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A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map

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Let G be a Lie group, $\mathcal{G}$ – its Lie algebra, $Z \subset G$ – a smooth hypersurface. By the (left) Gauss map of Z we shall mean the map $\gamma_Z : Z \to \mathbb{P}(\mathcal{G}^*)$ which takes $z \in Z$ to the left translation to unity of the hyperplane $T_z Z \subset T_z G$. (see Definition 1.2 below). We shall be interested in the case when G is a complex *algebraic* group and Z is an *algebraic* hypersurface. We shall consider as well the case of singular hypersurfaces Z. Then, of course, the map γ_Z will be defined only on the set of smooth points of Z; in other words, γ_Z will be a *rational* map from Z to $\mathbb{P}(\mathcal{G}^*)$. Note that $\dim Z = \dim \mathbb{P}(\mathcal{G}^*)$. Thus we have the following natural problem.

Problem 0.1. Classify algebraic hypersurfaces $Z \subset G$ such that $\gamma_Z : Z \dashrightarrow \mathbb{P}(\mathcal{G}^*)$ is a birational isomorphism.

The present paper is devoted to the solution of this problem for the case when G is an algebraic torus $(\mathbb{C}^*)^m$. It turns out (see Theorem 1.3 below) that equations of such hypersurfaces are exactly (up to killing inessential quasi-homogeneities) A-discriminants, introduced and studied in [7–10]. The class of A-discriminants includes classical discriminants and resultants of forms in several variables, determinants of square matrices, hyperdeterminants of multi-dimensional matrices [7]. In papers [8–10] Newton polytopes of A-discriminants were described. The Newton polytope of a (Laurent) polynomial $F(x_1, \ldots, x_m)$, see [8] is not an intrinsic invariant of the hypersurface $\{F=0\}$, but an extrinsic invariant of its embedding into the algebraic torus $(\mathbb{C}^*)^m$. So is the Gauss map. Therefore, non-trivial structure of the Newton polytope for A-discriminants reflects non-trivial way of embedding of corresponding hypersurfaces into the torus.

The Gauss map for hypersurfaces in $(\mathbb{C}^*)^m$ will be called the logarithmic Gauss map, since explicit formulas for it involve logarithmic derivatives. Let us give some examples of hypersurfaces $Z \subset (\mathbb{C}^*)^m$ with birational γ_Z, provided by Theorem 1.3.

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Example 0.2. a) The hypersurface $Z \subset (\mathbb{C}^*)^{n^2}$ consisting of $(n \times n)$-matrices $\|a_{ij}\|$, $a_{ij} \in \mathbb{C}^*$ such that

$$\det \begin{bmatrix} 1 & 1 \cdots 1 \\ 1 & a_{11} \cdots a_{1n} \\ 1 & a_{n1} \cdots a_{nn} \end{bmatrix} = 0,$$

has birational logarithmic Gauss map.

b) The hypersurface $Z \subset (\mathbb{C}^*)^{n-1}$ consisting of $(a_1, \ldots, a_{n-1})$ such that the polynomial $x^n + x^{n-1} + a_1 x^{n-2} + \ldots + a_{n-1}$ has a multiple root, posesses birational logarithmic Gauss map.

The proof of Theorem 1.3 is based on ideas of the paper [1] by Horn published in 1889. The point is that A-discriminantal hypersurfaces are singularity sets for general A-hypergeometric functions [5], [6], [8]. As shown in [6], there is a basis in the space of A-hypergeometric functions consisting of so called Γ-series which are in fact, hypergeometric series in the sense of Horn. More precisely, Horn [1] has called a power series $\Sigma C_{v_1, \ldots, v_m} x_1^{v_1} \ldots x_m^{v_m}$ *hypergeometric* if the ratios

$$R_i(v_1, \ldots, v_m) = \frac{C_{v_1, \ldots, v_{i-1}, v_i + 1, v_{i+1}, \ldots, v_m}}{C_{v_1, \ldots, v_m}}$$

are rational functions in $v_1, \ldots, v_m$. It is known that power series converge "up to first singularity". On the other hand, knowledge of the functions R_i enables one to determine the growth of coefficients and hence the convergence radius in any given direction. This yields an uniformisation of the set of singularities which we call Horn uniformisation (see [1]; a brief exposition of Horn's theory can be found in [4], Sect. 5.7). The cocycle conditions (2.3) satisfied by R_i imply that the Horn uniformisation is inverse to the logarithmic Gauss map (see Remark 2.4 below).

So, the present paper could be written a hundred years ago. It relies heavily on [1]. The reason for presenting this "group-theoretical" characterization of A-discriminants is its simplicity and the fact that it suggests non-commutative generalizations, where the torus is replaced by an arbitrary reductive algebraic group G. The study of "non-commutative discriminants" i.e., solution of problem 0.1 should perhaps lead to some non-commutative generalization of hypergeometric functions.

This paper arose as a by-product of the work on general hypergeometric functions and discriminants carried on by Gelfand, Zelevinsky and the author [5–10]. The author is grateful to Gelfand and Zelevinsky for introducing him into this beautiful subject.

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1 *A*-discriminantal hypersurfaces and main theorem

Let $A \subset \mathbb{Z}^{n-1}$ be a finite subset, $N = |A|$. We suppose that A affinely generates over $\mathbb{Z}$ the lattice $\mathbb{Z}^{n-1}$. To each point $\omega = (\omega_1, \ldots, \omega_{n-1}) \in \mathbb{Z}^{n-1}$ we associate the Laurent monomial $x^\omega = x_1^{\omega_1} \ldots x_{n-1}^{\omega_{n-1}}$ in $n-1$ variables. Denote by $\mathbb{C}^A$ the space of Laurent polynomials of the form $f(x)=\sum_{\omega\in A} a_\omega x^\omega$, $a_\omega\in\mathbb{C}$. Consider the subvariety

$$\nabla_A^0=\{f\in\mathbb{C}^A:\exists x\in(\mathbb{C}^*)^{n-1} \text{ s.t. } f(x)=d_xf=0\}\subset\mathbb{C}^A$$

of polynomials which have a critical point with critical value zero. Let ∇_A be the closure of ∇_A^0. The algebraic variety ∇_A is called the A-discriminantal variety. In the case when ∇_A is a hypersurface, the equation of ∇_A is called the A-discriminant and denoted Δ_A (see [7]–[10]).

The torus $(\mathbb{C}^*)^n=\{(\lambda_1,\ldots,\lambda_n)\in\mathbb{C}^n:\lambda_i\neq 0\}$ acts on $\mathbb{C}^A$ by the formula

$$((\lambda_1,\ldots,\lambda_n)f)(x_1,\ldots,x_{n-1})=\lambda_nf(\lambda_1x_1,\ldots,\lambda_{n-1}x_{n-1}).$$

preserving ∇_A. This action comes from a homomorphism of tori $\varphi:(\mathbb{C}^*)^n\to(\mathbb{C}^*)^A$ $\varphi(\lambda_1,\ldots,\lambda_n)_\omega=\lambda_n\lambda_1^{\omega_1}\ldots\lambda_{n-1}^{\omega_{n-1}}$. Let $\varphi^*:\mathbb{Z}^A\to\mathbb{Z}^n$ be the dual homomorphism of character lattices, and denote $L_A=\operatorname{Ker}\varphi^*\subset\mathbb{Z}^A$. The rank of L_A equals $N-n$, where $N=|A|$.

Lemma 1.1. *L_A is the lattice of all affine relations among elements of A i.e., of families $(a_\omega)_{\omega\in A}$, $a_\omega\in\mathbb{Z}$ such that $\sum_{\omega\in A} a_\omega\omega=0$, $\Sigma a_\omega=0$.*

Let $T(L_A)=\operatorname{Spec}\mathbb{C}[L_A]=\operatorname{Hom}(L_A,\mathbb{C}^*)$ be the algebraic torus whose character lattice is L_A. Thus we have an exact sequence of tori

$$1\to(\mathbb{C}^*)^n\xrightarrow[\varphi]{}(\mathbb{C}^*)^A\xrightarrow[p]{}T(L_A)\to 1\,. \tag{1.1}$$

The variety $\nabla_A\cap(\mathbb{C}^*)^A$, being $(\mathbb{C}^*)^n$-invariant, has the form $p^{-1}(\tilde\nabla_A)$ for some subvariety $\tilde\nabla_A\subset T(L_A)$, which is uniquely defined. The subvariety $\tilde\nabla_A$ will be called the *reduced* A-discriminantal variety. Clearly, $\operatorname{codim}\nabla_A=\operatorname{codim}\tilde\nabla_A$. We shall be only interested in the case when ∇_A and $\tilde\nabla_A$ are hypersurfaces.

Definition 1.2. Let G be an algebraic group with Lie algebra $\mathscr{G}$, $Z\subset G$ be an algebraic hypersurface. For each $g\in G$ let $l_g:h\to gh$ be the left translation by g. The left Gauss map of the hypersurface Z is the rational map $\gamma_Z:Z\to\mathbb{P}(\mathscr{G}^*)$, taking a smooth point $z\in Z$ to the hyperplane $d(l_z^{-1})(T_zZ)\subset T_eG=\mathscr{G}$.

Now we state our main result.

Theorem 1.3. *Let $G=(\mathbb{C}^*)^m$ be an algebraic torus, $Z\subset G$ – an algebraic irreducible hypersurface. The Gauss* map *$\gamma_Z:Z\to\mathbb{P}^{m-1}$ is birational if an only if there exist $n>0$, a finite subset $A\subset\mathbb{Z}^{n-1}$ as above and an isomorphism of tori $G\to T(L_A)$ taking Z to the reduced A-discriminantal hypersurface $\tilde\nabla_A$.*

The part "if" will be proved in Sect. 2, part "only if" – in Sect. 3.

2 The Horn uniformisation

We conserve the notations of Sect. 1. Let us number elements of A: $A=\{\omega^{(1)},\ldots,\omega^{(N)}\}$, where each $\omega^{(j)}$ is $(\omega_1^{(j)},\ldots,\omega_{n-1}^{(j)})\in\mathbb{Z}^{n-1}$. Correspondingly identify $\mathbb{Z}^A$ and $\mathbb{C}^A$ with $\mathbb{Z}^N$ and $\mathbb{C}^N$. Put $m=N-n$. Choose a $\mathbb{Z}$-basis $\{a^{(1)},\ldots,a^{(m)}\}$ of the lattice L_A, where each $a^{(p)}$ is $(a_1^{(p)},\ldots,a_N^{(p)})\in\mathbb{Z}^N$. Thus the torus $T(L_A)$ is identified with $(\mathbb{C}^*)^m$. Define a rational map $h:\mathbb{P}^{m-1}\to(\mathbb{C}^*)^m$ taking a point with homogeneous coordinates $\lambda=(\lambda_1:\ldots:\lambda_m)$ to $(\Psi_1(\lambda),\ldots,\Psi_m(\lambda))$, where

$$\Psi_k(\lambda_1,\ldots,\lambda_m)=\prod_{j=1}^N\left(\sum_{p=1}^m a_j^{(p)}\lambda_p\right)^{a_j^{(k)}}. \tag{2.1}$$

Since for each $a=(a_1,\dots,a_n)\in L_A$ we have $\Sigma a_j=0$, each Ψ_k is homogeneous of degree 0. We cal the map h the Horn uniformisation.

Theorem 2.1. a) *Under the identification* $(\mathbb{C}^*)^m \simeq T(L_A)$ *defined by the chosen base of* L_A, *the image of* $h:\mathbb{P}^{m-1}\to(\mathbb{C}^*)^m$ *is identified with the reduced A-discriminantal variety* $\tilde{\nabla}_A\subset T(L_A)$.

b) *If* $\tilde{\nabla}_A$ *is a hypersurface, then* $h:\mathbb{P}^{m-1}\to\tilde{\nabla}_A$ *is a birational isomorphism. The inverse to h coincides with the logarithmic Gaus* map $\gamma_{\tilde{\nabla}_A}$.

Proof. Let us define h in more invariant terms. Consider the map (rational)

$$h':L_A\otimes\mathbb{C}\to\mathrm{Hom}(L,\mathbb{C}^*)=T(L)$$

such that for $(a_1,\dots,a_n)\in L_A\otimes\mathbb{C}$, $a_j\in\mathbb{C}$ and $(b_1,\dots,b_n)\in L_A$, $b_j\in\mathbb{Z}$ we have

$$h'(a_1,\dots,a_n)(b_1,\dots,b_n)=\prod_{j=1}^{N} a_j^{b_j}.$$

Since $\Sigma b_j=0$ for each $b\in L_A$, h' descends to a rational map $\mathbb{P}(L_A\otimes\mathbb{C})\to T(L_A)$ which we also denote h'.

Lemma 2.2. *Under the identification* $L_A\otimes\mathbb{C}\to\mathbb{C}^m$ *defined by the chosen basis,* h' *is identified with h.* □

Now consider a commutative diagram

$$\begin{array}{ccc} L_A\otimes\mathbb{C} & \xrightarrow{h} & T(L_A)=\mathrm{Hom}(L_A,\mathbb{C}^*) \\ \downarrow & & \uparrow \\ \mathbb{C}^A=\mathbb{Z}^A\otimes\mathbb{C} & \xrightarrow[H]{} & (\mathbb{C}^*)^A=\mathrm{Hom}(\mathbb{Z}^A,\mathbb{C}^*), \end{array}$$

where H is given by the same formula as h': $H(a_1,\dots,a_n)(b_1,\dots,b_n)=\Pi a_j^{b_j}$ for each $(a_1,\dots,a_n)\in\mathbb{C}^A$, $(b_1,\dots,b_n)\in\mathbb{Z}^A$.

Consider the action of $(\mathbb{C}^*)^n$ on $V=(\mathbb{C}^A)^*$ dual to (1.1). Let Y be the closure of the orbit of the point $e=(1,\dots,1)$ under this action. This is a conical (i.e., $\mathbb{C}^*$-invariant) subvariety in $(\mathbb{C}^A)^*=V$. It is easy to see (and shown in [7]) that $\nabla_A\subset V^*$ is the conical variety projectively dual to Y; see [11]. Consider the diagram of rational maps

$$\check{Y}=\nabla_A\xleftarrow{\alpha}\mathbb{P}(N^*_{Y/V})\xrightarrow{\beta}Y,$$

where $N^*_{Y/V}$ is the conormal bundle to Y. If $\check{Y}$ is a hypersurface, α is birational (cf. [11]).

Also consider maps

$$N^*_{Y/V}\xleftarrow{\varepsilon}(\mathbb{C}^*)^n\times(L_A\otimes\mathbb{C})\xrightarrow{\delta}\mathbb{C}^A,$$

where ε is the extension by $(\mathbb{C}^*)^n$-equivariance of the identification $L_A\otimes\mathbb{C}\simeq N^*_{Y/V}$, and δ is induced by the action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^A\supset L_A\otimes\mathbb{C}$ (see [5]). It is clear that ε is a birational isomorphism, and the image of δ is ∇_A (this is the first definition of the singularity set of A-hypergeometric system from [5]). Part a) and the first half of part b) of Theorem 2.1 follow, therefore from the following lemma which we leave to the reader.

Lemma 2.3. *The diagram*

$$\begin{array}{ccccc} & & \mathbb{C}^A & \xrightarrow{H} & \mathbb{C}^A \\ & \overset{\delta}{\nearrow} & & & \uparrow \\ (\mathbb{C}^*)^n \otimes (L_A \otimes \mathbb{C}) & & & & \\ & \underset{\varepsilon}{\searrow} & & & \\ & & N^*_{Y/V} & \xrightarrow{\alpha} & \tilde{V}_A \end{array}$$

is commutative. □

The second half of part b) of Theorem 2.1 follows from the identities

$$\frac{\partial \log \Psi_i}{\partial \lambda_j} = \frac{\partial \log \Psi_j}{\partial \lambda_i}, \qquad \forall i, j = 1, \ldots, m \tag{2.2}$$

which can be verified immediately from formula (2.1). Having established (2.2), consider the Jacobian matrix $J(\lambda_1, \ldots, \lambda_m) = \|\partial \log \Psi_i / \partial \lambda_j\|$. Since Ψ_j and hence $\log \Psi_j$ are homogeneous of degree 0, we have

$$\sum_{i=1}^{m} \lambda_i \frac{\partial \log \Psi_j}{\partial \lambda_i} = 0, \qquad \forall j = 1, \ldots, m.$$

In other words, the matrix $J(\lambda_1, \ldots, \lambda_m)$ annihilates the vector $(\lambda_1, \ldots, \lambda_m)^t$. Since the image of h is a hypersurface, the kernel of $J(\lambda_1, \ldots, \lambda_m)$ for generic $(\lambda_1, \ldots, \lambda_m)$ is one-dimensional and hence generated by $(\lambda_1, \ldots, \lambda_m)^t$.

On the other hand, the matrix J is symmetric by (2.2). Therefore $\operatorname{Im} J(\lambda)$ is the orthogonal complement $\left(\text{with respect to the form } \sum_{i=1}^{m} x_i^2\right)$ to $\operatorname{Ker} J(\lambda) = \mathbb{C} \cdot (\lambda_1, \ldots, \lambda_m)^t$. In other words, $\lambda_1, \ldots, \lambda_m$ are coefficients of the linear equation of the image of $J(L)$. This image is nothing but the tangent space to $\log \tilde{V}_A$ at the point $\log h(\lambda)$ (where by log we denote the logarithmic map from the torus $T(L_A)$ to its Lie algebra). Thus we have established that the logarithmic Gauss map is inverse to h. Theorem 2.1 is proved, as well as part "if" of Theorem 1.3.

Remark 2.4. Let $\delta = (\delta_1, \ldots, \delta_N) \in \mathbb{C}^N$ be a vector and $\phi_\delta(v) = \sum_{a \in L_A} \frac{v^{\delta + a}}{\prod_{j=1}^{N} \Gamma(\delta_j + a_j + 1)}$ be the corresponding Γ-series [5–6] in variables $v = (v_1, \ldots, v_N)$. Let $I \subset \{1, \ldots, N\}$ be a base i.e., a n-element subset such that points $\omega_j, j \in I$ are affinely independent. If $\delta_j \in \mathbb{Z}$ for $j \notin I$ then as shown in [6], $\phi_\delta(v)$ has a non-empty domain of convergence. Moreover, if the base I and the basis $\{a^{(1)}, \ldots, a^{(m)}\}$ of L_A are compatible in the sense of [6], then, introducing variables $x_p = v^{a^{(p)}}$ we can write $\phi_\delta(v)$ in the form $v^\theta \cdot \sum_{v \in \mathbb{Z}_+^m} c_v x^v$ where the latter series converges for small $|x_p|$. The series $\sum c_c x^v$ is hypergeometric series in the sense of Horn. Let $e_i \in \mathbb{Z}^m$ be the i-th standard basis vector. Introduce the notation $(x)_a = \Gamma(x + a) / \Gamma(x)$. Then

$$R_i(v) = c_{v + e_i} / c_v = \prod_{j=1}^{n} \left(1 + \delta_j + \sum_{p=1}^{m} a_j^{(p)} v_p\right)^{-1}_{a_j^{(i)}}$$

is a rational function in v. We have

$$\Psi_i(v_1, \ldots, v_m) = \lim_{t \to \infty} R_i(t v_1, \ldots, t v_m) \quad \text{for all} \quad v_i \in \mathbb{C}.$$

The relations (2.2) are obtained, by passing to the limit, from the cocycle conditions

$$R_i(v)R_j(v+e_i)=R_j(v)R_i(v+e_j) \tag{2.3}$$

which follow directly from the definition of R_i. The part "only if" of Theorem 1.3 which will be proved in the next section, can be seen as an infinitesimal analog of the classical theorem due to Birkeland and Ore [2, 3] which states that each rational solution of (2.3) is cohomologous (as a cocycle) to such where all irreducible factors of all R_i are linear polynomials.

Let us write explicity the primitive of the closed 1-form $\sum \log \Psi_i d\lambda_i$. Denote

$$f_j(\lambda)=\sum_{p=1}^{m} a_j^{(p)}\lambda_p$$

Proposition 2.5. *We have* $\log \Psi_i = \partial S/\partial \lambda_i$ *where*

$$S(\lambda)=\sum_{j=1}^{n} f_j(\lambda)\log f_j(\lambda)=\sum_{k=1}^{m}\lambda_k \log \Psi_k(\lambda).$$

Proof. The equality $\log\Psi_i=(\partial/\partial\lambda_i)(\sum \lambda_k \log \Psi_k)$ follows from (2.2) and the homogeneity conditions $\sum_k \lambda_k(\partial \Psi_i/\partial\lambda_k)=0$, $i=1,\dots,m$.

Further, we have from (2.1)

$$\sum_{k=1}^{m}\lambda_k\log\Psi_k=\sum_{k=1}^{m}\lambda_k\sum_{j=1}^{n}a_j^{(k)}\log f_j=\sum f_j \log f_j \text{ as required.} \quad \square$$

The function S can be called entropy for obvious reason. It is homogeneous of degree 1. Proposition 2.5 means that the hypersurface $\log \tilde{V}$ is projectively dual to the graph of S in $\mathbb{C}^{m+1}$ (whereas the A-discriminantal hypersurface $\tilde{V}$ itself is projectively dual to a toric subvariety in the projective space). Entropy – like expressions of the form $\sum a \log a$ or $\prod a^a$ are ubiquitous in the theory of discriminants (see, for example, the formula for boundary coefficients of principal A-determinant in [10], Theorem 3A.2). To find a conceptual (say, probabilistic) explanation of this fact is a challenging problem.

3 Part "only if" of Theorem 1.3

Let $Z\subset(\mathbb{C}^*)^m$ be hypersurface such that the logarithmic Gauss map $\varphi_Z: Z\to\mathbb{P}^{m-1}$ is birational. Denote by $\Psi:\mathbb{P}^{m-1}\to Z$ the inverse rational map. It is given by m rational functions $\Psi_i(\lambda_1,\dots,\lambda_m)$, $i=1,\dots,m$, homogeneous of degree 0. Let us prove the relations (2.2) for Ψ_j.

Let $J(\lambda_1,\dots,\lambda_m)$ denote the Jacobian Matrix $\|\partial\log\Psi_i/\partial\lambda_j\|$ (depending on λ). Denote by log the (multi-valued) map from $(\mathbb{C}^*)^m$ to $\mathbb{C}^m$ taking $(x_1,\dots,x_m)$ to $(\log x_1,\dots,\log x_m)$. The tangent space to $\log(Z)$ at the point $z=\log\Psi(\lambda)$ is the image of $J(\lambda)$, and the Kernel of $J(\lambda)$ contains the vector $(\lambda_1,\dots,\lambda_m)^t$, (due to homogeneity of $\Psi_i(\lambda)$).

The hyperplane $\operatorname{Im} J(\lambda)$ is determined by its linear equation i.e., a linear function $\mu_1\xi_1+\dots+\mu_m\xi_m$ vanishing for $(\xi_1,\dots,\xi_m)\in\operatorname{Im}J(\lambda)$.

To say that Ψ is inverse to the logarithmic Gauss map is equivalent to saying that for any $\lambda=(\lambda_1,\dots,\lambda_m)$ the above vector $\mu=(\mu_1,\dots,\mu_m)$ is proportional to λ. But

μ generates the 1-dimensional vector space $\operatorname{Ker} J^t(\lambda)$. Hence we have relations

$$\sum_{i=1}^{m} \lambda_i \frac{\partial \log \Psi_i}{\partial \lambda_k} = 0, \quad \forall k \tag{3.1}$$

$$\sum_{k=1}^{m} \lambda_k \frac{\partial \log \Psi_i}{\partial \lambda_k} = 0, \quad \forall i, \tag{3.2}$$

where (3.2) is the homogeneity condition. This means that $J(\lambda)$ and its transposed $J^t(\lambda)$ both annihilate $\lambda^t = (\lambda_1, \ldots, \lambda_m)^t$.

Let us prove the equality

$$\log \Psi_k = \frac{\partial}{\partial \lambda_k}\left(\sum_{i=1}^{m} \lambda_i \log \Psi_i\right) \tag{3.3}$$

(cf. Proposition 2.5).

Indeed, in the summands of (3.1) with $i \neq k$ we can interchange λ_i and $\partial/\partial\lambda_k$, and for $i = k$ we have

$$\frac{\partial}{\partial \lambda_k}(\lambda_k \log \Psi_k) = \lambda_k \frac{\partial \log \Psi_k}{\partial \lambda_k} + \log \Psi_k$$

whence (3.3). The Relations (2.2) follow from (3.3).

The following proposition is essentially due to Horn [1].

Proposition 3.1. *Suppose that rational functions $\Psi_i(\lambda_1, \ldots, \lambda_m)$, $i = 1, \ldots, m$ are homogeneous of degree 0 and such that $\frac{\partial \log \Psi_i}{\partial \lambda_j} = \frac{\partial \log \Psi_j}{\partial \lambda_i}$, $i, j = 1, \ldots, m$. Then there exist $N > 0$, an integral $(m \times N)$-matrix $\|a_i^{(p)}\|$, $i = 1, \ldots, N$, $p = 1, \ldots, m$ and constants $\xi_1, \ldots, \xi_m \in \mathbb{C}^*$ such that each $\xi_k \Psi_k$ has the form* (2.1).

Proof. Let $f_1, \ldots, f_N$ be all irreducible polynomials entering in the factorization of the numerator or denominator of at least one Ψ_i. Polynomials f_j are homogeneous. Put $d_j = \deg f_j$. So, we have

$$\Psi_k(\lambda_1, \ldots, \lambda_m) = \varepsilon_k \prod_{j=1}^{N} f_j(\lambda)^{a_{kj}}, \quad \varepsilon_k \in \mathbb{C}^*, \quad a_{kj} \in \mathbb{Z}.$$

Hence

$$\frac{\partial \log \Psi_k}{\partial \lambda_r} = \sum_{j=1}^{N} \frac{a_{kj}(\partial f_j/\partial \lambda_r)}{f_j} = \frac{\partial \log \Psi_r}{\partial \lambda_k} = \sum_{j=1}^{N} \frac{a_{rj}(\partial f_j/\partial \lambda_k)}{f_j}. \tag{3.4}$$

In each fraction in each of the sums in (3.4) the numerator has the degree less than that of the denominator. Hence

$$a_{kj} \frac{\partial f_j}{\partial \lambda_r} = a_{rj} \frac{\partial f_j}{\partial \lambda_k}, \quad \forall k = 1, \ldots, m, \quad j = 1, \ldots, N. \tag{3.5}$$

In other words, all (2×2) – minors of the $(2 \times m)$ matrix

$$\begin{bmatrix} a_{1j}, & a_{2j}, \ldots, a_{mj} \\ \dfrac{\partial f_j}{\partial \lambda_1}, & \dfrac{\partial f_j}{\partial \lambda_2}, \ldots, \dfrac{\partial f_j}{\partial \lambda_m} \end{bmatrix} \quad \text{vanish for each } j.$$

Thus the numerical vector $(a_{1j}, \ldots, a_{mj})$ is proportional (with the coefficient from the field $\mathbb{C}(\lambda_1, \ldots, \lambda_m)$ of rational functions) to the vector of polynomials $\left(\frac{\partial f_j}{\partial \lambda_1}, \ldots, \frac{\partial f_j}{\partial \lambda_m}\right)$. So, we have

$$\frac{\partial f_j}{\partial \lambda_p} = \delta_j(\lambda) a_{pj} \tag{3.6}$$

for some polynomials $\delta_j(\lambda)$. By homogenity of f_j we have

$$\sum_{p=1}^{m} \lambda_p \frac{\partial f_j}{\partial \lambda_p} = d_j f_j .$$

By (3.6) this means

$$\left(\sum \lambda_p a_{pj}\right) \delta_j(\lambda) = d_j f_j .$$

If $d_j > 1$ then $\deg \delta_j > 0$ and this contradicts the irreducibility of f_j. Hence $d_j = 1$ for all j. So f_j are homogeneous linear functions, say

$$f_j(\lambda_1, \ldots, \lambda_m) = \sum_{p=1}^{m} b_{pj} \lambda_p .$$

From (3.6) we find that $b_{pj} = \delta_j \cdot a_{pj}$ where $\delta_j \in \mathbb{C}^*$. Hence

$$\Psi_k(\lambda) = \left(\prod \delta_j^{a_{kj}}\right) \varepsilon_k \prod_{j=1}^{N} \left(\sum_{p=1}^{m} a_{pj} \lambda_p\right)^{a_{kj}}$$

as required. □

Now let us finish the proof of Theorem 1.3. Since Z is a hypersurface, the rank of the matrix $\|a_i^{(p)}\|$, $i = 1, \ldots, N$, $p = 1, \ldots, m$, equals m i.e., the rows $a^{(p)} \in \mathbb{Z}^N$ are linearly independent over $\mathbb{Q}$. Denote by $L \subset \mathbb{Z}^N$ the free Abelian subgroup generated by $a^{(p)}$. Then L is a primitive lattice, i.e. $L = (L \otimes \mathbb{Q}) \cap \mathbb{Z}^N$. In fact, otherwise the map Ψ given by (2.1) would be a ν-sheeted cover of A, where $\nu = [(L \otimes \mathbb{Q}) \cap \mathbb{Z}^n : L]$. By homogeneity of Ψ_j, for each $a \in L$ we have $\sum_{j=1}^{N} a_j = 0$. So the claim is a consequence of the following lemma.

Lemma 3.2. *Let $L \subset \mathbb{Z}^N$ be a primitive subgroup of rank m such that $\sum_{j=1}^{N} a_j = 0$ for any $a \in L$. Then there is a finite set $A = \{\omega_1, \ldots, \omega_N\} \subset \mathbb{Z}^{N-m}$ such that L is the lattice of affine relations among elements of A, and A affinely generates $\mathbb{Z}^{N-m-1}$ over $\mathbb{Z}$.*

Proof. Put $E = \mathbb{Z}^N/L$. It is a free abelian group rank $N - m$. Let $\omega_j \in E$ be the image of the i-th basis vector of $\mathbb{Z}^N$. Since each $a \in L$ has $\sum a_j = 0$, all ω_j lie on some affine hyperplane $\{h(u) = 1\}$, where $h: \mathbb{Z}^{N-m} \to \mathbb{Q}$ is a homomorphism. Since ω_j generate E as an Abelian group, this hyperplane is primitive, i.e., in fact we have $h: \mathbb{Z}^{N-m} \to \mathbb{Z}$. So, identifying $\{u \in E : h(u) = 1\}$ with $\mathbb{Z}^{N-m-1}$ we find the required conditions. □

Theorem 1.3 is proved.

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