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On the existence of positive solutions of semilinear elliptic equations with Dirichlet boundary conditions*

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1 Introduction

In this paper we study the existence of positive continuous solutions to the following semilinear elliptic equation with Dirichlet boundary conditions:

$$\tfrac{1}{2}\Delta u + F(x,u) = -g(x) \quad \text{for } x\in D\,, \tag{1.1}$$

$$u(x) = \phi(x) \quad \text{for } x\in\partial D\,, \tag{1.2}$$

$$\lim_{\substack{|x|\to\infty \\ x\in D}} u(x) = \alpha \quad \text{when } D \text{ is unbounded}\,, \tag{1.3}$$

where Δ denotes the Laplacian in $\mathbb{R}^d$ ($d \geqq 3$), D is a regular domain in $\mathbb{R}^d$ with boundary ∂D, ϕ is a non-negative continuous real-valued function defined on ∂D, α is a non-negative constant such that when ∂D is unbounded, $\lim_{|x|\to\infty,\, x\in\partial D}\phi(x) = \alpha$, g is a non-negative Green-tight function on D, and F is a real-valued Borel measurable function defined on $D\times(0,b)$ for some $b\in(0,\infty]$ such that $F(x,\cdot)$ is continuous on $(0,b)$ for each $x\in D$ and $-U(x)u \leqq F(x,u) \leqq V(x)f(u)$ for all $(x,u)\in D\times(0,b)$, where U and V are non-negative Green-tight functions on D and f is a non-negative Borel measurable function defined on $(0,b)$. In order for our theorems on existence of positive solutions for (1.1)–(1.3) to apply, we shall need further restrictions on f, g and ϕ, which are specified precisely in Theorems 1.1 and 1.2.

Here a positive solution means a solution that is strictly positive on D and non-negative on $\bar{D}$. We paraphrase (1.3) by saying that u has limit α at infinity when D is unbounded. It is implicit here that the convergence is uniform as $|x|\to\infty$. As with all solutions of partial differential equations discussed in this paper, solutions of (1.1) are to be interpreted in the sense of distributions [4, Chap. 2]. The factor of

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$\frac{1}{2}$ appears in front of Δ in (1.1) because our method of proof uses Brownian motion which has $\frac{1}{2}\Delta$ as its infinitesimal generator. The domain D may be bounded or unbounded, it may even be all of $\mathbb{R}^d$. The regularity of D is in the usual sense of the Dirichlet problem. This is equivalent to the condition that Brownian motion hits the complement of D immediately after time zero when started from any point on the boundary ∂D [3, Chap. 4]. The notion of a Green-tight function on D is defined precisely in Sect. 2 below and examples of conditions that imply Green-tightness are given there. In particular, it is shown that a function w is Green-tight on D if and only if $1_D w$ is in the space K_d^∞ of Kato class functions on $\mathbb{R}^d$ that satisfy a uniform integrability condition with respect to the free space Green's function at infinity when D is unbounded. We note here that a function w that is Green-tight on D satisfies

$$\|w\|_D \equiv \sup_{x\in D} \int_D \frac{|w(y)|}{|x-y|^{d-2}}\,dy < \infty . \tag{1.4}$$

An example of the kind of function $F(x, u)$ satisfying the assumptions above is a finite linear combination of functions of the form $w(x)u^p$ where w is a Green-tight function on D and $p \geqq 1$.

We shall consider two cases, namely,

(i) either g is strictly positive on some set of positive Lebesgue measure in D, or ϕ is strictly positive on the whole of ∂D and $\alpha > 0$ if D is unbounded,

(ii) D is a Lipschitz domain with compact boundary, $g \equiv 0$ on D, and either ϕ is not identically zero on $\partial D \neq \emptyset$ or D is unbounded and $\alpha > 0$.

To state our main result for these two cases, we need the following estimates. Let G denote the Green's function for one half of the Laplacian on the domain D. Then [3, p. 181],

$$G(x, y) = 0 \quad \text{for } x\in\partial D,\ y\in D , \tag{1.5}$$

$$0 \leqq G(x, y) \leqq c_1 |x-y|^{2-d} \quad \text{for all } x, y\in D , \tag{1.6}$$

where $c_1 = \Gamma\left(\frac{d}{2}-1\right)\Big/ 2\pi^{d/2}$. In particular, if D is unbounded,

$$\lim_{\substack{|x|\to\infty\\ x\in D}} G(x, y) = 0 \quad \text{for each } y\in D . \tag{1.7}$$

For case (ii) we shall need the following "3G-inequality" which holds for a Lipschitz domain D with compact boundary:

$$\frac{G(x, y)G(y, z)}{G(x, z)} \leqq \frac{c_2}{2}(|x-y|^{2-d} + |y-z|^{2-d}) \quad \text{for all } x, y, z\in D , \tag{1.8}$$

where c_2 is a positive constant that depends on D. The inequality (1.8) was proved by Cranston, Fabes and Zhao [5] for a bounded Lipschitz domain. The proof employed in Herbst and Zhao [10, Theorem B.2], for $\Delta - \lambda$, $\lambda > 0$, can be adapted to establish it also for an unbounded Lipschitz domain in $\mathbb{R}^d (d \geqq 3)$ with compact boundary. It is an open question as to whether this inequality holds for any Lipschitz domain.

From the fact that g is Green-tight on D and the properties of G, it can be shown in a similar manner to that in the proof of Theorem 3.2 in [4] (with the assumption of Green-tightness on D in place of the assumption of being in the class $J \cap L^1(D)$ used there), that

$$(1.9) \qquad (Gg)(x) \equiv \int_D G(x, y)g(y)dy, \quad x \in \bar{D},$$

defines a bounded continuous function on $\bar{D}$ such that Gg is zero on ∂D and Gg goes to zero at infinity if D is unbounded. Furthermore, by Proposition 2.10 of [4], Gg satisfies the equation:

$$(1.10) \qquad \tfrac{1}{2}\Delta(Gg) = -g \quad \text{in } D.$$

Now for $i = 1, 2$, let

$$(1.11) \qquad C_i(f, V, D) = \frac{1}{2}\sup\left\{\varepsilon \in (0, b): \sup_{0 < y \leq \varepsilon} \frac{f(y)}{y} \leq \frac{1}{2c_i \|V\|_D}\right\}.$$

Here $\sup \emptyset \equiv 0$ and $1/0 \equiv +\infty$. We shall only be interested in those cases where $C_i(f, V, D) > 0$ for at least one $i \in \{1, 2\}$. Observe that in the special case where $f(u) = u^p$ for $p > 1$, or $f(u) = u$ and $\|V\|_D$ is sufficiently small, then $C_i(f, V, D)$ is strictly positive for $i = 1, 2$.

In the sequel, the conditions stated in the first paragraph of this paper are assumed to hold. In particular, ϕ, α and g are non-negative. So that we may treat the cases of D bounded and unbounded simultaneously, we define ∂D_δ to equal ∂D if D is bounded, and if D is unbounded, it is defined to equal $\partial D \cup \{\delta\}$, where δ denotes the point at infinity in the one-point compactification of $\mathbb{R}^d$ and ∂D_δ has the topology induced by this compactification. When D is unbounded, we extend ϕ continuously to ∂D_δ by defining $\phi(\delta) = \alpha$.

Let $C_b(\bar{D})$ denote the Banach space of continuous, bounded, real-valued functions defined on $\bar{D}$ with the supremum norm. The notation $\|\cdot\|_\infty$ will denote the supremum norm of a real-valued function over its domain of definition. In particular, $\|\phi\|_\infty = \sup_{x \in \partial D_\delta} |\phi(x)|$ and $\|Gg\|_\infty = \sup_{x \in \bar{D}} |(Gg)(x)|$.

Theorem 1.1 *Suppose the following conditions hold:*

(a) *either $g > 0$ on some set of positive Lebesgue measure in D or $\phi > 0$ on the whole of ∂D_δ,*

(b) $\|Gg\|_\infty + \|\phi\|_\infty < C_1(f, V, D)$.

Then the Dirichlet boundary value problem specified by (1.1)–(1.3) *has a solution $u \in C_b(\bar{D})$ such that $u > 0$ on D.*

Theorem 1.2 *Assume that D is a Lipschitz domain with compact boundary and $g \equiv 0$ on D. Suppose that the following conditions hold:*

(a) *$\phi \not\equiv 0$ on ∂D_δ,*

(b) $\|\phi\|_\infty < C_2(f, V, D)$.

Then the Dirichlet boundary value problem specified by (1.1)–(1.3) *has a solution $u \in C_b(\bar{D})$ such that $u > 0$ on D.*

Remark 1. Observe that the above theorems only have content if $C_i(f, V, D) > 0$. The cases where $F(x, u) \leq 0$ or $g \equiv 0$ or $D = \mathbb{R}^d$ are interesting special cases.

Remark 2. For the existence of solutions, the situation where $F(x, u) \leq 0$ is simpler than that where $F(x, u)$ may be positive, because the former corresponds to killing a Brownian motion, whereas the latter allows for creation and one has to avoid explosion of mass.

Remark 3. In general, we do not know that there is uniqueness for the solutions found in Theorems 1.1 and 1.2. However, if $F(x, u)$ is monotone decreasing as a function of $u \in (0, b)$ for each fixed $x \in D$, then it follows from a maximum principle argument (cf. [7], Theorem 0.5) that for a given ϕ, α and g, there is uniqueness of continuous bounded solutions to (1.1)–(1.3).

Remark 4. Theorem 1.2 covers the case treated previously by Zhao [18] where D is an unbounded Lipschitz domain with compact boundary, $g \equiv 0$ on D and $\phi \equiv 0$ on ∂D.

Our method of proof uses an implicit probabilistic representation for positive solutions of (1.1)–(1.3) (see (3.18)), together with Schauder's fixed point theorem. Whilst it may be possible to translate our proof into a purely analytic one, perhaps with some further restrictions, we feel that it is our method of proof, with its use of a probabilistic representation, that is especially interesting. To keep the exposition as transparent as possible, we have not attempted to optimize the bound on $\|Gg\|_\infty + \|\phi\|_\infty$. Further elaboration on our method of proof is postponed until after the probabilistic representation (3.18) has been introduced.

There is a wealth of literature on semilinear elliptic equations. However, we could not find results that entirely subsumed ours. In particular, we allow F to be locally unbounded and to take positive as well as negative values. We shall confine our comments on the literature to some papers illustrating applications and to papers giving probabilistic approaches to the solution of semilinear elliptic equations. With $D = \mathbb{R}^d$, $g \equiv 0$ and $F(x, u) = -k(x)u + w(x)u^p$ where $p > 1$, $k \geqq 0$ and slightly stronger assumptions are made on k and w than Green-tightness on D, (1.1) has been studied by Kenig and Ni [11] and other authors in connection with the problem of conformal deformation of metrics in Riemannian geometry. Equations of the form (1.1) with a certain class of non-positive F's which includes those of the form $F(x, u) = -w(x)u^p$, w a non-negative, bounded Borel function and $p \in (1, 2]$, have appeared in connection with the study of superprocesses (see Dynkin [7] for example). In [9], Glover and McKenna used techniques of probabilistic potential theory to develop a refinement of the usual method of sub- and super-solutions for solving semilinear elliptic and parabolic differential equations on $\mathbb{R}^d$. Ma and Song [13] adapted the method of Glover and McKenna [9] to elliptic equations in bounded domains with Dirichlet, Neumann, or mixed boundary conditions. When the results of [9] and [13] are applied to the situation in (1.1)–(1.3) with a view to finding non-negative solutions, the hypotheses of these papers in particular require that $F = F(u)$ and on each fixed compact u-interval there is a positive constant A such that

$$-Au \leqq F(u) \leqq 0 . \tag{1.12}$$

The lower constraint on F is slightly weaker than ours in the case treated by Glover and McKenna of $D = \mathbb{R}^d$. However, this leads to the possibility that their solution may be identically zero when $g \equiv 0$. In fact, in [2] we show that our method can be adapted to recover the existence of non-negative solutions of (1.1)–(1.3) under the conditions of Glover and McKenna [9] or Ma and Song [13]. Freidlin [8] has used an implicit probabilistic representation together with a contraction mapping argument (or Picard iteration) to obtain continuous solutions for (degenerate) elliptic quasilinear equations with Dirichlet boundary conditions, where the coefficients are bounded Lipschitz continuous and the boundary is regular. In [18], Zhao used an implicit probabilistic representation together with Schauder's fixed point theorem to obtain positive solutions of the equation (1.1)–(1.3) under some conditions with $g \equiv 0$, $\phi \equiv 0$, $F(x, u) = V(x) f(u)$ where V is Green-tight on D, f is continuous and neither V nor f need be non-negative, and D is an unbounded Lipschitz domain with compact boundary. Both Freidlin and Zhao used the implicit probabilistic representation, but Zhao used the potentially broader mechanism of Schauder's fixed point theorem rather than a contraction argument. The present work uses some ideas from Zhao's paper, but additional features are needed to handle non-zero g and ϕ. Furthermore, in Theorem 1.1, we relax the Lipschitz and compact boundary condition on D to that of a regular domain.

In a separate paper [1] we consider the equation (1.1) with Neumann and mixed boundary conditions in place of the Dirichlet conditions considered here. In particular, the boundary conditions may include nonlinear terms depending on x and u.

2 Green-tight and Kato class functions

In this section, prior to Proposition 2.3, D may be any domain in $\mathbb{R}^d$ where $d \geqq 3$. For Proposition 2.3 we assume that D is a regular domain, as in the Introduction.

Definition 2.1 A function w is Green-tight on D if and only if w is a real-valued Borel measurable function defined on D such that the family of functions $\{w(\cdot)/|x - \cdot|^{d-2}, x \in D\}$ is uniformly integrable, i.e., w satisfies

$$\lim_{\substack{m(A) \to 0 \\ A \subset D}} \left\{ \sup_{x \in D} \int_A \frac{|w(y)|}{|x - y|^{d-2}} dy \right\} = 0 , \tag{2.1}$$

and (if D is unbounded),

$$\lim_{M \to \infty} \left\{ \sup_{x \in D} \int_{\substack{|y| > M \\ y \in D}} \frac{|w(y)|}{|x - y|^{d-2}} dy \right\} = 0 , \tag{2.2}$$

where m denotes Lebesgue measure on $\mathbb{R}^d$. Note that the limit in (2.1) is uniform in sets $A \subset D$ such that $m(A) \to 0$. It follows easily from (2.1)–(2.2) that (1.4) holds for any w that is Green-tight on D.

We shall relate the class of Green-tight functions on D to the class K_d^∞ defined below. For this we need the following lemma.

Lemma 2.1 *A real-valued Borel measurable function w defined on D is Green-tight on D if and only if $1_D w$ is Green-tight on $\mathbb{R}^d$.*

Proof. Without loss of generality, we may assume $D \neq \mathbb{R}^d$. For any Borel set $A \subset D$, by Fatou's lemma we have

$$\sup_{x \in \bar{D}} \int_A \frac{|w(y)|}{|x - y|^{d-2}} dy = \sup_{x \in D} \int_A \frac{|w(y)|}{|x - y|^{d-2}} dy . \tag{2.3}$$

For each $z \in \mathbb{R}^d \setminus \bar{D}$, there is $x_z \in \partial D$ such that $|x_z - z| = \inf_{x \in D} |x - z|$. Then for each $y \in D$,

$$|x_z - y| \leqq |x_z - z| + |z - y| \leqq 2|z - y| , \tag{2.4}$$

and so

$$(2.5)\quad \int_A \frac{|w(y)|}{|z-y|^{d-2}}dy \leqq 2^{d-2}\int_A \frac{|w(y)|}{|x_z-y|^{d-2}}dy \leqq 2^{d-2}\sup_{x\in D}\int_A \frac{|w(y)|}{|x-y|^{d-2}}dy\,.$$

Combining (2.3) and (2.5), we obtain

$$(2.6)\qquad \sup_{z\in\mathbb{R}^d}\int_A \frac{|w(y)|}{|z-y|^{d-2}}dy \leqq 2^{d-2}\sup_{x\in D}\int_A \frac{|w(y)|}{|x-y|^{d-2}}dy\,.$$

The desired equivalence follows immediately from this and the definition of Green-tightness on D and $\mathbb{R}^d$. □

The Kato class K_d consists of all those real-valued Borel measurable functions w defined on $\mathbb{R}^d$ such that

$$(2.7)\qquad \lim_{r\downarrow 0}\left[\sup_{x\in\mathbb{R}^d}\int_{|x-y|\leqq r}\frac{|w(y)|}{|x-y|^{d-2}}dy\right]=0\,.$$

A real-valued Borel measurable function w defined on $\mathbb{R}^d$ is in K_d^{loc} if and only if $1_B w\in K_d$ for each bounded ball B in $\mathbb{R}^d$. The class K_d^∞ is defined by

$$(2.8)\qquad K_d^\infty=\left\{w\in K_d^{\mathrm{loc}}:\ \lim_{M\to\infty}\left[\sup_{x\in\mathbb{R}^d}\int_{|y|>M}\frac{|w(y)|}{|x-y|^{d-2}}dy\right]=0\right\}.$$

Remark. It is easy to see that K_d^{loc} can be replaced with K_d in the above definition of K_d^∞.

The following lemma was stated in [18] without a complete proof. The lemma above provides the missing step.

Lemma 2.2 *A real-valued Borel measurable function w defined on D is Green-tight on D if and only if $1_D w$ is in the class K_d^∞.*

Proof. It was shown in [18], Proposition 2, that a function is Green-tight on $\mathbb{R}^d$ if and only if it is in K_d^∞. Combining this with Lemma 2.1 above yields the desired result. □

Remark. If D is bounded, a real-valued Borel measurable function w defined on D is Green-tight on D if and only if $1_D w$ is in K_d or equivalently K_d^{loc}. Examples of functions in K_d and K_d^{loc} may be found in [15]. In particular, it follows from Example E on page 456 of [15] that when D is bounded, any function in $L^p(D)$ for $p>d/2$ is Green-tight on D. If D is unbounded, by [18], Proposition 1, a sufficient condition for a real-valued Borel measurable function w to be Green-tight on D is that $1_D w\in K_d^{\mathrm{loc}}$ and there is $L>0$ such that

$$(2.9)\qquad |(1_D w)(x)|\leqq \frac{\psi(|x|)}{|x|^2}\quad \text{for all } |x|\geqq L\,,$$

where ψ is a positive function defined on the interval $[L,\infty)$ such that $\int_L^\infty r^{-1}\psi(r)\,dr<\infty$.

Let $C_0(\bar{D})$ denote the subspace of $C_b(\bar{D})$ given by

$$\left\{u\in C_b(\bar{D}): u(x)=0 \text{ on } \partial D,\ \text{ and if } D \text{ is unbounded, } \lim_{\substack{|x|\to\infty\\ x\in D}} u(x)=0\right\}.$$

For any function w that is Green-tight on D, let

(2.10) $\Gamma_w = \{v : D \to \mathbb{R},\ v \text{ is Borel measurable and } |v(x)| \leqq |w(x)| \text{ for all } x \in D\}$.

The following proposition plays a key role in our proof of Theorems 1.1 and 1.2. Here we revert to the assumption that D is a regular domain in $\mathbb{R}^d$, $d \geqq 3$.

Proposition 2.3 *For any Green-tight function w on D, the family of functions $\{Gv : v \in \Gamma_w\}$ is uniformly bounded and equicontinuous in $C_0(\bar{D})$, and consequently, it is relatively compact in $C_0(\bar{D})$.*

Proof. The proof given in [18] for an unbounded Lipschitz domain carries over without modification, since the Lipschitz assumption in [18] was only used to ensure that the domain was regular and the unboundedness assumption did not play a role. □

For further properties of Kato class and Green-tight functions, we refer the reader to [4], [15] and [18].

3 Proof of Theorem 1.1

Let ϕ, g satisfy the hypotheses of Theorem 1.1. For each $x \in \bar{D}$, let $\{X(t), t \geqq 0\}$ under the probability measure P^x be a d-dimensional Brownian motion starting from x. We use E^x to denote expectation under P^x and let $\tau_D = \inf\{t > 0 : X(t) \notin D\}$. When D is unbounded, on $\{\tau_D = \infty\}$ we let $X(\tau_D) = \delta$ and then $\phi(X(\tau_D)) = \phi(\delta) \equiv \alpha$.

For any real-valued Borel measurable function w defined on D such that $\|w\|_D < \infty$, for each $x \in \bar{D}$, P^x-a.s. the following stopped Feynman-Kac functional is well defined, positive and finite for all $t \geqq 0$ [4, Chap. 3],

(3.1) $$e_w(t) \equiv \exp\left(\int_0^{t \wedge \tau_D} w(X(s))\,ds\right) .$$

We shall consider such functionals with $w = -U$, V, etc. Now for each $x \in \bar{D}$ let

(3.2) $$h_0(x) = E^x\left[\int_0^{\tau_D} e_{-U}(t)\, g(X(t))\,dt\right] ,$$

(3.3) $$h_1(x) = E^x\left[\int_0^{\tau_D} g(X(t))\,dt\right] ,$$

(3.4) $$h_2(x) = E^x[\phi(X(\tau_D))] ,$$

(3.5) $$h(x) = h_1(x) + h_2(x) .$$

Note that $h_1 = Gg$, and by our convention,

(3.6) $$h_2(x) = E^x[\phi(X(\tau_D)); \tau_D < \infty] + \alpha P^x(\tau_D = \infty) .$$

It is well known (cf. [3, Sect. 4.4]) that h_2 is a bounded continuous function on $\bar{D}$ which is harmonic in D, equals ϕ on the boundary of D (since D is regular), and if D is unbounded has limit of $\phi(\delta) = \alpha$ at infinity. Combining these properties with those of Gg we see that h is a bounded continuous function on $\bar{D}$ which agrees with

ϕ on ∂D, has limit of α at infinity if D is unbounded, and satisfies the following equation:

$$\tfrac{1}{2}\Delta h = -g \quad \text{in } D . \tag{3.7}$$

If g is positive on a set of positive Lebesgue measure in D, then since such a set is of positive potential for Brownian motion killed on exit from D, it follows that $h_1 > 0$ on D and similarly, since also P^x-a.s., $e_{-U}(t) > 0$ for all $t \geqq 0$, we have $h_0 > 0$ on D. On the other hand, if $\phi > 0$ on ∂D_δ, then $h_2 \geqq \gamma_0 > 0$ on $\bar{D}$, where

$$\gamma_0 \equiv \min\{\phi(x) : x \in \partial D_\delta\} . \tag{3.8}$$

Furthermore, $h_1 \leqq \|Gg\|_\infty$ and $h_2 \leqq \|\phi\|_\infty$ on $\bar{D}$. Let

$$\beta = c_1 \|U\|_D, \quad \gamma_1 = \|Gg\|_\infty + \|\phi\|_\infty , \tag{3.9}$$

and define

$$\Lambda = \{u \in C_b(\bar{D}) : h_0 + e^{-\beta}\gamma_0 \leqq u \leqq 2\gamma_1 \text{ on } \bar{D}\} . \tag{3.10}$$

Note that $h_0 + e^{-\beta}\gamma_0 > 0$ on D since either $h_0 > 0$ on D or $\gamma_0 > 0$, by assumption (a) of the theorem. Now, by assumption (b), $2\gamma_1 < 2C_1(f, V, D)$ and so for $u \in \Lambda$, using the definition (1.11) of $C_1(f, V, D)$ and properties of F, we have that

$$q_u(x) \equiv \frac{F(x, u(x))}{u(x)} \tag{3.11}$$

is well defined for all $x \in D$, and

$$-U(x) \leqq q_u(x) \leqq \frac{V(x) f(u(x))}{u(x)} \leqq \frac{V(x)}{2c_1\|V\|_D} \equiv \tilde{V}(x) \quad \text{on } D , \tag{3.12}$$

and so

$$\|q_u\|_D \leqq \|U\|_D + \|\tilde{V}\|_D \leqq \|U\|_D + \frac{1}{2c_1} . \tag{3.13}$$

By (1.6) we have for any $u \in \Lambda$ and $x \in D$,

$$\begin{aligned} E^x\left[\int_0^{\tau_D} \tilde{V}(X(t))\,dt\right] &= \int_D G(x, y)\,\tilde{V}(y)\,dy \\ &\leqq c_1 \int_D \tilde{V}(y)|x - y|^{2-d}\,dy \\ &\leqq c_1 \|\tilde{V}\|_D \\ &= \tfrac{1}{2} , \end{aligned} \tag{3.14}$$

and similarly,

$$E^x\left[\int_0^{\tau_D} U(X(t))\,dt\right] \leqq c_1 \|U\|_D = \beta . \tag{3.15}$$

By (3.12), (3.14)–(3.15), together with Jensen's inequality applied on the left and Khasminskii's lemma [4, Sect. 3.2] on the right, we have the following estimate for each $x \in D$:

$$e^{-\beta} \leqq E^x[e_{-U}(\tau_D)] \leqq E^x[e_{q_u}(\tau_D)] \leqq E^x[e_{\tilde{V}}(\tau_D)] \leqq 2 . \tag{3.16}$$

Furthermore, by using ordinary calculus, Fubini's theorem for non-negative integrands, and the Markov property, we have for all $x \in D$,

$$\begin{aligned} E^x\left[\int_0^{\tau_D} e_{q_u}(t)\, g(X(t))\, dt\right] &\leqq E^x\left[\int_0^{\tau_D} e_{\tilde{V}}(t)\, g(X(t))\, dt\right] \\ &= E^x\left[\int_0^{\tau_D} (e_{\tilde{V}}(t)-1)\, g(X(t))\, dt\right] + E^x\left[\int_0^{\tau_D} g(X(t))\, dt\right] \\ &= E^x\left[\int_0^{\tau_D}\left(\int_0^t e_{\tilde{V}}(s)\, \tilde{V}(X(s))\, ds\right) g(X(t))\, dt\right] + h_1(x) \\ &= E^x\left[\int_0^{\tau_D} e_{\tilde{V}}(s)\, \tilde{V}(X(s))\left(\int_s^{\tau_D} g(X(t))\, dt\right) ds\right] + h_1(x) \\ &= E^x\left[\int_0^{\tau_D} e_{\tilde{V}}(s)\, \tilde{V}(X(s))\, E^{X(s)}\left[\int_0^{\tau_D} g(X(t))\, dt\right] ds\right] + h_1(x) \\ &= E^x\left[\int_0^{\tau_D} e_{\tilde{V}}(s)\, \tilde{V}(X(s))\, h_1(X(s))\, ds\right] + h_1(x) \\ &\leqq \|h_1\|_\infty E^x[e_{\tilde{V}}(\tau_D)] \\ &\leqq 2\|h_1\|_\infty , \end{aligned} \tag{3.17}$$

where the last inequality follows from (3.16). Combining (3.16), (3.17), and using the regularity of D, we see that the following expression is well defined and finite for all $x \in \bar{D}$:

$$(Tu)(x) \equiv E^x\left[\int_0^{\tau_D} e_{q_u}(t)\, g(X(t))\, dt\right] + E^x[e_{q_u}(\tau_D)\, \phi(X(\tau_D))] . \tag{3.18}$$

It is convenient at this point to explain the strategy of our method of proof. If $u \in \Lambda$ were a solution of (1.1)–(1.3), it would also be a positive solution of

$$\tfrac{1}{2}\Delta u + q_u u = -g \quad \text{in } D . \tag{3.19}$$

By thinking of q_u as a known function of x, results [4] for the linear reduced Schrödinger equation would yield the representation $u = Tu$ in $\bar{D}$. Thus u would be a fixed point of T. The idea of our proof is to reverse this procedure, that is, to show that the mapping T has a fixed point in Λ and then to verify using the results of [4] that this fixed point is a solution of (3.19) (and hence of (1.1)), together with the boundary conditions (1.2) and (1.3).

For $u \in \Lambda$ and each $x \in \bar{D}$, let

$$v_1(x) = E^x\left[\int_0^{\tau_D} e_{q_u}(t)\, g(X(t))\, dt\right], \tag{3.20}$$

$$v_2(x) = E^x[e_{q_u}(\tau_D)\, \phi(X(\tau_D))] . \tag{3.21}$$

Using (3.12), (3.16), (3.17), the non-negativity of g, and the regularity of D, we can show that on $\bar{D}$,

$$h_0 \leqq v_1 \leqq 2\|h_1\|_\infty = 2\|Gg\|_\infty, \tag{3.22}$$

$$e^{-\beta}\gamma_0 \leqq v_2 \leqq 2\|\phi\|_\infty. \tag{3.23}$$

Now, by ordinary calculus, Fubini's theorem (the required absolute integrability being implied by $\|q_u\|_D < \infty$), the Markov property, and the regularity of D, we have for each $x \in \bar{D}$,

$$\begin{aligned}
v_1(x) - h_1(x) &= E^x\left[\int_0^{\tau_D} (e_{q_u}(t) - 1) g(X(t))\,dt\right] \\
&= E^x\left[\left(\int_0^{\tau_D}\int_0^t \left(-\frac{d}{ds}(e_{q_u}(s))^{-1}\right) ds\right) e_{q_u}(t) g(X(t))\,dt\right] \\
&= E^x\left[\int_0^{\tau_D}\left(\int_0^t q_u(X(s))(e_{q_u}(s))^{-1}\,ds\right) e_{q_u}(t) g(X(t))\,dt\right] \\
&= E^x\left[\int_0^{\tau_D} q_u(X(s))(e_{q_u}(s))^{-1}\left(\int_s^{\tau_D} e_{q_u}(t) g(X(t))\,dt\right) ds\right] \\
&= E^x\left[\int_0^{\tau_D} q_u(X(s))\, E^{X(s)}\left[\int_0^{\tau_D} e_{q_u}(t) g(X(t))\,dt\right] ds\right] \\
&= E^x\left[\int_0^{\tau_D} q_u(X(s))\, v_1(X(s))\,ds\right] \\
&= G(q_u v_1).
\end{aligned} \tag{3.24}$$

It follows from the boundedness of v_1, (3.12), and the fact that U and V are Green-tight on D, that $q_u v_1$ is Green-tight on D and hence by the same reasoning as for Gg in Sect. 1, $G(q_u v_1) \in C_0(\bar{D})$ and it satisfies (1.10) with $q_u v_1$ in place of g there. Combining (3.24) with the properties of $G(q_u v_1)$ and h_1 we see that $v_1 \in C_0(\bar{D})$ and it satisfies the following equation:

$$\tfrac{1}{2}\Delta v_1 + q_u v_1 = -g \quad \text{in } D. \tag{3.25}$$

Similarly, for v_2 we have for each $x \in \bar{D}$,

$$\begin{aligned}
v_2(x) - h_2(x) &= E^x[\{e_{q_u}(\tau_D) - 1\}\phi(X(\tau_D))] \\
&= E^x\left[\int_0^{\tau_D} q_u(X(t)) \exp\left(\int_t^{\tau_D} q_u(X(s))\,ds\right)\phi(X(\tau_D))\,dt\right] \\
&= E^x\left[\int_0^{\tau_D} q_u(X(t)) E^{X(t)}[e_{q_u}(\tau_D)\,\phi(X(\tau_D))]\,dt\right] \\
&= G(q_u v_2).
\end{aligned} \tag{3.26}$$

It follows that $v_2 \in C_b(\bar{D})$, it agrees with ϕ on ∂D, it has limit α at infinity if D is unbounded, and it satisfies the following equation:

$$\tfrac{1}{2}\Delta v_2 + q_u v_2 = 0 \quad \text{in } D. \tag{3.27}$$

Now $Tu = v_1 + v_2$ and so from the above properties of v_1 and v_2 we conclude that $Tu \in C_b(\bar{D})$,

$$h_0 + e^{-\beta}\gamma_0 \leqq Tu \leqq 2\gamma_1 \quad \text{on } \bar{D}\,, \tag{3.28}$$

and $v = Tu$ is a solution of

$$\tfrac{1}{2}\Delta v + q_u v = -g \quad \text{in } D\,, \tag{3.29}$$

$$v = \phi \quad \text{on } \partial D\,, \tag{3.30}$$

$$\lim_{\substack{|x|\to\infty \\ x\in D}} v(x) = \alpha \quad \text{if } D \text{ is unbounded}\,. \tag{3.31}$$

In particular, for any $u \in \Lambda$, $Tu \in \Lambda$, and so

$$T\Lambda \subset \Lambda\,. \tag{3.32}$$

By (3.24), (3.26), we have for each $u \in \Lambda$,

$$(Tu)(x) - h(x) = G(q_u\, Tu) \quad \text{for all } x \in \bar{D}\,. \tag{3.33}$$

From (3.12) and (3.28) we have

$$|q_u(x)(Tu)(x)| \leqq \gamma_1\left(2U(x) + \frac{V(x)}{c_1\,\|V\|_D}\right) \quad \text{for all } x \in D\,. \tag{3.34}$$

Since U and V are Green-tight on D, it follows from (3.34) and Proposition 2.3 that as u ranges over all functions $u \in \Lambda$, the right member of (3.33) defines a family of functions in $C_0(\bar{D})$ that is uniformly bounded and equicontinuous. Now h is bounded and uniformly continuous on $\bar{D}$ with limit of α at infinity if D is unbounded. Combining the above, it follows by a simple modification of the Ascoli-Arzela theorem that $T\Lambda$ is relatively compact in $C_b(\bar{D})$.

Suppose that $u \in \Lambda$ and $\{u_n\} \subset \Lambda$ with $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Since $F(x,\cdot)$ is continuous on $(0, b)$ for each $x \in D$, $h_0 + e^{-\beta}\gamma_0 > 0$ on D, and $2\gamma_1 < 2C_1(f, V, D) \leqq b$, it follows from (3.10)–(3.11) that as $n \to \infty$,

$$q_{u_n} \to q_u \quad \text{pointwise on } D\,. \tag{3.35}$$

Now by (3.12), for any $v \in \Lambda$,

$$\int_0^{\tau_D} |q_v(X(s))|\, ds \leqq \int_0^{\tau_D} (U(X(s)) + \tilde{V}(X(s)))\, ds\,, \tag{3.36}$$

where for $x \in D$, by (3.14)–(3.15),

$$E^x\left[\int_0^{\tau_D} (U(X(s)) + \tilde{V}(X(s)))\, ds\right] \leqq \beta + \frac{1}{2}\,. \tag{3.37}$$

It follows from (3.35), and the dominated convergence afforded by (3.36)–(3.37), that for each $x \in D$ and $t \in [0, \infty]$,

$$\int_0^{t\wedge\tau_D} q_{u_n}(X(s))\, ds \to \int_0^{t\wedge\tau_D} q_u(X(s))\, ds \quad P^x\text{-a.s. as } n \to \infty\,. \tag{3.38}$$

Now for $v \in \Lambda$, by (3.12) we have for each $x \in D$ and $t \in [0, \infty]$,

$$e_{q_v}(t \wedge \tau_D) \leqq e_{\tilde{V}}(t \wedge \tau_D) . \tag{3.39}$$

By (3.16)–(3.17), we have for each $x \in D$,

$$E^x[e_{\tilde{V}}(\tau_D)] \leqq 2 , \tag{3.40}$$

and

$$E^x\left[\int_0^{\tau_D} e_{\tilde{V}}(t) g(X(t))\, dt\right] \leqq 2 \|h_1\|_\infty . \tag{3.41}$$

Then by (3.18) and the dominated convergence afforded by (3.38)–(3.41) and the boundedness of ϕ, we have that $(Tu_n)(x) \to (Tu)(x)$ as $n \to \infty$ for each $x \in \bar{D}$ (note that $Tu = Tu_n$ on ∂D by the regularity of ∂D). Since $T\Lambda$ is relatively compact, the pointwise convergence implies uniform convergence, and so we have $\|Tu_n - Tu\|_\infty \to 0$ as $n \to \infty$. Thus, we have proved that T is a compact and continuous mapping from the non-empty, convex, closed, bounded set $\Lambda \subset C_b(\bar{D})$ into itself. Hence by the Schauder fixed point theorem (see Theorem 2.A, Sect. 2.6 of [17]), there exists a function $u_0 \in \Lambda$ such that $Tu_0 = u_0$. It then follows from (3.29)–(3.31) that u_0 is a solution of (1.1)–(1.3). □

4 Proof of Theorem 1.2

Theorem 1.2 can be proved in a very similar manner to Theorem 1.1. The main difference is that in the definition of Λ, γ_0 is replaced by h_2 (3.6). The fact that $h_2 > 0$ on D is ensured by the maximum principle and assumption (a) of Theorem 1.2, since $\phi \not\equiv 0$ being continuous is either strictly positive on a set of positive capacity in the Lipschitz boundary ∂D (for Brownian motion absorbed on ∂D), or D is unbounded and $\alpha \equiv \phi(\delta) > 0$.

We indicate below the modifications that need to be made to the proof of Theorem 1.1 in order to prove Theorem 1.2. Replace c_1, $C_1(f, V, D)$, γ_0 and Λ, by c_2, $C_2(f, V, D)$, h_2, and

$$\tilde{\Lambda} = \{u \in C_b(\bar{D}) : e^{-\beta} h_2 \leqq u \leqq 2\gamma_1 \text{ on } \bar{D}\} , \tag{4.1}$$

respectively. (Note that g is assumed to be identically zero.) The key difference in the proof involves showing that T maps $\tilde{\Lambda}$ into itself. For this, we need to use conditioned Brownian motion on D, together with the estimate (1.8). For the development of this, we observe that since D is a Lipschitz domain with compact boundary, ∂D_δ is the Martin boundary of D. Fix x_0 in D and let K denote the associated Martin kernel for D defined by

$$K(x, z) = \lim_{\substack{y \to z \\ y \in D}} \frac{G(x, y)}{G(x_0, y)} \quad \text{for all } x \in D,\ z \in \partial D_\delta . \tag{4.2}$$

By taking limits in (1.8) we obtain:

$$\frac{G(x, y) K(y, z)}{K(x, z)} \leqq \frac{c_2}{2} (|x - y|^{2-d} + |y - z|^{2-d}) \quad \text{for all } x, y \in D, z \in \partial D_\delta , \tag{4.3}$$

where the last term in the right member is zero if $z = \delta$. For each $(x, z) \in D \times \partial D_\delta$, let P_z^x and E_z^x denote the probability and expectation, respectively, under which X is Brownian motion starting at x conditioned to converge to z as $t \uparrow \tau_D$ [6]. Now for $u \in \tilde{\Lambda}$ we have

$$(4.4)\qquad E_z^x\left[\int_0^{\tau_D} \tilde{V}(X(t))\,dt\right] = \int_D \frac{G(x, y)\tilde{V}(y)K(y, z)}{K(x, z)}\,dy$$

$$\leq \frac{c_2}{2}\left[\int_D \frac{\tilde{V}(y)}{|x - y|^{d-2}}\,dy + \int_D \frac{\tilde{V}(y)}{|y - z|^{d-2}}\,dy\right]$$

$$\leq c_2 \|\tilde{V}\|_D$$

$$= \tfrac{1}{2},$$

where $\tilde{V} = \dfrac{V}{2c_2\|V\|_D}$. Similarly,

$$(4.5)\qquad E_z^x\left[\int_0^{\tau_D} U(X(t))\,dt\right] \leq \beta \equiv c_2\|U\|_D < \infty .$$

Then in the same manner that (3.16) was derived, but with E_z^x in place of E^x, we have for each $x \in D$ and $z \in \partial D_\delta$,

$$(4.6)\qquad e^{-\beta} \leq E_z^x[e_{-U}(\tau_D)] \leq E_z^x[e_{q_u}(\tau_D)] \leq E_z^x[e_{\tilde{V}}(\tau_D)] \leq 2 .$$

Note that by integrating out over the conditioning variable z, we see that (4.6) also holds with E^x in place of E_z^x.

For each $x \in D$, let $H(x, dz)$ denote the harmonic measure on ∂D_δ relative to x, i.e., for each Borel set $A \subset \partial D_\delta$,

$$(4.7)\qquad H(x, A) = E^x[1_A(X(\tau_D))]$$

$$= E^x[1_A(X(\tau_D)); \tau_D < \infty] + 1_A(\delta)P^x(\tau_D = \infty) .$$

Then for each $x \in D$,

$$(4.9)\qquad h_2(x) = \int_{\partial D_\delta} \phi(z)H(x, dz) ,$$

and [4, Chap. 5],

$$(4.10)\qquad v_2(x) = \int_{\partial D_\delta} E_z^x[e_{q_u}(\tau_D)]\phi(z)H(x, dz) .$$

It then follows from (4.6) and the regularity of D that

$$(4.11)\qquad e^{-\beta}h_2 \leq v_2 \leq 2\|\phi\|_\infty \quad \text{on } \bar{D} .$$

Thus we have as in the proof of Theorem 1.1 that T maps $\tilde{\Lambda}$ into itself. The rest of the proof of Theorem 1.2 proceeds as for Theorem 1.1. □

5 Extensions

The results of this paper can be extended to second order uniformly elliptic operators of divergence form, in place of the Laplacian. Properties of the Green's functions for such operators on regular domains can be found in [12] and [16]. The associated 3G inequality on bounded Lipschitz domains can be found in [5] and it can be extended to unbounded Lipschitz domains with compact boundary using the method in [10].

Some of the results in this paper, especially for the case where D is bounded, can be extended to dimensions $d = 1, 2$, by making suitable modifications according to the difference in the potential theory between higher ($d \geqq 3$) and lower ($d \leqq 2$) dimensional spaces.

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