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Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of Riemannian manifolds

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0 Introduction

Let X be a complex manifold of dimension n and $u: X \to \mathbb{R}$ a twice differentiable plurisubharmonic function. The (homogeneous, complex) Monge-Ampère equation for u is

$$(\partial\bar{\partial}u)^n = 0, \tag{0.1}$$

or, in local coordinates $z_1, \ldots, z_n$ on X,

$$\det \partial^2 u / \partial z_j \partial \bar{z}_k = 0.$$

When $n=1$, the above equations reduce to Laplace's equation $\Delta u = 0$, and, indeed, the Monge-Ampère equation is the most natural extension of the Laplace equation to higher dimensional complex manifolds. It first appeared in a paper by Bremermann [4]. For an extensive reference about work on the Monge-Ampère equation see a survey paper by Bedford [1].

Here we shall address the following question. To what extent is a solution u of (0.1), or even X, determined if certain global conditions are imposed on u? We shall consider plurisubharmonic solutions $u(z)$ of (0.1) that go to infinity as $z \in X$ diverges in X, or, more precisely, such that for any $c \in \mathbb{R}$

$$\{z \in X : u(z) \leqq c\} \quad \text{is compact}. \tag{0.2}$$

In this case u is an exhaustion function of X. A little more generally we shall also consider bounded exhaustion functions u, i.e. when (0.2) is required to hold only for $c < \sup u < \infty$. In this generality there are too many solutions u. For example if $X = Y \times Z$ with Y compact and Z Stein, any smooth plurisubharmonic exhaustion function v on Z defines a solution $u(y,z) = v(z)$. To eliminate such examples we shall assume X itself is Stein. Then, however, it turns out that there are no everywhere smooth exhaustion functions u that would solve (0.1) (see Theorem 1.1). This

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naturally leads us to admit u that have some type of singularities on a set $M \subset X$. In fact, as Theorem 1.1 shows, this set M must be related to the minimum set of u.

That a global condition, type of singularity, and the Monge-Ampère equation may uniquely characterize complex manifolds X and functions u on them was first observed by Stoll. In [12] he considered the situation when M reduces to a point, in which case the natural (in some sense, minimal) singularity to prescribe is a logarithmic pole. He proved that in this case X is biholomorphic to $\mathbb{C}^n$ and u is equivalent to $\log|z|$. See also [5, 15]. (Actually, Stoll did not a priori assume that M was a single point, but showed that logarithmic singularity already implies this.)

Later, Patrizio and Wong considered the other extreme, when the singular set M is an n real dimensional manifold. Here the natural (minimal) singularity to assume is a "square root singularity", in a sense to be made precise in Chap. 2, see [11]. They conjectured that in certain cases the mere knowledge of the differentiable manifold M determines X and u (viz. when M is diffeomorphic to a simply connected compact symmetric space of rank 1). They could settle this conjecture only under the assumption that a more precise information about the singular behavior of u is available. Also, they constructed examples of X and u with M a compact symmetric space of rank 1, or a torus. Apart from that, the main contribution of [11] is the description of the rich geometry that is determined by u. (That there is an interesting geometry associated with a solution of the Monge-Ampère equation was, of course, first discovered by Stoll and Burns.)

Further examples of solutions of the Monge-Ampère equation with square root singularity were found by the first named author, see [10]. In those examples M is a hyperbolic manifold, and the function u is a *bounded* exhaustion function of X.

The primary objects of study in this paper are unbounded exhaustion functions u on Stein manifolds X that satisfy (0.1) and have square root singularity along a smooth manifold M, $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} X$. Following [11], we shall introduce a Kähler metric on X and its restriction, a Riemannian metric on M. By studying these metrics, we shall prove that X and u are determined (up to biholomorphism) by the metric on M (even when u is bounded). This extends the result of [11] that applies when M with the above metric becomes a compact symmetric space of rank 1. This result can also be regarded as defining canonical complexifications of Riemannian manifolds (with real analytic metrics).

We shall also prove that when u is unbounded, the metric on M must be nonnegatively curved. From this we shall conclude that when M is diffeomorphic to a torus, X and u are almost uniquely determined: they must be one of the examples found by Patrizio and Wong. However, the original conjecture mentioned above does not hold: as shown in a subsequent paper of the second named author, there is a two-parameter family of inequivalent examples with singularity set diffeomorphic to the sphere S^2. (See note added in proof.)

1 The singularity

Theorem 1.1. *On a Stein manifold there is no nonconstant, bounded or unbounded smooth plurisubharmonic exhaustion function that satisfies the Monge-Ampère equation* (0.1).

Proof. Suppose u is a nonconstant smooth exhaustion function on a Stein manifold X and (0.1) is satisfied. It can be assumed that $\min u = 0$. Put

$$M = \{z \in X : u(z) = 0\}.$$

Since u is nonconstant, there is a number λ such that $0<\lambda<\sup u$. Put $D=\{z\in X: u(z)<\lambda\}$. Let v be a positive, strictly plurisubharmonic unbounded exhaustion function of X. If $\varepsilon>0$ is sufficiently small, $\varepsilon v<u$ on ∂D. Hence by the minimum principle of Bedford and Taylor, see [3], $\varepsilon v<u$ on D. (In [3] the minimum principle is proved only on domains in $\mathbb{C}^n$, however, the proof, especially in the case of smooth functions, applies with $\mathbb{C}^n$ replaced by any complex manifold.) Thus we came to a contradiction since on M $\varepsilon v>u=0$.

Thus we see that an (exhaustive) solution u of the Monge-Ampère equation (0.1) must have singularities; moreover, the singular set is connected with the minimum set M of u. In fact, it turns out that the (singular) behaviour of u at M is by and large determined by M.

Theorem 1.2. *Let u_1 and u_2 be nonnegative, continuous, plurisubharmonic (bounded or unbounded) exhaustion functions on a Stein manifold X. Assume that their minimum sets agree:*

$$\{u_1=0\}=\{u_2=0\}=M(\neq\emptyset),$$

and both u_1 and u_2 are smooth outside M and satisfy the Monge-Ampère equation (0.1) *there. Then there is a positive constant C such that in a neighbourhood of M*

$$u_1/C\leqq u_2\leqq Cu_1. \tag{1.1}$$

Proof. Let $\lambda>0$ be less than $\sup u_2$ and $D=\{z\in X: u_2(z)<\lambda\}$. There is a $C>0$ such that

$$u_2<Cu_1 \quad \text{on} \quad \partial D.$$

For any $\varepsilon>0$ M has arbitrarily small neighbourhoods U_ε such that on ∂U_ε

$$u_2<Cu_1+\varepsilon. \tag{1.2}$$

The minimum principle now implies that (1.2) holds on $D\backslash U_\varepsilon$, hence also on $D\backslash M$. Letting $\varepsilon\to 0$ the second inequality of (1.1) is obtained. The first is proved by reversing the roles of u_1 and u_2.

In this paper we shall assume that the minimum set M is an n real dimensional smooth submanifold ($n=\dim_{\mathbb{C}} X$). When, in addition, M is real analytic and totally real, one can construct a u_1 as above that is comparable to the distance to M. (For that one may have to shrink X to a small neighbourhood of M.) This follows from Example 2.1 if M is diffeomorphic to a compact quotient of $\mathbb{R}^n$ by a discrete group of isometries; and in general from the construction in [14]. Therefore, by Theorem 1.2, *any* u_1 as above is comparable to the distance to M.

In the sequel we shall assume a slightly stronger property of the singularity at M; namely that u^2 is smooth (even on M). This corresponds with the assumption in Stoll's theorem from [12], where the singularity of u is assumed to be such that e^u is smooth. As we shall see, our smoothness assumption connects equation (0.1) with Riemannian geometry on M. A less restrictive class of singularities for u would lead us to Finsler geometry on M, a direction we shall not pursue here.

2 Formulation of the problem and the main results

Considerations in chapter 1 lead us to the following set up. Let X be a complex manifold, M a smooth submanifold, $\dim_{\mathbb{R}} M=\dim_{\mathbb{C}} X$; u a nonnegative bounded or unbounded exhaustion function of X, $\{u=0\}=M$. We shall assume that u^2 is smooth and strictly plurisubharmonic, and that u satisfies the Monge-Ampère equation

$$(\partial\bar{\partial}u)^n=0 \quad \text{on} \quad X\backslash M. \tag{2.1}$$

In this case we shall say that the triple (X,M,u) is a Monge-Ampère model (of bounded type, if u is bounded; no further specifications if u is unbounded).

Example 2.1. Let Γ be a discrete group of motions of $\mathbb{R}^n$ such that $\mathbb{R}^n/\Gamma$ is a compact manifold. Γ also acts on $\mathbb{C}^n$ and $\mathbb{C}^n/\Gamma$ is a complex manifold. It is easy to check that the function $\left(\sum_{j=1}^{n}(\operatorname{Im} z_j)^2\right)^{1/2}$ on $\mathbb{C}^n$ is Γ invariant so that it descends to a function u_0 on $\mathbb{C}^n/\Gamma$. A straightforward computation shows that $(\mathbb{C}^n/\Gamma, \mathbb{R}^n/\Gamma, u_0)$ is a Monge-Ampère model (see [13]).

We would like to classify all Monge-Ampère models with given "center" M. This could not be carried through completely, but for certain types of centers M we do have definitive results. Here are the main results.

Theorem 2.2. *Suppose that the center M of a Monge-Ampère model (X,M,u) is diffeomorphic to a compact flat manifold. Then X is biholomorphic to a manifold $\mathbb{C}^n/\Gamma$ as in Example 2.1, and, moreover, the biholomorphism can be chosen so that under it u and u_0 of Example 2.1 correspond.*

Theorem 2.3. *If (X,M,u) is a Monge-Ampère model then M admits a Riemannian metric of nonnegative curvature.*

Thus, when $n=2$, only the torus, the Klein bottle, the sphere, and the projective plane can occur as centers of Monge-Ampère models. That they do occur as centers follows from Example 2.1 for the first two surfaces and from [11] for the last two.

Theorem 2.3 can be made more precise if, following [11] we introduce a Kähler metric on X whose Kähler form is $-i\partial\bar{\partial}u^2$. Call this metric h and its restriction to M g.

Theorem 2.4. *Let (X,M,u) be a Monge-Ampère model of bounded type, $R=\sup u$. Then the sectional curvatures of the metric g are $\geqq -\pi^2/(4R^2)$.*

The estimate in the theorem is sharp, although we shall not prove this. When $n=2$, the cases where equality in the above estimate is attained can be described. According to [13], 2 dimensional models for which the curvature of g equals $-\pi^2/(4R^2)$ in one point of M are in one-to-one correspondence with 2 dimensional compact space forms of curvature $-\pi^2/(4R^2)$. When $n>2$, cases of equality are not so easy to characterize but it is still true that compact space forms give rise to models that show the sharpness of the curvature estimate.

In Theorems 2.2 and 2.3 the aim was to find Monge-Ampère models once M and the zero order behaviour of u at M is known (i.e., that $u=0$ on M). If also the first order behaviour of u at M is given, then the model can always be reconstructed. Observe that first order behaviour of u at M translates to second order behaviour of u^2, which contains enough information to determine the metric g on M. We shall prove

Theorem 2.5. *Suppose (X,M,u) and (X',M',u') are two Monge-Ampère models (possibly of bounded type). Let g, g' be the metrics on M respectively M' associated with u, u'. Assume (M,g) and (M',g') are isometric and* $\sup u=\sup u'$. *Then there is a biholomorphic map $F: X\to X'$ such that $u=u'\circ F$.*

This theorem is proved in [11] in the special case when (X, M, u) is a model constructed from a compact symmetric space of rank 1.

3 Reduction to the tangent bundle

The aim of this chapter is to show that any Monge-Ampère model (X, M, u) is equivalent to a model (X_0, M, u_0) where X_0 is an open subset of TM, endowed with a certain complex structure, which complex structure and u_0 are determined by the metric g on M.

Thus, let (X, M, u) be a Monge-Ampère model (eventually of bounded type), $\sup u = R$. Since u^2 is strictly plurisubharmonic and u satisfies (2.1), $\operatorname{rk} \partial\bar{\partial} u = n-1$ on $X \backslash M$. Therefore $\ker \partial\bar{\partial} u$ is a smooth rank 1 subbundle of the complex vector bundle $T(X \backslash M)$, and defines a smooth foliation of $X \backslash M$ by Riemann surfaces (see [2]), called the Monge-Ampère foliation. The characteristic property of this foliation is that ∂u, restricted to any leaf L is a holomorphic section of $(T^{1,0}X)^*|_L$.

On the other hand, if (M, g) is any compact – or even complete – Riemannian manifold, in particular, if it is the center of a Monge-Ampère model, $TM \backslash M$ also carries a natural foliation by Riemann surfaces defined as follows. For $\tau \in \mathbb{R}$ denote by $N_\tau : TM \to TM$ the smooth mapping defined by multiplication by τ in the fibers. If $\gamma : \mathbb{R} \to M$ is a geodesic, define an immersion $\psi_\gamma : \mathbb{C} \to TM$ by

$$\psi_\gamma(\sigma + i\tau) = N_\tau \dot{\gamma}(\sigma). \tag{3.1}$$

If for two geodesics, γ, δ $\psi_\gamma(\mathbb{C} \backslash \mathbb{R})$ and $\psi_\delta(\mathbb{C} \backslash \mathbb{R})$ intersect each other, then γ and δ are the same geodesic traversed with different velocities, hence $\psi_\gamma(\mathbb{C}) = \psi_\delta(\mathbb{C})$. Therefore the images of $\mathbb{C} \backslash \mathbb{R}$ under the mappings ψ_γ define a smooth foliation of $TM \backslash M$ by surfaces. Moreover, each leaf has a complex structure that it inherits from $\mathbb{C}$ via ψ_γ. (One can check that if $\psi_\gamma(\mathbb{C} \backslash \mathbb{R}) = \psi_\delta(\mathbb{C} \backslash \mathbb{R})$ then ψ_γ and ψ_δ define identical complex structures on this leaf.) The leaves, along with their complex structure extend across M but, of course, on M the foliation becomes singular. We shall refer to the above foliation as the Riemann foliation.

Returning to the model (X, M, u) and metric g on M as before, denote by $2E$ the smooth function on TM which is g-length squared (i.e., E is the "energy"), and put

$$T^R M = \{v \in TM : 2E(v) < R^2\}. \tag{3.2}$$

Theorem 3.1. *Let (X, M, u) be a Monge-Ampère model, eventually of bounded type, $R = \sup u \leqq \infty$. With notation as above, there exists a diffeomorphism $\phi : T^R M \to X$ such that*

1° ϕ maps leaves of the Riemann foliation biholomorphically onto leaves of the Monge-Ampère foliation;

2° $2E = (u \circ \phi)^2$.

This theorem is essentially contained in [11]. Although it is never formulated there as a theorem, the steps to prove it, that we shall briefly present here since [11] has not been published, are all contained in [11]. See also [13].

So, let (X, M, u) be as in the theorem; the Kähler metric h is defined on X by

$$h(V, W) = \langle -i\partial\bar{\partial} u^2, JV \wedge \bar{W} \rangle, \qquad V, W \in \mathbb{C} \otimes T_p X$$

(here J is the almost complex tensor of X and $\bar{W}$ denotes the conjugate of W). g is the restriction of h to TM. Let Y denote the gradient field of u^2.

Proposition 3.2. *The length of Y is $2u$.*

Proof. A corresponding statement with u^2 replaced by $\exp u$ (both in the definition of the metric and that of Y) is proved in Stoll [12]. The computations for u^2 are completely analogous and will be omitted. Also, Patrizio and Wong in [11] do the computations for any function $F(u)$ of u.

Proposition 3.3. *Y is tangential to the leaves of the Monge-Ampère foliation, that is, trajectories of Y lie on leaves.*

Proof. See the analogous proposition in [12], or in [11].

Proposition 3.4. *The leaves of the Monge-Ampère foliation are totally geodesic, flat submanifolds of $X\backslash M$.*

Proof. Total geodesy (in the case of $\exp u$) was discovered by Burns, see [5], or [12]. The same proof applies in our case, see [11]. Flatness is also contained in [11] but the computations there can simply be bypassed by the following considerations. The metric on a leaf L is given by the Kähler form $-i\partial\bar{\partial}(u^2|_L)$. Since ∂u is holomorphic along L, $u|_L$ is harmonic. Hence locally a holomorphic coordinate $\sigma+i\tau$ can be introduced on L such that $u(\sigma+i\tau)=\tau$. In this coordinate $-i\partial\bar{\partial}(u^2|_L)=d\sigma\wedge d\tau$, whence the metric is Euclidean.

Proposition 3.5. *If $0<a<b<R$, the distance between the level surfaces $\{u=a\}$ and $\{u=b\}$ is $b-a$. The distance minimizing unit speed geodesics between these two hypersurfaces are the trajectories of the normalized gradient field $Y/\sqrt{h(Y,Y)}=Y/(2u)$.*

Proof. (See also [11].) Let $\gamma:[0,T]\to X$ be a curve, $u(\gamma(0))=a$, $u(\gamma(T))=b$, such that $\dot{\gamma}(t)$ is of unit length for $0\leqq t\leqq T$. Then

$$\frac{d}{dt}u(\gamma(t))=\frac{du^2(\gamma(t))}{dt}\Big/(2u(\gamma(t)))$$
$$=\frac{h(\dot{\gamma}(t),Y(\gamma(t)))}{2u(\gamma(t))}\leqq 1$$

by Proposition 3.2. Hence for the length T of γ we have

$$T\geqq\int_0^T\frac{d}{dt}u(\gamma(t))dt=b-a.$$

Equality occurs if $\dot{\gamma}(t)$ is parallel to $Y(\gamma(t))$ for every t, whence γ is a trajectory of the normalized gradient field.

Letting $a\to 0$ we obtain

Corollary 3.6. *Trajectories of $Y/(2u)$ are unit speed geodesics $\gamma(t)$ issuing from points of M in directions perpendicular to M. For every t, $0<t<R$, they minimize distance between M and $\gamma(t)$, and this distance is $u(\gamma(t))$.*

Let us introduce the normal bundle $\mathcal{N}M\subset TX|_M$ of M in X, and

$$\mathcal{N}^RM=\{v\in\mathcal{N}M:h(v,v)<R^2\}.$$

The exponential map exp is defined on some open subset of $\mathcal{N}M$ containing M and maps it into X.

Proposition 3.7. *The exponential map is defined on $\mathcal{N}^RM$ and maps $\mathcal{N}^RM$ diffeomorphically onto X.*

Proof (cf. [11]). Corollary 3.6 and the exhaustive property of u show that exp maps $\mathcal{N}^R M$ into X. Because the trajectory of the normalized gradient field starting in any point p of $X \setminus M$ eventually arrives at a point $m(p) \in M$, and then perpendicularly to M, the exponential mapping is onto X and one-to-one. Also, exp is smooth (this holds generally). What needs to be proved is that its inverse is also smooth.

Denote by $-v(p)$ the velocity vector in $m(p)$ of the minimizing unit speed geodesic from p to $m(p)$. The inverse e of the exponential map is given by

$$X \ni p \mapsto e(p) = u(p)v(p) \in \mathcal{N} M .$$

It is known that in general the exponential map is a diffeomorphism on a neighbourhood of M in $\mathcal{N} M$; hence e is smooth on a neighborhood of a tube

$$X_a = \{p \in X : u(p) \leqq a\}, \qquad a > 0 .$$

On $X \setminus X_a$ the normalized gradient field is smooth, so if for $p \in X \setminus X_a$ $\alpha(p)$ denotes the point where the trajectory through p hits ∂X_a, then $\alpha : X \setminus X_a \to \partial X_a$ is a smooth mapping. Since

$$e(p) = \frac{u(p)}{u(\alpha(p))} e(\alpha(p)),$$

it follows that e is smooth on $X \setminus X_a$, hence on the whole of X.

Proposition 3.8. *M is a Lagrangean submanifold of X, i.e. $JTM = \mathcal{N} M$.*

Proof (cf. [11]). By a simple result due to Harvey and Wells [7], in a neighbourhood of any point $0 \in M$ one can introduce holomorphic coordinates on X, $z_1, \ldots, z_n$, such that

$$u^2(z) = \sum_{j=1}^{n} (1-\lambda_j)(\operatorname{Re} z_j)^2 + \sum_{j=1}^{n} (1+\lambda_j)(\operatorname{Im} z_j)^2 + O(|z|^3), \tag{3.3}$$

with $\lambda_j \geqq 0$. Simple computation shows that the Monge-Ampère equation (2.1) can be rewritten for u^2 as

$$u^2(\partial\bar\partial u^2)^n = n\partial u^2 \wedge \bar\partial u^2 \wedge (\partial\bar\partial u^2)^{n-1}. \tag{3.4}$$

Plugging (3.3) into this, we conclude $\lambda_j = 1$ $(j = 1, \ldots, n)$, i.e.

$$u^2(z) = 2\sum (\operatorname{Im} z_j)^2 + O(|z|^2).$$

Hence the metric h at 0 agrees with the Euclidean metric and $T_0 M$ agrees with $\mathbb{R}^n \subset \mathbb{C}^n$, which is a Lagrangean subspace: $J\mathbb{R}^n$ is orthogonal to $\mathbb{R}^n$.

Proposition 3.9. *M is totally geodesic.*

Proof (cf. [11]). In coordinates as above, write

$$u^2(z) = 2\sum (\operatorname{Im} z_j)^2 + \operatorname{Re}\left(\sum a_{jkl} z_j z_k z_l + \sum b_{jkl} \bar z_j z_k z_l\right) + O(|z|^4). \tag{3.5}$$

Replacing z_j by z'_j where

$$z_j = z'_j - \sum_{k,l=1}^{n} b_{jkl} z'_k z'_l$$

we can assume that $b_{jkl} = 0$. Substituting (3.5) into (3.4) we find that $a_{jkl} = 0$. Thus

$$u^2(z) = 2\sum (\operatorname{Im} z_j)^2 + O(|z|^4).$$

This implies that M and $\mathbb{R}^n \subset \mathbb{C}^n$ agree at 0 to third order, and not only the metric h but also the connection of h at 0 is the same as the Euclidean connection. Therefore the restriction of the connection of h to M at 0 is the same as the connection of the metric $h|_M$ at 0, i.e. M is totally geodesic.

By virtue of Propositions 3.7 and 3.8, we have a diffeomorphism $\phi: T^R M \to X$ defined by

$$\phi(v) = \exp Jv, \qquad v \in T^R M. \tag{3.6}$$

Corollary 3.6 implies that $u(\phi(v)) = \sqrt{h(v,v)}$, i.e. 2° of Theorem 3.1 holds. To prove 1° one more step is needed:

Proposition 3.10 (cf. [11]). *Let $\gamma: \mathbb{R} \to M$ be a, say, unit speed geodesic. Let*

$$\varphi_\gamma(\sigma + i\tau) = \exp JN_\tau \dot{\gamma}(\sigma) = \phi(N_\tau \dot{\gamma}(\sigma)), \tag{3.7}$$

$\sigma + i\tau \in \mathbb{C}$, $|\tau| < R$. Then φ_γ is a holomorphic mapping into X.

Recall that on the right hand side of (3.7) $N_\tau: TM \to TM$ is multiplication by τ in the fibers of TM.

Proof. For brevity, put $\varphi_\gamma = \varphi$. We need to show that

$$J\frac{\partial\varphi}{\partial\sigma} = \frac{\partial\varphi}{\partial\tau},$$

or, equivalently,

$$J\frac{\partial\varphi}{\partial\tau} = -\frac{\partial\varphi}{\partial\sigma}. \tag{3.8}$$

We shall prove (3.8) when $\sigma = 0$ and $\tau \geqq 0$. Put

$$\partial\varphi(i\tau)/\partial\tau = \vartheta(\tau), \qquad (\partial\varphi(\sigma + i\tau)/\partial\sigma)_{\sigma=0} = V(\tau).$$

By Proposition 3.3 the geodesic $\varphi(i\tau)$ $(0 \leqq \tau < R)$ lies on a leaf L of the Monge-Ampère foliation. Since L is a complex submanifold of X, $J\vartheta$ is tangential to L. Also, since ϑ is parallel along $\varphi(i\tau)$, and the metric h is Kähler, $J\vartheta$ is parallel, too. Recall that L is flat and totally geodesic (Proposition 3.4), which implies that $J\vartheta$ is a Jacobi field along $\varphi(i\tau)$. It satisfies the intial conditions

$$J\vartheta(0) = -\dot{\gamma}(0), \qquad \nabla_{\vartheta(0)} J\vartheta = 0. \tag{3.9}$$

Next consider V. From its definition as a variation of geodesics it follows that it is a Jacobi field along $\varphi(i\tau)$, and

$$V(0) = \dot{\gamma}(0). \tag{3.10}$$

Since the vector fields $\partial\varphi(\sigma + i\tau)/\partial\sigma$ and $\partial\varphi(\sigma + i\tau)/\partial\tau$ commute, we also have

$$\nabla_{\vartheta(0)} V = \nabla_{V(0)} \frac{\partial\varphi(\sigma + i\tau)}{\partial\tau} = \nabla_{V(0)} J\dot{\gamma}(\sigma), \qquad J\nabla_{V(0)}\dot{\gamma}(\sigma) = 0, \tag{3.11}$$

where we used that J is parallel and that γ is a geodesic not only in M, but in X, too, so that $\dot{\gamma}$ is parallel.

Now (3.9), (3.10), and (3.11) show that the Jacobi fields $J\vartheta$ and $-V$ satisfy the same initial conditions, hence they agree for every τ, $0 \leqq \tau < R$; that is, (3.7) holds.

Proof of Theorem 3.1. As observed before, ϕ defined by (3.6) is a diffeomorphism: $T^R M \to X$ that satisfies 2°. Also $\phi \circ \psi_\gamma = \varphi_\gamma$ is holomorphic for unit speed geodesics γ by Proposition 3.10, and then by a simple change of variables for any geodesic γ. Hence ϕ is holomorphic on the leaves of the Riemann foliation.

Choose now a geodesic $\gamma: \mathbb{R} \to M$. The corresponding leaf map ψ_γ is defined on a domain

$$D = \{\sigma + i\tau \in \mathbb{C} : |\tau| < R/\sqrt{h(\dot\gamma, \dot\gamma)}\}.$$

Put $D_+ = \{\sigma + i\tau \in D : \tau > 0\}$ and $\varphi_\gamma = \phi \circ \psi_\gamma$. As in the proof of Proposition 3.10, there is a leaf L of the Monge-Ampère foliation such that $L \cap \varphi_\gamma(D_+)$ contains a trajectory of the field Y. Since both L and $\varphi_\gamma(D_+)$ are holomorphically immersed connected Riemann surfaces, it follows by a simple argument involving analytic continuation that $L = \varphi_\gamma(D_+)$, which then completes the proof.

4 Complex structures on tangent bundles

Theorem 3.1 reduces the study of Monge-Ampère models to the study of certain complex structures on the tangent bundle of the center. Namely, the pull back of the complex structure of X by ϕ will be an "adapted" complex structure on $T^R M$ in the sense of the following definition:

Definition 4.1. Let (M, g) be a compact Riemannian manifold, E the energy function on TM, $0 < R \leqq \infty$, and $T^R M$ defined as in (3.2). A (smooth) complex structure on $T^R M$ will be called adapted if the leaves of the Riemann foliation (with their complex structure defined in Chap. 3) are complex submanifolds in this structure.

Because of Theorem 3.1, Theorem 2.5 follows from

Theorem 4.2. *Given a compact Riemannian manifold (M, g) and R, $0 < R \leqq \infty$, there is at most one adapted complex structure on $T^R M$.*

More precisely, we shall see how to reconstruct the almost complex tensor J on $T^R M$ in terms of Jacobi fields on M. We mention here that when (M, g) is analytic, and R is sufficiently small, then there does exist a (by Theorem 4.2) unique adapted complex structure on $T^R M$ (see [14]). In fact, it seems that analyticity of (M, g) is necessary in order to have an adapted complex structure on $T^R M$; but we could not completely prove that. According to Corollary 5.8 M must have a real analytic structure such that the geodesics of g in this structure are real analytic curves, and more analyticity will follow from results in Chap. 5, but this is still not enough to conclude that g is analytic (equivalently, that in any eventually bounded Monge-Ampère model u has to be real analytic).

Because of Theorem 3.1, Theorem 2.4, and therefore also Theorem 2.3, follow from

Theorem 4.3. *Let (M, g) be a compact Riemannian manifold such that on $T^R M$ there is an adapted complex structure for some R, $0 < R \leqq \infty$. Then the sectional curvatures of g are bounded from below by $-\pi^2/(4R^2)$.*

It is not hard to show that when g has constant negative curvature $-\pi^2/(4R^2)$ then $T^R M$ does admit an adapted complex structure. For this a construction in [10] has to be slightly extended.

Similarly, Theorem 2.2 follows from Theorem 3.1, Theorem 4.2, and

Theorem 4.4. *Suppose that on the entire tangent bundle of a compact Riemann manifold (M, g) there exists an adapted complex structure, and M is diffeomorphic to a flat manifold. Then g is flat.*

Theorem 4.2 is proved in the next chapter, while the proof of Theorems 4.3 and 4.4 is concluded in Chap. 7.

We shall end this chapter by fixing notation concerning the symplectic structure on the tangent bundle $\pi: TM \to M$ of a Riemannian manifold (M, g). The canonical (or tautological) one form Θ on TM is defined by

$$\langle \Theta, v \rangle = g(z, \pi_* v), \qquad v \in T_z(TM). \tag{4.1}$$

$\Omega = d\Theta$ is a symplectic form, and the geodesic flow $\phi_t: TM \to TM$ ($t \in \mathbb{R}$) is the Hamiltonian flow induced by the Hamiltonian E (see [8] as a general source of information about geodesics).

5 Proof of Theorem 4.2

We shall begin with some remarks about tangent bundles of complex manifolds X. The real tangent bundle TX and the complex $(1,0)$ tangent bundle $T^{1,0}X$ are isomorphic as real vector bundles. If J is the almost complex tensor on X, an isomorphism is given by

$$TX \ni \xi \mapsto \xi^{1,0} = \tfrac{1}{2}(\xi - iJ\xi) \in T^{1,0}X. \tag{5.1}$$

For a holomorphic function F on X we have $\xi F = \xi^{1,0} F$.

We shall need the following simple

Proposition 5.1. *Let $U \subset \mathbb{C}$ be an open set and for $-\varepsilon < t < \varepsilon$ $f_t: U \to X$ holomorphic mappings that depend differentiably on t. Let*

$$\xi = df_t/dt|_{t=0};$$

this is a section of $f_0^* TX$. Then $\xi^{1,0}$ is a holomorphic section of $f_0^* T^{1,0}X$.

Proof. Let F be any holomorphic function defined on a neighbourhood (in X) of some point of $f_0(U)$. To show that $\xi^{1,0}$ is holomorphic it suffices to prove that $\xi^{1,0}F$ is holomorphic. Now

$$\xi^{1,0}F = \xi F = \left.\frac{d(F \circ f_t)}{dt}\right|_{t=0},$$

which is clearly holomorphic.

Proof of Theorem 4.2. We have to show that the almost complex tensor J that defines an adapted structure on $T^R M \backslash M$ is determined by the metric g. Let $z \in T^R M \backslash M$, and $\bar{\xi} \in T_z(TM)$ be the tangent vector at z of a smooth curve $z(t) \in TM$, $z(0) = z$. (In $\bar{\xi}$ the bar, of course, does not mean complex conjugation.) Denote by $g_t: \mathbb{R} \to M$ the geodesic such that $\dot{\gamma}_t(0) = z(t)$ (dot meaning differentiation with respect to the parametrization of the geodesic). For every t the mapping ψ_{γ_t} defined by

$$\psi_{\gamma_t}(\sigma + i\tau) = N_\tau \dot{\gamma}_t(\sigma)$$

(cf. (3.1)) are holomorphic from open subsets of $\mathbb{C}$ to TM (e.g., ψ_{γ_0} is defined on $\{\sigma+i\tau:|\tau|<R/\sqrt{2E(z)}\}$), and $\psi_{\gamma_t}(i)=z(t)$. Put

$$\xi=\frac{d}{dt}\psi_{\gamma_t}\bigg|_{t=0}.$$

Then $\xi(i)=\bar{\xi}$ and by Proposition 5.1 $\xi^{1,0}$ is a holomorphic section of $T^{1,0}(TM)$ along ψ_{γ_0}. Vector fields like ξ will be called parallel (along ψ_{γ_0}, or along a leaf of the Riemann foliation). ξ is obviously determined by $\bar{\xi}$, and the correspondence $\bar{\xi}\mapsto\xi$ is linear. We can also identify $\xi|\mathbb{R}$. Indeed,

$$\xi(\sigma)=\frac{d}{dt}N_0\dot{\gamma}_t(\sigma)\bigg|_{t=0}=\frac{d}{dt}\gamma_t(\sigma)\bigg|_{t=0}, \tag{5.2}$$

hence $\xi|_{\mathbb{R}}$ is a Jacobi field along γ_0.

In the special situation when $\bar{\xi}$ is tangent to the Riemann foliation, γ_t can be chosen to be reparametrizations of γ_0; then also ξ will be tangent to the Riemann foliation and $\xi|_{\mathbb{R}}$ will be a Jacobi field parallel to $\dot{\gamma}_0$. Also $J\bar{\xi}$ is tangent to the Riemann foliation, hence the restriction of the corresponding vector field ξ' to $\mathbb{R}$ will be another Jacobi field parallel to $\dot{\gamma}_0$.

Choose now n vectors $\bar{\xi}_1,\ldots,\bar{\xi}_n\in T_z(TM)$ so that the vectors $\bar{\xi}_1^{1,0},\ldots,\bar{\xi}_n^{1,0}$ span $T_z^{1,0}(TM)$. Then $\bar{\xi}_1, J\bar{\xi}_1,\ldots,\bar{\xi}_n, J\bar{\xi}_n$ span $T_z(TM)$ over $\mathbb{R}$. We can assume that $\bar{\xi}_n$ is tangent to the Riemann foliation. To these vectors construct parallel vector fields ξ_j along ψ_{γ_0} as above. By taking linear combinations of the $\bar{\xi}_j$'s ($j=1,\ldots,n-1$) with $\bar{\xi}_n$ and $J\bar{\xi}_n$ we can achieve that the Jacobi fields $\xi_j|\mathbb{R}$ ($j=1,\ldots,n-1$) are all orthogonal to the geodesic γ_0 (such fields are called *normal* Jacobi fields).

Choose next n further vectors $\bar{\eta}_1,\ldots,\bar{\eta}_n\in T_z(TM)$ so that together with $\bar{\xi}_j$ they form a basis, and extend them to parallel vector fields $\eta_1,\ldots,\eta_n$ along ψ_{γ_0}. We can again assume that $\bar{\eta}_n$ is tangential to the Riemann foliation (in fact, since J is known on the tangents of the Riemann foliation, we can put $\bar{\eta}_n=J\bar{\xi}_n$), and that the Jacobi fields $\eta_j|\mathbb{R}$ are normal ($j=1,\ldots,n-1$). We shall now show how to express $J\bar{\xi}_j$ in terms of the basis $(\bar{\xi}_j,\bar{\eta}_k)$. The holomorphic sections $\xi_1^{1,0},\ldots,\xi_n^{1,0}$ are independent in the point $i\in\mathbb{C}$ (which corresponds to $z\in TM$), hence they are pointwise independent except on a discrete subset of $D=\{\sigma+i\tau:|\tau|<R/\sqrt{2E(z)}\}$. Therefore there are meromorphic functions f_{jk} on D such that

$$\eta_k^{1,0}=\sum_{j=1}^{n} f_{jk}\xi_j^{1,0}\qquad(k=1,\ldots,n). \tag{5.3}$$

Because of the independence of the vector fields $\xi_j^{1,0}|\mathbb{R}$, except for a discrete set on $\mathbb{R}$, the Jacobi fields $\xi_1|\mathbb{R},\ldots,\xi_n|\mathbb{R}$ are also independent, so there are smooth real valued functions φ_{jk} defined on $\mathbb{R}\backslash$a discrete set such that

$$\eta_k|_{\mathbb{R}}=\sum_{j=1}^{n}\varphi_{jk}\xi_j. \tag{5.4}$$

Hence

$$\eta_k^{1,0}|_{\mathbb{R}}=\sum_{j=1}^{n}\varphi_{jk}\xi_j^{1,0}|_{\mathbb{R}}.$$

It follows that

$$f_{jk}|_{\mathbb{R}}=\varphi_{jk}. \tag{5.5}$$

Substituting $i \in D$ into (5.3) and separating real and imaginary parts we obtain

$$\bar{\eta}_k = \sum_{j=1}^{n} (\operatorname{Re} f_{jk}(i))\bar{\xi}_j + \sum_{j=1}^{n} (\operatorname{Im} f_{jk}(i)) J\bar{\xi}_j . \tag{5.6}$$

Since both $2n$-tuples $(\bar{\xi}_j, \bar{\eta}_k)$ and $(\bar{\xi}_j, J\bar{\xi}_h)$ are independent, the matrix $(\operatorname{Im} f_{jk}(i))$ must be invertible; let

$$(e_{jk}) = (\operatorname{Im} f_{jk}(i))^{-1} . \tag{5.7}$$

Then from (5.6)

$$J\bar{\xi}_h = \sum_{k=1}^{n} e_{kh}\left(\bar{\eta}_k - \sum_{j=1}^{n} \operatorname{Re} f_{jk}(i)\bar{\xi}_j\right). \tag{5.8}$$

Formula (5.8) defines the action of J purely in terms of the geometry of M. Indeed, given $\bar{\xi}_j, \bar{\eta}_k$, the fields ξ_j, η_k are determined by (M, g); hence by (5.4) so are the functions φ_{jk}. (5.5) now determines f_{jk} as (unique) meromorphic extensions of φ_{jk} (to D), and e_{jk} are then determined by (5.7). This proves Theorem 4.2.

Remark 5.2. Because $\xi_1, \ldots, \xi_{n-1}, \eta_1, \ldots, \eta_{n-1}$ restricted to $\mathbb{R}$ are normal, while $\xi_n|\mathbb{R}$ and $\eta_n|\mathbb{R}$ are tangential Jacobi fields, in (5.4) we have $\varphi_{nk} \equiv \varphi_{jn} \equiv 0$, hence $f_{nk} \equiv f_{jn} \equiv 0$ and $e_{nk} \equiv e_{jn} \equiv 0$ for $1 \leq j, k \leq n-1$.

Theorem 5.3. *If $T^R M$ is endowed with an adapted complex structure and Θ is the canonical* 1*-form on TM (see Chap. 4) then for $z \in T^R M \setminus M$*

$$V_z = \ker \Theta_z \cap \ker (dE)_z \subset T_z(TM) \tag{5.9}$$

is a J invariant subspace.

Proof. First observe that $\dim V_z = 2n-2$. Indeed, choose local coordinates $\{q_j\}$ on M and corresponding local coordinates $\{q_j, p_j\}$ on TM (i.e. a vector $v = \Sigma p_j \partial/\partial q_j \in T_q M$ has coordinates coming from the coordinates of $q \in M$, and p_j). Then

$$\Theta = \Sigma g_{ij}(q) p_i dq_j$$

and

$$E = \tfrac{1}{2} \Sigma g_{ij}(q) p_i p_j ,$$

so that

$$dE = \Sigma g_{ij}(q) p_i dp_j + \tfrac{1}{2} \Sigma p_i p_j dg_{ij}(q) .$$

Thus $\ker \Theta$ and $\ker dE$ are transverse (off M), whence $\dim V_z = 2n-2$.

In what follows, we shall use the same notation as in the proof of Theorem 4.2. By virtue of Remark 5.2, (5.8) becomes

$$J\bar{\xi}_h = \sum_{k=1}^{n-1} e_{kh}\left(\bar{\eta}_k - \sum_{j=1}^{n-1} \operatorname{Re} f_{jk}(i)\bar{\xi}_j\right), \qquad h = 1, \ldots, n-1 . \tag{5.10}$$

Thus to prove the theorem it suffices to prove that $\bar{\xi}_1, \ldots, \bar{\xi}_{n-1}, \bar{\eta}_1, \ldots, \bar{\eta}_{n-1}$, belong to the space V_z in (5.9). Let $\bar{\xi}$ be any of the above vectors, ξ the corresponding parallel vector field along ψ_{γ_0}. Then $\xi|\mathbb{R}$ is a normal Jacobi field along γ_0, and z is tangent to γ_0. Hence, by (4.1) and (5.2)

$$\langle \Theta, \bar{\xi} \rangle = g(z, \pi_* \bar{\xi}) = g(z, \xi(\pi(z))) = 0 \tag{5.11}$$

(note that $\pi = N_0$!)

On the other hand, since $\xi|\mathbb{R}=(d\gamma_t/dt)_{t=0}$ is normal, the velocity of γ_t is the same as that of γ_0 plus $0(t^2)$, i.e.

$$\left(\frac{d}{dt}\right)_{t=0} E(\dot{\gamma}_t)=0.$$

Since $\dot{\gamma}_t(0)=z(t)$, this implies

$$\bar{\xi}E=\left(\frac{d}{dt}\right)_{t=0} E(z(t))=0. \tag{5.12}$$

(5.11) and (5.12) mean that $\bar{\xi}\in V_z$, and we are done.

Remark 5.4. Above we have established that if for a vector field ξ, parallel along a leaf of the Riemann foliation, the restriction $\xi|\mathbb{R}$ is a normal Jacobi field, then $\bar{\xi}\in V_z$. The converse also holds: if $\bar{\xi}\in V_z$ then $\xi|\mathbb{R}$ is a normal Jacobi field. To see this, observe that if $\xi|\mathbb{R}$ is a tangential Jacobi field then $\bar{\xi}$ is tangential to the Riemann foliation, whence the claim follows by counting dimensions.

Corollary 5.5. *On TM*

$$\bar{\partial}E-\partial E=i\Theta \tag{5.13}$$

and

$$\partial\bar{\partial}E=\frac{i}{2}\Omega. \tag{5.14}$$

Proof. Let $z\in TM\backslash M$. Because of Theorem 5.3, if

$$\bar{\xi}\in\ker\Theta_z\cap\ker(dE)_z \tag{5.15}$$

then

$$\langle\bar{\partial}E-\partial E,\bar{\xi}\rangle=0=\langle i\Theta,\bar{\xi}\rangle. \tag{5.16}$$

Next, with notation as before, we have

$$E(\psi_{\gamma_0}(\sigma+i\tau))=E(N_\tau\dot{\gamma}_0(\sigma))=\tau^2E(z).$$

Since ψ_{γ_0} is holomorphic,

$$\psi^*_{\gamma_0}(\bar{\partial}E-\partial E)=E(z)(\bar{\partial}\tau^2-\partial\tau^2)=2iE(z)\tau d\sigma.$$

Therefore, if

$$\bar{\eta}=\psi_{\gamma_0*}v,\qquad v=a\frac{\partial}{\partial\sigma}+b\frac{\partial}{\partial\tau}\in T_i\mathbb{C} \tag{5.17}$$

then

$$\begin{aligned}\langle\bar{\partial}E-\partial E,\bar{\eta}\rangle&=\left\langle 2iE(z)d\sigma, a\frac{\partial}{\partial\sigma}+b\frac{\partial}{\partial\tau}\right\rangle\\&=2iE(z)a.\end{aligned} \tag{5.18}$$

A simple computation gives $\pi_*\bar{\eta}=a\dot{\gamma}_0(0)=az$, so that

$$\langle i\Theta,\bar{\eta}\rangle=ig(z,\pi_*\bar{\eta})=2iE(z)a. \tag{5.19}$$

Since $T_z(TM)$ is spanned by vectors ξ as in (5.16) and $\bar{\eta}$ as in (5.17), comparing (5.16), (5.18), and (5.19), (5.13) follows (first only in points $z \in TM \backslash M$, and then, by continuity, all over TM).

Finally (5.14) is obtained by taking d of both sides of (5.13).

Theorem 5.6. *E is strictly plurisubharmonic in an adapted complex structure on $T^R M$. $\sqrt{E}$ is plurisubharmonic and satisfies the Monge-Ampère equation $(\partial\bar{\partial}\sqrt{E})^n = 0$ on $T^R M \backslash M$.*

Proof. First we shall prove strict plurisubharmonicity of E in points $z \in M$. Let $0 \neq v \in T_z(TM)$. Strict plurisubharmonicity means that for some (or any) complex curve C through z in the direction v $E|_C$ is strictly subharmonic in z. If the two-plane spanned by v, Jv is transverse to M, then so is C, and $E|_C$ has a nondegenerate minimum in z. Hence $E|_C$ is strictly subharmonic in z. Otherwise we can take C to be the closure of a leaf of the Riemann foliation determined by a geodesic γ. Then $E(\psi_\gamma(\sigma + i\tau)) = \text{const}\, \tau^2$, whence $E|_C$ is strictly subharmonic everywhere.

Thus E is strictly plurisubharmonic along M. Since by 5.14 $\partial\bar{\partial}E$ is nowhere degenerate, E must stay strictly plurisubharmonic all through $T^R M$.

Next we shall prove $(\partial\bar{\partial}\sqrt{E})^n = 0$. Because of (3.4) this is equivalent to

$$2E(\partial\bar{\partial}E)^n = n\partial E \wedge \bar{\partial}E \wedge (\partial\bar{\partial}E)^{n-1}. \tag{5.20}$$

In local coordinates as in Theorem 5.3, using also Corollary 5.5, the right hand side can be written as

$$\begin{aligned}
&\frac{n}{2}(\partial E + \bar{\partial}E) \wedge (\bar{\partial}E - \partial E) \wedge (\partial\bar{\partial}E)^{n-1} = \frac{n}{2}\, dE \wedge i\Theta \wedge \left(\frac{i}{2}\Omega\right)^{n-1} \\
&= n\left(\frac{i}{2}\right)^n \left(\Sigma g_{ij} p_i dp_j + \frac{1}{2}\Sigma p_i p_j dg_{ij}\right) \wedge (\Sigma g_{ij} p_i dq_j) \\
&\quad \wedge (\Sigma g_{ij} dp_i \wedge dq_j + \Sigma p_i dg_{ij} \wedge dq_j)^{n-1} \\
&= n\left(\frac{i}{2}\right)^n \Sigma g_{ij} p_i dp_j \wedge \Sigma g_{ij} p_i dq_j \wedge (\Sigma g_{ij} dp_i \wedge dq_j)^{n-1},
\end{aligned}$$

since dg_{ij} contains only dq_k terms. In any particular point we can assume $g_{ij} = \delta_{ij}$, $p_1 = 1$, $p_j = 0$ $(j = 2, \ldots, n)$, and in that point the right hand side of (5.20) is then

$$n\left(\frac{i}{2}\right)^n dp_1 \wedge dq_1 \wedge (\Sigma dp_j \wedge dq_j)^{n-1} = \left(\frac{i}{2}\right)^n (\Sigma dp_j \wedge dq_j)^n.$$

This proves (5.20).

Finally it suffices to prove plurisubharmonicity of $\sqrt{E}$ off M. There $\sqrt{E}$ is smooth and $\partial\bar{\partial}\sqrt{E}$ can have only one nonpositive eigenvalue, which by virtue of the Monge-Ampère equation is 0. Thus $-i\partial\bar{\partial}\sqrt{E}$ is positive semidefinite, and we are done.

Theorem 5.6 shows that finding Monge-Ampère models and finding adapted complex structures are completely equivalent.

Adapted complex structures on $T^R M$ have an important (although fairly obvious) symmetry:

Theorem 5.7. *If $T^R M$ is endowed with an adapted complex structure, then the diffeomorphism $N_{-1}: TM \to TM$ is antiholomorphic.*

Proof. It suffices to prove that N_{-1} is antiholomorphic on $TM\backslash M$. Thus, let $z \in TM\backslash M$, $\bar{\xi} \in T_z(TM)$, $w = N_{-1}z(=-z)$, $\bar{\kappa} = N_{-1*}\bar{\xi} \in T_w(TM)$. The problem is to relate $J\bar{\kappa}$ to $N_{-1*}J\bar{\xi}$. Accordingly, we shall have to go through the steps of the proof of Theorem 4.2 and see how the objects connected with $w, \bar{\kappa}$ are related to the corresponding objects connected with $z, \bar{\xi}$. We shall adopt the notation of the proof of Theorem 4.2.

Let $\delta : \mathbb{R} \to M$ be the geodesic determined by $w : \dot{\delta}(0) = w$. Then $\delta(\sigma) = \gamma(-\sigma)$, so that

$$\psi_\delta(\sigma + i\tau) = N_\tau \dot{\delta}(\sigma) = -N_\tau \dot{\gamma}(-\sigma) = \psi_\gamma(-\sigma - i\tau).$$

If κ denotes the parallel extension of $\bar{\kappa}$ along ψ_δ then this implies

$$\kappa(\sigma + i\tau) = \xi(-\sigma - i\tau). \tag{5.21}$$

With $\bar{\xi}_j$, $\bar{\eta}_k$, ξ_j, η_k as before, define

$$\bar{\kappa}_j = N_{-1*}\bar{\xi}_j, \qquad \bar{\lambda}_k = N_{-1*}\bar{\eta}_k,$$

and let κ_j, λ_k be the parallel extensions of $\bar{\kappa}_j$ resp. $\bar{\lambda}_k$. As in (5.21), we have

$$\kappa_j(\sigma + i\tau) = \xi_j(-\sigma - i\tau), \qquad \lambda_k(\sigma + i\tau) = \eta_k(-\sigma - i\tau). \tag{5.22}$$

If now h_{jk} denote the meromorphic functions on D such that

$$\lambda_k^{1,0} = \sum_{j=1}^{n} h_{jk} \kappa_j^{1,0} \qquad (k = 1, \ldots, n)$$

[cf. (5.3)], then (5.3) and (5.22), restricted to $\tau = 0$ imply $h_{jk}(\sigma) = f_{jk}(-\sigma)$, hence

$$h_{jk}(\sigma + i\tau) = f_{jk}(-\sigma - i\tau).$$

Therefore, as in (5.6)

$$\begin{aligned} \bar{\lambda}_k &= \sum_{j=1}^{n} (\operatorname{Re} h_{jk}(i))\bar{\kappa}_j + \sum_{j=1}^{n} (\operatorname{Im} h_{jk}(i)) J\bar{\kappa}_j \\ &= \sum_{j=1}^{n} \operatorname{Re} f_{jk}(-i)\bar{\kappa}_j + \sum_{j=1}^{n} \operatorname{Im} f_{jk}(-i) J\bar{\kappa}_j. \end{aligned} \tag{5.23}$$

Now f_{jk} is real on the real line [cf. (5.5)], so that by the reflection principle

$$\operatorname{Re} f_{jk}(-i) = \operatorname{Re} f_{jk}(i), \qquad \operatorname{Im} f_{jk}(-i) = -\operatorname{Im} f_{jk}(i).$$

Substituting this into (5.23) and solving for $J\bar{\kappa}_j$ we obtain

$$J\bar{\kappa}_h = -\sum_{k=1}^{n} e_{kh} \left(\bar{\lambda}_k - \sum_{j=1}^{n} \operatorname{Re} f_{jk}(i) \bar{\kappa}_j \right).$$

Comparing this with (5.8):

$$J\bar{\kappa}_h = -N_{-1*} J\bar{\xi}_h,$$

i.e.

$$JN_{-1*} = -N_{-1*}J,$$

which is precisely the statement that N_{-1} is antiholomorphic.

Corollary 5.8. *M is a real analytic submanifold of the complex manifold $T^R M$, and the geodesics $\gamma : \mathbb{R} \to M$ are real analytic mappings.*

Proof. M is real analytic since it is the fixed set of an antiholomorphic mapping. The geodesics γ are real analytic since they are restrictions of holomorphic mappings $\psi_\gamma: D \to TM$ to $\mathbb{R}$.

6 Special Jacobi fields

The aim of the present chapter is to show that when the parallel vector fields $\xi_1, \ldots, \xi_n$, $\eta_1, \ldots, \eta_n$ in the proof of Theorem 4.2 are carefully chosen, the meromorphic matrix (f_{jk}) in (5.3) becomes very simple and will be directly connected with the sectional curvatures of g. Much of what is to be discussed here would be substantially simpler in the case $n=2$.

Proposition 6.1. *Suppose ξ is a parallel vector field along a leaf of the Riemann foliation. Then for any $s \in \mathbb{R}$*

$$N_{s*}\xi = \xi, \tag{6.1}$$

and

$$\phi_{s*}\xi = \xi. \tag{6.2}$$

(Recall that ϕ_s is the geodesic flow.)

Proof. Let γ_t be a family of geodesics such that $\xi = (d\psi_{\gamma_t}/dt)_{t=0}$. From

$$\psi_{\gamma_t}(\sigma + i\tau) = N_\tau \dot{\gamma}_t(\sigma)$$

[cf. (3.1)] follow

$$N_s \psi_{\gamma_t}(\sigma + i\tau) = N_{s\tau}\dot{\gamma}_t(\sigma) = \psi_{\gamma_t}(\sigma + is\tau), \tag{6.3}$$

$$\phi_s \psi_{\gamma_t}(\sigma + i\tau) = N_\tau \dot{\gamma}_t(\sigma + s\tau) = \psi_{\gamma_t}(\sigma + s\tau + i\tau). \tag{6.4}$$

Taking $(d/dt)_{t=0}$ of these equations we obtain

$$N_{s*}\xi(\sigma + i\tau) = \xi(\sigma + is\tau), \tag{6.5}$$

$$\phi_{s*}\xi(\sigma + i\tau) = \xi(\sigma + s\tau + i\tau), \tag{6.6}$$

which prove (6.1), (6.2).

Proposition 6.2. *Let ξ, η be parallel vector fields along a leaf of the Riemann foliation. If for a number $\sigma_0 + i\tau_0 \in \mathbb{C}\setminus\mathbb{R}$*

$$\Omega(\xi(\sigma_0 + i\tau_0), \eta(\sigma_0 + i\tau_0)) = 0,$$

then $\Omega(\xi, \eta) \equiv 0$ along the whole leaf.

Proof. Because ϕ_s is a Hamiltonian flow, it leaves Ω invariant. Simple computations also show that $N_s^*\Omega = s\Omega$. Since by (6.3), (6.4) the orbit of the point $\psi_\gamma(\sigma_0 + i\tau_0)$ under repeated applications of N_s and ϕ_s is the whole leaf, Proposition 6.1 implies $\Omega(\xi, \eta) \equiv 0$.

In the sequel we shall need the notion of the connection map. The metric g defines the Levi-Civita connection on TM, i.e. a splitting of $T_z(TM)$ $(z \in TM)$ into vertical and horizontal subspaces. The vertical subspace is $T_{\pi(z)}M$, which canonically sits in $T_z(TM)$; the datum of a complementary horizontal subspace is equivalent to the datum of a projection $T_z(TM) \to T_{\pi(z)}M$. The collection of these projections for every $z \in TM$ is the connection map $K: T(TM) \to TM$. K is related to Ω by the formula

$$\Omega(\bar{\xi}, \bar{\eta}) = g(\pi_* \bar{\xi}, K\bar{\eta}) - g(\pi_* \bar{\eta}, K\bar{\xi}), \tag{6.7}$$

see [8]. We shall also need

Proposition 6.3. *Let ξ be a parallel vector field along a leaf of the Riemann foliation determined by $\bar{\xi} \in T_z(TM)$, $z \in TM\backslash M$. Then the covariant derivative of the Jacobi field $\xi|\mathbb{R}$ at σ is $K\xi(\sigma+i)$.*

Proof. With notation as in the proof of Theorem 4.2 the covariant derivative in question is

$$K\frac{d\xi(\sigma)}{d\sigma} = K\frac{d}{d\sigma}\left(\frac{d\gamma_t(\sigma)}{dt}\right)\bigg|_{t=0} = K\frac{d}{dt}\dot{\gamma}_t(\sigma)|_{t=0} = K\xi(\sigma+i).$$

After these generalities, let us return to an adapted complex structure on $T^R M$. Choose a point $z \in T^R M\backslash M$, and let γ be the geodesic determined by z (i.e. $\dot{\gamma}(0) = z$). Select an orthonormal n-tuple of vectors $v_1, \ldots, v_n$ in $T_{\gamma(0)}M$, and lift them to horizontal vectors $\bar{\xi}_1, \ldots, \bar{\xi}_n \in T_z(TM)$. These vectors define parallel vector fields $\xi_1, \ldots, \xi_n$ along $\psi_\gamma: D \to TM$, where $D = \{\sigma + i\tau : |\tau| < R/\sqrt{2E(z)}\}$. $\bar{\xi}_j$ being horizontal, (6.7) implies $\Omega(\bar{\xi}_j, \bar{\xi}_k) = 0$. By Proposition 6.2, therefore

$$\Omega(\xi_j, \xi_k) = 0, \quad 1 \leqq j, k \leqq n. \tag{6.8}$$

Proposition 6.4. *The $2n$ vectors ξ_j, $J\xi_k$ are independent on $D\backslash\mathbb{R}$. Hence on $D\backslash\mathbb{R}$ the vectors $\xi_j^{1,0} = \frac{1}{2}(\xi_j - iJ\xi_j)$ are independent over $\mathbb{C}$.*

Proof. First observe that since Ω is of type (1, 1) (cf. Corollary 5.5) (6.8) implies

$$\Omega(J\xi_j, J\xi_k) = 0. \tag{6.9}$$

Suppose now that a nontrivial combination $\xi = \Sigma a_j \xi_j$ belongs to the span of $J\xi_k$ in a point $\sigma + i\tau \in D\backslash\mathbb{R}$. Since the vectors ξ_j are independent off the real axis, $\xi \neq 0$. On the other hand, in this point $\langle -2i\partial\bar{\partial}E, \xi \wedge J\xi \rangle = \Omega(\xi, J\xi) = \Sigma a_j \Omega(\xi, J\xi_j) = 0$ by (6.9), contradicting the strict plurisubharmonicity of E.

As discussed above, $T_{\pi(z)}M$ can be regarded as a subspace of $T_z(TM)$. Thus $v_j = \pi_* \bar{\xi}_j \in T_{\pi(z)}M$ determines a vertical vector $\bar{\eta}_j \in T_z(TM)$:

$$\pi_* \bar{\eta}_j = 0, \quad K\bar{\eta}_j = v_j. \tag{6.10}$$

Extend $\bar{\eta}_j$ to a parallel vector field η_j along ψ_γ.

Proposition 6.5. *The $2n$ vectors ξ_j, η_k are independent in points $\sigma + i\tau \in D\backslash\mathbb{R}$.*

Proof. By (6.10) and $\pi_* \bar{\xi}_j = v_j$, $K\bar{\xi}_j = 0$, the vectors $\bar{\xi}_j$, $\bar{\eta}_k$ are independent. Therefore so are their parallel extensions ξ_j, η_k, off $\mathbb{R}$.

Proposition 6.6. $\Omega(\eta_j, \eta_k) \equiv 0$, $1 \leqq j, k \leqq n$.

Proof. By (6.7), (6.10), and Proposition 6.2, $0 = \Omega(\bar{\eta}_j, \bar{\eta}_k) = \Omega(\eta_j, \eta_k)$. q.e.d.

Consider now the n-tuples

$$\Xi = (\xi_1, \ldots, \xi_n), \quad H = (\eta_1, \ldots, \eta_n),$$

and holomorphic n-tuples

$$\Xi^{1,0} = (\xi_1^{1,0}, \ldots, \xi_n^{1,0}), \quad H^{1,0} = (\eta_1^{1,0}, \ldots, \eta_n^{1,0}).$$

Proposition 6.4 implies that $\eta_k^{1,0}$ is in the span of $\xi_j^{1,0}$, at least off $\mathbb{R}$, but more generally, where the $\xi_j^{1,0}$'s are independent: on D minus a discrete set S of reals. We shall need the fact that $0 \notin S$. Indeed, $\xi_j(0) = v_j \in T_{\pi(z)}M \subset T_{\pi(z)}(TM)$, while $J\xi_k(0)$'s are vertical. Hence the $2n$ vectors $\xi_j(0)$ and $J\xi_k(0)$ are independent, and therefore so are $\xi_j^{1,0}(0)$.

Thus we can express $\eta_k^{1,0}$ as meromorphic linear combinations of $\xi_j^{1,0}$, i.e., there is an $n \times n$ matrix valued meromorphic function f on D such that

$$H^{1,0} = \Xi^{1,0} f. \tag{6.11}$$

The poles of f are restricted to $S \subset \mathbb{R}$. As in the proof of Theorem 4.2 we have that $f|\mathbb{R}$ is real valued (cf. (5.4), (5.5)), hence, taking real parts

$$H(\sigma) = \Xi(\sigma) f(\sigma), \qquad \sigma \in \mathbb{R} \backslash S. \tag{6.12}$$

Lemma 6.7. *If $\sigma + i\tau \in D$, $\sigma > 0$ then $\operatorname{Im} f(\sigma + i\tau)$ is a symmetric, positive definite matrix.*

The proof will require several steps.

Proposition 6.8. *If $\sigma + i\tau \in D$, $\sigma \neq 0$ then $\operatorname{Im} f(\sigma + i\tau)$ is invertible.*

Proof. Suppose there is a nonzero column vector $v = (v_j) \in \mathbb{R}^n$ such that $\operatorname{Im} f(\sigma + i\tau)v = 0$, i.e., $\mu = (\mu_k) = f(\sigma + i\tau)v \in \mathbb{R}^n$. From (6.11), in the point $\sigma + i\tau$

$$\sum \eta_j^{1,0} v_j = H^{1,0} v = \Xi^{1,0} \mu = \sum \xi_k^{1,0} \mu_k .$$

Taking real parts

$$\sum \eta_j v_j = \sum \xi_k \mu_k ,$$

in contradiction with Proposition 6.5.

For $\sigma \in \mathbb{R} \backslash S$ we can view $\Xi(\sigma)$, $H(\sigma)$ as linear mappings $\mathbb{R}^n \to T_{\gamma(\sigma)}M$ given by

$$\mathbb{R}^n \ni (w_j) = w \mapsto \Xi(\sigma) w = \sum_1^n w_j \xi_j(\sigma) \in T_{\gamma(\sigma)}$$

and similarly for $H(\sigma)$. This is consistent with (6.12). We shall denote covariant differentiation along γ by a prime, e.g. $H'(\sigma)$. In the case of a mapping of $\mathbb{R}$ into some fixed space (like $f|\mathbb{R} : \mathbb{R} \backslash S \to \operatorname{End} \mathbb{R}^n$) prime will also be used to denote ordinary differentiation with respect to σ, as $f'(\sigma)$.

Proposition 6.9. $f(0) = 0$, $f'(0) = Id$.

Proof. Put $\sigma = 0$ in (6.12). Since $\eta_j(0) = \pi_* \tilde{\eta}_j = 0$ (cf. (6.10)), and $\xi_j(0) = \pi_* \tilde{\xi}_j = v_j$, $1 \leq j \leq n$, which are independent, we obtain $f(0) = 0$.

On the other hand, covariant differentiation of (6.12) in $\sigma = 0$ yields

$$H'(0) = \Xi(0) f'(0) .$$

By Proposition 6.3, $\eta_j'(0) = K\tilde{\eta}_j = v_j = \xi_j(0)$, i.e., $H'(0) = \Xi(0)$, so that $f'(0) = Id$, as claimed.

Denote the adjoint of a linear mapping $\mathbb{R}^n \to T_{\gamma(\sigma)}M$ by star (adjoint defined using the euclidean scalar product on $\mathbb{R}^n$ and the Riemann metric on $T_{\gamma(\sigma)}$).

Proposition 6.10. *For $\sigma \in \mathbb{R} \backslash S$*

$$\Xi'^*(\sigma)\Xi(\sigma) - \Xi^*(\sigma)\Xi'(\sigma) = H'^*(\sigma)H(\sigma) - H^*(\sigma)H'(\sigma) = 0 .$$

Proof. According to (6.7) and (6.8)

$$0=\Omega(\xi_j,\xi_k)=g(\xi_j,\xi_k')-g(\xi_j',\xi_k), \tag{6.13}$$

where we have omitted the argument σ. Using the standard orthonormal basis $\{e_j\}$ of $\mathbb{R}^n$, we have

$$g(\xi_j,\xi_k')=g(\Xi e_j,\Xi' e_k)=(\Xi'^*\Xi e_j,e_k),$$

(,) standing for Euclidean scalar product. Thus (6.13) can be written

$$0=((\Xi'^*\Xi-\Xi^*\Xi')e_j,e_k),\qquad 1\leq j,k\leq n,$$

i.e.

$$\Xi'^*\Xi-\Xi^*\Xi'=0\quad\text{on}\quad \mathbb{R}\backslash S.$$

Similarly, by Proposition 6.6

$$0=\Omega(\eta_j,\eta_k),$$

whence we obtain exactly as in the case of Ξ that

$$H'^*H-H^*H'=0\quad\text{on}\quad \mathbb{R}\backslash S. \tag{6.14}$$

q.e.d.

Proposition 6.11. *$f(\sigma)$ is a symmetric matrix for small $\sigma\in\mathbb{R}$.*

Proof. If (6.12) is substituted into (6.14), straightforward calculations show

$$f'^*\Xi^*\Xi f-f^*\Xi^*\Xi f'=0. \tag{6.15}$$

Now we bring the Jacobi equation

$$\xi''(\sigma)+R_\gamma(\sigma)\xi(\sigma)=0 \tag{6.16}$$

into play. Here ξ is any Jacobi field along γ and $R_\gamma(\sigma):T_{\gamma(\sigma)}\to T_{\gamma(\sigma)}$ is the "curvature operator along γ": if $\mathscr{R}$ is the curvature tensor of g, then

$$R_\gamma(\sigma)v=\mathscr{R}(v,\dot\gamma(\sigma))\dot\gamma(\sigma),\qquad v\in T_{\gamma(\sigma)}. \tag{6.17}$$

(6.16) holds for $\xi(\sigma)=\xi_j(\sigma)$ and $\xi(\sigma)=\eta_j(\sigma)$. Hence

$$0=\Xi''+R_\gamma\Xi, \tag{6.18}$$

$$0=H''+R_\gamma H=(\Xi f)''+R_\gamma\Xi f=2\Xi' f'+\Xi f'', \tag{6.19}$$

where we used (6.18).

Next compute

$$(\Xi^*\Xi f')'=\Xi^{*\prime}\Xi f'+\Xi^*\Xi' f'+\Xi^*\Xi f'',$$

which, by (6.19) and Proposition 6.10 equals

$$\Xi^{*\prime}\Xi f'-\Xi^*\Xi' f'=0.$$

Hence $\Xi^*(\sigma)\Xi(\sigma)f'(\sigma)$ is a constant mapping $\mathbb{R}^n\to\mathbb{R}^n$ for small $\sigma\in\mathbb{R}$. Now $\Xi(0)$ maps the orthonormal basis $\{e_j\}$ of $\mathbb{R}^n$ into an orthonormal basis $\{v_j\}$ of $T_{\gamma(0)}$, so that $\Xi^*(0)\Xi(0)=Id$. Since, according to Proposition 6.9, $f'(0)=Id$, we have

$$\Xi^*(\sigma)\Xi(\sigma)f'(\sigma)=Id.$$

Substituting this into (6.15) we obtain $f^*(\sigma)-f(\sigma)=0$, which proves our claim.

Remark 6.12. Much of the above computation comes from [6].

We are now ready to prove Lemma 6.7.

Proof of Lemma 6.7. From Proposition 6.11 follows by analytic continuation that f is symmetric in every point of D, therefore so is $\operatorname{Im} f$. For small positive τ, using Proposition 6.9

$$\begin{aligned}\operatorname{Im} f(i\tau) &= \operatorname{Im}(f(0)+i\tau f'(0))+O(\tau^2)) \\ &= \tau Id + O(\tau^2),\end{aligned}$$

which is obviously positive definite. Since $\operatorname{Im} f$ is nondegenerate on $D\backslash\mathbb{R}$ by Proposition 6.8, its signature cannot change on $\{\sigma+i\tau\in D:\tau>0\}$. Therefore $\operatorname{Im} f(\sigma+i\tau)$ is positive definite for $\tau>0$.

7 The curvature estimates

We shall use the notation introduced in the previous chapter.

Proposition 7.1. *On $\mathbb{R}\backslash S$ the curvature operator can be expressed as*

$$R_\gamma = \tfrac{1}{2}\Xi(f'''f'^{-1} - \tfrac{3}{2}(f''f'^{-1})^2)\Xi^{-1}. \tag{7.1}$$

Remark 7.2. When $n=1$, Ξ and Ξ^{-1} above cancel and the right hand side becomes the Schwarz derivative of f. Then Proposition 7.1 reduces to the well known relationship between the ratio of two solutions of the *scalar* Sturm-Liouville equation $y''+Ry=0$ and R itself. The proof of the matrix valued version is much the same:

Proof. From (6.19)

$$\Xi' = -\tfrac{1}{2}\Xi f'' f'^{-1}. \tag{7.2}$$

Differentiating (6.19) and using (7.2) and (6.18):

$$\begin{aligned}0 &= 2\Xi''f' + 3\Xi' f'' + \Xi f''' \\ &= -2R_\gamma \Xi f' - \tfrac{3}{2}\Xi f'' f'^{-1} f'' + \Xi f''',\end{aligned}$$

whence (7.1) follows.

Corollary 7.3. $R_\gamma(0) = \frac{1}{2}\Xi(0) f'''(0)\Xi(0)^{-1}$.

Proof. Since ξ_j is a horizontal vector, by Proposition 6.3, $\xi_j'(0)=K\xi_j=0$. Thus $\Xi'(0)=0$, whence (7.2) gives

$$f''(0)=0. \tag{7.3}$$

According to Proposition 6.9, $f'(0)=Id$, so that (7.1) reduces to the equation in Corollary 7.3.

As we saw in the proof of Proposition 6.11, $\Xi^*(0)\Xi(0)=Id$, i.e., $\Xi(0)$ is an isometry, so estimation of the curvature operator $R_\gamma(0)$ is the same as estimation of $f'''(0)$. To get the desired estimate we shall use Lemma 6.7. We shall also need certain facts about representations of holomorphic functions in the upper half plane.

Proposition 7.4. *Let F be an $n \times n$ matrix valued holomorphic function on the upper half plane $\mathbb{C}^+ = \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$ that has a holomorphic extension to a neighbourhood of* 1. *Suppose for every $\zeta \in \mathbb{C}^+$ $\operatorname{Im} F(\zeta)$ is a symmetric, positive definite matrix, whereas for $\zeta \in \mathbb{R}$ near* 1 $\operatorname{Im} F(\zeta) = 0$. *Then there is an $n \times n$ symmetric matrix $\mu = (\mu_{jk})$ whose entries are real valued, signed Borel measures on $\mathbb{R}$ such that*

1° $1 \notin \operatorname{supp} \mu_{jk}$;

2° $\displaystyle\int_{-\infty}^{\infty} \frac{|d\mu_{jk}(x)|}{1+x^2} < \infty$;

3° *μ is positive semidefinite in the sense that for any $(w_j) \in \mathbb{R}^n$ the measure $\Sigma w_j w_k \mu_{jk}$ is nonnegative;*

4° $\displaystyle F'(\zeta) = A + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(x)}{(\zeta - x)^2} \quad (\zeta \in \mathbb{C}^+)$,

where A is a symmetric, positive semidefinite constant matrix.

Proof. Let F_{jk} denote the entries of F. Since the diagonal entries $\operatorname{Im} F_{jj}$ are nonnegative harmonic functions on $\mathbb{C}^+$, Fatou's representation theorem (transplanted from the disc to $\mathbb{C}^+$; cf. [9]) implies the existence of nonnegative measures μ_{jj} and nonnegative numbers A_{jj} (the latters corresponding to mass at ∞) such that $\int d\mu_{jj}(x)/(1+x^2) < \infty$ and

$$\operatorname{Im} F_{jj}(\zeta) = A_{jj} \operatorname{Im}\zeta + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im}\zeta}{(\operatorname{Re}\zeta - x)^2 + (\operatorname{Im}\zeta)^2} \, d\mu_{jj}(x).$$

For any j, k

$$|\operatorname{Im} F_{jk}(\zeta)| \leqq (\operatorname{Im} F_{jj}(\zeta) \operatorname{Im} F_{kk}(\zeta))^{1/2},$$

because $\operatorname{Im} F(\zeta)$ is semidefinite. Since both $\operatorname{Im} F_{jj}$ and $\operatorname{Im} F_{kk}$ belong to the Hardy space $h^1(\mathbb{C}^+)$ corresponding to the weight $(1+x^2)^{-1}$, it follows that $\operatorname{Im} F_{jk} \in h^1(\mathbb{C}^+)$; hence we also have a Poisson representation

$$\operatorname{Im} F_{jk}(\zeta) = A_{jk} \operatorname{Im}\zeta + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im}\zeta}{(\operatorname{Re}\zeta - x)^2 + (\operatorname{Im}\zeta)^2} \, d\mu_{jk}(x), \tag{7.4}$$

with the μ_{jk}'s satisfying 2° of Proposition 7.4. Taking $\partial/\partial z$ of (7.4) we obtain 4°. The matrix $A = (A_{jk})$ is symmetric positive semidefinite since

$$A = \lim_{y \to +\infty} \operatorname{Im} F(iy)/y.$$

Finally, since $d\mu(x)$ is the weak limit of $\operatorname{Im} F(x+iy)$ as $y \to +0$, 1° and 3° are also satisfied. For more details, cf. [13].

Proposition 7.5. *Let F be as in Proposition 7.4. Assume, in addition, that $F'(1) = a\mathrm{Id}$, $F''(1) = b\mathrm{Id}$, $a, b \in \mathbb{R}$. Then for $w = (w_j) \in \mathbb{R}^n$*

$$\sum_{j,k=1}^{n} F'''_{jk}(1) w_j w_k \geqq \frac{3b^2}{2a} \sum_{1}^{n} w_j^2. \tag{7.5}$$

Proof. Put $v = \Sigma w_j w_k \mu_{jk}$. This is a nonnegative measure. Put also $G = \Sigma F_{jk} w_j w_k$. Then the left hand side of (7.5) is $G'''(1)$. Since by analytic continuation the formula

$4°$ in Proposition 7.5 holds in a neighbourhood of 1, by differentiation we obtain

$$a\sum w_j^2 = G'(1) \geqq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(x)}{(1-x)^2} \tag{7.6}$$

$$b\sum w_j^2 = G''(1) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(x)}{(1-x)^3} \tag{7.7}$$

$$G'''(1) = \frac{6}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(x)}{(1-x)^4}. \tag{7.8}$$

By Schwarz's inequality

$$\int_{-\infty}^{\infty} \frac{d\nu(x)}{(1-x)^4} \int_{-\infty}^{\infty} \frac{d\nu(x)}{(1-x)^2} \geqq \left(\int_{-\infty}^{\infty} \frac{d\nu(x)}{(1-x)^3} \right)^2.$$

Estimating respectively replacing the integrals here by what is obtained from (7.6)...(7.8), we get (7.5).

Proof of Theorem 4.3. Choose a point $p \in M$ and two orthogonal vectors $v, z \in T_pM$. We want to estimate the sectional curvature $K(v,z)$ of g in the direction determined by v, z. It can be assumed that $z \in T^R M$, i.e. that

$$R/\sqrt{2E(z)} = r > 1.$$

Issue a geodesic γ from p such that $\dot\gamma(0) = z$, and perform the construction in Chap. 6 to obtain the matrix valued meromorphic function f on

$$D = \{\zeta + i\tau \in \mathbb{C} : |\tau| < r\},$$

with poles on a set $S \subset \mathbb{R} \setminus \{0\}$. Assuming for simplicity that the length of v is 1, $K(v,z)$ is related to the curvature operator $R_\gamma(0)$ by (see (6.17))

$$\begin{aligned} K(v,z) &= g(\mathcal{R}(v, \dot\gamma(0))\dot\gamma(0), v)/(2E(z)) \\ &= g(R_\gamma(0)v, v)/(2E(z)). \end{aligned} \tag{7.9}$$

By Corollary 7.3

$$\min_{g(v,v)=1} g(R_\gamma(0)v, v) = \frac{1}{2} \min \left\{ \sum_{j,k=1}^{n} f_{jk}'''(0) w_j w_k : (w_j) \in \mathbb{R}^n, \sum w_j^2 = 1 \right\}, \tag{7.10}$$

since $\Xi(0)$ is an isometry. Thus we have to estimate $f'''(0)$.

Let $\log\zeta$ for $\zeta \in \mathbb{C}^+$ denote the branch of logarithm such that $0 \leqq \operatorname{Im} \log\zeta < \pi$, and put

$$F(\zeta) = f\left(\frac{r}{\pi} \log\zeta\right).$$

By Lemma 6.7, this F satisfies the assumptions of Proposition 7.4. Moreover,

$$F'(1) = \frac{r}{\pi} f'(0) = \frac{r}{\pi} Id,$$

$$F''(1) = \frac{r^2}{\pi^2} f''(0) - \frac{r}{\pi} f'(0) = -\frac{r}{\pi} Id,$$

$$F'''(1) = \frac{r^3}{\pi^3} f'''(0) + \frac{2r}{\pi} f'(0) = \frac{r^3}{\pi^3} f'''(0) + \frac{2r}{\pi} Id,$$

by the chain rule, Proposition 6.9, and (7.3). Thus F even satisfies the extra assumptions of Proposition 7.5 with $a=r/\pi$, $b=-r/\pi$. According to this proposition, if $\Sigma w_j^2=1$:

$$\frac{3r}{2\pi} \leqq \sum F_{jk}'''(0) w_j w_k \leqq \frac{r^3}{\pi^3} \sum f_{jk}'''(0) w_j w_k + \frac{2r}{\pi}$$

whence

$$-\frac{\pi^2}{2r^2} \leqq \sum f_{jk}'''(0) w_j w_k.$$

By (7.10)

$$g(R_y(0)v, v) \geqq -\frac{\pi^2}{4r^2} = -\frac{\pi^2}{4R^2} 2E(z),$$

and by (7.9)

$$K(v, z) \geqq -\frac{\pi^2}{4R^2}.$$

Thus the theorem is proved.

Proof of Theorem 4.4. In this case $R=\infty$ so that g is nonnegatively curved. It is well known that on a compact flat manifold any nonnegatively curved metric is itself flat, as claimed by the theorem. (The proof of this fact goes as follows: Let $(T^n, \tilde{g})$ be a torus that covers (M, g), see [16]. Choose n independent harmonic one forms $\omega_1, \ldots, \omega_n$ on T^n, and let $X_1, \ldots, X_n$ be dual vector fields. Weizenböck's formula gives

$$\Delta \tilde{g}(X_j, X_j) = 2 \operatorname{Ric} \tilde{g}(X_j, X_j) + 2\tilde{g}(\nabla\omega_j, \nabla\omega_j) \geqq 0,$$

i.e. $\tilde{g}(X_j, X_j)$ is subharmonic, hence constant. Therefore the left hand side above is 0, and $\nabla\omega_j=0$. Thus ω_j is parallel and so is X_j. The existence of n independent parallel vector fields then implies the flatness of $\tilde{g}$; hence also g is flat.)

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Note added in proof. Recently we have learned that related problems have been investigated by V. Guillemin and M. Stenzel. They consider certain complex structures on (co)tangent bundles of Riemannian manifolds. Although their formal definitions are different from ours, they recover the same complex manifolds X and functions u as we do. See their joint papers "Grauert tubes and the homogeneous complex Monge-Ampère equation, I, II", and also the thesis of Stenzel. In another development, a version of [11] has appeared in the meantime, in this journal, **289**, 355–382 (1991).