t(A)\leqq\sum_j\sum\{R_x^p: x\in Q_j\}\leqq\sum_j c_3c_2^pr_j^p\leqq c_3c_2^p(m_p(fA)+\varepsilon).$$

Since $\varepsilon>0$ and $t>0$ were arbitrary, this yields $m_p(A)\leqq c_3c_2^pm_p(fA)$.

4.23. **Corollary.** *A* BLD *map preserves the Hausdorff dimension of every set.*

4.24. *The Branch Set.* The branch set B_f of a K-QR map $f: G \to R^n$ can be rather complicated for $n \geqq 3$, even if K is relatively small. For example, there is a universal K and K-QR maps $f: R^3 \to R^3$ such that B_f consists of an arbitrarily large number of rays from the origin [Ri_1, p. 264]. Our next result shows that this cannot happen for BLD maps. Note that for $n=2$ the branch set B_f of a QR map $f: G \to R^2$, and hence that of a BLD map, is a discrete set of points in G. However, even for $n=2$ our result imposes metric conditions on B_f in the BLD case. These conditions do not hold for plane QR maps.

The concept of a uniform domain was introduced in [MS]. We recall the definition. A domain D in R^n is c-uniform if for each pair of points $x_1, x_2 \in D$ there is a rectifiable path $\alpha: [0, l(\alpha)] \to D$, parametrized by arc length, joining x_1 to x_2 with

$$l(\alpha) \leqq c|x_1 - x_2|,$$

$$d(\alpha(t), \partial D) \geqq \frac{1}{c} \min(t, l(\alpha) - t), \qquad t \in [0, l(\alpha)].$$

For a survey of different characterizations see [$\mathrm{Vä}_4$].

4.25. **Theorem.** *If $f: R^n \to R^n$ is L-BLD, then $\complement B_f$ and $\complement f B_f$ are c-uniform domains with $c = c(n, L)$.*

Proof. Since the topological dimension of B_f is at most $n-2$, $D = \complement B_f$ is connected for every discrete and open map $f: R^n \to R^n$. Let $L \geqq 1$ and let H_L be the family of the branch sets of all L-BLD maps $f: R^n \to R^n$. By [$\mathrm{Vä}_4$, 3.6] it suffices to show that H_L is stable in the sense of [$\mathrm{Vä}_4$, 3.1]. This means that (1) H_L is invariant under similarities of R^n and that (2) the family $H_L^2 = \{A \in H_L : \{0, e_1\} \subset \partial A\}$ is compact in the Hausdorff metric of $\dot{R}^n = R^n \cup \{\infty\}$. Since $\phi \circ f \circ \phi^{-1}$ is L-BLD whenever f is L-BLD and ϕ is a similarity, the condition (1) is clear. To prove (2), consider a sequence of L-BLD maps $f_j: R^n \to R^n$ such that their branch sets B_{f_j} converge to a set A and $\{0, e_1\} \subset B_{f_j}$. We may assume that $f_j(0) = 0$. By 4.7 we may assume that (f_j) converges uniformly in compact sets to an L-BLD map $f: R^n \to R^n$. Now [MR, 3.2] yields $B_f = A$. Hence $A \in H_L$. The proof for $f B_f$ is similar.

4.26. *Remark.* An elaboration of the preceding proof shows that if G is c-uniform and if $f: G \to R^n$ is L-BLD, then $G \setminus B_f$ is c_1-uniform with $c_1 = c_1(n, L, c)$.

4.27. *The Hausdorff Dimension of B_f.* Suppose that $f: G \to R^n$ is BLD with $B_f \neq \emptyset$. Then $\dim_H f B_f \geqq n-2$, since this is true for all discrete and open maps [MRV_2, 3.4]. By 4.23, this implies $\dim_H B_f \geqq n-2$. The corresponding QR result is only known for $n=2$ and 3 [MR, 2.20]. To the other direction, $\dim_H f B_f \leqq c = c(n, K) < n$ for every K-QR map f [Sa, 5.13]. Hence

$$\dim_H B_f = \dim_H f B_f \leqq c = c(n, L^{2(n-1)})$$

for L-BLD maps. Note that $\dim_H B_f = n-2 = \dim_H f B_f$ for all discrete and open maps f with $B_f \neq \emptyset$ in the plane.

We next show that if $n \geqq 3$, then $\dim_H B_f$ can be arbitrarily close to n. Suppose first that $n=3$. Let $m \geqq 2$ be an integer, and let $g: R^3 \to R^3$ be the winding map, defined by $g(r, \phi, x_3) = (r, m\phi, x_3)$ in the cylindrical coordinates of R^3. Then B_g is the line $Z = \{x \in R^3 : x_1 = x_2 = 0\}$. Let $0 < \varepsilon < 2$. Applying [GV, 20] we choose a K-QR homeomorphism, i.e. a K-quasiconformal map, $h: R^3 \to R^3$ sending Z to a curve E with $\dim_H E = 3-\varepsilon$. By [TV, 7.12] we can choose h so that the map $h|R^3\backslash Z$ onto $R^3\backslash E$ is L-bilipschitz with some $L=L(K)$ in the quasihyperbolic metrics of $R^3\backslash Z$ and $R^3\backslash E$. The map $f=hgh^{-1}: R^3 \to R^3$ is QR with $B_f=E$. We show that f is BLD. It suffices to show that $|f'(x)|$ is a.e. bounded away from 0 and ∞.

If $y_0 \in E$ and $r>0$, we let as usual $L(y_0,f,r)$ and $l(y_0,f,r)$ denote the maximum and minimum, respectively, of $|f(y)-f(y_0)|$ over $y \in S(y_0,r)$. Since $|g(x)-g(x_0)| = |x-x_0|$ for all $x_0 \in Z$ and $x \in R^3$, we obtain from [MS, 2.16] that

$$L(y_0,f,r) \leqq H l(y_0,f,r) \tag{4.28}$$

with $H=H(K)$.

Suppose that y is a point in $R^3\backslash E$ at which $f'(y)$ exists. Choose $y_0 \in E$ with $|y-y_0|=d(y,E)=r$, and then $y_1 \in E$ with $|y_1-y_0|=r$. Since $f|E=\mathrm{id}$, we have $l(y_0,f,r) \leqq |y_0-y_1|=r$. Hence (4.28) yields

$$d(f(y),E) \leqq |f(y)-f(y_0)| \leqq L(y_0,f,r) \leqq Hr.$$

Since g is locally m-bilipschitz in $R^3\backslash Z$, $f|R^3\backslash E$ is locally L_1-bilipschitz in the quasihyperbolic metric of $R^3\backslash E$ with $L_1=mL^2$. This implies

$$\frac{|f'(y)|d(y,E)}{d(f(y),E)} \leqq L_1$$

and thus $|f'(y)| \leqq L_1 H$.

A lower bound $|f'(y)| \geqq q = q(K)$ is obtained similarly considering the numbers L^*, l^* instead of L, l. Hence f is BLD.

If $n>3$, we write $R^n = R^3 \times R^{n-3}$ and define $f: R^n \to R^n$ by $f=f_0 \times \mathrm{id}$ where f_0 is the BLD map of R^3 defined above. Then f is BLD and $B_f = E \times R^{n-3}$; thus $\dim_H B_f = n-\varepsilon$.

4.29. *Quasiconformal Groups.* The method of 4.27 also gives new examples of quasiconformal (in fact, bilipschitz) groups not quasiconformally conjugate to Möbius groups. The first such example was given by Tukia [Tu]. Let $T: R^3 \to R^3$ be the rotation $T(r,\phi,x_3)=(r,\phi+2\pi/m,x_3)$ in the cylindrical coordinates. Then $G=\{\mathrm{id}, T, \ldots, T^{m-1}\}$ is a finite cyclic group of rotations of R^3. Let h be the homeomorphism of 4.27. Then $G_1 = hGh^{-1}$ is a group of bilipschitz maps of R^3. If $n \geqq 4$, we write $R^n = R^3 \times R^{n-3}$ and define the group $G_2 = \{f \times \mathrm{id}: f \in G_1\}$. Then G_2 is a finite bilipschitz group in R^n. The fixpoint set of each $g \in G_2$, $g \neq \mathrm{id}$, is $E \times R^{n-3}$. This set cannot be mapped onto R^{n-2} by a quasiconformal map of R^n; this is due to the nonrectifiability of E and to [Tu, Lemma 4]. On the other hand the fixpoint set of a Möbius map is a sphere or an affine subspace. Hence G_2 is not quasiconformally conjugate to a Möbius group. Note that for $m=2$ the group G_2 is algebraically isomorphic to Z_2, the simplest non-trivial group.

4.30. *Boundary Behavior.* Compared with QR maps, the boundary behavior of BLD maps is considerably simpler. For example, if G is a convex domain, every L-BLD map $f: G \to R^n$ is L-lipschitz, and has thus a continuous extension to $\bar{G}$.

More generally, a BLD map $f: G \to R^n$ has a limit at a boundary point $b \in \partial G$ if G is *rectifiably locally connected* at b. By this we mean that for every $\varepsilon > 0$ there is $\delta > 0$ such that each pair of points in $D \cap B(b, \delta)$ can be joined in D by a path of length less than ε. Indeed, we then have $|f(x) - f(y)| \leqq L\varepsilon$ for all $x, y \in D \cap B(b, \delta)$. For example, a Jordan domain in R^2 with rectifiable boundary is rectifiably locally connected. This can be proved either directly or by conformal mapping; cf. [Po, 10.3]. Also a quasiball D in R^n, i.e. D is the image of B^n or a half space under a quasiconformal map $g: R^n \to R^n$, is rectifiably locally connected at each boundary point. Indeed, for a domain $D \subset R^n$ we have: quasiball $\Rightarrow$ uniform domain $\Rightarrow$ quasiextremal distance domain $\Rightarrow$ quasiconvex $\Rightarrow$ rectifiably locally connected. For the concepts quasiextremal distance domain and quasiconvex as well as for the middle implications, see [GM]. The last implication is obvious, and the first one follows from [MS, 2.15]. Note that although the boundary of a quasiball in the plane is a Jordan curve, it need not be rectifiable; in fact, its Hausdorff dimension can be arbitrary close to 2.

On the other hand, if D is the unit disk in R^2 minus the positive real axis, then it is easy to see that for each $L > 1$ there is an L-BLD map $f: D \to R^n$ without continuous extension to the boundary. More interesting examples can be constructed using the following idea: Suppose that $A_1, A_2 \subset R^2$ are open C^2 arcs of infinite length, and that $g: A_1 \to A_2$ is a length preserving homeomorphism. For $x \in A_1$ let $N_1(x)$ be the line through x, orthogonal to A_1. There is a neighborhood U of A_1 such that (1) $N_1(x) \cap U$ is connected for every $x \in A_1$ and (2) $N_1(x) \cap N_1(y) \cap U = \emptyset$ for $x \neq y$. Then we extend g to a continuous map $f: U \to R^2$ so that for each $x \in A_1$, $f|N_1(x) \cap U$ is an isometry into the line $N_2(g(x))$ orthogonal to A_2. The restriction of f to a smaller neighborhood G of A_1 is an L-BLD homeomorphism onto a neighborhood of A_2. Moreover, L can be chosen to be arbitrarily close to one.

For example, we can choose A_1 and A_2 to be the graphs of the functions $\phi_1(t) = t \sin(1/t)$ and $\phi_2(t) = \sin(1/t)$, $0 < t < 1$, respectively. Then G can be chosen to be a Jordan domain, and $f: G \to R^2$ is a bounded injective BLD map with no limit at the origin. Alternatively, letting A_2 be a half line, we obtain an unbounded BLD map of a Jordan domain.

We close this paper with a simple removability result.

4.31. **Theorem.** *Suppose that G is a domain in R^n and that E is a closed subset of G with $m_{n-1}(E) = 0$. If $f: G \setminus E \to R^n$ is L-BLD, then f has an L-BLD extension $f^*: G \to R^n$.*

Proof. Since $m_{n-1}(E) = 0$, $G \setminus E$ is a domain. By the same reason almost every line segment in G orthogonal to a coordinate plane omits E. Hence $G \setminus E$ is rectifiably locally connected at each point $b \in E$. Thus f has a continuous extension f^* to G. By the above segment property f^* is ACL. Finally, since $m_n(E) = 0$, the double inequality $|h|/L \leqq |f^{*\prime}(x)h| \leqq L|h|$ holds a.e. in G. Since $J(x, f^*) \geqq 0$ a.e. as well, f^* is L-BLD.

4.32. *Remark.* The same proof shows that 4.31 is also true for locally L-lipschitz maps.

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