Math. Ann. 298, 89–100 (1994)

Mathematische Annalen


# Geometric methods for solving Codazzi and Monge-Ampère equations

**U. Pinkall[1,*], A. Schwenk-Schellschmidt[2,**] and U. Simon[1,**]**

[1] FB Mathematik, Technische Universität Berlin, Strasse des 17. Juni 135, D-10623 Berlin, Federal Republic of Germany
[2] FB Mathematik/Physik, TFH Berlin, D-13353 Berlin, Germany

Received March 13, 1993

## Introduction

In [Si-4] we studied the uniqueness problem for certain Monge-Ampère equations on a compact, simply connected, projectively flat manifold $M$. We now solve the corresponding existence problem by a systematic study of Codazzi-relations between projective and conformal structures in Sects. 1–3. In Sect. 4 we apply this to the given existence problem for certain Monge-Ampère operators on manifolds diffeomorphic to the unit sphere $S^n$. The key for our global study is the following statement in (1.7): Any projectively flat structure on $S^n$ coincides with the canonical projectively flat structure generated by the constant curvature metric. This lemma, together with the transformation Lemma 4.2 for Monge-Ampère operators, allows the transformation of the existence problem from projectively flat structures to constant curvature structures on $M$; for these spaces the existence problem was solved in [O-S].

As an application we present another affine version of the Euclidean Minkowski problem for ovaloids in Sect. 5. Blaschke already pointed out that the Minkowski problem is in fact an affine problem; see the references in the Blaschke commentary [Si-1, Sect. III.2]. For related local affine considerations see the recent paper [L-N-W].

## 1 Projective structures

Let $M$ be a $C^\infty$-manifold of dimension $n \geqq 2$ which is connected and oriented. We consider affine connections on $M$ which we always assume to be torsionless. Our

* Partially supported by DFG, SFB 288
** Partially supported by the DFG-project "Affine differential geometry" and the GADGET programme on the EC

definition of the sign of the curvature tensor $R$ and the Ricci tensor Ric, resp., of such an affine connection follows [K-N].

**1.1** *Projective equivalence.* Two affine connections $\nabla$, $\nabla^{\#}$ are called *projectively equivalent* if their unparametrized geodesics coincide. This is equivalent to the fact that there exists a 1-form $\theta$ on $M$ such that

$$\nabla^{\#}_v w - \nabla_v w = \theta(v)w + \theta(w)v \tag{1.1.1}$$

for all tangent vector fields $v$, $w$.

**1.2** *Ricci-symmetric connections.* (i) An affine connection is called *Ricci-symmetric* if its Ricci tensor Ric is symmetric. This is equivalent to the fact that $\nabla$ locally admits a parallel volume form $\omega$; that means $\nabla\omega = 0$. This form $\omega$ is unique up to a constant nonzero factor; we denote the equivalence class of such volume forms by $[\omega]$. If $M$ is simply connected the form $\omega$ is globally defined. (Some authors call $\nabla$ equiaffine instead of Ricci-symmetric. We avoid this terminology to prevent confusion with geometric quantities which are invariant under the equiaffine (unimodular) transformation group which we consider below). We call such a pair $\nabla$ and $[\omega]$ compatible if $\nabla\omega = 0$ for $\omega \in [\omega]$.
(ii) Let $\nabla$, $\nabla^{\#}$ be projectively equivalent and Ricci-symmetric. Then the 1-form $\theta$ is closed: $d\theta = 0$. If the parallel volume elements $\omega$ and $\omega^{\#}$ are locally related by $\omega^{\#} = \lambda\omega$ for a positive function $\lambda \in C^{\infty}(M)$, then $(n+1)\theta = d\log(\lambda)$. We define a positive function $\beta \in C^{\infty}(M)$ by $\beta^{n+1} = \lambda$. Then $\theta = d\log(\beta)$. (See e.g. [N-S, (9.4)]). If $M$ is simply connected, $\theta$ is exact which means that $\theta = d\log(\beta)$ holds globally.

**1.3** *Projective structures.* (i) A projective structure $\wp'$ on $M$ is given by an open covering $\{U_\alpha\}$ together with a family $\{\nabla_\alpha\}$ of torsionless affine connections, where $\nabla_\alpha$ is defined on $U_\alpha$, such that $\nabla_\alpha$ and $\nabla_\beta$ are projectively equivalent on $U_\alpha \cap U_\beta \neq \emptyset$. It is obvious when two such projective structures coincide.
(ii) It is known that, for a given projective structure $\wp'$ on $M$, one can find a torsionless affine connection $\nabla$ on $M$ such that, for each pair $\{U_\alpha, \nabla_\alpha\}$ from (i), $\nabla$ and $\nabla_\alpha$ are projectively equivalent; see [N-P-2, Proposition 1].
(iii) For any torsionless affine connection $\nabla$ and a given volume form $\omega$ on $M$ there exists a unique torsionless affine connection $\nabla^{\#}$ on $M$ with the following properties:
(1) $\nabla$ and $\nabla^{\#}$ are projectively equivalent;
(2) $\nabla^{\#}\omega = 0$.
See [Eis, p. 104] or [Schou, p. 288].
(iv) If $\wp'$ is a projective structure on $M$ and $\omega$ a volume form then there exists a unique Ricci-symmetric, torsionless affine connection $\nabla$ which induces the given projective structure $\wp'$ and satisfies $\nabla\omega = 0$; [N-P-2, Proposition 4].
(v) In the following, for a given projective structure $\wp'$ on $M$, we consider the subclass $\wp$ of torsionless, Ricci-symmetric affine connections which are mutually projectively equivalent.
(vi) Let $M$ be simply connected, let $\omega$ be a volume form on $M$ and $[\omega]$ the equivalence class from (1.2). Denote by $\Omega'$ the set of such equivalence classes of volume forms, and by $\Omega$ the set of positively oriented classes on $M$. Following (1.2.i) and (1.3.iv), there is a bijective correspondence $\wp \to \Omega$ defined by compatibility.
(vii) Let $\nabla \in \wp$. Then the projective curvature tensor $P$ of $\nabla$ is given by

$$(n-1)P(u,v)w := (n-1)R(u,v)w - \{\mathrm{Ric}(v,w)u - \mathrm{Ric}(u,w)v\} .$$

$P$ was introduced by H. Weyl. $P$ is an invariant of the class $\wp$.

**1.4** *Codazzi-tensors.* A symmetric (0.2) tensor $h$ is called a *Codazzi tensor* relative to an affine connection $\nabla$ if Codazzi equations are satisfied:

$$(\nabla_u h)(v,w) = (\nabla_v h)(u,w)\ .$$

We also say that $\{\nabla, h\}$ satisfy Codazzi equations or $\{\nabla, h\}$ form a Codazzi pair.

**1.5** *Projective flatness.* (i) $M$ carries a flat projective structure if there exists an atlas $\{U_\alpha, \varphi_\alpha\}$ on $M$ with the following properties: (a) A chart $(U, \varphi)$ maps the open set $U \subset M$ diffeomorphically onto an open set $\varphi(U) \in \mathbb{R}P^n$. (b) For $U_\alpha \cap U_\beta \neq \emptyset$, the transition function

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a restriction of a projective transformation of $\mathbb{R}P^n$ to $\varphi_\alpha(U_\alpha \cap U_\beta)$.
(ii) $\nabla \in \wp$ is called projectively flat on $M$ if $\nabla$ locally is projectively equivalent to a flat affine connection. Thus a projective structure $\{U_\alpha, \nabla_\alpha\}$ is projectively flat if each $\nabla_\alpha$ is projectively flat on $U_\alpha$.
(iii) Following a result of Weyl, a connection $\nabla \in \wp$ is projectively flat if and only if $\nabla$ satisfies the following two conditions:
(PF.1) $\mathrm{P} \equiv 0$ on $M$;
(PF.2) the Ricci tensor Ric of $\nabla$ satisfies Codazzi equations relative to $\nabla$.
See [Eis, p. 95], and [Schou, p. 289].
(vi) It is well known that, for $n \geqq 3$, (PF.1) implies (PF.2). In dimension $n = 2$ one has $P \equiv 0$ for any torsionfree connection, thus, for $n = 2$, (PF.2) is necessary and sufficient for $\nabla \in \wp$ to be projectively flat.

**1.6** *Projectively flat metric connections.* An affine connection $\nabla$ is called *metric* if $\nabla$ is the Levi-Civita connection $\nabla(h)$ of a semi-Riemannian metric $h$ on $M$. It is well known (and easily to be verified) that a connection is at the same time metric and projectivey flat if and only if the metric is of constant sectional curvature.

**1.7** *Projectively flat structures on $S^n$.* Let $M$ be a connected, simply connected and compact projectively flat manifold of dimension $n$. We sketch the proof of the well-known fact that $M$ must be projectively equivalent to $S^n$ equipped with the projective structure coming from the standard metric: Choose an open set $U \subset M$ and a projective diffeomorphism $\delta_0 : U \to V \subset S^n$. Using arguments familiar from the theory of analytic continuation in complex analysis and using the simply connectedness of $M$ we can extend $\delta_0$ to a local diffeomorphism $\delta : M \to S^n$ preserving the projective structure. By the compactness of $M$, $\delta$ is a covering map, hence (since $S^n$ is simply connected) a diffeomorphism.

In particular, up to projective diffeomorphisms, there is only one projectively flat structure on $S^n$.

Consider an ovaloid $x : M \to E$ embedded into $(n + 1)$-space $E$ and let $S^n \subset E$ be the unit sphere. The Gauss map $\mu$ induces on $S^n$ a metric of constant curvature 1, corresponding to the third fundamental form of $x$. The geodesics of this metric on $S^n$ are the great circles. The connection $\nabla(\mathrm{III})$ of the third fundamental form metric III is projectively flat and generates the canonical subclass $\wp$ on $M$, but different ovaloids generally induce different connections of this type on $M$. The Gauss map induces a diffeomorphism $\Phi : x(M) \to S^n(1)$. The images of the great circles under $\Phi^{-1}$ can be interpreted as shadow lines (see [Si-1, p. 51], for references).

## 2 Codazzi-tensors on projectively flat manifolds

We defined the notion of Codazzi tensors and a Codazzi pair $\{\nabla, h\}$ in (1.4). It is well known that Codazzi tensors locally can be generated by functions on constant curvature spaces [Fe]; for global results see [O-S]. Norden much earlier stated the local result in (2.3) below (see his appendix in [Schi, p. 232]).

**2.1** *Tensors generated by functions.* Let $f \in C^{\infty}(M)$ and $\nabla$ be a Ricci-symmetric, affine connection. We recall the notion of the covariant Hessian of a function

(2.1.1) $$\mathrm{Hess}(f)(v,w) := w(v(f)) - (\nabla_w v)f$$

for tangent fields $v, w$ on $M$ and define the symmetric (0.2) tensor field $h(f)$ by

(2.1.2) $$h(f)(v,w) := \mathrm{Hess}(f)(v,w) + \frac{1}{n-1} f\,\mathrm{Ric}(v,w)\,.$$

We say that $f$ generates $h(f)$. Examples are given in [O-S].

**2.2** *Affine harmonic functions.* Let $\nabla$ be Ricci-symmetric and projectively flat.
(i) If $F \in C^{\infty}(M)$ satisfies

(2.2.1) $$h(F) = \mathrm{Hess}(F) + \frac{1}{n-1} F\ \mathrm{Ric} \equiv 0$$

on $M$ then $F$ is said to be an *affine harmonic function* on $(M, \nabla)$. If $\nabla$ is given as above then locally there always exists a non-trivial solution $F$ of the system $h(F) \equiv 0$ because the integrability conditions are satisfied from the projective flatness of $\nabla$.
(ii) Let $f, f' \in C^{\infty}(M)$ generate the same tensor $h(f) = h(f')$. Then the difference $f - f'$ obviously satisfies (2.2.1).
(iii) The concept of affine-harmonic functions generalizes the notion of spherical harmonics of first order. Namely, on $S^n(c)$ with constant curvature $c$, the metric $g$ satisfies $\mathrm{Ric} = (n-1)cg$, thus (2.2.1) reads

$$\mathrm{Hess}(F) + cFg \equiv 0\,,$$

where the Hessian is now defined with respect to the Levi-Civita connection $\nabla(g)$. This last system characterizes the first eigenfunctions of the Laplacian $\Delta = \Delta(g)$ on $S^n(c)$.

**2.3 Norden's lemma.** *Let $\nabla$ be Ricci-symmetric and projectively flat on $M$. Then*
(i) *for any function $f \in C^{\infty}(M)$, the pair $\{\nabla, h(f)\}$ satisfies Codazzi equations;*
(ii) *if $M$ is simply connected and $h$ a given Codazzi tensor relative to $\nabla$ then locally there exists $f \in C^{\infty}(M)$ such that $h$ can be generated by $f$, which means $h = h(f)$; the generating function is unique up to an additive affine-harmonic function on $(M, \nabla)$.*

*Proof.* (i) and (ii) follow directly from the projective flatness of $\nabla$. This property gives the integrability condition for the equation $h = h(f)$ in (ii) for a given Codazzi tensor $h$.

## 3 Conformal and projective structures

In this section we assume the manifold $M$ to be connected, simply connected and oriented. On $M$ we consider the following two geometric structures and study their "compatibility", namely a conformal class of symmetric (0.2) tensor fields (which in this section might be degenerate) and a projective structure. In later applications the conformal class will be non-degenerate, which means that the class is a conformal structure defined by a class of semi-Riemannian metrics.

**3.1.a.** *Conformal class.* Let $\mathfrak{C}'$ be a conformal class on $M$, which means a class of mutually conformally related symmetric (0.2) tensor fields such that any two tensor fields $h$, $h^{\#}$ differ by a positive function $q \in C^{\infty}(M)$, say

$$h^{\#} = qh .$$

We identify elements of $\mathfrak{C}'$ which differ only by a constant positive factor and denote the set of equivalence class by $\mathfrak{C}$.

**3.1.b.** *Projective class.* Let $\wp$ be a class of torsionless, Ricci-symmetric, mutually projectively equivalent affine connections on $M$. As $M$ is simply connected, following (1.1) and (1.2.ii), for $\nabla, \nabla^{\#} \in \wp$ the 1-form $\theta$ in

$$\nabla^{\#}_v w - \nabla_v w = \theta(v)w + \theta(w)v$$

is exact and satisfies globally $\theta = d\log(\beta)$ for some positive function $\beta \in C^{\infty}(M)$. If $\omega$ is parallel relative to $\nabla \in \wp$ then $\omega^{\#} = \beta^{n+1}\omega$ is parallel to $\nabla^{\#} \in \wp$. We call $\beta$ a transition function within the projective class. Two affine connection $\nabla, \nabla^{\#} \in \wp$ determine their transition function uniquely modulo a constant factor.

**3.2** *Transformation of Codazzi equations.* Let $M$ be as above with given structures $\mathfrak{C}$ and $\wp$ as in (3.1).

(3.2.1) **Lemma.** *Let $\nabla, \nabla^{\#} \in \wp$ with transition function $\beta$, see* (3.1.b). *Assume that $h \in \mathfrak{C}$ and $h^{\#} := \beta h$. Then*

$$(\nabla^{\#}_u h^{\#})(v,w) - (\nabla^{\#}_v h^{\#})(u,w) = \beta\{(\nabla_u h)(v,w) - (\nabla_v h)(u,w)\} .$$

*Proof.* Straightforward calculation using (1.1.1) and (3.1).

(3.2.2) **Corollary.** *For $\mathfrak{C}$, $\mathfrak{C}'$ and $\wp$ as before, the following statements are equivalent:*
(i) *there exists a Codazzi pair $\{\nabla, h\} \in \wp \times \mathfrak{C}'$;*
(ii) *there exists a bijective mapping $B$: $\wp \to \mathfrak{C}$ such that any pair $\{\nabla, h\}$ where $h \in B(\nabla)$, satisfies Codazzi equations.*

*Proof.* We have to show that (i) implies (ii). Let $\{\nabla, h\}$ be the given pair and assume $\nabla^{\#} \in \wp$ to be arbitrary. From (3.1.b) there exists a positive transition function $\beta \in C^{\infty}(M)$ such that, for $\theta = d\log(\beta)$, the connections $\nabla$ and $\nabla^{\#}$ satisfy (1.1.1). Define the mapping $B$ by $B(\nabla) := h$ and $B(\nabla^{\#}) := \beta h =: h^{\#}$. We have $\theta = 0$ if and only if $\beta = \text{const} > 0$, thus $B$ is injective. But $B$ is also surjective, namely, for $h' \in \mathfrak{C}'$, there exists a positive function $\beta' \in C^{\infty}(M)$ such that $h' = \beta' h$. Now define $\nabla'$ by $\nabla'_v w := \nabla_v w + \theta'(v)w + \theta'(w)v$, where $\theta' := d\log(\beta')$ and $B(\nabla') = h'$. From (3.2.1) the pair $\{\nabla', h'\}$ satisfies Codazzi equations.

(3.2.3) **Definition and remarks.** (i) Let $\wp$ and $\mathfrak{C}$ be given as in (3.1). We call $\wp$ and $\mathfrak{C}$ *Codazzi-compatible* if there exists a pair $\{\nabla, h\} \in \wp \times \mathfrak{C}'$ satisfying Codazzi equations.
(ii) A pair $\{\nabla, h\}$, where $\nabla$ is an affine connection without torsion and $h$ a semi-Riemannian manifold, is sometimes called a *statistical manifold* if $\{\nabla, h\}$ is a Codazzi pair. If $M$ is simply connected and $\nabla$ is Ricci-symmetric, (3.2.2) immediately gives the possibility of constructing a whole class of statistical manifolds.

(iii) Codazzi equations play an important role in local and global differential geometry as systems of pde's. From the point of view of differential eqs. (3.2.1–3.2.2) admit a "transformation" of Codazzi equations. The following section gives an interesting example.

**3.3** *Bilinear forms generated by functions.* As before, let $M$ be simply connected. In Sect. 2 we recalled that, for a given projectively flat connection $\nabla$, any Codazzi-tensor $h$ can be generated by functions. We study the transformation of tensors of the form $h(f)$ in (2.1.2) under a projective change of the connection, without assuming the class $\wp$ from (3.1.b) being projectively flat.

(3.3.1) **First transformation lemma.** *Let $M$ be connected, simply connected and $\nabla$, $\nabla^\# \in \wp$ be related by the transition function $\beta$ as in* (3.1.b). *For $f \in C^\infty(M)$, define $h(f)$ with respect to $\nabla$ as in* (2.1.2), *and $h^\#(f)$ with respect to $\nabla^\#$, correspondingly. Then*

$$h^\#(\beta f) = \beta h(f) .$$

*Proof.* The technical proof is straightforward with the following steps:
(i) The curvature tensors of $\nabla$, $\nabla^\#$, respectively, satisfy (see [Eis, Sect. 32]):

$$R^\#(u,v)w = R(u,v)w + \beta[\mathrm{Hess}(\beta^{-1})(v,w)u - \mathrm{Hess}(\beta^{-1})(u,w)v] .$$

(ii) The Ricci tensors satisfy:

$$\mathrm{Ric}^\#(v,w) = \mathrm{Ric}(v,w) + (n-1)\beta\,\mathrm{Hess}(\beta^{-1})(v,w) .$$

(iii) The covariant Hessians are related by:

$$(\mathrm{Hess}^\# f)(u,v) = (\mathrm{Hess}\, f)(u,v) - \beta^{-1}[d\beta(u)df(v) + d\beta(v)df(u)] .$$

(iv) For $\gamma \in C^\infty(M)$

$$(\mathrm{Hess}(\gamma f))(u,v) = \gamma(\mathrm{Hess}\, f)(u,v) + f(\mathrm{Hess}\,\gamma)(u,v) + d\gamma(u)df(v) + d\gamma(v)df(u) .$$

(v) From (iii) and (iv) we verify

$$(\mathrm{Hess}^\#(\beta f))(u,v) = \beta(\mathrm{Hess}\, f)(u,v) - f\beta^2(\mathrm{Hess}(\beta^{-1}))(u,v)$$

which together with (i) gives the assertion.

(3.3.2) *Codazzi equations as pde's.* Let $M$ be simply connected and $\nabla$, $\nabla^\# \in \wp$ again be related by a transition function $\beta$ as (3.1.b). Let $h$ be a symmetric (0.2) tensor field. Then:
(i) The function $f \in C^\infty(M)$ is a solution of the second order pde

(3.3.2.a) $$h(f) = h$$

if and only if the function $f^\# := \beta f$ is a solution of

(3.3.2.b) $$h^\#(f^\#) = \beta h .$$

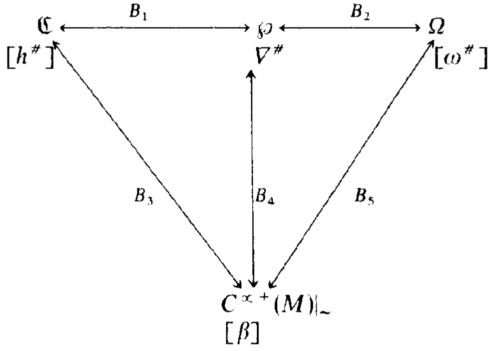
(ii) If $\nabla$ is projectively flat and $\{\nabla, h\}$ a Codazzi pair then both systems (3.3.2.a) and (3.3.2.b) are locally solvable; in particular we have:
$F$ is an affine-harmonic function on $(M, \nabla)$ if and only if $F^\# = \beta F$ is an affine-harmonic on $(M, \nabla^\#)$.

**3.4** *Diagram.* We summarize in the following diagram the relations we considered so far. Let $\wp$, $\mathfrak{C}$, $\Omega$ be compatible structures as in (3.2.3) and (1.2) and let $C^{\infty+}(M)|_\sim$ denote the set of all positive (transition) functions on $M$ which are identified if they differ only by a constant positive factor. Starting with arbitrary but fixed compatible elements $\nabla$, $[h]$, $[\omega]$ we define bijections $B_3, B_4, B_5$ by the following relations:

$$B_3 : h^{\#} := \beta h, \quad B_4 : \nabla_v^{\#} w := \nabla_v w + \theta(v)w + \theta(w)v$$
$$\text{with } \theta = d\log(\beta),\ B_5 : \omega^{\#} := \beta^{n+1}\omega .$$

The mappings $B_3, B_4, B_5$ induce bijective mappings $B_1, B_2$ such that the diagram below is commutative.

Then we have: $B_1$, $B_2$ are the bijections considered in (1.3) and (3.2.2), resp., which means that the images of the class $[\beta]$ of transition functions under $B_3$, $B_4$, $B_5$ are compatible in the sense of (1.2) and 3.2.3).




## 4 Monge-Ampère operators

Similarly to the study of Codazzi equations we investigate now certain Monge-Ampère equations. We consider the projective class $\wp$ defined in (3.1.b).

**4.1 Definition of a Monge-Ampère operator.** Let $\nabla \in \wp$ with parallel volume form $\omega$; recall from (1.2) that this form is unique up to a non-zero constant factor.
(i) We define a Monge-Ampère operator $\mathcal{M}$, acting on functions $f \in C^{\infty}(M)$, with respect to a local Gauss basis field $\partial_1, \ldots, \partial_n$:

$$\mathcal{M}(f) := \frac{\det(h(f)(\partial_i, \partial_j))}{(\omega(\partial_1, \ldots, \partial_n))^2} .$$

This definition is independent of the local representation. The connection $\nabla$ determines the operator $\mathcal{M}$ uniquely up to a positive constant factor.
(ii) Let $\varphi \in C^{\infty}(M)$ be a positive function. We call $f \in C^{\infty}(M)$ an *elliptic solution of*

$$\mathcal{M}(f) = \varphi$$

if the tensor $h(f) = \operatorname{Hess}(f) + \frac{1}{n-1} f \operatorname{Ric}$ is definite.

**4.2 Second transformation lemma.** *Let* $\nabla, \nabla^{\#} \in \wp$ *be given as in* (3.1.b) *with parallel volume forms* $\omega, \omega^{\#}$ *related by* $\omega^{\#} = \beta^{n+1}\omega$. *Then*

$$\mathcal{M}^{\#}(\beta f) = \beta^{-(n+2)}\mathcal{M}(f) ,$$

*where* $\mathcal{M}^{\#}$ *corresponds to* $\nabla^{\#}$.

The proof immediately follows from the transformation lemma for $h(f)$ in (3.3.1).

**4.3** *Equivalence of solutions.* Let $M$ be simply connected and the operators $\mathcal{M}, \mathcal{M}^{\#}$ defined as in (4.2). For $\varphi \in C^{\infty}(M)$ positive, define $\varphi^{\#} := \beta^{-(n+2)}\varphi$ which again is positive.

(i) $f$ solves the Monge-Ampère equation

$$\mathcal{M}(f) = \varphi \tag{4.3.1}$$

if and only if the function $f^{\#} := \beta f$ solves the corresponding equation

$$\mathcal{M}^{\#}(f^{\#}) = \varphi^{\#} . \tag{4.3.2}$$

Moreover, $f$ is an elliptic solution if and only if $f^{\#}$ is an elliptic solution.

(ii) We recall the situation that a given connection $\nabla \in \wp$ is projectively flat and metric. Then $\nabla$ is the Levi-Civita connection $\nabla(g)$ where $(M, g)$ is a space of constant curvature $c$. The Monge-Ampère equation now reads

$$\frac{\det(\mathrm{Hess}(f) + cfg)}{\det(g)} = \varphi . \tag{4.3.3}$$

The existence and uniqueness of global solutions on closed (compact without boundary) constant curvature manifolds was investigated in [O-S]. We recall in particular the case that $M$ is simply connected and $c > 0$ (see ibid., Theorem 3.4). In this case, (4.3.3) has an elliptic solution if and only if $\varphi$ satisfies the integrability condition

$$\int_M \varphi F \omega = 0$$

where $M = S^n(c)$, $F$ is a first eigenfunction of the Laplacian and $\omega = \omega(g)$ the Riemannian volume element. As a first eigenfunction $F$ satisfies the system of pde's

$$\mathrm{Hess}(F) + \frac{1}{n-1} F \,\mathrm{Ric} = \mathrm{Hess}(F) + cFg = 0$$

on $M$; hat means $F$ is affine-harmonic on $(M, \nabla(g))$.

(iii) Having the situation in (ii) as a motivation in mind we consider integrands of the type $\varphi F \omega$ under a projective change of the connection. Let $\nabla, \nabla^{\#} \in \wp$ be related by the transition function $\beta$ as in (3.1.b) and let $F, F^{\#}$, where $F^{\#} = \beta F$, be affine-harmonic functions, resp. (see (3.3.2.ii)). Let $\varphi, \varphi^{\#} \in C^{\infty}(M)$ be related by $\varphi^{\#} = \beta^{-(n+2)}\varphi$. Then

$$\varphi F \omega = \varphi^{\#} F^{\#} \omega^{\#}$$

or, in terms of Monge-Ampère operators:

$$\mathcal{M}(f) F \omega = \mathcal{M}^{\#}(f^{\#}) F^{\#} \omega^{\#} ,$$

where $f^{\#} = \beta f$ again.

**4.4 Global solutions of Monge-Ampère equations.** *Let $M$ be diffeomorphic to $S^n(1)$ and let $\nabla$ be a torsionfree, Ricci-symmetric, projectively flat connection on $M$ with parallel volume form $\omega$. Let $\varphi \in C^{\infty}(M)$ be positive.*

(i) Existence. *The Monge-Ampère equation*

$$\mathcal{M}(f) = \varphi$$

*has an elliptic solution f if and only if*

$$\int_M \varphi F\omega = 0$$

*for any affine-harmonic function F on* $(M, \nabla)$.

(ii) Uniqueness. *The difference* $f - f'$ *of any two elliptic solutions of* $\mathcal{M}(f) = \varphi$ *is an affine-harmonic function on* $(M, \nabla)$.

*Proof. Existence.* We consider the class $\wp$ of all torsionless affine connections on $M$ which are Ricci-symmetric and projectively equivalent to $\nabla$. The induced projective structure is flat on $M$. Following (1.7), there exists a metric $g$ of constant positive curvature $c > 0$ on $M$ such that $\nabla^{\#} := \nabla(g) \in \wp$. As $M$ is simply connected there exists a positive transition function $\beta \in C^{\infty}(M)$ for $\nabla$ and $\nabla^{\#}$ as in (3.1.b). Define as before the function $\varphi^{\#} := \beta^{-(n+2)}\varphi$ which is positive on $M$. (4.3.iii) implies that

$$\int_M \varphi^{\#} F^{\#} \omega^{\#} = \int_M \varphi F\omega = 0$$

for any first eigenfunction $F^{\#}$ of the Laplacian $\Delta(g)$ on $M$; these functions are exactly the affine-harmonic functions on $(M, g)$; see (4.3.ii). Thus

$$\mathcal{M}^{\#}(f^{\#}) = \varphi^{\#}$$

has an elliptic solution $f^{\#}$ on $(M, g)$; see (4.3.ii) and [O-S, Theorem 3.4]. This means that $\mathcal{M}^{\#}(f^{\#}) = \varphi^{\#}$ has an elliptic solution on $(M, \nabla^{\#})$. Then $\mathcal{M}(f) = \varphi$ has an elliptic solution $f$ on $(M, \nabla)$ from (4.3), namely $f = \beta^{-1} f^{\#}$.

*Uniqueness.* Let $f, f'$ be elliptic solutions of $\mathcal{M}(f) = \varphi$ in (4.3.1). Then $f^{\#} = \beta f$ and $f'^{\#} = \beta f'$ are elliptic solutions of $\mathcal{M}^{\#}(f^{\#}) = \varphi^{\#}$ in (4.3.2), and from [O-S, Theorem 3.4], their difference $f^{\#} - f'^{\#}$ is a first eigenfunction of the Laplacian. As above, then $f - f' = \beta^{-1}(f^{\#} - f'^{\#})$ is affine-harmonic on $(M, \nabla)$.

For a different proof of the uniqueness problem see [Si-4].

## 5 An affine version of the Minkowski problem

In the introduction we referred to known results about the relations between the Euclidean Minkowski problem and affine hypersurface theory. We will now apply the results of the foregoing section to this topic.

For an introduction to relative hypersurface theory we refer to one of the lecture notes [N] or [S-S-V] or to [N-P-1]. For the uniqueness part of the following theorem see also [Si-2, Si-3, Si-4].

We denote by $A$ a real affine space of dimension $n + 1$, $n \geqq 2$, with associated vector space $V$, $V^*$ its dual, and $\langle \, , \rangle : V^* \times V \to \mathbb{R}$ the canonical scalar product. Let

$$x : M \to A$$

be a non-degenerate hypersurface immersion with relative normalization $\{Y, y\}$. That means $Y$ denotes a conormal field $Y : M \to V^*$, $\langle Y, dx(v) \rangle = 0$ for any tangential field $v$, and $y : M \to Y$ is transversal to $x$ satisfying

$$\langle Y, y \rangle = 1 \quad \text{and} \quad \langle Y, dy(v) \rangle = 0 \, .$$

A relative normalization is called equiaffine if $y$ is the equiaffine normal of $x$ (see [S-S-V, Sect. 6.2]).

A relative normalization $\{Y, y\}$ induces a triple $\{\nabla, h, \nabla^*\}$ on $M$, where $\nabla, \nabla^*$ are two affine connections without torsion conjugate to the relative metric $h$. $\nabla$ is called the connection induced by $y$, $\nabla^*$ is called the conormal connection induced by $Y$ [S-S-V, Sect. 4.4]. Recall that the relative metric $h$ is generated by the class of relative support functions $\rho$ of $\{x, Y, y\}$, which means $h = h(\rho)$. Any two generating functions $\rho_1, \rho_2$, satisfying $h(\rho_1) = h(\rho_2)$, differ by an affine harmonic function $F := \rho_1 - \rho_2$ on $(M, \nabla^*)$, which can be represented in the form $F = \langle Y, b\rangle$, $b \in V$ (see (2.1) above; [S-S-V, Sect. 4.13]).

We state Radon's fundamental theorem of relative hypersurface theory in a modified version of [D-N-V]; see [S-S-V, (4.11)]:

**5.1 Fundamental theorem.** *Let $M$ be a connected, simply connected and oriented $C^\infty$-manifold of dimension $n \geqq 2$.*
Existence. *Let $\nabla^*$ be a given affine connection without torsion and Ricci-symmetric, and $h$ a given semi-Riemannian metric. There exists a non-degenerate hypersurface $x$ with relative normalization $\{Y, y\}$ such that $h$ is the relative metric and $\nabla^*$ the conormal connection of $\{x, Y, y\}$ if and only if*
(i) *$\nabla^*$ is projectively flat;*
(ii) *$h$ satisfies Codazzi equations relative to $\nabla^*$.*
Uniqueness. *The triple $\{x, Y, y\}$ is unique modulo affine equivalences of the affine space.*

We are now able to prove the following affine version of the Euclidean Minkowski problem.

**5.2. Theorem.** Affine version of the Minkowski problem. *Let $M$ be the unit sphere $S^n$ and let $\wp$ be the subclass of Ricci-symmetric, torsionless connections of the canonical projectively flat structure of $M$.*
Existence. *Let $\nabla^* \in \wp$ be given with parallel volume form $\omega^*$. There exists a hyperovaloid $x$ with $\nabla^*$ as equiaffine conormal connection if and only if any affine-harmonic function $F^*$ on $(M, \nabla^*)$ satisfies*

$$\int_M F^* \omega^* = 0 .$$

Uniqueness. *The hyperovaloid $x$ is uniquely determined modulo affine equivalences.*

*Proof.* Existence: Let $\nabla^* \in \wp$ be given. From (1.6–1.7) there exists a metric connection $\nabla^\circ \in \wp$, and the metric is of constant positive sectional curvature $c$. Following (3.1.b) there exists a positive transition function $\beta$ satisfying

$$\nabla^*_v w - \nabla^\circ_v w = (d\log(\beta))(v)w + (d\log(\beta))(w)v ;$$

moreover, from (4.3) the positive function $\varphi^\circ := \beta^{n+2}$ satisfies

$$\varphi^\circ F^\circ \omega^\circ = F^* \omega^* .$$

We know from (4.3) that the equation

$$M^*(\rho) = 1$$

has a solution $\rho$ if and only if

$$M^\circ(f^\circ) = \varphi^\circ$$

has a solution $f^\circ$ and $\rho = \beta f^\circ$. As $h^\circ$ has constant sectional curvature, the Levi-Civita connection $\nabla^\circ$ determines the metric

$$\mathrm{Ric}^\circ = c(n-1)h^\circ .$$

Moreover, for any $f \in C^\infty(M)$,

$$\mathrm{Hess}^\circ f + cfh^\circ = \mathrm{Hess}^\circ f + \frac{1}{n-1}\,\mathrm{Ric}^\circ f .$$

It is known [O-S, Theorem 3.4] that on spaces of constant positive curvature the Monge-Ampère equation $\mathcal{M}^\circ(f^\circ) = \varphi^\circ > 0$, which now is of the form in (4.3.3), has an elliptic solution $f^\circ$ if and only if

$$\int_M \varphi^\circ F^\circ \omega^\circ = 0$$

for any affine-harmonic function $F^\circ$ on $(M, \nabla^\circ)$; see (4.4). But this integrability condition is equivalent to

$$\int_M F^* \omega^* = 0$$

for any affine-harmonic function $F^*$ on $(M, \nabla^*)$; see (4.3.iii). Thus the assumptions gaurantee the existence of an elliptic solution $\rho := f$ of $M^*(\rho) = 1$. The function $\rho$ or $(-\rho)$ generates a positive definite tensor $h(\rho)$ which is a Codazzi tensor relative to $\nabla^*$. As $\nabla^*$ is projectively flat there exists a hypersurface immersion $x : M \to A$ with relative normalization $\{Y, y\}$ satisfying (5.1). The relative metric $h(\rho)$ is positive definite and $M \cong S^n$, thus $x(M)$ is a hyperovaloid. Consider the Riemannian volume form $\omega(h) := \omega(h(\rho))$ and the class $[\omega^*]$ of volume forms which satisfy $\nabla^* \omega^* = 0$. The Monge-Ampère equation

$$1 = \mathcal{M}^*(\rho) = \frac{\det(h(\rho)(\partial_i, \partial_j))}{(\omega^*(\partial_1, \ldots, \partial_n))^2}$$

implies that there exists $\omega^* \in [\omega^*]$ such that $\omega^* = \omega(h)$; this means that the relative normalization $\{Y, y\}$ is equiaffine (see [S-S-V, Sect. 6.2]). One easily verifies that $\nabla^*$ is the conormal connection of $\{x, Y, y\}$. This proves the existence part.

Uniqueness: Following (4.4), any two elliptic solutions $\rho, \rho^\#$ of $\mathcal{M}^*(\rho) = 1$ differ by a $\nabla^*$-affine-harmonic function, thus $h(\rho) = h(\rho^\#)$. But then $\{x, Y, y\}$ is unique modulo affine transformations of $A$; see (5.1).

## References


[D-N-V] Dillen, F., Nomizu, K., Vrancken, L.: Conjugate connections and Radon's theorem in affine differential geometry. Monatsh. Math. **109**, 221–235 (1990)

[Eis] Eisenhart, L.P.: Non-Riemannian geometry. Colloq. Publ., Am. Math. Soc. **VIII** (1927)

[Fe] Ferus, D.: A remark on Codazzi tensors in constant curvature space. In: Ferus, D. et. al. (eds.) Global differential geometry and global analysis. (Lect. Notes Math., vol. 838, p. 257) Berlin Heidelberg New York: Springer 1981

[K-N] Kobayashi, S., Nomizu, K.: Foundations of differential geometry, vol. I. New York: Interscience 1963

[L-N-W] Li, A.-M., Nomizu, K., Wang, C.-P.: A generalization of Lelieuvre's formula. Result. Math. **20**, 682–690 (1991)

[N] Nomizu, K.: Introduction to affine differential geometry, part I. Lecture Notes, MPI preprint MPI 88-37, 1988; Revised: Department of Mathematics, Brown University (1989)

[N-P-1] Nomizu, K., Pinkall, U.: On the geometry of affine immersions. Math. Z. **195**, 165–178 (1987); [Zbl. 629.53012]

[N-P-2] Nomizu, K., Pinkall, U.: On a certain class of homogeneous projectively flat manifolds. Tôhoku Math. J., II. Ser. **39**, 407–427 (1987)

[N-S] Nomizu, K., Simon, U.: Notes on Conjugate Connections. In: Dillen, F., Verstraelen, L. (eds.). Geometry and Topology of Submanifolds, IV. Proc. Conf. Diff. Geom. Vision, Leuven 1991, pp. 152–171. Singapore: World Scientific 1992

[O-S] Oliker, V., Simon, U.: Codazzi tensors and equations of Monge-Ampère type on compact manifolds of constant sectional curvature. J. Reine Angew. Math. **342**, 35–65 (1983)

[Schi] Schirokow, A.P., Schirokow, P.A.: Affine Differentialgeometrie. Leipzig: Teubner 1962, [Zbl.: 106.147; Russ. Original. Zbl. 85.367]

[Schou] Schouten, J. A.: Ricci-Calculus, 2nd. ed. Berlin Heidelberg New York: Springer 1954

[Si-1] Simon, U.: Zur Entwicklung der affinen Differentialgeometrie. In: Blaschke, W. Gesammelte Werke, vol. 4, Affine Differentialgeometrie. Differentialgeometrie der Kreis- und Kugelgruppen. Essen: Thales 1985

[Si-2] Simon, U.: The fundamental theorem in affine hypersurface theory. Geom. Dedicata **26**, 125–137 (1988)

[Si-3] Simon, U.: Global uniqueness for ovaloids in Euclidean and affine differential geometry. Tôkoku Math. J. **44**, 327–334 (1992)

[Si-4] Simon, U.: The uniqueness problem for Monge-Ampère equations on projectively flat manifolds. Bull. Belg. Math. Soc. **45**, 25–34 (1993)

[S-S-V] Simon, U., Schwenk-Schellschmidt, A., Viesel, H.: Introduction to the affine differential geometry of hypersurfaces. Lecture Notes, Science University of Tokyo 1991