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Strengthening pseudoconvexity of finite-dimensional Teichmüller spaces

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G.M. Henkin informed me on a question of M. Gromov: whether the Teichmüller space T_p for closed Riemann surfaces of a fixed genus p is strengthenly pseudoconvex, that means does there exist a plurisubharmonic function ϱ on T_p such that $\varrho \to 0$ for $x \to \infty$?

That is interesting by itself also, since, on one hand, the Teichmüller spaces are not strongly pseudoconvex and, on the other hand, in some questions, in particular in complex potential theory, in fact the strengthening pseudoconvexity is essential (see, for example [9]).

It turns out that the answer is affirmative for all finite-dimensional Teichmüller spaces $T(p,n)$ (which correspond to the Riemann surfaces of finite conformal type (p,n) with $m=3p-3+n>0$), moreover in a stronger form:

Theorem. *On any space $T(p,n)$ there exists a continuous plurisubharmonic function $\varrho(x)$ which tends to zero for $x \to \infty$ (in the Teichmüller-Kobayashi metric). Under holomorphic imbedding of $T(p,n)$ as a bounded subdomain in $\mathbb{C}^m$ this function becomes continuous on the closure of $T(p,n)$ also.*

Proof. Fix a base point in $T(p,n)$ and model $T(p,n)$ as Teichmüller space $T(\Gamma)$ of corresponding finitely generated Fuchsian group Γ of the first kind without torsion which acts discontinuously in $\Delta \cup \Delta^*$, where

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}, \quad \Delta^* = \{z \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : |z| > 1\}.$$

Let us denote by $B_2(\Gamma)$ the complex space of holomorphic Γ-automorphic forms of the weight -4 with the supports in Δ^*, i.e. of the holomorphic functions φ in Δ^* such that $(\varphi \circ \gamma)\gamma'^2 = \varphi$ for all $\gamma \in \Gamma$ and $\varphi(z) = O(|z|^{-4})$ for $z \to \infty$, with the norm $\|\varphi\| = \sup_{\Delta^*} (|z|^2-1)^2 |\varphi(z)|$; here $\dim B_2(\Gamma) = m$.

Consider now those $\varphi \in B_2(\Gamma)$ which are the Schwarzian derivatives

$$S_f = (f''/f')' - (f''/f')^2/2$$

of the univalent functions $f(z) = z + \sum_{q=1}^{\infty} a_q z^{-q}$ with quasiconformal extension onto $\bar{\Delta}$ (also compatible with Γ). These Schwarzians fill a bounded arcwise connected domain in $B_2(\Gamma)$ which is (up to biholomorphic isomorphism) a standard model of $T(\Gamma)$. Fixing a basis in $B_2(\Gamma)$, we obtain a corresponding bounded domain in $\mathbb{C}^m$ and then the coordinatewise convergence in $\mathbb{C}^m$ is equivalent to the convergence of the elements φ in $B_2(\Gamma)$. In what follows, we will not distinguish these two domains and $T(\Gamma)$.

It is well-known (see, for example [7, 8]) that a holomorphic function f in Δ^* with the expansion $f(z)=z+a_1z^{-1}+\ldots$ is univalent in Δ^* if and only if

$$\left|\sum_{p,q=1}^{\infty} \sqrt{pq}\,\alpha_{pq}x_px_q\right| \leqq \sum_{p=1}^{\infty} |x_p|^2, \tag{1}$$

where α_{pq} are the Grunsky coefficients of the function f, which are defined from the expansion

$$\log\frac{f(z)-f(\zeta)}{z-\zeta} = -\sum_{p,q=1}^{\infty} \alpha_{pq}z^{-p}\zeta^{-q}$$

(a branch of logarithm is taken which is equal to zero for $z=\zeta=\infty$), and $x=(x_1,x_2,\ldots)$ is an arbitrary element from the complex Hilbert space l^2, i.e. $\sum|x_p|^2=\|x\|^2<\infty$.

For the functions with k-quasiconformal extension onto $\bar{\Delta}$ the inequality (1) can be easily made more precise, namely

$$\left|\sum_{p,q=1}^{\infty} \sqrt{pq}\,\alpha_{pq}x_px_q\right| \leqq k\|x\|^2 \tag{2}$$

(in fact the factor k in the right-hand part of (2) may be changed by a smaller one, see [4]). On the other hand, according to Pommerenke's result [8], the validity of (2) (for all $x\in l^2$) ensures k'-quasiconformal extension of f to $\hat{\mathbb{C}}$ with some $k'\geqq k$, which depends on k only.

Due to our assumed normalization, the function f and its Grunsky coefficients α_{pq} are determined by Schwarzian $\varphi=S_f$ uniquely, therefore one can write $\alpha_{pq}(\varphi)$ and f_φ. Here all $\alpha_{pq}(\varphi)$ are holomorphic with respect to φ in $T(\Gamma)$ and continuous on closure of $T(\Gamma)$.

It is sufficient to take in (1) the elements $x\in l^2$ with $\|x\|=1$, and that will be assumed further; then (1) becomes of the form

$$\left|\sum_{p,q=1}^{\infty} \sqrt{pq}\,\alpha_{pq}(\varphi)x_px_q\right| \leqq 1, \quad \varphi\in\overline{T(\Gamma)}. \tag{1'}$$

For a given $x=(x_1,x_2,\ldots)\in l^2$ with $\|x\|=1$ we set

$$h_x(\varphi)=\sum_{p,q=1}^{\infty} \sqrt{pq}\,\alpha_{pq}(\varphi)x_px_q$$

and let

$$u(\varphi)=\sup_{\|x\|=1} |h_x(\varphi)|.$$

Any function $h_x(\varphi)$ is holomorphic on $T(\Gamma)$.

Our main goal is to show that the function

$$\varrho(\varphi)=\sup_{\|x\|=1} \log|h_x(\varphi)|$$

satisfies the statement of the theorem. Obviously, that from (1') $\varrho(\varphi)\leqq 0$ on $\overline{T(\Gamma)}$ and $\varrho(0)=-\infty$.

(a) *Continuity of ϱ on $T(\Gamma)$*. Let us notice, first of all, that by known results of complex analysis (based on the connection of the Grunsky matrice with Fredholm eigenvalues of the boundary curve $f(\partial\Delta)$) it follows only that ϱ is *lower*, but not upper semicontinuous (see [11, 5, 10]). But we need not use the upper semicontinuous regularization $\varrho^*(\varphi)=\limsup_{\psi\to\varphi}\varrho(\psi)$, because of the specific features of the finite-dimensional case give a possibility to establish the continuity of ϱ.

It is sufficient to establish the continuity of the function $u(\varphi)$ on $T(\Gamma)$. Let us denote

$$L_\varphi=f_\varphi(|z|=1), \quad G_\varphi^*=f_\varphi(\Delta^*)=\operatorname{ext}L_\varphi, \quad G_\varphi=\hat{\mathbb{C}}\setminus\overline{G^*}=\operatorname{int}L_\varphi .$$

Consider for the curve L_φ the smallest positive Fredholm eigenvalue λ_φ, which is defined by the equality

$$\frac{1}{\lambda_\varphi}=\sup\frac{|D_{G_\varphi}(h)-D_{G_\varphi^*}(h)|}{D_{G_\varphi}(h)+D_{G_\varphi^*}(h)}, \tag{3}$$

where $D_G(h)=\iint_G(|h_x|^2+|h_y|^2)dxdy$ is the Dirichlet integral and supremum is taken over all functions h continuous in $\hat{\mathbb{C}}$ and harmonic in $\hat{\mathbb{C}}\setminus L_\varphi$, with $0<D_{G_\varphi}(h)+D_{G_\varphi^*}(h)<\infty$; we denote the family of such functions by H_φ. (In the case of smooth curve that is an eigenvalue of homogeneus integral equation with the double-layer potential over L_φ, and $\lambda_\varphi=\infty$ for a circle.)

Due to known Schiffer-Kühnau result [10, 5], λ_φ is connected with $u(\varphi)$ by the equality

$$1/\lambda_\varphi=u(\varphi), \tag{4}$$

and we will prove the continuity of λ_φ. For this we will use certain considerations employed in Schober's paper [11].

Let us associate with the curve L_φ also the functional

$$\varkappa(G,G^*)=\sup_{h\in H_\varphi}\frac{D_{G_\varphi}(h)}{D_{G_\varphi^*}(h)},$$

then, by virtue of (3) and (4),

$$u(\varphi)=\frac{\max\{\varkappa(G_\varphi,G_\varphi^*),\varkappa(G_\varphi^*,G_\varphi)\}-1}{\max\{\varkappa(G_\varphi,G_\varphi^*),\varkappa(G_\varphi^*,G_\varphi)\}+1}.$$

We prove that for any sequence $\{\varphi_n\}\subset T(\Gamma)$, which converges to $\varphi_0\in T(\Gamma)$, the equality

$$\lim_{n\to\infty}u(\varphi_n)=u(\varphi_0)$$

holds. It is sufficient to establish that

$$\lim_{n\to\infty}\varkappa(G_n,G_n^*)=\varkappa(G_0,G_0^*), \qquad \lim_{n\to\infty}\varkappa(G_n^*,G_n)=\varkappa(G_0^*,G_0) \tag{5}$$

(the continuity of the functional $\varkappa$), where, for the sake of simplicity of notations, it is denoted $G_n=G_{\varphi_n}$, $G_n^*=G_{\varphi_n}^*$. Analogously, let be $L_n=L_{\varphi_n}$, $H_n=H_{\varphi_n}$, $f_n=f_{\varphi_n}$ $(n=0,1,2,\dots)$.

Consider the mappings

$$g_n^*=f_n\circ f_0^{-1}:G_0^*\to G_n^*, \qquad n=1,2,\dots,$$

which are conformal in G_0^*, and take their Teichmüller extremal extensions $\hat{g}_n^*$ to whole sphere $\hat{\mathbb{C}}$, compatible with $\Gamma_0=f_0\Gamma f_0^{-1}$ (on which the Teichmüller distance in $T(\Gamma)$ between φ_n and φ_0 is realized), $\hat{g}_n^*|G_0 = g_n$. Then their quasiconformality coefficients (maximal dilatations)

$$\varkappa(\hat{g}_n^*) = \|(|\partial_z \hat{g}_n^*| + |\partial_{\bar{z}} \hat{g}_n^*|)/(|\partial_z \hat{g}_n^*| - |\partial_{\bar{z}} \hat{g}_n^*|)\|_\infty \equiv \varkappa_n^*,$$

where, as usually, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ tend to 1 for $n \to \infty$.

Now fix $h_n \in H_n$ and define h_0 as $h_n \circ g_n^*$ in $\bar{G}_0^*$ and as harmonic extension of $h_n \circ g_n^*|L_0$ into G_0. Then $h_0 \in H_0$ and, due to conformality of g_n^* in G_0^*, we have $D_{G_0^*}(h_0) = D_{G_n^*}(h_n)$. Further, by the Dirichlet principle, $D_{G_0}(h_0) \leqq D_{G_0}(h_n \circ \hat{g}_n^*)$, and on the other hand, the quasiinvariance of the Dirichlet integral under quasiconformal mappings gives that $D_{D_0}(h_n \circ \hat{g}_n^*) \leqq K_n^* D_{G_n}(h_n)$. Hence we obtain

$$\varkappa(G_0^*, G_0) \geqq \frac{D_{G_0^*}(h_0)}{D_{G_0}(h_0)} = \frac{D_{G_n^*}(h_n)}{D_{G_0}(h_0)} \geqq \frac{D_{G_n^*}(h_n)}{D_{G_0}(h_n \circ g_n^*)} \geqq \frac{D_{G_n^*}(h_n)}{K_n^* D_{G_n}(h_n)}$$

and, consequently, for any $n = 1, 2, \ldots$

$$\varkappa(G_0^*, G_0) \geqq \frac{1}{K_n^*} \varkappa(G_n^*, G_n),$$

and since we have $\lim_{n \to \infty} K_n^* = 1$, we get

$$\varkappa(G_0^*, G_0) \geqq \limsup_{n \to \infty} \varkappa(G_n^*, G_n). \tag{6}$$

We now establish that

$$\liminf_{n \to \infty} \varkappa(G_n^*, G_n) \geqq \varkappa(G_0^*, G_0), \tag{7}$$

and that together with (6) gives us the second equality from (5).

Taking into account the definition of the functional $\varkappa$, we choose for a given $\varepsilon > 0$ a function $h_0 \in H_0$ such that

$$\frac{D_{G_0^*}(h_0)}{D_{G_0}(h_0)} > \varkappa(G_0^*, G_0) - \varepsilon.$$

Then we transform h_0, by analogy with the proof of the inequality (6), into the functions $h_n \in H_n$, setting h_n to be equal $h_0 \circ \hat{g}_n^{*-1}$ in $\bar{G}_n^*$ and in G_n be equal to harmonic function with boundary values $h_0 \circ g_n^{*-1}$ on L_n. Then we have

$$D_{G_n^*}(h_n) = D_{G_0^*}(h_0) \quad \text{and} \quad D_{G_n}(h_n) \leqq D_{G_n}(h_0 \circ \hat{g}_n^{*-1}) \leqq K_n^* D_{G_0}(h_0).$$

That gives now

$$\varkappa(G_n^*, G_n) \geqq \frac{D_{G_n^*}(h_n)}{D_{G_n}(h_n)} = \frac{D_{G_0^*}(h_0)}{D_{G_n}(h_n)} \geqq \frac{D_{G_0^*}(h_0)}{K_n^* D_{G_0}(h_0)}$$

$$\geqq \frac{1}{K_n^*} (\varkappa(G_0^*, G_0) - \varepsilon)$$

and, consequently,

$$\liminf_{n \to \infty} \varkappa(G_n^*, G_n) = \varkappa(G_0^*, G_0) - \varepsilon.$$

From this, since ε is arbitrary, it follows the validity of (7).

To obtain analogous relations for $\varkappa(G_n, G_n^*)$ we must construct the quasiconformal extensions $\hat{g}_n$ of conformal mappings from G_0 onto G_n with $K(\hat{g}_n) \to 1$ for $n \to \infty$. In compare with the mappings g_n^* considered above in this case a difference arises caused by the fact that any quasi-Fuchsian group

$$\Gamma_n = f_n \Gamma f_n^{-1} \qquad (n=0,1,2,\ldots)$$

(with invariant quasicircle L_n) uniformized two Riemann surfaces G_n/Γ_n and G_n^*/Γ_n, which are different if $f_n \neq \mathrm{id}$; we shall consider these surfaces as marked surfaces, i.e. shall distinguish their canonical dissections. While all the surfaces G_n^*/Γ_n are conformally equivalent, for G_n/Γ_n it is not true, and thus we need to exploit the another arguments.

It will be more convenient for us to pass from Δ and Δ^* to upper and lower halfplanes $U=\{\mathrm{Im}\, z>0\}$ and $U^*=\{\mathrm{Im}\, z<0\}$ correspondendly, using the transformation

$$\tau: z \mapsto i(1+z)/(1-z).$$

To avoid the complication of the notations, we will preserve for the corresponding group in $U \cup U^*$ and for the mappings $U^* \to G_n^*$ instead of $\tau\Gamma\tau^{-1}$ and $f_n \circ \tau^{-1}$ the above notations Γ and f_n.

Let us take a fixed canonical system of generators $\gamma_1, \ldots, \gamma_r$, $r=2p+n$, of the group Γ; they satisfy the relation

$$\prod_{j=1}^{p} [\gamma_j, \gamma_{p+j}] \prod_{j=2p+1}^{r} \gamma_j = 1,$$

where $\gamma_1, \ldots, \gamma_{2p}$ are hyperbolic and $\gamma_{2p+1}, \ldots, \gamma_r$ are parabolic elements (it is possible that $p=0$ or $n=0$), and define a canonical dissection of the surface U^*/Γ (and of U/Γ). The image of this system in Γ_n is the system of generators

$$\gamma_{f_n,1} = f_n \circ \gamma_1 \circ f_n^{-1}, \ldots, \gamma_{f_n,r} = f_n \circ \gamma_r \circ f_n^{-1}. \tag{8}$$

Consider the Fuchsian equivalents $\Sigma_n = w_n^{-1} \Gamma_n w_n$ of the groups Γ_n in U, where w_n are conformal mappings from U onto G_n; we normalize these mappings by conditions

$$w_n(i)=0, \quad w_n'(i)>0, \quad n=0,1,2,\ldots.$$

Now let

$$g_n = w_n \circ w_0^{-1} : G_0 \to G_n \tag{9}$$

and notice that w_n are homeomorphic in $\bar{U}$ and g_n homeomorphic in $\bar{G}_0$, also

$$\lim_{n\to\infty} \|w_n - w_0\|_{C(\bar{U})} = 0, \qquad \lim_{n\to\infty} \|g_n - g_0\|_{C(\bar{G}_0)} = 0. \tag{10}$$

In the group Σ_n the system of generators

$$\sigma_{n1} = w_n^{-1} \circ \gamma_{f_n,1} \circ w_n, \ldots, \qquad \sigma_{nr} = w_n^{-1} \circ \gamma_{f_n,r} \circ w_n \tag{11}$$

corresponds to the system (8).

Assume that the entries of all linear-fractional automorphisms of $\hat{\mathbb{C}}$ considered here are choosen so that determinants are equal to one and the traces are positive; for example, let

$$\gamma_j = \frac{a_j z + b_j}{c_j z + d_j}, \quad a_j d_j - b_j c_j = 1, \quad a_j + d_j > 0,$$

$$\sigma_{nj} = \frac{a_{nj} z + b_{nj}}{c_{nj} z + d_{nj}}, \quad a_{nj} d_{nj} - b_{nj} c_{nj} = 1, \quad a_{nj} + d_{nj} > 0$$

$$(j=1,\ldots,r).$$

We need to construct the quasiconformal homeomorphisms $t_n: U^*/\Gamma \to U^*/\Sigma_n$ (with $K(t_n)\to 1$ for $n\to\infty$) which realize geometrically the isomorphisms $t_n^*: \Gamma \to \Sigma_n$, and which are defined on the generators (8), (11) by

$$t_n^*(\gamma_j)=\sigma_{nj}, \quad j=1,\dots,r. \tag{12}$$

Geometrically that means the marking of the surfaces U^*/Γ and U^*/Σ_n or, equivalently, the choice of a homotopic class of homeomorphisms.

The existence of such quasiconformal homeomorphisms follows from the general theory of Teichmüller spaces, and the procedure of their construction is well-known (see, for example [1]). The groups Γ and Σ_n are normalized as needed, then for $p>0$ the quantities

$$a_1, b_1, c_1, \dots, a_{2p-2}, b_{2p-2}, c_{2p-2}, a_{2p+1}, c_{2p+1}, \dots, a_r, c_r; \tag{13}$$

$$a_{n1}, b_{n1}, c_{n1}, \dots, a_{n,2p-2}, b_{n,2p-2}, c_{n,2p-2}, a_{n,2p+1}, c_{n,2p+1}, \dots, a_{nr}, c_{nr} \tag{14}$$

and, for $p=0$ (and $n\geqq 3$) the quantities

$$a_{2p+1}, c_{2p+1}, \dots, a_r, c_r; \tag{13'}$$

$$a_{n,2p+1}, c_{n,2p+1}, \dots, a_{nr}, c_{nr} \tag{14'}$$

are taken as *Fricke coordinates* of the groups Γ and Σ_n $(n=0,1,\dots)$. Using these coordinates one can construct for any isomorphism t_n^* the Teichmüller extremal quasiconformal homeomorphism t_n which conjugates Γ and Σ_n, i.e., a mapping from U''/Γ onto U^*/Σ_n compatible with given markings of these surfaces. By virtue of (10) these Teichmüller mappings and their quasiconformality coefficients depend continuously from the coordinates (14) or (14').

On the other hand, as is well-known also (see [2, Chap. YI.A]), a choosing of a homotopic class of homeomorphisms $t_n: U^*/\Gamma \to U^*/\Sigma_n$ (modulo ideal boundary) is equivalent to the fixing of boundary values of the liftings $\tilde{t}_n$ to U^* of these homeomorphisms.

As a result, we get for any $n=1,2,\dots$ two mappings: conformal $w_n: U\to G_n$ and quasiconformal $f_n\circ \tilde{t}_n^{-1}: U^*\to G_n^*$, which are glued continuously on the unit circle, and hence these mappings form an unit quasiconformal automorphism of the whole sphere $\hat{\mathbb{C}}$.

It follows from the above also, that the extremal Teichmüller extension $\hat{g}_n$ to $\hat{\mathbb{C}}$ of the mapping g_n from (9), that is conformal in G_0, has quasiconformality coefficient $K(\hat{g}_n)$ for which $\lim_{n\to\infty} K(\hat{g}_n)=1$.

We may now consider functionals $\varkappa(G_n, G_n^*)$ and obtain for them the relations, analogous to (6), (7). Thus we state the validity of the first equality from (5) also. It completes the proof of continuity of the functions u and ϱ. Notice, that $\lim_{\varphi\to 0} \varrho(\varphi)=-\infty=\varrho(0)$.

We pass to proof of the remaining conclusions of the theorem.

(b) The *plurisubharmonity* of ϱ in $T(\Gamma)$ is now trivial, because it is the envelope $\sup_{\|x\|=1} \log|h_x(\varphi)|$ of plurisubharmonic functions and is upper semicontinuous.

(c) Let us show that

$$\varrho(\varphi)\to 0 \quad \text{for} \quad \varphi\to T(\Gamma) \quad (\varphi\in R(\Gamma)). \tag{15}$$

Assume that there exists a sequence $\{\varphi_n\}\subset T(\Gamma)$ which converges to a boundary point φ_0, such that

$$\lim_{n\to\infty} \varrho(\varphi_n)=a<0$$

(one can obtain this by passing to a subsequence). Then, taking $a_0 \in (a, 0)$ for $n \geqq n_0(a_0)$, we have $\varrho(\varphi_n) \leqq a_0 < 0$. Hence, for such n,

$$\sup_{\|x\|=1} |h_x(\varphi_n)| \leqq e^{a_0} < 1,$$

and then, using the Pommerenke's result mentioned after (2), we conclude that the correspondend functions f_{φ_n} admit k-quasiconformal extensions $\hat{f}_{\varphi_n}$ to the whole sphere $\hat{\mathbb{C}}$ with $k<1$, which depends only from a_0. From this by virtue of the compactness of the class of normalized k-quasiconformal automorphisms of the plane and since φ_n converge to φ_0, it follows immediately that the limit function $f_{\varphi_0}: \Delta^* \to \hat{\mathbb{C}}$ must be also a restriction to Δ^* of some k-quasiconformal automorphism of $\hat{\mathbb{C}}$, but it is impossible, because φ_0 is, by assumption, a boundary point of $T(\Gamma)$ and, consequently, $\partial f_{\varphi_0}(\Delta^*)$ can not be a quasicircle. The derivered contradiction proves (15).

(d) It remains to remark that after biholomorphic imbedding of $T(\Gamma)$ into $B_2(\Gamma)$, the Kleinian groups $f_{\varphi_0} \Gamma f_{\varphi_0}^{-1}$ with one invariant simply connected component $f_{\varphi_0}(\Delta^*)$ corresponding by the Bers-Maskit theorem [3, 6] to the boundary points $\varphi_0 \in \partial T(\Gamma)$, the functions f_{φ_0} are also univalent in Δ^*, so $\varrho(\varphi_0) \leqq 0$ is defined. Analogously to (c) we obtain that the case $\varrho(\varphi_0) < 0$ is excluded, so $\varrho(\varphi_0) = 0$ in all boundary points φ_0. Together with (15) this gives the continuity of ϱ on the boundary of $T(\Gamma)$ also. The theorem is proved completely.

The constructed function ϱ is continuous as a mapping $T(p, n) \to \mathbb{R}_- \cup \{-\infty\}$; it is equal to $-\infty$ in a single point from $T(p, n)$ only.

It is interesting to know, whether $\varrho(x)$ is maximal among all nonpositive plurisubharmonic functions on $T(\Gamma)$ with given logarithmic singularity at the origin and equal to zero on the boundary of $T(\Gamma)$.

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