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On Teichmüller modular forms

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Introduction

In this paper, we define **Teichmüller modular forms** (of degree $\geqq 3$) as global sections of the automorphic line bundles on the moduli space of algebraic curves, which are seen to be, over **C**, holomorphic functions on the Teichmüller space satisfying the automorphy conditions. Restricting to the jacobian locus, we obtain Teichmüller modular forms from Siegel modular forms. On the other hand, the Mumford form for algebraic curves of genus 3 induces a Teichmüller modular form of degree 3 and weight 9, which is expressed by the square root of a Siegel modular form but cannot be induced from Siegel modular forms.

Since the Teichmüller space of degree $\geqq 2$ is known to be not a homogeneous space [13], there is no Fourier expansion of Teichmüller modular forms. Our aim is to construct an analogue of Fourier expansion for Teichmüller modular forms. This expansion is expected to equal, over local fields, the expansion with respect to the Koebe coordinates (: the fixed points and the eigenvalues of generators of Schottky groups associated with Riemann surfaces and Mumford curves), and is expected to have the following properties: an analogy of the q-expansion principle and a factorization principle.

We state our conjecture more precisely. Let $x_{\pm i}, y_i$ ($i = 1, \ldots, g$) be variables, and let A_g be the ring of formal power series over $\mathbf{Z}\left[x_i, \prod_{j<k} 1/(x_j - x_k)\right]$ ($i, j, k \in \{\pm 1, \ldots, \pm g\}$) with variables $y_1, \ldots, y_g$. Let $p_{ij} \in A_g$ ($i, j = 1, \ldots, g$) be the universal periods of curves [5] which are expanding the infinite product expression of the multiplicative periods of Riemann surfaces and Mumford curves [7, 14] with respect to the Koebe coordinates. For any commutative ring R with unit element, let $T_{g,h}(R)$ (resp. $S_{g,h}(R)$) denote the space of Teichmüller (resp. Siegel) modular forms over R of degree $g \geqq 3$ and weight $h \geqq 0$. Then we

conjecture that there exists a functorial R-linear map

$$\kappa_R : T_{g,h}(R) \to A_g \hat{\otimes} R$$

satisfies the following:

(1) If R is $\mathbf{C}$ or a complete nonarchimedean valuation field, then for any $f \in T_{g,h}(R)$, $\kappa_R(f)$ is equal to, under a canonical trivialization, the expansion of f with respect to the Koebe coordinates.

(2) κ_R is injective.

(3) If R' is a commutative ring containing R and $f \in T_{g,h}(R')$ satisfies $\kappa_{R'}(f) \in A_g \hat{\otimes} R$, then $f \in T_{g,h}(R)$.

(4) For any $\varphi \in S_{g,h}(R)$, let $\varphi|_{\mathcal{M}_g}$ be the Teichmüller modular form obtained by restricting φ to the jacobian locus, and

$$F_R(\varphi) \in R\left[q_{ij}, \prod_{i<j} 1/q_{ij}\right][[q_{11}, \ldots, q_{gg}]] \qquad (1 \leqq i, j \leqq g)$$

be the Fourier (q)-expansion of φ constructed by Faltings [3]. Then

$$\kappa_R(\varphi|_{\mathcal{M}_g}) = F_R(\varphi)|_{q_{ij}=p_{ij}} \, .$$

(5) There exists a functorial R-linear map $\Psi : T_{g,h}(R) \to T_{g-1,h}(R)$ such that

$$\kappa_R(\Psi(f)) = \kappa_R(f)|_{y_g=0} \qquad (f \in T_{g,h}(R))$$

and that

$$\Psi(\varphi|_{\mathcal{M}_g}) = \Phi(\varphi)|_{\mathcal{M}_{g-1}} \qquad (\varphi \in S_{g,h}(R)) \, ,$$

where $\Phi : S_{g,h}(R) \to S_{g-1,h}(R)$ denotes the Siegel Φ-operator.

The main result of this paper (Theorem 3.2) says that such a expansion exists for Teichmüller modular forms over fields of characteristic $\neq 2$, where (5) is shown in a weakened version. To prove this, we consider the expansion of Teichmüller modular forms with respect to the Koebe coordinates over the power series field $k((z))$ (k: the base field), and show that the coefficients of the above expansion belong to k using the universal jacobian and the (global and local) Torelli theorem. Therefore, this proof cannot work if the base field is of characteristic 2 because the local Torelli theorem does not hold in this case.

1 Teichmüller modular forms

1.1 Here we recall a summary of moduli theory on algebraic curves and abelian schemes. Let g and n be positive integers, and let $\mathcal{M}_{g,n}$ denote the moduli stack of smooth and proper curves of genus g with symplectic level n structures [2]. Let $\pi : \mathcal{C} \to \mathcal{M}_{g,n}$ be the universal curve, and let λ be the invertible sheaf on $\mathcal{M}_{g,n}$ such that for any scheme S over $\mathcal{M}_{g,n}$,

$$\lambda(S) = \wedge^g \pi_*(\Omega_{\mathcal{C} \times S/S}) \, .$$

Let $\mathscr{X}_{g,n}$ denote the moduli stack of principally polarized abelian schemes of relative dimension g with symplectic level n structures. Let $\varrho:\mathscr{A} \to \mathscr{X}_{g,n}$ be the universal abelian scheme with zero section $\varepsilon:\mathscr{X}_{g,n} \to \mathscr{A}$, and let μ be the invertible sheaf on $\mathscr{X}_{g,n}$ such that for any scheme S over $\mathscr{X}_{g,n}$,

$$\mu(S) = \wedge^g \varrho_*(\Omega_{\mathscr{A}\times S/S}) = \wedge^g \varepsilon^*(\Omega_{\mathscr{A}\times S/S}).$$

Then the Torelli map $\tau:\mathscr{M}_{g,n} \to \mathscr{X}_{g,n}$ satisfies that $\tau^*(\mu) = \lambda$. Let ζ_n be a primitive n-th root of 1. Then as is shown in [1] and [2], the geometric fibers of $\mathscr{M}_g$ and $\mathscr{X}_g$ over $\mathbf{Z}[\zeta_n, 1/n]$ are all irreducible. The action of $\mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z})$ on symplectic level n structures of abelian schemes induces the action on $\mathscr{M}_{g,n}$ and $\mathscr{X}_{g,n}$, and hence we have

$$\mathscr{M}_g \cong \mathscr{M}_{g,n}/\,\mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z}), \qquad \mathscr{X}_g \cong \mathscr{X}_{g,n}/\,\mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z}),$$

where $\mathscr{M}_g = \mathscr{M}_{g,1}$ and $\mathscr{X}_g = \mathscr{X}_{g,1}$. If $n \geqq 3$, then it is shown in [11] that $\mathscr{M}_{g,n}$ and $\mathscr{X}_{g,n}$ can be represented by schemes smooth over $\mathbf{Z}[\zeta_n, 1/n]$ which we denote by $M_{g,n}$ and $X_{g,n}$ respectively. The multiplication by -1 on level n structures induces an involution on $\mathscr{M}_{g,n}$ and $M_{g,n}$ which we denote by ι.

Assume that $g \geqq 3$, let k be a field whose characteristic $\mathrm{char}(k)$ is not 2, and assume that $n \geqq 3$ is prime to $\mathrm{char}(k)$ and $k \ni \zeta_n$. Then by the Torelli theorem and a result of Oort and Steenbrink [12], $\tau: M_{g,n} \to X_{g,n}$ induces the isomorphism $(M_{g,n/k})/\langle\iota\rangle \tilde{\to} \tau(M_{g,n/k})$ which implies that for any h,

$$\Gamma(M_{g,n/k}, \lambda^{\otimes h})^{\langle\iota\rangle} = \Gamma(\tau(M_{g,n/k}), \mu^{\otimes h}),$$

where $M_{g,n/k} = M_{g,n} \otimes_{\mathbf{Z}[\zeta_n,1/n]} k$, and $V^{\langle\iota\rangle}$ denotes the fixed part of V under the action of ι. By a result of Faltings [1, Chap. 5], there exists the Satake compactification $\bar{X}_{g,n}$ of $X_{g,n}$ with invertible sheaf which is a unique extension of μ. We denote this invertible sheaf by μ again. Let $\overline{\tau(M_{g,n/k})}$ be the schematic closure of $\tau(M_{g,n/k})$ in $\bar{X}_{g,n} \otimes_{\mathbf{Z}[\zeta_n,1/n]} k$, and let $N_{g,n/k}$ be its normalization. Then $N_{g,n/k}$ is finite over $\overline{\tau(M_{g,n/k})}$, and contains $\tau(M_{g,n/k})$ as an open subset because $\tau(M_{g,n/k}) \cong (M_{g,n/k})/\langle\iota\rangle$ is normal. Any point of $\overline{\tau(M_{g,n/k})} - \tau(M_{g,n/k})$ corresponds to $J_1 \times \ldots \times J_m$ with canonical polarization and symplectic level n structure, where J_i $(i = 1, \ldots, m)$ are the jacobian varieties of algebraic curves of genus $g_i \geqq 1$ $(i = 1, \ldots, m)$ satisfying $\sum_{i=1}^{m} g_i \leqq g$ except $m = 1$ and $g_1 = g$. Hence under $g \geqq 3$, the codimension of $N_{g,n/k} - \tau(M_{g,n/k})$ in $N_{g,n/k}$ is greater than 1. Therefore, there exists an invertible sheaf $\bar{\mu}$ on $N_{g,n/k}$ which is a unique extension of μ, and for any h,

$$\Gamma(\tau(M_{g,n/k}), \mu^{\otimes h}) = \Gamma(N_{g,n/k}, \bar{\mu}^{\otimes h}).$$

Let $\bar{M}_{g,n/k}$ be the normalization of $N_{g,n/k}$ in the function field of $M_{g,n/k}$. Then $M_{g,n/k}$ is an open subset of $\bar{M}_{g,n/k}$ whose complement has codimension > 1 because $\bar{M}_{g,n/k}$ is finite over $N_{g,n/k}$. Hence there exists an invertible sheaf $\bar{\lambda}$

on $\bar{M}_{g,n/k}$ which is a unique extension of λ and is the pull back of $\bar{\mu}$, and for any h,

$$\Gamma(M_{g,n/k}\lambda^{\otimes h}) = \Gamma(\bar{M}_{g,n/k}, \bar{\lambda}^{\otimes h}).$$

Let Π be the group with generators s_i and t_i $(i = 1, \dots, g)$ satisfying the single relation $\prod_{i=1}^{g}(s_i t_i s_i^{-1} t_i^{-1}) = 1$, and let $\mathrm{Aut}(\Pi)^+$ be the group of automorphisms of Π preserving the cup product

$$H^1(\Pi, \mathbf{Z}) \times H^1(\Pi, \mathbf{Z}) \to H^2(\Pi, \mathbf{Z}) \cong \mathbf{Z}.$$

Then it is known that $\mathrm{Aut}(\Pi)^+$ is a subgroup of $\mathrm{Aut}(\Pi)$ of index 2 and that the homomorphism $\psi: \mathrm{Aut}(\Pi) \to \mathrm{Aut}(\mathbf{Z}^{2g})$ induced from the natural projection $\Pi \to \Pi/[\Pi, \Pi] \cong \mathbf{Z}^{2g}$ maps $\mathrm{Aut}(\Pi)^+$ onto $\mathrm{Sp}_{2g}(\mathbf{Z})$. For a positive integer n, we define the subgroup $\mathrm{Aut}(\Pi)^+_n$ of $\mathrm{Aut}(\Pi)^+$ by

$$\mathrm{Aut}(\Pi)^+_n = \psi^{-1}(\{G \in \mathrm{Sp}_{2g}(\mathbf{Z}) \mid G \equiv I_{2g} \bmod(n)\}).$$

Let $X = (R, \{a_i, b_i\}_{1 \leqq i \leqq g})$ be a Riemann surface R of genus g with generators $\{a_i, b_i\}$ of $\pi_1(R)$ (: the fundamental group of R with fixed base point) satisfying $\prod_{i=1}^{g}(a_i b_i a_i^{-1} b_i^{-1}) = 1$. Then the period matrix $\Omega(X)$ of X is given by

$$\Omega(X) = \left(\int_{a_i} \omega_j\right)_{i,j} \left(\left(\int_{b_i} \omega_j\right)_{i,j}\right)^{-1},$$

where $\{\omega_j\}_{1 \leqq j \leqq g}$ is a basis of $H^0(R, \Omega_R)$. Let $\mathscr{H}_g$ denote the Siegel upper half space of degree g, and let $\mathscr{T}_g$ denote the Teichmüller space of degree g which is the complex manifold as the moduli space of the above pairs $X = (R, \{a_i, b_i\})$ with $\Omega(X) \in \mathscr{H}_g$. Then $\mathrm{Aut}(\Pi)^+$ acts on $\mathscr{T}_g$ via the isomorphism $\iota: \Pi \xrightarrow{\sim} \pi_1(R)$ given by $\iota(s_i) = a_i$ and $\iota(t_i) = b_i$ $(i = 1, \dots, g)$, and for any n, the quotient stack $\mathscr{T}_g / \mathrm{Aut}(\Pi)^+_n$ is isomorphic to the analytic stack (orbifold) associated with $\mathscr{M}_{g,n} \otimes_{\mathbf{Z}[\zeta_n, 1/n]} \mathbf{C}$.

1.2 Let $g \geqq 3$ and $h \geqq 0$ be integers. For any commutative ring R with unit element, put

$$T_{g,h}(R) = \Gamma(\mathscr{M}_g \otimes R, \lambda^{\otimes h}),$$

and we call these elements **Teichmüller modular forms** of degree g and weight h over R. Let $S_{g,h}(R)$ denote the space $\Gamma(\mathscr{A}_g \otimes R, \mu^{\otimes h})$ of Siegel modular forms of degree g and weight h over R. Then the Torelli map $\tau: \mathscr{M}_g \to \mathscr{A}_g$ induces an R-linear homomorphism $\tau^*: S_{g,h}(R) \to T_{g,h}(R)$. For each $\varphi \in S_{g,h}(R)$, we denote $\tau^*(\varphi)$ by $\varphi|_{\mathscr{M}_g}$ and call it a Teichmüller modular form of **Siegel modular type**.

Assume that $g = 3$. Then by $(f, f') \to ff'$ (f, f': sections of $\pi_*(\Omega_{\mathscr{C}/\mathscr{M}_3})$), the symmetric product $S^2(\pi_*(\Omega_{\mathscr{C}/\mathscr{M}_3}))$ of $\pi_*(\Omega_{\mathscr{C}/\mathscr{M}_3})$ is isomorphic to $\pi_*(\Omega^{\otimes 2}_{\mathscr{C}/\mathscr{M}_3})$, and hence

$$\lambda^{\otimes 4} = \wedge^6 S^2(\pi_*(\Omega_{\mathscr{C}/\mathscr{M}_3})) \cong \wedge^6 \pi_*(\Omega^{\otimes 2}_{\mathscr{C}/\mathscr{M}_3}).$$

By this isomorphism and the canonical isomorphism $\lambda^{\otimes 13} \cong \wedge^6 \pi_*(\Omega^{\otimes 2}_{\mathscr{C}/\mathscr{M}_3})$ by Mumford [10, Theorem 5.10], we have a Teichmüller modular form $f_{3,9}$ of degree 3 and weight 9 over $\mathbf{Z}$ which has no zeros on $\mathscr{M}_3$. Since $S_{3,9}(\mathbf{Q}) = \{0\}$ and $S_{3,18}(\mathbf{Q})$ is 1-dimensional, $f_{3,9}$ is not of Siegel modular type and $(f_{3,9})^2$ is, up to $\mathbf{Q}^\times$, equal to the Siegel cusp form of degree 3 and weight 18:

$$\prod_{\vec{a},\vec{b}} \left\{ \sum_{\vec{n}\in\mathbf{Z}^3} \exp(\pi\sqrt{-1}(\vec{n}+\vec{a})\cdot\Omega\cdot(\vec{n}+\vec{a}) + 2\pi\sqrt{-1}(\vec{n}+\vec{a})\cdot\vec{b}) \right\},$$

where $\vec{a}$ and $\vec{b}$ run through $(1/2)\mathbf{Z}^3$ such that $4\vec{a}\cdot\vec{b} \equiv 0(2)$ and $\Omega \in \mathscr{H}_3$.

1.3 Proposition. *For any field k of characteristic $\neq 2$, $T_{g,h}(k)$ is a finite dimensional k-vector space.*

Proof. Let $n \geqq 3$ be prime to char(k), and put $k' = k(\zeta_n)$. Since $\bar{M}_{g,n/k'}$ is proper over k',

$$T_{g,h}(k) \subset \Gamma(M_{g,n/k'}, \lambda^{\otimes h}) = \Gamma(\bar{M}_{g,n/k'}, \bar{\lambda}^{\otimes h})$$

is finite dimensional over k'.

1.4 Proposition. *$T_{g,h}(\mathbf{C})$ consists of holomorphic functions $f : \mathscr{T}_g \to \mathbf{C}$ satisfying*

$$f(\varrho(X)) = \det(C\Omega(X) + D)^h f(X) \qquad (\varrho \in \mathrm{Aut}(\Pi)^+, X \in \mathscr{T}_g),$$

where $\psi(\varrho) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Proof. Let $F_{g,h}$ be the space of holomorphic functions $\mathscr{T}_g \to \mathbf{C}$ satisfying the above automorphy condition, which is the space of analytic sections of $\lambda^{\otimes h}$ on the analytic stack $(\mathscr{M}_g \otimes \mathbf{C})^{\mathrm{an}}$ associated with $\mathscr{M}_g \otimes \mathbf{C}$. Then $T_{g,h}(\mathbf{C}) \subset F_{g,h}$, and by the principle of GAGA, for $n \geqq 3$,

$$F_{g,h} \subset \Gamma((\bar{M}_{g,n/\mathbf{C}})^{\mathrm{an}}, \bar{\lambda}^{\otimes h}) = \Gamma(\bar{M}_{g,n/\mathbf{C}}, \bar{\lambda}^{\otimes h}),$$

and hence $F_{g,h} \subset T_{g,h}(\mathbf{C})$.

2 Schottky uniformization

2.1 In this section, we review the Schottky uniformization theory on algebraic curves over local fields and its universal version. Let K be $\mathbf{C}$ or a complete nonarchimedean valuation field with multiplicative valuation $|\ |$. Let $\mathrm{PGL}_2(K)$ act on $\mathbf{P}^1(K)$ by the Möbius transformation. A Schottky group $\Gamma \subset \mathrm{PGL}_2(K)$ of rank g is **marked** if selecting a sequence $\gamma_1, \dots, \gamma_g$ of its free generators such that there exist $2g$ open domains bounded by Jordan curves if $K = \mathbf{C}$ (resp. $2g$ open disks if K is nonarchimedean) $D_{\pm 1}, \dots, D_{\pm g} \subset \mathbf{P}^1(K)$ satisfying

$$\overline{D_i} \cap \overline{D_j} = \emptyset \ (i \neq j), \qquad \gamma_i(\partial D_{-i}) = \partial D_i \ (i = 1, \dots, g).$$

Then for each $i = 1, \dots, g$, we can take uniquely $\alpha_{\pm i} \in D_{\pm i}$ and $\beta_i \in K^\times$ with $|\beta_i| < 1$ such that

$$\gamma_i = \begin{pmatrix} \alpha_i & \alpha_{-i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta_i \end{pmatrix} \begin{pmatrix} \alpha_i & \alpha_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \bmod(K^\times).$$

We call $(\alpha_{\pm i}, \beta_i)_{1 \leqq i \leqq g}$ the **Koebe coordinate** of $(\Gamma; \gamma_1, \dots, \gamma_g)$. For a Schottky group Γ over K, let C_Γ denote the smooth and proper curve over K uniformized by Γ [6, 8] which is the quotient K-analytic space U_Γ/Γ, where

$$U_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma\left(\mathbf{P}^1(K) - \bigcup_{i=1}^{g} (D_i \cup \overline{D_{-i}})\right)$$

is an open subset of $\mathbf{P}^1(K)$ whose complement is the closure of fixed points of elements of $\Gamma - \{1\}$. For a marked Schottky group $(\Gamma; \gamma_1, \dots, \gamma_g)$, let Γ' be the Schottky group generated by $\gamma_1, \dots, \gamma_{g-1}$, and let $C_{\Gamma'}$ be the associated algebraic curve over K. Then one can show that under $\beta_g \to 0$, C_Γ becomes the degenerate algebraic curve obtained by identifying α_g and α_{-g} in $C_{\Gamma'}$.

2.2 Let x_i, x_{-i}, and y_i $(i = 1, \dots, g)$ be variables, put

$$A_g = \mathbf{Z}\left[x_i, \prod_{j<k} \frac{1}{x_j - x_k}\right][[y_1, \dots, y_g]] \quad (i, j, k \in \{\pm 1, \dots, \pm g\}),$$

and let Ω be its quotient field. Let f_i be the element of $\mathrm{PGL}_2(\Omega)$ given by

$$f_i = \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y_i \end{pmatrix} \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \bmod(\Omega^\times),$$

and let F be the subgroup of $\mathrm{PGL}_2(\Omega)$ generated by f_i $(i = 1, \dots, g)$. Then F is a free group with generators f_i. Let

$$[a, b; c, d] = \frac{(a-c)(b-d)}{(a-d)(b-c)}$$

denote the cross-ratio of four points. For $i, j = 1, \dots, g$, let $\psi_{ij} : F \to \Omega^\times$ be the map given by

$$\psi_{ij}(f) = \begin{cases} y_i & (\text{if } i = j \text{ and } f \in \langle f_i \rangle) \\ [x_i, x_{-i}; f(x_j), f(x_{-j})] & (\text{otherwise}). \end{cases}$$

Then it is easy to see that ψ_{ij} depends only on double coset classes $\langle f_i \rangle \backslash f / \langle f_j \rangle$ $(f \in F)$.

2.3 Proposition. *For any* $f = f_{\sigma(1)} f_{\sigma(2)} \cdots f_{\sigma(n)}$ $(\sigma(l) \in \{\pm 1, \dots, \pm g\})$ *satisfying* $\sigma(l) \neq -\sigma(l+1)$ $(l = 1, \dots, n-1)$, *put* $y = y_{\sigma(1)} y_{\sigma(2)} \cdots y_{\sigma(n)}$. *Then for* $i \neq \pm\sigma(1)$ *and* $j \neq \pm\sigma(n)$, $[x_i, x_{-i}; f(x_j), f(x_{-j})] \in 1 + yA_g$.

Proof. This is shown in the proof of Lemma 2.3 and Proposition 2.4 in [5].

2.4 Corollary. *For any $i, j = 1, \ldots, g$, the infinite product*

$$p_{ij}(\{f_1, \ldots, f_g\}) = \prod_f \psi_{ij}(f)$$

(f runs through all representatives of $\langle f_i\rangle\backslash F/\langle f_j\rangle$) is convergent in A_g, and if $i, j \leqq g-1$, then

$$p_{ij}(\{f_1, \ldots, f_g\})|_{y_g=0} = p_{ij}(\{f_1, \ldots, f_{g-1}\}).$$

2.5 By results of [7] and [14], it is known that $p_{ij} = p_{ij}(\{\gamma_1, \ldots, \gamma_g\}) \in A_g$ are convergent for the Koebe coordinate $(\alpha_{\pm i}, \beta_i)_{1\leqq i\leqq g}$ of a marked Schottky group $(\Gamma; \gamma_1, \ldots, \gamma_g)$ if K is nonarchimedean, or $K = \mathbf{C}$ and β_i $(i = 1, \ldots, g)$ are sufficiently small, and that the multiplicative periods of C_Γ are given by specializing p_{ij} with respect to the Koebe coordinate, i.e., the jacobian variety J_Γ of C_Γ is isomorphic as a K-analytic space to the quotient of $(K^\times)^g$ by its subgroup $\left\{\left(\prod_{j=1}^g q_{ij}{}^{n_j}\right)_{1\leqq i\leqq g} \middle| n_j \in \mathbf{Z}\right\}$, where $q_{ij} = p_{ij}|_{x_{\pm k}=\alpha_{\pm k}, y_k=\beta_k}$.

3 Canonical expansion

3.1 In this section, we construct a canonical expansion for Teichmüller modular forms over fields of characteristic $\neq 2$. Let A_g and Ω be as in 2.2, and let Y be the subgroup of $\mathbf{G}_m^g(\Omega)$ given by $\left\{\left(\prod_{j=1}^g p_{ij}{}^{n_j}\right)_{1\leqq i\leqq g} \middle| n_j \in \mathbf{Z}\right\}$. Let $\phi: Y \to X = \mathrm{Hom}(\mathbf{G}_m^g, \mathbf{G}_m)$ be the homomorphism given by

$$\phi\left(\left(\prod_{j=1}^g p_{ij}{}^{n_j}\right)_i\right)((x_i)_i) = \prod_{i=1}^g x_i{}^{n_i} \left((x_i)_i \in \mathbf{G}_m{}^g\right).$$

Then ϕ satisfies the axiom on polarizations given in [9]. Since A_g is regular, there exists a semi-abelian scheme G of relative dimension g over A_g with principal polarization ϕ whose multiplicative periods are p_{ij} [1, Chap. 3]. Let χ_i $(i = 1, \ldots, g)$ be the basis of X over $\mathbf{Z}$ defined by $\chi_i((x_j)_j) = x_i$. Then $\Gamma(G, \Omega_{G/A})$ is known to be a free A_g-module generated by $d\chi_i/\chi_i$ $(i = 1, \ldots, g)$ [1, Chap. 5], and hence $\wedge^g \Gamma(G, \Omega_{G/A_g})$ is an invertible A_g-module generated by

$$\omega = (d\chi_1/\chi_1) \wedge (d\chi_2/\chi_2) \wedge \ldots \wedge (d\chi_g/\chi_g).$$

Let k be a field of any characteristic, and let $K = k((z))$ be the complete valuation field over k with prime element z. Let $S_{g/K}$ be the set of marked Schottky groups of rank g which becomes a K-analytic subspace of $(\mathbf{P}^1(K)\times\mathbf{P}^1(K)\times K^\times)^g$ taking Koebe coordinates, and let $\xi_K: S_{g/K} \to \mathscr{M}_g \otimes K$ be the morphism defined by $\xi_K(\Gamma) = C_\Gamma$ $((\Gamma; \gamma_1, \ldots, \gamma_g) \in S_{g/K})$. Then for any ring homomorphism $u: A_g \to K$ such that $(u(x_{\pm i}), u(y_i))_{1\leqq i\leqq g} \in S_{g/K}$, there exists a canonical isomorphism $G \otimes_{A_g, u} K \cong J_\Gamma$, where Γ is the Schottky group over K with

Koebe coordinate $(u(x_{\pm i}), u(y_i))_{1 \leqq i \leqq g}$. Hence ω induces a canonical generator of $\wedge^g \Gamma(J_\Gamma, \Omega_{J_\Gamma/K})$ which we denote by ω again. For a $f \in T_{g,h}(k), \xi_K^*(f)\omega^{-h}$ is a K-analytic function on $S_{g/K}$. As is shown in [4], $S_{g/K}$ contains an open subset D_K which consists of $(\alpha_{\pm i}, \beta_i)_{1 \leqq i \leqq g} \in K^{3g}$ satisfying $\alpha_j \neq \alpha_k$ $(j \neq k)$ and

$$0 < |\beta_i| < \min\{|[\alpha_i, \alpha_{-i}; \alpha_j, \alpha_k]|; j, k \neq \pm i\} \quad (i = 1, \ldots, g),$$

and hence $\xi_K^*(f)\omega^{-h}$ is a K-analytic function on D_K. Therefore, there exists $\kappa_k(f) \in A_g\left[\prod_{i=1}^{g} 1/y_i\right] \hat{\otimes} K$ such that for any $(\alpha_{\pm i}, \beta_i)_{1 \leqq i \leqq g} \in D_K$,

$$\kappa_k(f)|_{x_{\pm i} = \alpha_{\pm i}, y_i = \beta_i} = \xi_K^*(f)\omega^{-h}.$$

3.2 Theorem. *For any field k of characteristic $\neq 2$ and $f \in T_{g,h}(k)$, $\kappa_k(f) \in A_g \hat{\otimes} k$, and $\kappa_k : T_{g,h}(k) \to A_g \hat{\otimes} k$ satisfies the following:*

(1) *κ_k is injective.*

(2) *Let $\iota : k \to k'$ be any homomorphism of fields. Then for any $f \in T_{g,h}(k)$,*

$$\kappa_{k'}(\iota(f)) = \iota(\kappa_k(f)).$$

(3) *Let k' be any field extension of k, and let f be any element of $T_{g,h}(k')$ such that $\kappa_{k'}(f) \in A_g \hat{\otimes} k$. Then $f \in T_{g,h}(k)$.*

(4) *For any $\varphi \in S_{g,h}(k)$,*

$$\kappa_k(\varphi|_{\mathscr{M}_g}) = F_k(\varphi)|_{q_{ij} = p_{ij}},$$

where $F_k(\varphi)$ denotes the Fourier expansion of φ with respect to the trivialization by ω^h.

(5) *For any $\varphi \in S_{g,h}(k)$,*

$$\kappa_k(\varphi|_{\mathscr{M}_g})|_{y_g = 0} = \kappa_k(\Phi(\varphi)|_{\mathscr{M}_{g-1}}).$$

3.3 In what follows, we prove Theorem 3.2. Let $n \geqq 3$ be prime to char(k), and put

$$A_k = A_g \hat{\otimes} k, \quad A_{k,n} = A_k[\zeta_n, (p_{ij})^{1/n} \, (i, j = 1, \ldots, g)].$$

By Proposition 2.3, for any $f \notin \langle f_i \rangle \langle f_j \rangle$,

$$[x_i, x_{-i}; f(x_j), f(x_{-j})]^{1/n} = \sum_{k=0}^{\infty} \binom{1/n}{k} \{[x_i, x_{-i}; f(x_j), f(x_{-j})] - 1\}^k$$

belongs to A_k, and hence $A_{k,n} = R[[y_1^{1/n}, \ldots, y_g^{1/n}]]$, where

$$R = A_k[\zeta_n, [x_i, x_{-i}; x_j, x_{-j}]^{1/n} \, (i \neq j)].$$

Let σ be a symplectic level n structure of $G \otimes_A A_{k,n}$. Then it is shown in [1] that there exists a unique morphism $\varphi_\sigma : \mathrm{Spec}(A_{k,n}) \to \tilde{X}_{g,n} \otimes k$ such that $\varphi_\sigma(\mathrm{Spec}(F_{k,n})) \in X_{g,n} \otimes k$ ($F_{k,n}$: the quotient field of $A_{k,n}$) and

$$(\mathscr{A}, \Sigma) \times_{X_{g,n}, \varphi_\sigma} \mathrm{Spec}(F_{k,n}) = (G \otimes_A F_{k,n}, \sigma),$$

where $(\mathscr{A}, \Sigma)$ denotes the universal abelian scheme with sympelctic level n structure over $X_{g,n}$.

3.4 Proposition. $\mathrm{Im}(\varphi_\sigma) \subset \overline{\tau(M_{g,n/k})}$.

Proof. Let I be the minimal ideal of $A_{k,n}$ such that φ_σ induces $\mathrm{Spec}(A_{k,n}/I) \to \overline{\tau(M_{g,n/k})}$. Then we will prove that $I = \{0\}$. On the contrary, assume that $I \neq \{0\}$. Let L be the power series field over the quotient field of R with variable z. Then there exist positive integers $m_1, \dots, m_g$ such that the R-linear homomorphism $\varrho: A_{k,n} \to L$ given by $\varrho(y_i^{1/n}) = z^{m_i}$ satisfies $\varrho(I) \neq \{0\}$. On the other hand, $G \otimes_{A,\varrho} L$ is the jacobian variety of the curve over L uniformized by the marked Schottky group with Koebe coordinate $(x_{\pm i}, z^{nm_i})_{1 \leqq i \leqq g}$, and hence the composite of the morphism $\mathrm{Spec}(L) \to \mathrm{Spec}(A_{k,n})$ induced from ϱ and φ_σ factors through $\tau(M_{g,n/k})$. Then $\mathrm{Ker}(\varrho) \supset I$, which is a contradiction. Therefore, $I = \{0\}$.

3.5 Corollary. *There exists a unique morphism* $\phi_\sigma : \mathrm{Spec}(A_{k,n}) \to N_{g,n/k}$ *such that* $\varphi_\sigma = \pi \circ \phi_\sigma$, *where* $\pi : N_{g,n/k} \to \overline{\tau(M_{g,n/k})}$ *denotes the normalization of* $\overline{\tau(M_{g,n/k})}$.

3.6 We continue the proof of Theorem 3.2. Since $\mathrm{char}(k) \neq 2$, the involution ι on $\mathscr{M}_{g,n}$ induces the decomposition $T_{g,h}(k) = T^+_{g,h}(k) \oplus T^-_{g,h}(k)$, where

$$T^{\pm}_{g,h}(k) = \{f \in T_{g,h}(k) \mid \iota(f) = \pm f\}.$$

First let

$$f \in T^+_{g,h}(k) \subset \Gamma(\tau(M_{g,n/k}), \lambda^{\otimes h}) = \Gamma(N_{g,n/k}, \bar{\mu}^{\otimes h}).$$

Then $\xi_K^*(f) = \phi_\sigma^*(f)|_{x_{\pm i} = \alpha_{\pm i}, y_i = \beta_i}$ for any $(\alpha_{\pm i}, \beta_i)_{1 \leqq i \leqq g} \in D_K$, and hence

$$\kappa_k(f) \in A_K\left[\prod_{i=1}^{g} 1/y_i\right] \cap A_{k,n} = A_k$$

because k is algebraically closed in K. Second let $f \in T^-_{g,h}(k)$. Then $f^2 \in T^+_{g,2h}(k)$, and hence $\kappa_k(f)^2 \in A_k$. Since $\kappa_k(f) \in A_K\left[\prod_{i=1}^{g} 1/y_i\right]$ and A_k is integrally closed in $A_K\left[\prod_{i=1}^{g} 1/y_i\right]$, we have $\kappa_k(f) \in A_k$. Therefore, $\kappa_k(f) \in A_k$ for any $f \in T_{g,h}(k)$.

In Theorem 3.2, assertions (2) and (4) follow from the construction of κ_k, (3) follows from (1) and (2), and (5) follows from (4) and Lemma 2.4. Therefore, to prove this theorem, it is enough to show (1). Let f be any element of $T_{g,k}(k)$ such that $\kappa_k(f) = 0$, and let Z be the closed subset of $\mathscr{M}_g \otimes k$ defined by $f = 0$. Take a field extension k' of k and a smooth and proper curve over $k'((z))$ with multiplicative reduction corresponding to the generic point η of $\mathscr{M}_g \otimes k$. Then by a result of Mumford [8], this curve can be uniformized by a certain Schottky group over $k'((z))$. Therefore, by the construction of κ_k, Z contains η. Hence by the irreducibility of $\mathscr{M}_g \otimes k$ [2], $Z = \mathscr{M}_g \otimes k$ which implies $f = 0$. This completes the proof of Theorem 3.2.

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