Math. Ann. 297, 133–146 (1993)

Mathematische Annalen
© Springer-Verlag 1993

On the topology of double coset manifolds

W. Singhof

Mathematisches Institut, Heinrich-Heine-Universität, Universitätsstrasse 1, D-40225 Düsseldorf, Germany

Received July 9, 1992; in revised form February 15, 1993

Mathematics Subject Classification (1991): 57T15, 57T35, 55R40

Introduction

We consider a compact Lie group G and two closed subgroups H and K of G. The abstract product $K \times H$ operates on G by $g \cdot (k, h) = k^{-1}gh$. If this operation happens to be free, the quotient $K\backslash G/H$ is a compact manifold, which we call a double coset manifold. These manifolds have attracted the attention of several differential geometers: Gromoll and Meyer [GrM] for instance described an exotic 7-sphere as a double coset manifold. Therefore the class of double coset manifolds is strictly larger than that of homogeneous spaces since Borel observed long ago that a homogeneous space which is homeomorphic to a sphere is actually diffeomorphic to it.

A serious study of double coset manifolds was made by Eschenburg [E1–E4]. He showed that some of them admit Riemannian metrics with strictly positive sectional curvature; he obtained a classification of certain types of double coset manifolds and computed the cohomology of some of them.

The objective of the present paper is to show that in many respects the topology of double coset manifolds is as easy to handle as that of homogeneous spaces.

To compute the cohomology of G/H, the best way is to look at the fibration

$$G/H \to BH \to BG .$$

In Sect. 2 we observe that, for double coset manifolds, there is a diagram

$$(1)\qquad \begin{array}{ccc} K\backslash G/H & \longrightarrow & BH \\ \downarrow & & \downarrow \\ BK & \longrightarrow & BG \end{array}$$

which is, up to homotopy, a fibre square. The Eilenberg–Moore spectral sequence of (1) collapses quite often and

$$(2)\qquad H^*(K\backslash G/H) \cong \mathrm{Tor}_{H^*(BG)}(H^*(BK), H^*(BH))$$

under reasonable torsion hypotheses.

Borel and Hirzebruch [BH] based their computation of the characteristic classes of homogeneous manifolds on a description of the tangent bundle of G/H which goes as follows: H operates on G/H from the left fixing $o = eH$. Therefore we obtain a representation ι of H on the tangent space $T_o(G/H)$, and the tangent bundle of G/H is $G \times_H T_o(G/H)$. It is obvious that this description cannot be immediately generalized to double coset manifolds since in general there is no natural group operation on $K\backslash G/H$. However, in Sect. 3, we derive a formula for the tangent bundle of $K\backslash G/H$ which is similar in spirit and which is perfectly sufficient for the computation of Pontrjagin classes and Stiefel–Whitney classes in Sect. 4.

These considerations show how some properties of homogeneous spaces generalize to the larger class of double coset manifolds while others don't: For instance, when H is a maximal torus of G then G/H admits a complex structure and is stably parallelizable [BH]. We find that $K\backslash G/H$ is at least stably almost complex when K and H are tori but that it may have non-vanishing Pontrjagin classes.

A fundamental property of homogeneous spaces is that their Euler characteristic is given in terms of the orders of the Weyl groups by

$$\chi(G/H) = \begin{cases} |W(G)|/|W(H)|, & \text{if } \operatorname{rk} H = \operatorname{rk} G\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3}$$

This formula generalizes in the most straightforward way:

$$\chi(K\backslash G/H) = \begin{cases} \dfrac{|W(G)|}{|W(K)|\cdot|W(H)|}, & \text{if } \operatorname{rk} K + \operatorname{rk} H = \operatorname{rk} G\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4}$$

We prove formula (4) in Sect. 5 using the isomorphism (2).

In Sect. 6 we take a closer look at (2). We find in particular that

$$\operatorname{Tor}^{s,t}_{H^*(BG)}(H^*(BK), H^*(BH)) = 0 \quad \text{for } s < \operatorname{rk} K + \operatorname{rk} H - \operatorname{rk} G\,. \tag{5}$$

(In the special case of homogeneous spaces, this result is due to Baum [Ba].) As a consequence, for $\operatorname{rk} K + \operatorname{rk} H = \operatorname{rk} G$, we have

$$H^*(K\backslash G/H) = H^*(BK) \otimes_{H^*(BG)} H^*(BH)$$

as algebras under the usual torsion hypotheses. In fact, we obtain a vanishing result which is much better than (5). It implies the vanishing of characteristic classes in high dimensions and shows that, if $\operatorname{rk} K + \operatorname{rk} H < \operatorname{rk} G$, all Pontrjagin numbers of $K\backslash G/H$ are zero so that $K\backslash H/G$ is rationally null-cobordant. Finally, in Sect. 7, following a suggestion of the referee, we treat a concrete example.

1 General properties of double coset manifolds

(1.1) Let G be any group and H, K subgroups of G. Then the following 5 conditions are equivalent:

1. $K \times H$ operates freely from the right on G by

$$g \cdot (k, h) = k^{-1} g h\,.$$

2. K operates freely from the left on G/H.
3. H operates freely from the right on $K\backslash G$.

4. G operates freely from the right on $(K\backslash G)\times(G/H)$ by

$$(Kg, g'H)\cdot x = (Kgx, x^{-1}g'H) .$$

5. If $h \in H$ and $k \in K$ are elements which are conjugated in G then $h = k = 1$.

If these conditions are satisfied, we will say that the subgroups H, K *satisfy the double coset condition.*

From now on we will assume that G is a compact connected Lie group and that H and K are closed subgroups. Then, if H, K satisfy the double coset condition, we obtain a compact differentiable manifold $K\backslash G/H$ which, by conditions 2., 3., and 4., is the basis of 3 principal fibrations with structure groups K, H, and G respectively. In particular, there is the principal G-bundle

$$K\backslash G \times G/H \to K\backslash G/H$$

$$(Kg, g'H) \mapsto Kgg'H .$$

The following facts are easy consequences of the various forms of the double coset condition:

(1.2) Let H, K satisfy the double coset condition and denote the Lie algebras of H and K by $\mathfrak{h}$ and $\mathfrak{k}$, respectively. Then

$$\mathfrak{h} \cap \mathrm{Ad}(g)\cdot\mathfrak{k} = 0 \quad \text{for } g \in G .$$

(1.3) Suppose that H' is conjugated to H and K' conjugated to K. Then $K\backslash G/H$ and $K'\backslash G/H'$ are diffeomorphic.

(1.4) If H and K are connected and T_H, T_K are their maximal tori, then the subgroups H, K satisfy the double coset condition if and only if T_H, T_K do. If this is the case, we may assume by (1.3) that T_H and T_K are both contained in a maximal torus T_G of G.

(1.5) If H, K satisfy the double coset condition then

$$\mathrm{rk}\, H + \mathrm{rk}\, K \leqq \mathrm{rk}\, G .$$

(1.6) If H, K (and G, as always) are connected with $\mathrm{rk}\, H + \mathrm{rk}\, K = \mathrm{rk}\, G$ and if H, K satisfy the double coset condition then $K\backslash G/H$ is 1-connected.

(1.7) Very often, double coset manifolds appear in the following disguise: Let U be a closed subgroup of $G \times G$ operating freely on G via

$$g\cdot(x, y) = x^{-1}gy \quad \text{for } g \in G , \quad (x, y) \in U .$$

Then the quotient G/U is diffeomorphic to the double coset manifold $U\backslash(G \times G)/\Delta G$ with $\Delta G = \{(g, g) | g \in G\}$. I am grateful to J.-H. Eschenburg for this remark.

2 The cohomology of double coset manifolds

(2.1) We fix a universal G-bundle with total space EG and base space $BG = (EG)/G$; as classifying spaces for H and K, we take $BH := (EG)/H$ and $BK := (EG)/K$. Then there are fibrations

$$G/H \to BH \to BG ,$$

$$G/K \to BK \to BG .$$

The inclusion of the fibre $G/H \to BH$ is a classifying map for the principal H-bundle $G \to G/H$. Composing the inclusion $G/K \to BK$ with the diffeomorphism $Kg \mapsto g^{-1}K$ of $K \backslash G$ onto G/K, we obtain a classifying map for the principal K-bundle $G \to K\backslash G$. Assume now that H, K satisfy the double coset condition. The principal bundles $K\backslash G \to K\backslash G/H$ and $G/H \to K\backslash G/H$ admit classifying maps $K\backslash G/H \to BH$ and $K\backslash G/H \to BK$, and since these two bundles have as pull-backs the principal bundles $G \to G/H$ resp. $G \to K\backslash G$, we have the following two diagrams which are commutative up to homotopy:

$$\begin{array}{ccc} G/H & \searrow & \\ \downarrow & & BH\,, \\ K\backslash G/H & \nearrow & \end{array} \qquad \begin{array}{ccc} K\backslash G & \searrow & \\ \downarrow & & BK\,. \\ K\backslash G/H & \nearrow & \end{array}$$

In the sequel, unnamed maps between the spaces appearing in (2.1) are the natural maps just introduced.

(2.2) Proposition. *The diagram*

$$(*) \qquad \begin{array}{ccc} K\backslash G/H & \longrightarrow & BH \\ \downarrow & & \downarrow \\ BK & \longrightarrow & BG \end{array}$$

is, up to homotopy, a fibre square, and the composition

$$K\backslash G/H \to BG$$

is a classifying map for the principal G-bundle $K\backslash G \times G/H \to K\backslash G/H$ *introduced in* (1.1).

That $(*)$ is, up to homotopy, a fibre square, is to mean the following: The diagram $(*)$ is commutative up to homotopy, and if X is the actual fibre product of BK and BH over BG, then there is a homotopy equivalence $X \to K\backslash G/H$ which makes the following diagram homotopy commutative:

$$(**) \qquad \begin{array}{ccccc} & \nearrow & BK & \nwarrow & \\ X & & \longrightarrow & & K\backslash G/H \\ & \searrow & BH & \swarrow & \end{array}$$

The proof of (2.2) is completely elementary:
a) It is obvious that $(K\backslash G) \times_H G$ and $K\backslash G \times G/H$ are G-diffeomorphic. This means that the composition

$$K\backslash G/H \to BH \to BG$$

is a classifying map for the G-bundle $K\backslash G \times G/H \to K\backslash G/H$. The same thing holds for the composition

$$K\backslash G/H \to BK \to BG\,.$$

As a consequence, $(*)$ is commutative.
b) It is easy to verify that the fibre product X of BK and BH over BG may be identified with

$$(K\backslash G \times EG \times G/H)/G$$

where G operates on $K\backslash G \times EG \times G/H$ by

$$(Kg, z, g'H) \cdot x = (Kgx, zx, x^{-1}g'H) .$$

Since G operates freely on $K\backslash G \times G/H$, we obtain a homotopy equivalence

$$(K\backslash G \times EG \times G/H) \to K\backslash G/H$$
$$[Kg, z, g'H] \mapsto Kgg'H .$$

c) The commutativity of $(**)$ is easily seen by considering the induced principal bundles. This completes the proof of (2.2).

(2.3) The fibre square $(*)$ has an Eilenberg–Moore spectral sequence (E_r) which converges to $H^*(K\backslash G/H)$ and has E_2-term

$$E_2^{**} = \operatorname{Tor}^{**}_{H^*(BG)}(H^*(BK), H^*(BH)) .$$

This is a spectral sequence in the second quadrant. As for the Serre spectral sequence, its differentials d_r are of degree $(r, -r+1)$, and the modules $E_\infty^{s,t}$ with $s+t=n$ form the quotients of a filtration of $H^n(K\backslash G/H)$. In Sect. 6 we shall find a rather small parallelogram B in the (s, t)-plane such that $E_r^{s,t} = 0$ for (s, t) outside B. It is quite frequent that this spectral sequence collapses and that there are no additive extension problems. (See [GuM, HMS, Mu, Wo]; reference [Mu] is the most convenient to extract the information we need.) We obtain:

(2.4) Theorem. *Let G be a compact connected Lie group and let H, K be closed connected subgroups satisfying the double coset condition. We consider cohomology with coefficients in a principal ideal domain R and assume that $H^*(BG)$, $H^*(BH)$, and $H^*(BK)$ are polynomial algebras with generators of even dimensions. Then*

(1) $$H^*(K\backslash G/H) \cong \operatorname{Tor}_{H^*(BG)}(H^*(BK), H^*(BH))$$

as graded R-modules. Moreover, the natural maps from $K\backslash G/H$ to BK and BH induce a ring homomorphism

(2) $$\rho : H^*(BK) \otimes_{H^*(BG)} H^*(BH) \to H^*(K\backslash G/H)$$

which is injective. The isomorphism (1) *may be chosen in such a way that the direct summand of $H^*(K\backslash G/H)$ corresponding to* Tor^0 *equals* $\operatorname{im} \rho$.

The isomorphism (1) is a special case of the main theorem of [Mu]. Given (1), the additional assertions are just an interpretation of the edge homomorphism of the Eilenberg–Moore spectral sequence. This interpretation is an easy consequence of naturality properties. (See [Sm, Propositions 1.4 and 1.4′] for the case that R is a field; the same arguments apply in our situation.)

(2.5) *Remark.* If 2 is invertible in the ring R, then (1) is an isomorphism of R-algebras [Hu].

3 The tangent bundle of double coset manifolds

(3.1) As observed in (1.1), the manifold $K\backslash G/H$ is the basis of the three principal bundles G/H, $K\backslash G$ and $K\backslash G \times G/H$.

If V, W, and U are finite-dimensional representations of H, K, and G, respectively, we obtain vector bundles

$$\begin{array}{lccc} \alpha_H(V): & (K\backslash G)\times_H V & \to & K\backslash G/H\ , \\ \alpha_K(W): & (G/H)\times_K W & \to & K\backslash G/H\ , \\ \alpha_G(U): & (K\backslash G\times G/H)\times_G U & \to & K\backslash G/H \end{array}$$

(with operations of H, K, and G on the first factors as in (1.1)).

(3.2) Proposition. *The tangent bundle of $K\backslash G/H$ fits into the following canonical short exact sequence of vector bundles:*

$$0\to\alpha_H(\mathrm{Ad}_H)\oplus\alpha_K(\mathrm{Ad}_K)\to\alpha_G(\mathrm{Ad}_G)\to\tau(K\backslash G/H)\to 0\ .$$

Proof. Let $X=K\backslash G/H$ and denote the total space of $\alpha_G(\mathrm{Ad}_G)$ by E. For $x=KgH\in X$, let E_x be the fibre of E over x. We obtain isomorphisms

$$\psi_g,\tilde{\psi}_g:\mathfrak{g}\to E_x$$

by $\psi_g(u)=[Kg,H,u]$ and $\tilde{\psi}_g(u)=[K,gH,u]$. If we denote the total space of $\alpha_H(\mathrm{Ad}_H)$ by E' and that of $\alpha_K(\mathrm{Ad}_K)$ by E'', we get morphisms of vector bundles $f'\colon E'\to E$ and $f''\colon E''\to E$ by

$$f'[Kg,v]=[Kg,H,v]\ ,\quad f''[gH,w]=[K,gH,w]\ .$$

Then, for $x=KgH$, we have

$$f'_x(E'_x)=\psi_g(\mathfrak{h})\ ,\quad f''_x(E''_x)=\tilde{\psi}_g(\mathfrak{k})\ .$$

Therefore, f' and f'' are injective, and since

$$\psi_g(\mathfrak{h})\cap\tilde{\psi}_g(\mathfrak{k})=0$$

by (1.2), we conclude that $\alpha_H(\mathrm{Ad}_H)\oplus\alpha_K(\mathrm{Ad}_K)$ may be identified with the subbundle F of E with fibre

$$F_x=\psi_g(\mathfrak{h})\oplus\tilde{\psi}_g(\mathfrak{k})\quad\text{for } x=KgH\ .$$

It remains to identify $\tau(X)$ with E/F. We consider the submersion $p\colon G\times G\to X$ with $p(g,g')=Kgg'H$ and its derivative $p_*\colon TG\times TG\to TX$. To describe the kernel of p_*, we introduce the isomorphism of vector bundles

$$\begin{array}{ccc} G\times G\times\mathfrak{g}\times\mathfrak{g} & \xrightarrow{\lambda} & TG\times TG \\ (g,g',u,u') & \mapsto & (u\cdot g,g'\cdot u')\in T_gG\times T_{g'}G\ . \end{array}$$

Obviously, $\ker p_*\subseteq\ker m_*$ where $m\colon G\times G\to G$ is the multiplication, and

$$\ker(m_*\circ\lambda)=\{(g,g',u,-\mathrm{Ad}(gg')^{-1}u\,|\,g,g'\in G,u\in\mathfrak{g}\}\ .$$

Moreover, $\ker p_*\supseteq\ker\tilde{p}_*$ where $\tilde{p}\colon G\times G\to K\backslash G\times G/H$ is the natural projection, and

$$\ker(\tilde{p}_*\circ\lambda)=\{(g,g',w,v)|g,g'\in G,v\in\mathfrak{h},w\in\mathfrak{k}\}\ .$$

By (1.2), we conclude that $\ker(m_*\circ\lambda)\cap\ker(\tilde{p}_*\circ\lambda)=0$. Hence, for dimensional reasons, $\ker(p_*\circ\lambda)=N$ where N is the subset of $G\times G\times\mathfrak{g}\times\mathfrak{g}$ consisting of the elements

$$(g,g',w+u,v-\mathrm{Ad}(gg')^{-1}u)$$

with $u\in\mathfrak{g}$, $v\in\mathfrak{h}$, $w\in\mathfrak{k}$.

Therefore we may identify TX with $\tilde{E} = (G \times G \times \mathfrak{g} \times \mathfrak{g})/\sim$ where $\sim$ is the equivalence relation generated by

$$(g, g', u, u') \sim (kg\tilde{g}, \tilde{g}^{-1}g'h, \mathrm{Ad}(k)u, \mathrm{Ad}(h^{-1})u')$$

for $\tilde{g} \in G$, $k \in K$, $h \in H$ and

$$(g, g', u, u') \sim (g, g', u + w + \tilde{u}, u' + v - \mathrm{Ad}(gg')^{-1}\tilde{u})$$

for $\tilde{u} \in \mathfrak{g}$, $v \in \mathfrak{h}$, $w \in \mathfrak{k}$.

Define $\sigma\colon E = (K\backslash G \times G/H) \times_G \mathfrak{g} \to \tilde{E}$ by

$$\sigma[Kg, g'H, u] = [g, g', \mathrm{Ad}(g)u, \mathrm{Ad}(g')^{-1}u] .$$

Then σ is an epimorphism of vector bundles with kernel F.

(3.3) *Remark.* If V is a representation of H, we have also the vector bundle $\alpha'_H(V)$ over BH with total space $EG \times_H V$. If $\varphi_H\colon K\backslash G/H \to BH$ is the natural map, we have

$$\alpha_H(V) = \varphi_H^*(\alpha'_H(V)) ;$$

similarly for K and G. Hence (3.2) may be rephrased as follows:

$$\tau(K\backslash G/H) \oplus \varphi_K^*\alpha'_K(\mathrm{Ad}_K) \oplus \varphi_H^*\alpha'_H(\mathrm{Ad}_H) \cong \varphi_G^*\alpha'_G(\mathrm{Ad}_G) .$$

Therefore, if G, H, and K are connected, we have $w_1(K\backslash G/H) = 0$.

(3.4) Corollary. *If H and K are connected then $K\backslash G/H$ is orientable.*

(3.5) Corollary. *If H and K are tori then $K\backslash G/H$ is stably almost complex.*

Proof. By (3.3) we know that $\tau(K\backslash G/H)$ is stably equivalent to $\varphi_G^*\alpha'_G(\mathrm{Ad}_G)$, and by commutativity of the diagram $(*)$ in (2.2) this bundle is isomorphic to $\varphi_H^*(\beta)$ with $\beta = \alpha'_H(\mathrm{Ad}_G|H)$. Obviously β is stably equivalent to the bundle underlying a complex vector bundle.

4 Characteristic classes of double coset manifolds

The computation of the Pontrjagin classes of homogeneous spaces in [BH] is based on the following easy lemma.

(4.1) Let T be a maximal torus of G and $\psi\colon BT \to BG$ the natural map. Let ρ be a real representation of G. The weights ($\neq 0$) of the complexification $\rho_{\mathbb{C}}$ of ρ are of the form $\pm b_i$ $(1 \leqq i \leqq k)$ where the b_i are considered as elements of $H^2(BT)$. As in (3.3), we have the vector bundle $\alpha'_G(\rho)$ over BH. Its total Pontrjagin class satisfies

$$\psi^* p(\alpha'_G(\rho)) = \prod_i (1 + b_i^2) .$$

In particular, if $\pm\beta_i$ are the roots of G, we have

$$\psi^* p(\alpha'_G(\mathrm{Ad}_G)) = \prod_i (1 + \beta_i^2) .$$

Together with (3.3), this leads immediately to the following description of the total Pontrjagin class of $K\backslash G/H$:

(4.2) Theorem. *Let H, K be closed connected subgroups of G satisfying the double coset condition. Choose maximal tori T_G, T_H, T_K of G, H, K and let $\psi_G\colon BT_G \to BG$, ψ_H, ψ_K be the natural maps. Moreover, we have the natural maps $\varphi_G, \varphi_H, \varphi_K$ from $K\backslash G/H$ to BG, BH, BK. Let $\pm\beta_i \in H^2(BT_G)$, $\pm\gamma_j \in H^2(BT_H)$, and $\pm\delta_k \in H^2(BT_K)$ be the roots of G, H, K. Then there exist (invertible) elements $b \in H^*(BG)$, $c \in H^*(BH)$, and $d \in H^*(BK)$ such that*

$$(1)\qquad p(K\backslash G/H) = \varphi_K^*(d^{-1})\varphi_G^*(b)\varphi_H^*(c^{-1}),$$

$$(2)\qquad \psi_G^*(b) = \prod_i (1+\beta_i^2),\quad \psi_H^*(c) = \prod_j (1+\gamma_j^2),\quad \psi_K^*(d) = \prod_k (1+\delta_k^2).$$

(4.3) *Example.* Let $G = \mathrm{SU}(4)$, let H be the 2-dimensional torus $\{\mathrm{diag}(s, t, s^{-1}t^{-1}, 1) \mid s, t \in S^1\}$ and let K be the 1-dimensional torus $\{\mathrm{diag}(t, t^{-1}, t, t^{-1}) \mid t \in S^1\}$. Then H, K satisfy the double coset condition. The cohomology of the 12-dimensional manifold $K\backslash G/H$ and its Pontrjagin classes can be easily calculated by (2.4) and (4.2); we find that

$$p_1(K\backslash G/H) \neq 0.$$

Recall that, on the contrary, if T is a torus in any Lie group G, the homogeneous space G/T is stably parallelizable.

(4.4) *Example.* Let $G = \mathrm{Sp}(3)$, $H = \mathrm{Sp}(1)\times\mathrm{Sp}(1) \subset \mathrm{Sp}(2) \subset \mathrm{Sp}(3)$ and $K = \Delta\mathrm{Sp}(1)$, where $\Delta\colon \mathrm{Sp}(1) \to \mathrm{Sp}(3)$ is the diagonal embedding. Then H, K satisfy the double coset condition. The manifold $K\backslash G/H$ was called M_{12} by Eschenburg [E2]; he showed that M_{12} has the same cohomology as $Y := \mathrm{Sp}(3)/(\mathrm{Sp}(1))^3$ and conjectured that M_{12} and Y are not diffeomorphic. We find that $H^4(M_{12})$ is a free abelian group with 2 generators y and z and $p_1(M_{12}) = 8y$. Since Y is stably parallelizable [SW], M_{12} is not diffeomorphic to Y. (This was also shown by Stolz.) Moreover, $p_1(M_{12}) \bmod 24$ is a homotopy invariant [Hi]. Therefore M_{12} is not even homotopy equivalent to Y.

(4.5) *Remark.* As in [BH], there is a result on Stiefel–Whitney classes which is formally very similar to (4.2): One has to replace maximal tori by maximal 2-tori and the squares of the positive roots by the 2-roots.

5 The Euler characteristics of double coset manifolds

(5.1) Theorem. *Let G be a compact connected Lie group, and let H, K be closed connected subgroups of G satisfying the double coset condition. Then*

$$\chi(K\backslash G/H) = \begin{cases} \dfrac{|W(G)|}{|W(K)|\cdot|W(H)|} & \text{if } \mathrm{rk}\,K + \mathrm{rk}\,K = \mathrm{rk}\,G, \\ 0 & \text{if } \mathrm{rk}\,K + \mathrm{rk}\,H < \mathrm{rk}\,G. \end{cases}$$

Proof. By the multiplicativity of χ in fibrations, it suffices to assume that K, H are tori. Furthermore, we may assume that there is a maximal torus T of G with $K, H \subseteq T$. Then there is a further torus $S \subseteq T$ such that the map

$$K \times H \times S \to T$$

$$(k, h, s) \mapsto khs$$

is an isomorphism. There are then natural numbers n, r, s with

$$1 \leqq r < s \leqq n$$

such that

$$H^*(BT) = \mathbb{Q}[x_1, \ldots, x_n] =: \Sigma ,$$

$$H^*(BK) = \mathbb{Q}[x_1, \ldots, x_r] =: \Lambda ,$$

$$H^*(BH) = \mathbb{Q}[x_s, \ldots, x_n] =: P$$

and that the homomorphisms which are induced by the inclusions are the natural projections $\Sigma \to \Lambda$ and $\Sigma \to P$. (Of course, cohomology is always with rational coefficients in this proof.)

It is well known that $\tilde{\Sigma} := H^*(BG)$ is a subring of Σ such that Σ is a free $\tilde{\Sigma}$-module with $w := |W(G)|$ generators $b_1, \ldots, b_w$, where b_j is in dimension $2m_j$, say. By (2.4),

$$H^*(K\backslash G/H) \cong \mathrm{Tor}_{\tilde{\Sigma}}(\Lambda, P) .$$

Our conventions concerning Tor will be those of [Ba]. To compute $\mathrm{Tor}_{\tilde{\Sigma}}(\Lambda, P)$, we use a free $\tilde{\Sigma}$-resolution of Λ obtained as follows: As a free Σ-resolution of Λ we take the Koszul complex

$$(*) \qquad 0 \to R^{(-(n-r))} \to \ldots \to R^{(-1)} \to R^{(0)} \to \Lambda \to 0$$

where $R^{(-p)}$ is the free Σ-module with basis $\{a_I\}$ with

$$I = (i_1, \ldots, i_p) , \quad 1 \leqq i_1 < \ldots < i_p \leqq n - r ;$$

the element a_I has degree $2p$. Now $(*)$ is also a free $\tilde{\Sigma}$-resolution of Λ, and $\mathrm{Tor}_{\tilde{\Sigma}}(\Lambda, P)$ is the homology of the complex

$$(**) \qquad 0 \to R^{(r-n)} \otimes_{\tilde{\Sigma}} P \to \ldots \to R^{(-1)} \otimes_{\tilde{\Sigma}} P \to R^{(0)} \otimes_{\tilde{\Sigma}} P \to 0 .$$

The term $R^{(-p)} \otimes_{\tilde{\Sigma}} P$ is the graded $\mathbb{Q}$-vector space $\bigoplus_{|I|=p} \bigoplus_{j=1}^{w} a_I b_j P$ which is concentrated in even dimensions.

Since $H^i(K\backslash G/H) = 0$ for large values of i, the complex $(**)$ is exact in high dimensions. For any graded $\mathbb{Q}$-vector space $A = (A_{2\nu})_{\nu \geqq 0}$, we introduce the notation

$$A_{[N]} := \bigoplus_{\nu=0}^{N} A_{2\nu} .$$

Then $\mathrm{Tor}_{\tilde{\Sigma}}(\Lambda, P)$ is, for $N \gg 0$, the homology of

$$0 \to (R^{(r-n)} \otimes_{\tilde{\Sigma}} P)_{[N]} \to \ldots \to (R^{(-1)} \otimes_{\tilde{\Sigma}} P)_{[N]} \to (R^{(0)} \otimes_{\tilde{\Sigma}} P)_{[N]} \to 0 .$$

Since every vector space $(R^{(-p)} \otimes_{\tilde{\Sigma}} P)_{[N]}$ has finite dimension, we conclude:

$$\chi(K\backslash G/H) = \sum_{p=0}^{n-r} (-1)^p \dim_{\mathbb{Q}} (R^{(-p)} \otimes_{\tilde{\Sigma}} P)_{[N]} .$$

We put

$$\tilde{P} := \bigoplus_{j=1}^{w} b_j P , \quad \alpha := \dim \tilde{P}_{[N]} .$$

Then we have

$$\dim (R^{(-p)} \otimes_{\tilde{g}} P)_{[N]} = \binom{n-r}{p} \cdot \left(\alpha - \sum_{v=0}^{p-1} \dim \tilde{P}_{2N-2v}\right)$$

and therefore

$$\begin{aligned}
\chi(K\backslash G/H) &= \sum_{p=0}^{n-r} (-1)^p \binom{n-r}{p} \left(\alpha - \sum_{v=0}^{p-1} \dim \tilde{P}_{2N-2v}\right) \\
&= \sum_{p=0}^{n-r} (-1)^{p+1} \binom{n-r}{p} \left(\sum_{v=0}^{p-1} \dim \tilde{P}_{2N-2v}\right) \\
&= \sum_{v=0}^{n-r-1} \left[\sum_{p=v+1}^{n-r} (-1)^{p+1} \binom{n-r}{p}\right] \dim \tilde{P}_{2N-2v} \\
&= \sum_{v=0}^{n-r-1} (-1)^v \binom{n-r-1}{v} \dim \tilde{P}_{2N-2v} .
\end{aligned}$$

Now we have

$$\dim \tilde{P}_{2v} = \sum_{j=1}^{w} \dim (b_j P)_{2v} = \sum_{j=1}^{w} \dim P_{2v-2m_j} = \sum_{j=1}^{w} \binom{n-2+v-m_j}{n-s}$$

and hence

$$\chi(K\backslash G/H) = \sum_{j=1}^{w} \sum_{v=0}^{n-r-1} (-1)^v \binom{n-r-1}{v} \binom{n-s-m_j+N-v}{n-s} .$$

Finally, there is the following identity for binomial coefficients [Ri, p. 16]:

$$\sum_v (-1)^v \binom{p}{v} \binom{t+p-v}{m+p} = \binom{t}{m} .$$

This yields

$$\chi(K\backslash G/H) = \sum_{j=1}^{w} \binom{r+1-s+N-m_j}{r+1-s} = \begin{cases} 0 & \text{if } s > r+1 , \\ w & \text{if } s = r+1 . \end{cases}$$

(5.2) Corollary. *Let U be a closed connected subgroup of $G \times G$ operating freely on G via*

$$g \cdot (x, y) = x^{-1} g y \quad \textit{for } g \in G \quad \textit{and} \quad (x, y) \in U .$$

Then

$$\chi(G/U) = \begin{cases} |W(G)|/|W(U)| & \textit{if } \operatorname{rk} U = \operatorname{rk} G , \\ 0 & \textit{if } \operatorname{rk} U < \operatorname{rk} G . \end{cases}$$

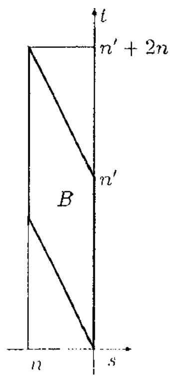
This follows immediately from (5.1) and (1.7).

6 More about the cohomology of double coset manifolds

(6.1) In this section, G is a compact connected Lie group, and H, K are closed connected subgroups of G satisfying the double coset condition. We denote by T_H and T_K maximal tori of H and K, respectively. All cohomology groups are with coefficients in a principal ideal domain R, and we assume throughout the section that $H^*(BG)$, $H^*(BK)$, and $H^*(BH)$ are polynomial algebras over R with generators in even dimensions.

(6.2) Lemma. *We consider the graded R-algebras $\Lambda := R[x_1, \ldots, x_k]$ and $\Lambda' := R[x_1, \ldots, x_{k-1}]$ with $\deg x_i = 2$ and the natural projection $p: \Lambda \to \Lambda'$. Let Σ be a graded commutative R-algebra and $\varphi: \Sigma \to \Lambda$ a homomorphism of graded R-algebras. Via φ (resp. $p \circ \varphi$), we make Λ (resp. Λ') into a Σ-module. Let M be a further graded Σ-module, and assume that the following two conditions hold:*

a) $\mathrm{Tor}_\Sigma^{s,t}(\Lambda, M) = 0$ *for* $t \gg 0$.
b) *There exist integers $n, n' \geqq 0$ such that the set of pairs (s, t) with $\mathrm{Tor}_\Sigma^{s,t}(\Lambda', M) \neq 0$ is contained in the following closed parallelogram B:*



Then, for every pair (s, t) with $\mathrm{Tor}_\Sigma^{s,t}(\Lambda, M) \neq 0$, we have $(s, t) \in B$ and $s > -n$.

Proof. We abbreviate $\mathrm{Tor}_\Sigma^{s,t}(\Lambda, M)$ by $\mathrm{Tor}^{s,t}$ and $\mathrm{Tor}_\Sigma^{s,t}(\Lambda', M)$ by $'\mathrm{Tor}^{s,t}$. Denoting the kernel of p by I, we get a short exact sequence of Σ-modules

$$0 \to I \to \Lambda \xrightarrow{p} \Lambda' \to 0 . \tag{1}$$

Since $I = x_k \Lambda$, we have

$$\mathrm{Tor}_\Sigma^{s,t}(I, M) \cong \mathrm{Tor}^{s,t-2} . \tag{2}$$

Hence (cf. [Ma, p. 299]) the Tor-sequence of (1) reads

$$\ldots \to {'\mathrm{Tor}}^{s-1,t} \to \mathrm{Tor}^{s,t-2} \xrightarrow{\omega} \mathrm{Tor}^{s,t} \xrightarrow{p_*} {'\mathrm{Tor}}^{s,t} \to \ldots \tag{3}$$

Now assume that $s \leqq -n$. By hypothesis b), we conclude from (3):

$$\omega: \mathrm{Tor}^{s,t} \to \mathrm{Tor}^{s,t+2} \quad \text{is injective for all } t \in \mathbb{Z} .$$

This, together with hypothesis a), implies that $\mathrm{Tor}^{s,t} = 0$. Similar arguments yield that $\mathrm{Tor}^{s,t} = 0$ if $t < -2s$ or $t > -2s + n'$.

(6.3) Lemma. $\mathrm{Tor}_{H^*(BG)}^{i,*}(H^*(BT_K), H^*(BT_H))$

$$\cong H^*(T_K \backslash K) \otimes_R \mathrm{Tor}_{H^*(BG)}^{i,*}(H^*(BK), H^*(BH)) \otimes_R H^*(H/T_H) .$$

Proof. Immediate generalisation of the proof of [Ba, Lemma 7.2].

(6.4) Proposition. *Let us write*

$$n := \operatorname{rk} G - \operatorname{rk} K - \operatorname{rk} H ,$$
$$m := \dim K\backslash G/H ,$$
$$n' := m - n .$$

Then we have $n' \geqq 0$, *and the pairs* (s, t) *with*

$$\operatorname{Tor}^{s,t}_{H^*(BG)}(H^*(BK), H^*(BH)) \neq 0$$

are contained in the parallelogram B *of* (6.2). *In particular,*

$$\operatorname{Tor}^{s,*}_{H^*(BG)}(H^*(BK), H^*(BH)) = 0 \quad \textit{for } s < \operatorname{rk} K + \operatorname{rk} H - \operatorname{rk} G .$$

Proof. By (6.3) we may assume that K and H are themselves tori. We proceed by induction on $\operatorname{rk} K + \operatorname{rk} H$: For $\operatorname{rk} K + \operatorname{rk} H = 0$, the assertion is true by our assumption on the structure of $H^*(BG)$. For the inductive step, we may assume that $\dim K > 0$. Choose a torus K' in K with $\dim K' + 1 = \dim K$. Then apply (6.2): to check hypothesis a) of (6.2), we use of course (2.4).

For a pair of subgroups of maximal rank, (6.4) means the following:

(6.5) Theorem. *Assume that* $\operatorname{rk} K + \operatorname{rk} H = \operatorname{rk} G$. *Then*

$$H^*(K\backslash G/H) = H^*(BK) \otimes_{H^*(BG)} H^*(BH)$$

as an algebra. In particular:

1. *The odd Betti numbers of* $K\backslash G/H$ *are zero.*
2. *If* H *and* K *are tori, the algebra* $H^*(K\backslash G/H)$ *is generated by* $H^2(K\backslash G/H)$.

Finally, combining our computation of Pontrjagin classes with (6.4), we obtain a vanishing result for characteristic classes and a result on Pontrjagin numbers; in the special case of homogeneous spaces, the latter was obtained in [HiS] using the Atiyah–Bott–Singer index formula.

(6.6) Proposition. *Characteristic classes of* $K\backslash G/H$ *in* $H^i(K\backslash G/H; R)$ *vanish for*

$$i > \dim(K\backslash G/H) - (\operatorname{rk} G - \operatorname{rk} K - \operatorname{rk} H) .$$

(6.7) Proposition. *If* $\operatorname{rk} K + \operatorname{rk} H < \operatorname{rk} G$, *all Pontrjagin numbers of* $K\backslash G/H$ *vanish, and the oriented manifold* $K\backslash G/H$ *is rationally null-cobordant.*

7 An example

(7.1) To illustrate the results of the previous sections, we consider the special situation $G = U(n)$ and $K = U(m)$, $m \leqq n$, with the standard inclusion $U(m) \subseteq U(n)$. For the moment, H is still allowed to be an arbitrary closed connected subgroup of $U(n)$ such that H and $U(m)$ satisfy the double coset condition and such that $H^*(BH; R)$ is a polynomial algebra with generators of even degrees. To compute $\operatorname{Tor}_{H^*(BU(n))}(H^*(BU(m)), H^*(BH))$, we use of course the Koszul resolution of $H^*(BU(m)) = R[c_1, \ldots, c_m]$ over $H^*(BU(n)) = R[c_1, \ldots, c_n]$, with $\deg(c_i) = 2i$.

We consider the R-algebra

$$A := \Lambda_R(e_{m+1}, \ldots, e_n) \otimes_R H^*(BH),$$

where $\Lambda_R(e_{m+1}, \ldots, e_n)$ is the exterior algebra generated by indeterminates $e_{m+1}, \ldots, e_n$ and define a bigrading on A by

$$\deg(e_j \otimes 1) = (-1, 2j) \quad \text{for } j = m+1, \ldots, n,$$

$$\deg(1 \otimes x) = (0, \deg(x)) \quad \text{for } x \in H^*(BH).$$

A differential d of degree $(1, 0)$ on A is defined by

$$d(1 \otimes x) = 0 \quad \text{for } x \in H^*(BH),$$

$$d(e_j \otimes 1) = 1 \otimes \varphi^*(c_j),$$

where $\varphi^*: H^*(BU(n)) \to H^*(BH)$ is the homomorphism induced by the inclusion $H \to U(n)$.

The homology $H(A, d)$ is again a bigraded R-algebra; we consider it as simply graded by the total degree, i.e. by adding the two components of the bidegree. Then

$$H^*(U(m)\backslash U(n)/H; R) \cong H(A, d). \tag{1}$$

In general, (1) is an isomorphism of graded R-modules; if R is however a field with $\operatorname{char} R \neq 2$, it is an isomorphism of algebras.

(7.2) Two subgroups H, K of $U(n)$ satisfy the double coset condition if and only if for all $x \in K\backslash\{1\}$ and all $y \in H\backslash\{1\}$ the eigenvalues of x are different from those of y. Given two natural numbers n and k, the subgroups $K = U(n-1)$ and $H = \Delta_n U(k)$ of $U(kn)$ therefore satisfy the double coset condition; $\Delta_n: U(k) \to U(kn)$ is the n-fold diagonal mapping.

(7.3) Let us now further specialize the situation of (7.1) by taking $G = U(2n)$, $K = U(n-1)$ and $H = \Delta_n(U(2))$. We write $H^*(BH) = R[x_1, x_2]$ with $\deg(x_i) = 2i$. An easy calculation shows that

$$\varphi^*(c_k) = \sum_{\nu + 2\mu = k} \binom{n-\mu}{\nu} \binom{n}{\mu} x_1^\nu x_2^\mu. \tag{2}$$

A completely explicit computation of the cohomology of

$$X_n := U(n-1)\backslash U(2n)/\Delta_n U(2)$$

using (1) and (2) is of course rather cumbersome and unilluminating. However, for $R = \mathbb{Q}$, it is easy to see that the direct summand

$$B^* := H^*(BU(n-1)) \otimes_{H^*(BU(2n))} H^*(B\Delta_n U(2))$$

of $H^*(X_n)$, which is important because it contains the characteristic classes of X_n, is generated by the elements

$$x_1^\nu x_2^\mu, \quad \nu + \mu < n,$$

as a $\mathbb{Q}$-vector space.

In particular, $B^i = 0$ for $i > 4n - 4$. For the rational Pontrjagin classes, this implies

$$p_i(X_n) = 0 \quad \text{for } i \geqq n.$$

(7.4) Let us finally consider the very special case $n = 2$. The manifold X_2 has dimension 11. Because of (6.4), we have

$$H^*(X_2) = \mathrm{Tor}^0 \oplus \mathrm{Tor}^{-1} .$$

According to (7.3), $\mathrm{Tor}^0 = B^*$ has the basis $1, x_1, x_2$; and $x_1^2 = -2x_2$. Furthermore, we have

$$\mathrm{Tor}^0 = H^{\mathrm{even}}(X_2), \quad \mathrm{Tor}^{-1} = H^{\mathrm{odd}}(X_2) .$$

Hence, by Poincaré duality,

$$H^*(X_2; \mathbb{Q}) = \mathbb{Q}[a_2, a_7]/(a_2^3, a_7^2)$$

with $\deg a_i = i$. Finally, we see from (4.2) that

$$p_1(X_2) \neq 0 .$$

References

[Ba] Baum, P.F.: On the cohomology of homogeneous spaces. Topology **7**, 15–38 (1968)

[BH] Borel, A., Hirzebruch, F.: Characteristic classes and homogeneous spaces, I, II, III. Am. J. Math. **80**, 458–538 (1958); **81**, 315–382 (1959); **82**, 491–504 (1960)

[E1] Eschenburg, J.-H.: New examples of manifolds with strictly positive curvature. Invent. Math. **66**, 469–480 (1982)

[E2] Eschenburg, J.-H.: Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen. Schriftenr. Math. Inst. Univ. Münster, 2. Ser. **32** (1984)

[E3] Eschenburg, J.-H.: Cohomology of biquotients. Manuscr. Math. **75**, 151–166 (1992)

[E4] Eschenburg, J.-H.: Inhomogeneous spaces of positive curvature. Differ. Geom. Appl. **2**, 123–132 (1992)

[GrM] Gromoll, D., Meyer, W.T.: An exotic sphere with nonnegative sectional curvature. Ann. Math. **100**, 401–406 (1974)

[GuM] Gugenheim, V.K.A.M., May, J.P.: On the theory and applications of differential torsion products. Mem. Am. Math. Soc. **142** (1974)

[Hi] Hirzebruch, F.: A Riemann-Roch theorem for differentiable manifolds. Séminaire Bourbaki, exposé 177 (Février 1959)

[HiS] Hirzebruch, F., Slodowy, P.: Elliptic genera, involutions, and homogeneous spin manifolds. Geom. Dedicata **35**, 309–343 (1990)

[Hu] Huebschmann, J.: Perturbation theory and small models for the chains of certain induced fibre spaces. Habilitationsschrift, Heidelberg, 1984

[HMS] Husemoller, D., Moore, J.C., Stasheff, J.: Differential homological algebra and homogeneous spaces. J. Pure Appl. Algebra **5**, 113–185 (1974)

[Ma] MacLane, S.: Homology. Berlin Heidelberg New York: Springer 1963

[Mu] Munkholm, H.J.: The Eilenberg-Moore spectral sequence and strongly homotopy multiplicative maps. J. Pure Appl. Algebra **5**, 1–50 (1974)

[Ri] Riordan, J.: Combinatorial identities. New York: Wiley 1968

[SW] Singhof, W., Wemmer, D.: Parallelizability of homogeneous spaces, II. Math. Ann. **274**, 157–176 (1986); **276**, 699–700 (1987)

[Sm] Smith, L.: Homological algebra and the Eilenberg-Moore spectral sequence. Trans. Am. Math. Soc. **129**, 58–93 (1967)

[Wo] Wolf, J.: The cohomology of homogeneous spaces. Am. J. Math. **99**, 312–340 (1977)