Math. Ann. 275, 573–582 (1986)

Mathematische Annalen
© Springer-Verlag 1986

A Note on Coherent G-Sheaves

Mark Roberts

Mathematics Institute, University of Warwick, Coventry CV4 7AL, England

Introduction

Sheaves defined on G-spaces and carrying actions of the group G which cover the actions on the base spaces have been studied by a number of authors [3, 6, 8], chiefly in the case when G is a finite group or acts properly discontinuously. In this note we consider sheaves on certain complex analytic G-spaces, X, where G is a reductive complex Lie group, for which complex analytic "categorical" quotient spaces X/G exist. These spaces are described in more detail in Sect. 3. The modules of sections of coherent analytic sheaves are Fréchet spaces and, as described in Sect. 2, the group actions on sheaves discussed in this paper are required to be holomorphic, in the sense that the induced actions on the spaces of sections are holomorphic. The point of this is that a general result concerning holomorphic group actions of reductive complex Lie groups on Fréchet spaces, given in Sect. 1, enables us to "average" sections of the G- sheaf over G. This is an important tool which is available for all G-sheaves (of abelian groups, say) when G is finite, but which requires extra, topological, structure on the sheaves for more general G.

The chief result of this paper is that the germs of invariant sections of a coherent analytic G-sheaf, $\mathscr{S}$, on an appropriate G-space X form a coherent analytic sheaf, denoted $\pi_*^G\mathscr{S}$, on X/G. The proof of this is given in Sect. 3. In Sect. 4 we identify the stalks of $\pi_*^G\mathscr{S}$, which by definition are germs of invariant sections of $\mathscr{S}$, but which we prove to correspond bijectively to invariant germs of sections of $\mathscr{S}$.

1. Fréchet Space Representations of Reductive Complex Lie Groups

Let F be a Fréchet space and $\mathscr{A}(F)$ the set of invertible linear endomorphisms of F. Composition of maps gives a group structure to $\mathscr{A}(F)$. Let G be a reductive complex Lie group.

Definition. A holomorphic representation of G on F is a group homomorphism $\tau: G \to \mathscr{A}(F)$ such that the associated map $\underset{(g,f)\to\tau(g)(f)}{G \times F \to F}$ is analytic.

For an account of analytic functions between Fréchet spaces see [2]. Note that if τ is a holomorphic representation of G on F then for each $f \in F$ the map $\underset{g \to \tau(g)(f)}{G \to F}$ is analytic.

Let $\tau : G \to \mathscr{A}(F)$ be a holomorphic representation of G and H a subgroup of G. An element f in F is said to be H finite if it is contained in a finite dimensional H stable subspace of F.

Theorem 1.1. (a) *The set of G finite elements is dense in F.*

(b) *There exists a unique continuous linear operator $L: F \to F$ satisfying:*

(i) $L(\tau(g)(f)) = L(f)$ *for all g in G*

(ii) $L^2 = L$

(iii) $L(f) = f$ *if and only if* $\tau(g)(f) = f$ *for all g in G.*

These properties imply that $\tau(g) L(f) = L(f)$ and so L is a projection from F onto the subspace of G invariant vectors, F^G.

Proof. (a) Let K be a maximal compact subgroup of G [9]. The representation τ restricts to a continuous representation of K on F and, by a result of Harish-Chandra (e.g. [4]), the set of K finite elements in F is dense. Thus it is sufficient to show that any K finite element is G finite.

Let f be K finite and let V be a K stable finite dimensional subspace of F containing f. The representation τ induces a continuous representation $K \to \mathrm{GL}(V)$ which extends to a unique holomorphic representation $\varrho : G \to \mathrm{GL}(V)$ because G is the "universal complexification" of K [9].

For each $v \in V$, the function $\alpha_v : g \to \tau(g)(v) - \varrho(g)(v)$ is an analytic function on G, taking values in F and vanishing on K. If $k \in K$, the function $\hat{\alpha}_v : \eta \to \alpha_v(\mathrm{k}(\exp_G \eta))$ is defined on the Lie algebra, $\mathfrak{g}$, of G and vanishes on the subalgebra $\mathfrak{k}$ given by K. Since $\mathfrak{g}$ is the complexification of $\mathfrak{k}$ it follows that $\hat{\alpha}_v$ vanishes on the whole of $\mathfrak{g}$ and so $\alpha_v = 0$ on kG_0 (where G_0 is the identity component of G) and hence on the whole of G. Thus, for all v in V, $\tau(g)(v) = \varrho(g)(v)$ and V is G stable under the action given by τ; that is f is G finite.

(b) Averaging over K defines a continuous linear endomorphism L of F [4, Sect. 3.3]:

$$L(f) = \int_K \tau(k) \cdot f dk .$$

This operator satisfies (ii) and (i) and (iii) with G replaced by K. The argument of (a) applied to the analytic functions $f - \tau(g) f$ and $L(f) - L(\tau g) f)$ on G shows that (ii) and (iii) also hold for G. Finally the uniqueness of L follows from the facts that its restriction to each finite dimensional G- stable subspace is unique [12] and that G-finite elements are dense in F. □

The first part of the theorem is a straightforward generalization of a result of Richardson [14] who considered the case when F is the space of holomorphic functions on a complex manifold and τ is induced from an action of G on the manifold.

2. Analytic G-Sheaves

Let G be a reductive complex Lie group and X a complex G-space, by which we mean a complex space with a holomorphic action of G,

$$\sigma : G \times X \to X \qquad (g, x) \mapsto g \cdot x .$$

Let $\mathscr{U}$ denote the set of open subsets of X and $\mathscr{U}_G$ that of open subsets which are G-stable.

Definition. An analytic G-sheaf on X is a sheaf $\mathscr{S}$ with the following structure.

(i) For each $U \in \mathscr{U}$ the module of sections $\mathscr{S}(U)$ is a Fréchet space and for any pair $U, V \in \mathscr{U}$ with $U \subset V$ the restriction homomorphism $\varrho_{U,V} : \mathscr{S}(V) \to \mathscr{S}(U)$ is continuous.

(ii) For each $g \in G$ and $U \in \mathscr{U}$ there is a module homomorphism

$$\varphi_{g,U} : \mathscr{S}(U) \to \mathscr{S}(g \cdot U) .$$

If $g, h \in G$ then $\varphi_{g,hU} \circ \varphi_{h,U} = \varphi_{gh,U}$.

(iii) If $U \in \mathscr{U}$ satisfies $\sigma^{-1}(U) \supset \Gamma \times W$ where $W \in \mathscr{U}$ and Γ is an open subset of G, then the map

$$\Gamma^{-1} \times \mathscr{S}(U) \to S(W) \qquad (g, s) \mapsto \varrho_{W, g \cdot U}(g \cdot s)$$

is holomorphic. In particular this implies that if $U \in \mathscr{U}_G$ the representation of G on $\mathscr{S}(U)$ is holomorphic.

A homomorphism of analytic G-sheaves is a sheaf homomorphism $\alpha : \mathscr{S}_1 \to \mathscr{S}_2$, between analytic G-sheaves, satisfying the following conditions for each $U \in \mathscr{U}$.

(i) The module homomorphism $\alpha(U) : \mathscr{S}_1(U) \to \mathscr{S}_2(U)$ is continuous.

(ii) For each $g \in G$, $\alpha(gU) \circ \varphi_{g,U} = \psi_{g,U} \circ \alpha(U)$, where φ and ψ denote the group actions on $\mathscr{S}_1$ and $\mathscr{S}_2$ respectively.

Analytic G-sheaves together with their homomorphisms form a category. Every analytic G-sheaf is a Fréchet sheaf as defined in [11] and also a G-sheaf as in [8]. The extra requirement, the analyticity of the group action, seems natural in the context of analytic geometry.

Examples. (1) Sheaves of germs of sections of analytic G-bundles are analytic G-sheaves.

(2) Any coherent sheaf on X is a Fréchet sheaf [7]. If in addition it satisfies conditions (ii) and (iii) of the definition it will be called a coherent analytic G-sheaf. The kernel, image and cokernel of an analytic G-sheaf homomorphism between coherent analytic G-sheaves are coherent analytic G-sheaves. Conversely, the following result shows that every coherent analytic G-sheaf is locally presented by sheaves of germs of sections of analytic G-bundles, provided X satisfies a certain condition.

Proposition 2.1. *Let X be a complex G-space that is covered by G-stable open Stein subspaces and let $\mathscr{S}$ be a coherent analytic G-sheaf on X. Then for every point $x \in X$*

there exist a G-stable neighbourhood U of x, finite dimensional representations V, W of G and an exact sequence:

$$\mathcal{O}_X(U)\otimes W\to\mathcal{O}_X(U)\otimes V\to\mathcal{S}(U)\to 0$$

Proof. By restricting to a G-stable Stein neighbourhood of x, it can be supposed that X is itself Stein. Since $\mathcal{S}$ is coherent there exists a finite set of sections $\{s_j\}_{j=1}^r\subset\mathcal{S}(X)$ and a neighbourhood Y of x such that the set of restrictions $\{\varrho_{Y,X}(s_j)\}_{j=1}^r$ generates $\mathcal{S}(Y)$ as an $\mathcal{O}_X(Y)$ module. Since $\operatorname{Hom}(\mathcal{O}_X^r,\mathcal{O}_X^r)$ is a coherent sheaf the space of sections $\operatorname{Hom}(\mathcal{O}_X(Y)^r,\mathcal{O}_X(Y)^r)$ is a Fréchet space and so the map

$$\Phi:\operatorname{Hom}(\mathcal{O}_X(Y)^r,\mathcal{O}_X(Y)^r)\to\mathcal{S}(Y)^r$$

$$\alpha=(\alpha_{i,j})\to\alpha\cdot s=\left(\sum_{j=1}^r\alpha_{ij}\,\varrho_{Y,X}(s_j)\right)$$

is a surjective homomorphism of Fréchet spaces which is therefore open by the Open Mapping Theorem. The subset $\operatorname{Aut}(\mathcal{O}_X(Y)^r)$ of invertible homomorphisms is open in the source and therefore its image under Φ is also open. Hence if $\varrho:\mathcal{S}(X)^r\to\mathcal{S}(Y)^r$ is the map given by restriction, $\varrho^{-1}(\Phi(\operatorname{Aut}(\mathcal{O}_X(Y)^r)))$ is an open subset of $\mathcal{S}(X)^r$ containing $(s_1,\dots,s_r)$. This set consists of elements of $\mathcal{S}(X)^r$ whose components have restrictions forming generating sets for $\mathcal{S}(Y)$. The group G acts on $\mathcal{S}(X)^r$ by the diagonal action and by Theorem 1.1 (a) the set of G-finite vectors is dense. Thus there exists a finite set of G-finite global sections $\{s_j\}_{j=1}^r\subset\mathcal{S}(X)$ such that the restrictions $\{\varrho_{Y,X}(s_j)\}_{j=1}^r$ generate $\mathcal{S}(Y)$.

Let V be a finite dimensional G-stable subspace of $\mathcal{S}(X)$ containing $\{s_j\}_{j=1}^r$ and define a G-sheaf homomorphism

$$\varphi:\mathcal{O}_X\otimes V\to\mathcal{S}$$

at the level of stalks by

$$\varphi\left(\sum_{i=1}^r f_{i,y}\otimes v_i\right)=\sum_{i=1}^r f_{i,y}v_{i,y}$$

where $v_{i,y}$ is the germ of $v_i\in V\subset\mathcal{S}(X)$ at $y\in X$. By construction φ is surjective over Y. By the G invariance of V it will therefore be surjective on $U:=G\cdot Y$. Thus there is a surjective equivariant module homomorphism

$$\varphi:\mathcal{O}_X(U)\otimes V\to\mathcal{S}(U)\to 0\,.$$

The proof of the proposition is completed by repeating this construction with $\ker\varphi$ replacing $\mathcal{S}$. □

If $U\in\mathcal{U}_G$ then there is a holomorphic linear action of G on $\mathcal{S}(U)$. The fixed point set $\mathcal{S}(U)^G$ is the space of invariant sections of $\mathcal{S}$ over U. From Theorem 1.1 we immediately obtain the following result which formalises the idea of averaging a section of $\mathcal{S}$ over G.

Proposition 2.2. *Let X be a complex G-space and $\mathcal{S}$ an analytic G-sheaf on X. Then for each G-stable open subset U of X there exists a unique G-invariant projection L_U of $\mathcal{S}(U)$ onto $\mathcal{S}(U)^G$. The uniqueness of this operator implies:*

(i) *if* $U_1 \subset U_2$ *are two G-stable open subsets of X and* $\varrho_{1,2}:\mathscr{S}(U_2)\to\mathscr{S}(U_1)$ *is the restriction map, then*

$$L_U \circ \varrho_{1,2} = \varrho_{1,2} \circ L_U,$$

(ii) *if* $\alpha:\mathscr{S}_1 \to \mathscr{S}_2$ *is a G-sheaf homomorphism between coherent G-sheaves then, for each open G-stable subset U of X, the induced map*

$$\alpha(U):\mathscr{S}_1(U)\to\mathscr{S}_2(U)$$

commutes with the projections onto $\mathscr{S}_1(U)^G$ *and* $\mathscr{S}_2(U)^G$. □

3. Sheaves of Invariant Sections

This section contains the main result of this paper, which states that the germs of invariant sections of a coherent analytic G-sheaf on a complex G-space X form a coherent sheaf over the quotient space X/G, under conditions which ensure that a suitable quotient space exists. These conditions are the following.

C1. The G-space X has an open covering by G-stable Stein subspaces.
C2. The closure of any G-orbit in X contains a unique closed G-orbit.

Examples of spaces satisfying these conditions occur in the work of Bialynicki-Birula and Sommese on $\mathbb{C}^*$ actions on Kähler manifolds [1].

Let X be a complex G-space satisfying C1 and C2 and let Y be the set of closed G-orbits in X. By C2 there is a well defined map $\pi: X\to Y$ given by letting $\pi(x)$ equal the closed orbit in $\overline{G\cdot x}$. The set Y is given the quotient topology induced from X and is made into a ringed space by defining

$$\mathcal{O}_Y(U) := \mathcal{O}_X(\pi^{-1}(U))^G$$

for every open subset U of Y. It follows immediately from the work of Snow [16] that $(Y, \mathcal{O}_Y)$ is a complex space and π an analytic map with the following universal property.

If Z is a complex space and $\varrho: X\to Z$ is a G-invariant analytic map then there is an analytic map $\sigma: Y\to Z$ such that $\varrho = \sigma\circ\pi$.

Thus Y is the *categorical quotient* of X by G [12, 16], and will be denoted by X/G. If $\xi\in X/G$ then the unique closed orbit in $\pi^{-1}(\xi)$ will be denoted by $T(\xi)$.

Snow also proved a slice theorem for such G-spaces. If x is a point in X with a closed orbit then the isotropy subgroup G_x is a reductive complex Lie group and there exists a finite dimensional representation R of G_x, a G_x-stable locally closed subvariety, S, of R, containing 0, and an equivariant analytic map $\tau: S, 0\to X, x$, satisfying:

(i) τ is an isomorphism onto its image,
(ii) $S\times_{G_x} G$ is isomorphic to a π saturated open neighbourhood of x.

Since reductive complex Lie groups have unique compatible reductive algebraic group structures and holomorphic representations of the Lie groups correspond bijectively to rational representations of the algebraic group [10], the slice S can be taken to be an open subset (in the transcendental topology) of an affine algebraic variety with a rational action of the reductive algebraic group.

A consequence of the slice theorem is that, in the neighbourhood of a closed orbit, a G-sheaf is completely determined by its restriction to a slice through the orbit. This can be seen by looking at the induced action of G on the sheaf space $\tilde{\mathscr{S}}$ of $\mathscr{S}$ given by

$$g \cdot \eta = \varrho_{g \cdot x, g \cdot U}(g \cdot s),$$

where $g \in G$, $x \in X$, $\eta \in \mathscr{S}_x$, U is a neighbourhood of x and s is an element of $\mathscr{S}(U)$ satisfying $\varrho_{x,U}(s) = \eta$. In particular there is an action of G_x on $\mathscr{S}_x$ for each x and if $T = G \cdot x$ is a closed orbit and $S, 0 \to X, x$ is a slice through T, then

$$\tilde{\mathscr{S}}\,|_{S \times_{G_x} G} \cong (\tilde{\mathscr{S}}\,|_S) \times_{G_x} G .$$

It follows from this that invariant sections of $\mathscr{S}$, and invariant germs of sections along T, are determined by their restrictions to S.

We continue to suppose that X satisfies C1 and C2 and so has a quotient space X/G.

Definitions. (1) If $\mathscr{S}$ is a G-sheaf on X define the sheaf $\pi_*^G\mathscr{S}$ on X/G by

$$\pi_*^G\mathscr{S}(U) := \mathscr{S}(\pi^{-1}(U))^G$$

for all open subsets U of X/G. The restriction homomorphisms of $\pi_*^G\mathscr{S}$ are those of $\mathscr{S}$ restricted to the spaces of invariant sections. Technically this only defines $\pi_*^G\mathscr{S}$ as a presheaf, but it is easily seen to be a sheaf.

(2) If $\varphi : \mathscr{S}_1 \to \mathscr{S}_2$ is a G-sheaf homomorphism then the sheaf homomorphism $\pi_*^G\varphi : \pi_*^G\mathscr{S}_1 \to \pi_*^G\mathscr{S}_2$ is defined by

$$\pi_*^G\varphi(U) := \varphi(\pi^{-1}(U))\,|_{\mathscr{S}_1(\pi^{-1}(U))^G} .$$

With these definitions π_*^G is a functor from the category of G-sheaves to that of sheaves on X/G.

Theorem 3.1. *If X is a complex G-space satisfying* C1 *and* C2 *and $\mathscr{S}$ is a coherent analytic G-sheaf on X, then $\pi_*^G\mathscr{S}$ is a coherent sheaf on X/G.*

Proof. By Proposition 2.1, if $\xi \in X/G$ there exists a G-stable neighbourhood V of $T(\xi)$ over which $\mathscr{S}$ has a presentation by sheaves of germs of sections of G-bundles:

$$\mathscr{S}_2 \to \mathscr{S}_1 \to \mathscr{S}\,|_V \to 0 .$$

For any neighbourbood U of ξ in X/G such that $\pi^{-1}(U) \subset V$ there is a commutative diagram:

$$\begin{array}{ccccccc}
\mathscr{S}_2(\pi^{-1}(U)) & \to & \mathscr{S}_1(\pi^{-1}(U)) & \to & \mathscr{S}(\pi^{-1}(U)) & \to & 0 \\
\text{J}\big\downarrow L & & \text{J}\big\downarrow L & & \text{J}\big\downarrow L & & \\
\mathscr{S}_2(\pi^{-1}(U))^G & \to & \mathscr{S}_1(\pi^{-1}(U)^G & \to & \mathscr{S}(\pi^{-1}(U))^G & \to & 0
\end{array}$$

where L is the averaging operator. The top row of this diagram is exact and a diagram chase shows that this implies that the bottom row is also exact. It follows that the sequence of sheaves,

$$\pi_*^G\mathscr{S}_1 \to \pi_*^G S_2 \to \pi_*^G(\mathscr{S}\,|_V) \to 0$$

is exact at ξ and the coherence of $\pi_*^G\mathcal{S}$ at ξ will be proved if it can be shown that the $\pi_*^G\mathcal{S}_j$ are coherent. This is done in the following lemma, which is a complex analytic analogue of a result of Poenaru [13].

Lemma 3.2. *Let X be a complex G-space satisfying* C1 *and* C2 *and $\mathcal{S}$ the sheaf of germs of sections of an analytic G-bundle over X. Then $\pi_*^G\mathcal{S}$ is a coherent sheaf on X/G.*

Proof. To prove coherence at $\xi \in X/G$ it can be supposed, by the slice theorem, that $(X, T(\xi))$ is of the form $(G\times_H Y, G\times_H\{y_0\})$ where H is a reductive subgroup of G and Y is a rational H-space which has y_0 as a fixed point. Let $\mathcal{S}|_Y$ denote the H-bundle obtained by restricting $\mathcal{S}$ to Y. Then $\mathcal{S}$ is completely determined by $\mathcal{S}|_Y$ and $\pi_*^G\mathcal{S}$ is coherent at ξ if and only if $\pi_*^H(\mathcal{S}|_Y)$ is coherent at $\pi(y_0)$. The spaces of sections of $\pi_*^H(\mathcal{S}|_Y)$ over sufficiently small neighbourhoods, U, of $\pi(y_0)$ can be identified with the spaces of equivariant analytic mappings $C_H^\omega(\pi^{-1}(U), W)$, where W is the fibre of the original G-bundle at a point on $T(\xi)$. It is sufficient to show that these spaces are finitely generated over $\mathcal{O}_{Y/H}(U)=\mathcal{O}_Y(\pi^{-1}(U))^H$. The proof now follows closely that of Poenaru.

Let W^* denote the dual of W. For $f\in C_H^\omega(\pi^{-1}(U), W)$ define $\alpha f\in\mathcal{O}_{Y\times W}(\pi^{-1}(U)\times W^*)^H$ by

$$\alpha f(y,\eta)=\langle f(y),\eta\rangle \quad \text{for} \quad y\in\pi^{-1}(U),\ \eta\in W^*.$$

In the opposite direction, if $h\in\mathcal{O}_{Y\times W}(\pi^{-1}(U)\times W^*)^H$ define $\beta h\in C_H^\omega(\pi^{-1}(U), W)$ by

$$\beta h(y)=\left(\frac{\partial h}{\partial\eta_1}(y,0),\ldots,\frac{\partial h}{\partial\eta_r}(y,0)\right)$$

for $\eta=(\eta_1,\ldots,\eta_r)\in W^*$. Then $\beta\alpha f=f$ for all $f\in C_H^\omega(\pi^{-1}(u), W)$.

There exist polynomial germs $\sigma_1,\ldots,\sigma_k\in\mathcal{O}_{Y\times W}(Y\times W^*)^H$ such that, if $\sigma=(\sigma_1,\ldots,\sigma_k)\colon Y\times W^*\to\mathbb{C}^k$, then

$$\mathcal{O}_{Y\times W}(\pi^{-1}(U)\times W^*)^H\cong(\sigma^*\mathcal{O}_{\mathbb{C}^k}^k)(\pi^{-1}(U)\times W^*).$$

Let $h\in\mathcal{O}_{Y\times W}(\pi^{-1}(U)\times W^*)^H$ and choose $\tilde{h}\in\mathcal{O}_{\mathbb{C}^k}(\mathbb{C}^k)$ such that $h=\sigma^*\tilde{h}$. Then

$$\beta h=\sum_{i=1}^{k}\left(\frac{\partial\tilde{h}}{\partial z_j}\circ\sigma|_{\pi^{-1}(U)\times\{0\}}\right)\cdot\beta\sigma_i$$

(where the z_i are co-ordinates for $\mathbb{C}^k$) and it follows that the equivariant polynomial mappings $\{\beta\sigma_i\}_{i=1}^k$ generate $C_H^\omega(\pi^{-1}(U), W)$ over $\mathcal{O}_Y(\pi^{-1}(U))^H$. □

4. Germs of Invariant Sections

In this section we show that "invariant germs of sections" of a coherent analytic G-sheaf $\mathcal{S}$ on a G-space X can be identified with "germs of invariant sections" of $\mathcal{S}$. If T is a closed G-orbit in X then there is an induced action of G on the stalk $\mathcal{S}_T$ defined by

$$g\cdot\eta=\varrho_{T,g\cdot U}(g\cdot s),$$

where $g \in G$, $\eta \in \mathcal{S}_T$, U is a neighbourhood of T and s is an element of $\mathcal{S}(U)$ satisfying $\varrho_{T,U}(s) = \eta$. The fixed point set of this action, consisting of invariant germs of sections of $\mathcal{S}$ along T, is denoted $\mathcal{S}_T^G$.

Proposition 4.1. *Let $\mathcal{S}$ be a coherent analytic G-sheaf on a complex G-space X satisfying* C1 *and* C2 *and let $\xi \in X/G$. Then*

$$(\pi_*^G \mathcal{S})_\xi \cong (\mathcal{S}_{T(\xi)})^G.$$

Proof. Let $\mathcal{U}$ be a system of neighbourhoods of ξ. Then

$$(\pi_*^G \mathcal{S})_\xi = \lim_{\mathcal{U}} (\pi_*^G \mathcal{S})(U) = \lim_{\mathcal{U}} \mathcal{S}(\pi^{-1}(U))^G.$$

Restriction defines a module homomorphism,

$$\varphi : (\pi_*^G \mathcal{S})_\xi \to \mathcal{S}_{T(\xi)}{}^G.$$

Let $s_1, s_2 \in \mathcal{S}(\pi^{-1}(U))^G$ for some $U \in \mathcal{U}$ satisfy

$$\varrho_{T(\xi),\pi^{-1}(U)}(s_1) = \varrho_{T(\xi),\pi^{-1}(U)}(s_2).$$

Then there exists a neighbourhood W of $T(\xi)$, contained in $\pi^{-1}(U)$, such that

$$\varrho_{W,\pi^{-1}(U)}(s_1) = \varrho_{W,\pi^{-1}(U)}(s_2).$$

This implies that

$$\varrho_{G \cdot W,\pi^{-1}(U)}(s_1) = \varrho_{G \cdot W,\pi^{-1}(U)}(s_2)$$

and the injectivity of φ follows from the fact that any G-stable neighbourhood of $T(\xi)$ contains $\pi^{-1}(U)$ for some $U \in \mathcal{U}$.

The surjectivity of φ is given by the following lemma.

Lemma 4.2. *Let $\mathcal{S}$ and X be as above. If T is a closed orbit in X and $\eta \in \mathcal{S}_T^G$ then there exist a G-stable neighbourhood U of T and $s \in \mathcal{S}(U)^G$ such that $\varrho_{T,U}(s) = \eta$.*

Proof. By the slice theorem we can assume that T is a fixed point of the G action on X. We first show that the lemma is true if G is replaced by K, a maximal compact subgroup of G.

If U is any neighbourhood of T in X there is a K-stable neighbourhood V of T with compact closure $\bar{V} \subset U$. Using this and the definition of the group action on $\mathcal{S}_T$ it can be seen that for every $\eta \in \mathcal{S}_T^G$ and $g \in K$ there exist a K-stable neighbourhood U_g of T and $s \in \mathcal{S}(U_g)$ such that $\varrho_{T,U_g}(s) = \eta$ and $g \cdot s = s$.

Let $\{g_i\}_{i=1}^k$ be a finite set of elements of K generating a dense subgroup of K [5, 2.12, Example 4] and set $U = \bigcap_{i=1}^k U_{g_i}$. Then U is a K-stable open neighbourhood of T and there exist $s_i \in \mathcal{S}(U)$ such that $g_i \cdot s_i = s_i$ and $\varrho_{T,U}(s_1) = \ldots = \varrho_{T,U}(s_k) = \eta$. By shrinking U if necessary we can suppose $s_1 = \ldots = s_k = s$, say.

Clearly $g \cdot s = s$ for all g in the group generated by $g_1, \ldots, g_k$; but this is a dense subgroup of K and so, by the continuity of the action of K on $\mathcal{S}(U)$, $g \cdot s = s$ for all g in K.

Now we show that $s \in \mathcal{S}(U)$ extends naturally to a G-invariant element, s, of $\mathcal{S}(V)$, where $V = G \cdot U$, satisfying $\varrho_{g \cdot U, V}(s) = g \cdot s$ for all $g \in G$. This is true if and only if for any pair of elements $(g,h)\in G\times G$,

$$\varrho_{gU\cap hU,gU}(g\cdot s)=\varrho_{gU\cap hU,hU}(h\cdot s).$$

Since K intersects every component of G there is a continous family of pairs of group elements, $(g_t,h_t):[0,1]\to G\times G$, with $(g_0,h_0)\in K\times K$ and $(g_1,h_1)=(g,h)$. For each t in $[0,1]$ let V_t be a K-stable open neighbourhood of T such that $\bar{V}_t$ is compact and is contained in $g_tU\cap h_tU$. Then $\bar{V}_t$ is contained in $g_sU\cap h_sU$ for all s in a neighbourhood of t and using the compactness of $[0,1]$ we can find a K-stable open neighbourhood V of T with compact closure $\bar{V}\subset g_tU\cap h_tU$ for all t in $[0,1]$.

Let $\Gamma=\{(g',h')\in G\times G: g'U\cap h'U\supset\bar{V}\}$. Then Γ is an open neighbourhood of $K\times K\cup\{(g_t,h_t):t\in[0,1]\}$ in $G\times G$ and

$$\varphi:\Gamma\to\mathscr{S}(V)\qquad (g',h')\mapsto\varrho_{V,g'U}(g'\cdot s)-\varrho_{V,h'U}(h'\cdot s)$$

is a holomorphic map which is constant on $K\times K$ and hence also on the connected components of Γ intersecting $K\times K$. It follows that $\varrho_{V,gU}(g\cdot s)=\varrho_{V,hU}(h\cdot s)$ as required. □

Final Remark. Coherent algebraic G-sheaves on an algebraic variety with a rational action of a reductive algebraic group can also be treated by methods analogous to those used in this paper, the algebraic analogue of the Stein G-space being the affine G-space. Instead of holomorphic actions on the spaces of sections, $\mathscr{S}(U)$, there are "dual actions" in the sense of Mumford [12]. In particular, suppose X is an algebraic variety with a rational action of the reductive algebraic group G and let X_{ss} denote the set of semistable points in X with respect to some linearisation of the action and $\pi: X_{ss}\to X_{ss}/G$ its quotient space [12]. Then, if $\mathscr{S}$ is a coherent G-sheaf on X and $\mathscr{S}_{ss}$ is its restriction to X_{ss} we get that $\pi^G_*\mathscr{S}_{ss}$ is a coherent sheaf on X_{ss}/G.

Acknowledgements. The results of this paper formed part of the author's Ph.D. thesis, written at the University of Liverpool with the support of a S.E.R.C. grant. The thesis also contained an application to a problem in equivariant singularity theory (see also [15]). The author is grateful to his supervisor, C.T.C. Wall, and to A. Wasserman for helpful conversations.

References

1. Bialynicki-Birula, A., Sommese, A.J.: Quotients by $\mathbb{C}^*$ and SL (2, $\mathbb{C}$) actions. Trans. Am. Math. Soc. **279**, 773–800 (1983)
2. Bochnak, J., Siciak, J.: Analytic functions in topological vector spaces. Stud. Math. **39**, 77–112 (1971)
3. Borel, A.: Seminar on transformation groups. Ann. Math. Stud. **46** (1960)
4. Borel, A.: Representation de groupes localement compacts. Lect. Notes Math. 276, Berlin, Heidelberg, New York: Springer 1972
5. Brocker, T., tom Dieck, T.: Representations of compact Lie groups. Berlin, Heidelberg, New York, Tokyo: Springer 1985
6. Conner, P.E.: Lecture on the action of a finite group. Lect. Notes Math. 70, Berlin, Heidelberg, New York: Springer 1968
7. Grauert, H., Remmert, R.: Theorie der Steinschen Räume. Berlin, Heidelberg, New York: Springer 1977
8. Grothendieck, A.: Sur quelques points d'algèbre homologique. Tohoku Math. J. (2 Ser.) **9**, 119–221 (1957)

9. Hochschild, G.: The structure of Lie groups. San Francisco, London, Amsterdam: Holden-Day 1965
10. Hochschild, G., Mostow, G.D.: Representations and representative functions of Lie group. III. Ann. Math. **70**, 85–100 (1959)
11. Kaup, L.: Eine Künnethformel für Fréchetgarben. Math. Z. **97**, 158–168 (1967)
12. Mumford, D., Fogarty, J.: Geometric invariant theory (2nd ed.). Berlin, Heidelberg, New York: Springer 1982
13. Poenaru, V.: Singularites C^∞ en presence de symétrie. Lect. Notes Math. 510, Berlin, Heidelberg, New York: Springer 1976
14. Richardson, R.N.: Principal orbit types for reductive groups acting on Stein manifolds. Math. Ann. **208**, 323–331 (1974)
15. Roberts, R.M.: Characterisations of finitely determined equivariant map germs. Math. Ann. **275**, 583–597 (1986)
16. Snow, D.M.: Reductive group actions on Stein spaces. Math. Ann. **259**, 79–97 (1982)

Received December 21, 1983; in revised form April 9, 1986