Geodesics in non-holonomic geometry.

Von

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Synopsis:

1. Introduction.

The stability of geodesics in Riemannian space was first discussed simultaneously and independently by Levi-Civita[1]) and myself[2]). The tensorial equation obtained may be written

$$(1.1) \qquad \overline{\overline{\eta}}^r + G^r_{.msn} x^{m'} \eta^s x^{n'} = 0\,,$$

where η^r is the infinitesimal vector joining a point on the fundamental geodesic to the corresponding point of a neighbouring geodesic, $x^{r'} = \frac{dx^r}{ds}$, where x^r is the coordinate system, $G^r_{.msn}$ is the mixed curvature tensor

[1]) „Sur l'écart géodésique," Math. Annalen 97 (1926), p. 291—320. Cf. also Levi-Civita, „The Absolute Differential Calculus," English Translation (1927).

[2]) „On the Geometry of Dynamics," Phil. Trans. Roy. Soc., A, 226 (1926), p 31—106. That paper will be referred to as GD. The dynamical problem of „Stability in the action sense," discussed in GD., Chap. IX, is precisely the geometrical problem of the stability of geodesics in Riemannian space.

of the manifold, and the operation denoted by superimposed bars is defined (for any vector given along a curve) by

$$(1.2)\qquad \begin{cases} \overline{X}^r = \dfrac{dX^r}{ds} + \left\{ {r \atop mn} \right\} X^m \dfrac{dx^n}{ds}, \\ \overline{\overline{X}}^r = \dfrac{d\overline{X}^r}{ds} + \left\{ {r \atop mn} \right\} \overline{X}^m \dfrac{dx^n}{ds}. \end{cases}$$

The question of stability for systems subject to non-holonomic constraints was not discussed in these papers. Subsequently a number of papers have appeared dealing with geodesic stability and non-holonomic constraints[3]).

The method of Vranceanu, in dealing with non-holonomic systems, differs essentially from mine, as developed in GD. and in the present paper. He has employed a set of vectors *in* the element in which motion subject to the constraint is possible, whereas I have employed a set of vectors *normal* to that element. We have therefore been led to different analytical expressions for the equations of motion of a non-holonomic conservative system (or, geometrically, the differential equations of constrained geodesics), his equations consisting[4]) for a system with N coordinates and M constraints, of $(2N-M)$ equations of the first order for the N coordinates and the $(N-M)$ components of the unit tangent vector along his fundamental vectors of constraint, whereas my equations[5]) consist of a set of N equations of the second order for the N coordinates, M particular first integrals of these equations being known, viz., the equations of constraint. The expressions given by him are invariant with respect to coordinate transformations and tensorial with respect to transformations of the constraint vectors, while the expressions which I shall develop are tensorial with respect to coordinate transformations and invariant with respect to transformations of the constraint vectors.

In the paper last cited, Vranceanu discusses the stability of geodesics. It appears to me, however, that his argument is incorrect, for he assumes that the infinitesimal displacement from the fundamental geodesic to the

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[3]) G. Vranceanu, „Sur les espaces non holonomes," Comptes Rendus **183** (1926), p. 852—854; „Sur le calcul différentiel absolu pour les variétés non holonomes," ibid., p. 1083—1085; „Sopra una classe di sistemi anolonomi," Rend. Acc. Lincei (6) **3** (1926), p. 548—553; „Sopra le equazioni del motó di un sistema anolonomo," ibid. **4** (1926), p. 508—511; „Sopra la stabilità geodetica", ibid. **5** (1927), p. 107—110. T. Boggio, „Sullo scostamento geodetico," Rend. Acc. Lincei (6) **4** (1926), p. 255—261. U. Crudeli, „Su lo scostamento geodetico elementare; procedimento di estensione della equazione di Jacobi ad una qualsiasi varietà riemanniana," Rend. Acc. Lincei (6) **5** (1927), p. 248—251. E. Cartan, „Sur l'écart géodésique et quelques notions connexes," Rend. Acc. Lincei (6) **5** (1927), p. 609—613.

[4]) „Sopra la stabilità geodetica," p. 107.

[5]) GD., p. 58, Theorem XVII (A).

neighbouring geodesic is consistent with the constraints. In general, this cannot continue to be true along the geodesic[6]).

In the present paper the equations of constrained geodesics are derived from a variational principle, and equations for geodesic separation are obtained. The mode of development is geometrical, the dynamical significance being indicated in § 8. The line-element is assumed to be positive definite.

2. The constraint.

Let N be the number of dimensions of the manifold and M the number of constraints.

The following conventions will be observed:

1. Small italic indices unrepeated imply a range of values from 1 to N, and, when repeated in a single term, summation over that range.

2. Greek indices unrepeated imply a range of values from 1 to M, and, when repeated[7]) in a single term, summation over that range.

3. Those indices (other than powers), which do not imply tensorial character with respect to coordinate transformations, are enclosed in brackets[8]).

4. The Christoffel symbol of the second kind is denoted by $\left\{ {r \atop mn} \right\}$, instead of $\left\{ {mn \atop r} \right\}$ as in GD.[9]).

Let x^r be the coordinate system, and let the line-element be given by

$$ds^2 = g_{mn}\, dx^m\, dx^n, \tag{2.1}$$

which form we suppose to be positive definite. Let the constraint be given by

$$A_{(\varrho)m}\, dx^m = 0, \tag{2.2}$$

which we suppose, in general, to possess no integrable combination. Equations (2.2) can always be replaced by

$$B_{(\varrho)m}\, dx^m = 0, \tag{2.3}$$

where $B^r_{(\varrho)}$ are M mutually orthogonal unit vectors[10]). These vectors will be called the *constraint vectors* and the linear M-space defined by them *the M-space of constraint*. Any curve satisfying (2.3) will be called a

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6) Cf. Cartan, loc. cit.

7) It is not necessary that one should occur as a superscript and the other as a subscript.

8) This rule is not observed in the case of $\Gamma^{r}_{\cdot mn}$ defined in (4.31).

9) The present notation is used by Eisenhart, Riemannian Geometry (1926).

10) GD., p. 54.

constrained curve. A curve is a constrained curve if and only if it is perpendicular at every point to every vector lying in the M-space of constraint.

It is of course well known that the non-integrable character of the equations (2.3) implies that the congruences $B^r_{(\varrho)}$ are not normal congruences; the fact that there exists no integrable combination implies that there is no normal congruence having at every point a direction contained in the M-space of constraint. If, on the other hand, there did exist an integrable combination, then the totality of curves passing through any assigned point of the manifold and satisfying that equation at every point would build up a subspace of $(N-1)$ dimensions, from which it would be impossible to depart if one always followed a constrained curve. But under the given conditions of non-holonomicity it will be possible in general to reach any point of the manifold by travelling along a constrained curve from an arbitrarily assigned point.

It is possible to write down a single function whose vanishing is necessary and sufficient for the satisfaction of the constraint by a given curve. Let λ^r denote the unit vector tangent to the curve, and let ω be defined as the positive quantity given by

$$\omega^2 = B_{(\varrho)m} B_{(\varrho)n} \lambda^m \lambda^n = (B_{(1)m}\lambda^m)^2 + (B_{(2)m}\lambda^m)^2 + \ldots + (B_{(M)m}\lambda^m)^2. \tag{2.4}$$

Thus ω vanishes if and only if all the equations of constraint are satisfied. It is not difficult to see the geometrical meaning of ω. Let B^r denote any unit vector in the M-space of constraint, so that

$$B^r = \theta_{(\varrho)} B^r_{(\varrho)}, \qquad \theta_{(\varrho)}\theta_{(\varrho)} = 1, \tag{2.41}$$

and let φ denote the angle between λ^r and B^r. Then

$$\cos\varphi = B_r \lambda^r = \theta_{(\varrho)} B_{(\varrho)r} \lambda^r. \tag{2.42}$$

If, by variation of $\theta_{(\varrho)}$, we vary the direction of B^r, we have

$$\delta \cos\varphi = B_{(\varrho)r} \lambda^r \delta\theta_{(\varrho)}, \tag{2.43}$$

$$\theta_{(\varrho)} \delta\theta_{(\varrho)} = 0; \tag{2.44}$$

if $\cos\varphi$ has a stationary value for all such variations, we must have

$$B_{(\varrho)r} \lambda^r = k\theta_{(\varrho)}, \tag{2.5}$$

where k is undetermined. Multiplying by $\theta_{(\varrho)}$ and summing as indicated, we find, by (2.42),

$$k = \cos\varphi. \tag{2.6}$$

If we square (2.5) and sum for ϱ from 1 to M, we find

$$\omega^2 = k^2 = \cos^2\varphi. \tag{2.7}$$

Thus the stationary values of $\cos\varphi$ are $\pm\omega$, the two signs corresponding to opposed unit vectors B^r in the M-space of constraint. In fact, ω *is the cosine of the angle which the vector* λ^r *makes with the M-space of constraint*, this angle being defined as the smallest angle made by λ^r with any vector in the M-space of constraint. We shall call ω the *constraint function*. Its importance is due to the fact that its value is independent of the choice of orthogonal unit vectors $B^r_{(\varrho)}$ in the M-space of constraint.

If we write

$$C_{mn} = B_{(\varrho)m} B_{(\varrho)n} \tag{2.8}$$

then C_{mn} is a symmetrical covariant tensor of the second order with respect to coordinate transformations, and is independent of the choice of the constraint vectors. We shall call C_{mn} the *constraint tensor*. Any vector λ^r, which satisfies the constraints, satisfies

$$C_{mn}\lambda^n = 0. \tag{2.9}$$

3. Equations of constrained geodesics derived from a variational principle.

We shall now define a *constrained geodesic*[11]) as a constrained curve whose length is stationary for all varations which vanish at the end points and satisfy the equations of constraint.

It is to be noted that it is the *variation*, and not the *varied curve*, that satisfies the equations of constraint. This definition is adopted because constrained geodesics, so defined, have a dynamical significance. From a purely geometrical point of view there are two possible generalisations of the concept of surface geodesics when the constraint is non-holonomic, viz., curves defined as above, and curves of stationary length with respect to variations which make the varied curve satisfy the equations of constraint. We shall reserve the name *constrained geodesics* for curves defined in the former sense.

For any variation δx^r with fixed end points we have[12])

$$\delta\int ds = -\int g_{mn}\varkappa^m \delta x^n\, ds, \tag{3.1}$$

where $\varkappa^r$ is the first curvature vector,

$$\varkappa^r = \frac{d^2x^r}{ds^2} + \begin{Bmatrix} r \\ mn \end{Bmatrix}\frac{dx^m}{ds}\frac{dx^n}{ds}. \tag{3.11}$$

[11]) Geodesics in non-holonomic geometry have been defined by Vranceanu, Comptes Rendus **183** (1926), p. 854, but he has not connected his definition with a variational principle. However it is not difficult to derive his equations from the variational principle here adopted. The curves defined by him are therefore identical with those of the present paper, but the notation is entirely different.

[12]) Proc. London Math. Soc. (2) **25** (1926), p. 253.

In order that a constrained curve may be geodesic, it is necessary and sufficient that $\delta \int ds$ should vanish for all variations consistent with

$$(3.12) \qquad B_{(\varrho)n} \delta x^n = 0,$$

and vanishing at the end points. Hence we must have

$$(3.2) \qquad \varkappa_n \delta x^n \equiv k_{(\varrho)} B_{(\varrho)n} \delta x^n,$$

where $k_{(\varrho)}$ are undetermined. Hence

$$(3.21) \qquad \varkappa_r = k_{(\varrho)} B_{(\varrho)r}, \qquad \varkappa^r = k_{(\varrho)} B^r_{(\varrho)}.$$

Now by differentiation of the equations of constraint

$$(3.3) \qquad B_{(\sigma)m} \frac{dx^m}{ds} = 0,$$

which are satisfied along the curve, we obtain

$$(3.31) \qquad B_{(\sigma)m} \varkappa^m + B_{(\sigma)mn} \frac{dx^m}{ds} \frac{dx^n}{ds} = 0,$$

where $B_{(\sigma)mn}$ is the covariant derivative of $B_{(\sigma)m}$. Hence, substituting for $\varkappa^m$ from (3.21), and remembering that the constraint vectors, being unit vectors and mutually orthogonal, satisfy

$$(3.32) \qquad B_{(\sigma)m} B^m_{(\varrho)} = \delta^\sigma_\varrho = \begin{cases} 1, & \text{for } \sigma = \varrho, \\ 0, & \text{for } \sigma \neq \varrho, \end{cases}$$

we find

$$(3.33) \qquad k_{(\sigma)} = - B_{(\sigma)mn} \frac{dx^m}{ds} \frac{dx^n}{ds}.$$

Substitution in (3.21) gives

$$(3.4) \qquad \varkappa^r = - B^r_{(\varrho)} B_{(\varrho)mn} \frac{dx^m}{ds} \frac{dx^n}{ds}.$$

We may state the following result:

Theorem I. *Every constrained geodesic satisfies the equations*

$$(3.5) \qquad \frac{d^2 x^r}{ds^2} + \left(\left\{ {r \atop m\, n} \right\} + B^r_{(\varrho)} B_{(\varrho)mn} \right) \frac{dx^m}{ds} \frac{dx^n}{ds} = 0.$$

Geometrically, the first curvature vector of a constrained geodesic lies in the M-space of constraint, and has components $-B_{(\varrho)mn} \frac{dx^m}{ds} \frac{dx^n}{ds}$ *in the directions of the vectors* $B^r_{(\varrho)}$ *respectively.*

We shall call (3.5) the *first form* of the equations of constrained geodesics.

Along any curve we have

$$(3.51) \qquad \frac{d}{ds} \left(B_{(\varrho)m} \frac{dx^m}{ds} \right) = B_{(\varrho)m} \varkappa^m + B_{(\varrho)mn} \frac{dx^m}{ds} \frac{dx^n}{ds},$$

which vanishes if the curve satisfies (3.5). Thus we have the result:

Theorem II. *The differential equations* (3.5) *have the first integrals*

$$B_{(\varrho)m}\frac{dx^m}{ds} = \text{constant}. \tag{3.52}$$

The differential equations (3.5) define a unique curve passing through an arbitrarily assigned point in an arbitrarily assigned direction. From the theorem just established we see that all solutions of (3.5), whose initial directions satisfy the equations of constraint, are constrained geodesics.

It may be remarked that the equations

$$\frac{d^2x^r}{du^2} + \left(\begin{Bmatrix} r \\ m\,n \end{Bmatrix} + B^r_{(\varrho)}B_{(\varrho)mn}\right)\frac{dx^m}{du}\frac{dx^n}{du} = 0 \tag{3.6}$$

do not in general possess the first integral

$$g_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = \text{constant}. \tag{3.61}$$

For we find

$$\frac{d}{du}\left(g_{rs}\frac{dx^r}{du}\frac{dx^s}{du}\right) = 2\,g_{rs}\left(\frac{d^2x^r}{du^2} + \begin{Bmatrix} r \\ m\,n \end{Bmatrix}\frac{dx^m}{du}\frac{dx^n}{du}\right)\frac{dx^s}{du}, \tag{3.62}$$

so that for a curve satisfying (3.6) we have

$$\frac{d}{du}\left(g_{rs}\frac{dx^r}{du}\frac{dx^s}{du}\right) = -2\,B_{(\varrho)s}\frac{dx^s}{du}B_{(\varrho)mn}\frac{dx^m}{du}\frac{dx^n}{du}. \tag{3.63}$$

This will vanish, giving the first integral (3.61), if the equations of constraint are satisfied, but not in general.

4. Second form of the equations of constrained geodesics.

The equations (3.5) for constrained geodesics suffer from the defect that they are not independent of the choice of the orthogonal unit constraint vectors in the M-space of constraint. We shall now obtain equations which have the requisite independence.

Since all along any constrained curve we have

$$B_{(\varrho)m}\frac{dx^m}{ds} = 0, \tag{4.1}$$

it follows that a constrained geodesic (which satisfies (3.5)) also satisfies

$$\frac{d^2x^r}{ds^2} + \left(\begin{Bmatrix} r \\ m\,n \end{Bmatrix} + B^r_{(\varrho)}B_{(\varrho)mn} + B^r_{(\varrho).n}B_{(\varrho)m}\right)\frac{dx^m}{ds}\frac{dx^n}{ds} = 0, \tag{4.2}$$

where

$$B^r_{(\varrho).n} = g^{rs}B_{(\varrho)sn}. \tag{4.21}$$

If we take the covariant derivative of the constraint tensor C_{mn} defined by (2.8) and then raise the first subscript, we have (changing the indices)

$$C^r_{.mn} = B^r_{(\varrho)}B_{(\varrho)mn} + B^r_{(\varrho).n}B_{(\varrho)m}. \tag{4.25}$$

Thus we have the result:

Theorem III. *Every constrained geodesic satisfies*

(4.3) $$\frac{d^2 x^r}{ds^2} + \Gamma^r_{.mn} \frac{dx^m}{ds} \frac{dx^n}{ds} = 0,$$

where

(4.31) $$\Gamma^r_{.mn} = \left\{ {r \atop mn} \right\} + C^r_{.mn}.$$

Since the constraint tensor is independent of the particular choice of constraint vectors, it follows that we have in (4.3) a form having the requisite independence. We shall call (4.3) the *second form* of the equations of constrained geodesics.

Unlike the first form (3.5), the second form (4.3) does not possess the first integrals (3.52). We find, for any curve satisfying (4.3),

(4.4) $$\begin{aligned} \frac{d}{ds}\left(B_{(\varrho)r} \frac{dx^r}{ds}\right) &= - B_{(\varrho)r} C^r_{.mn} \frac{dx^m}{ds} \frac{dx^n}{ds} + B_{(\varrho)rn} \frac{dx^r}{ds} \frac{dx^n}{ds}, \\ &= [B_{(\varrho)mn} - B_{(\varrho)r} (B^r_{(\sigma)} B_{(\sigma)mn} + B^r_{(\sigma).n} B_{(\sigma)m})] \frac{dx^m}{ds} \frac{dx^n}{ds}, \\ &= - B_{(\varrho)r} B^r_{(\sigma).n} B_{(\sigma)m} \frac{dx^m}{ds} \frac{dx^n}{ds}. \end{aligned}$$

This will not vanish in general, but will vanish at a point if the equations of constraint are satisfied at that point. We may state the result:

Theorem IV. *The differential equations* (4.3) *possess the particular first integrals*

(4.5) $$B_{(\varrho)m} \frac{dx^m}{ds} = 0.$$

Constrained geodesics are a particular class of curves belonging to the more general system defined by

(4.6) $$\frac{d^2 x^r}{du^2} + \Gamma^r_{.mn} \frac{dx^m}{du} \frac{dx^n}{du} = 0.$$

I have not discovered the geometrical meaning of this system.

5. Parallel (Γ) propagation.

The quantities $\Gamma^r_{.mn}$ are not the components of a tensor; if we transform from the coordinate system x^r to the coordinate system $'x^r$, the formulae of transformation are easily seen to be

(5.1) $$'\Gamma^r_{.mn} = \Gamma^c_{.ab} \frac{\partial' x^r}{\partial x^c} \frac{\partial x^a}{\partial' x^m} \frac{\partial x^b}{\partial' x^n} + \frac{\partial^2 x^c}{\partial' x^m \partial' x^n} \frac{\partial' x^r}{\partial x^c}.$$

Thus, if a contravariant vector X^r is given along any curve, the quantities

(5.2) $$\tilde{X}^r = \frac{dX^r}{ds} + \Gamma^r_{.mn} X^m \frac{dx^n}{ds}$$

are the components of a contravariant vector. The coefficients $\Gamma^r_{.mn}$ are competent to define a type of parallel propagation[13]), which we shall call *parallel* (Γ) *propagation*, defined by the equations

$$\tilde{X}^r = 0. \tag{5.21}$$

This is to be distinguished from the ordinary parallel propagation,

$$\overline{X}^r \equiv \frac{dX^r}{ds} + \left\{ {r \atop m\,n} \right\} X^m \frac{dx^n}{ds} = 0. \tag{5.22}$$

We have, in fact,

$$\tilde{X}^r = \overline{X}^r + C^r_{.mn} X^m \frac{dx^n}{ds}. \tag{5.23}$$

It is obvious that the unit tangent vector to a constrained geodesic undergoes parallel (Γ) propagation. If λ^r denotes the unit tangent vector, the equations of a constrained geodesic may be written in either of the forms

$$\tilde{\lambda}^r = 0, \tag{5.3}$$

$$\overline{\lambda}^r = -C^r_{.mn} \lambda^m \lambda^n. \tag{5.31}$$

An interesting feature of the present work is the natural appearance in a discussion immediately suggested by classical dynamical theory of a type of geometry whose physical interpretation is usually connected with electromagnetic theory in the general theory of relativity. Equations (4.3), however, are not as general as the paths of Eisenhart and Veblen[14]), because our $\Gamma^r_{.mn}$ are defined by (4.31), in which $C^r_{.mn}$ is not an arbitrary mixed tensor of the type indicated.

It is to be remembered that $C^r_{.mn}$ is not, in general, symmetrical in m and n; hence $\Gamma^r_{.mn}$ is not, in general, symmetrical in these indices.

6. Geodesic stability for a general correspondence.

In the present section there is first discussed a problem slightly more general than that of the stability of constrained geodesics. The work up to and including equation (6.37) is a discussion of the stability of the curves defined by the equations

$$\frac{d^2x^r}{du^2} + \Gamma^r_{.mn} \frac{dx^m}{du} \frac{dx^n}{du} = 0 \qquad (\Gamma^r_{.mn} \neq \Gamma^r_{.nm}), \tag{6.1}$$

with the particular first integral

$$g_{mn} \frac{dx^m}{du} \frac{dx^n}{du} = 1. \tag{6.11}$$

[13]) This definition differs from the definition of parallel propagation given by Vranceanu, Comptes Rendus 183 (1926), p. 853, which applies only to vectors satisfying the constraints.

[14]) Proc. Nat. Acad. Sci. 8 (1922), p. 19.

The facts that $\Gamma^r_{.mn}$ are defined by (4.31) and that the curves under discussion are constrained curves are not introduced until after (6.37). In the work prior to that equation $\Gamma^r_{.mn}$ may be any functions of position, subject only to the condition that (6.11) is true. We shall refer to any curve satisfying (6.1) and (6.11) as a *path*. On account of the existence of (6.11), we may put $u = s$ in (6.1).

Let C and C^* (Fig. 1) be two neighbouring paths, and let a correspondence be established in some manner between the points of C and the points of C^*. Let O^* correspond to O, and let P^* correspond to P. Regarding O, O^* as fixed and P, P^* as variable, let us write $s^* = O^* P^*$, $s = O P$. The correspondence between the points of the two paths may be expressed by a relation of the form

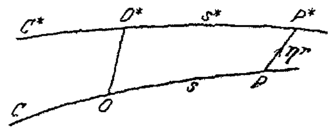

Fig. 1.

$$(6.2) \qquad s^* = s + f(s).$$

Now, if λ^r denotes the unit vector tangent to C and η^r the infinitesimal displacement PP^*, we have[15])

$$(6.21) \qquad f(s) = s^* - s = \int_O^P g_{mn} \lambda^m \bar{\eta}^n \, ds$$

Hence

$$(6.22) \qquad f'(s) = g_{mn} \lambda^m \bar{\eta}^n,$$

and

$$(6.23) \qquad f''(s) = g_{mn} \lambda^m \bar{\bar{\eta}}^n + g_{mn} \bar{\lambda}^m \bar{\eta}^n.$$

Let x^r denote the coordinates of P. Then the coordinates of P^* are $(x^r + \eta^r)$. Since C^* is a path, we have

$$(6.3) \qquad \frac{d^2}{ds^{*2}}(x^r + \eta^r) + \left(\Gamma^r_{.mn} + \eta^s \frac{\partial}{\partial x^s} \Gamma^r_{.mn}\right)\left(\frac{dx^m}{ds^*} + \frac{d\eta^m}{ds^*}\right)\left(\frac{dx^n}{ds^*} + \frac{d\eta^n}{ds^*}\right) = 0.$$

Assuming the function $f(s)$ and its derivatives to be infinitesimals of the order of η^r, and neglecting squares and higher powers of such infinitesimals, we have

$$(6.31) \qquad \frac{d}{ds^*} = [1 - f'(s)] \frac{d}{ds},$$

$$(6.32) \qquad \frac{d^2}{ds^{*2}} = [1 - 2f'(s)] \frac{d^2}{ds^2} - f''(s) \frac{d}{ds}.$$

Thus (6.3) may be written

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[15]) Cf. Proc. London Math. Soc. (2) 25 (1926), p. 252.

$$(6.33)\qquad [1-2f'(s)]\Big[\frac{d^2}{ds^2}(x^r+\eta^r)+\Big(\Gamma^r_{.mn}+\eta^s\frac{\partial}{\partial x^s}\Gamma^r_{.mn}\Big)$$
$$\Big(\frac{dx^m}{ds}+\frac{d\eta^m}{ds}\Big)\Big(\frac{dx^n}{ds}+\frac{d\eta^n}{ds}\Big)\Big]-f''(s)\frac{dx^r}{ds}=0,$$

or, since C is a path,

$$(6.34)\qquad \frac{d^2\eta^r}{ds^2}+(\Gamma^r_{.mn}+\Gamma^r_{.nm})\frac{d\eta^m}{ds}\lambda^n+\eta^s\lambda^m\lambda^n\frac{\partial}{\partial x^s}\Gamma^r_{.mn}-f''(s)\lambda^r=0.$$

Now

$$(6.35)\qquad \tilde{\eta}^r=\frac{d\eta^r}{ds}+\Gamma^r_{.mn}\eta^m\lambda^n,$$

and

$$(6.36)\qquad \tilde{\tilde{\eta}}^r=\frac{d^2\eta^r}{ds^2}+\eta^m\lambda^n\lambda^s\frac{\partial}{\partial x^s}\Gamma^r_{.mn}+\Gamma^r_{.mn}\lambda^n(\tilde{\eta}^m-\Gamma^m_{.ab}\eta^a\lambda^b)$$
$$+\Gamma^r_{.mn}\eta^m(\tilde{\lambda}^n-\Gamma^n_{.ab}\lambda^a\lambda^b)+\Gamma^r_{.mn}\tilde{\eta}^m\lambda^n,$$

or, since $\tilde{\lambda}^n=0$,

$$(6.365)\qquad \tilde{\tilde{\eta}}^r=\frac{d^2\eta^r}{ds^2}+2\Gamma^r_{.mn}\tilde{\eta}^m\lambda^n$$
$$+\eta^m\lambda^n\lambda^s\Big(\frac{\partial}{\partial x^s}\Gamma^r_{.mn}-\Gamma^r_{.an}\Gamma^a_{.ms}-\Gamma^r_{.ma}\Gamma^a_{.ns}\Big).$$

Substituting for $d^2\eta^r/ds^2$ in (6.34), we obtain

$$(6.37)\qquad \boxed{\tilde{\tilde{\eta}}^r-2S^r_{.mn}\tilde{\eta}^m\lambda^n+F^r_{.msn}\lambda^m\eta^s\lambda^n-f''(s)\lambda^r=0,}$$

where

$$(6.38)\qquad S^r_{.mn}=\frac{1}{2}(\Gamma^r_{.mn}-\Gamma^r_{.nm})=\frac{1}{2}(C^r_{.mn}-C^r_{.nm}),$$

the torsion tensor of Cartan, and

$$(6.39)\qquad F^r_{.msn}=\frac{\partial}{\partial x^s}\Gamma^r_{.nm}-\frac{\partial}{\partial x^n}\Gamma^r_{.sm}+\Gamma^a_{.nm}\Gamma^r_{.sa}-\Gamma^a_{.sn}\Gamma^r_{.na},$$

a curvature tensor of the manifold with respect to $\Gamma^r_{.mn}$.

Equation (6.37) is the *first tensorial equation for the disturbance vector* η^r. These equations are linear differential equations of the second order.

When we think of C and C^* as constrained geodesics, instead of general paths, we must add the conditions that they should be constrained curves. These conditions are

$$(6.4)\qquad B_{(\varrho)n}\lambda^n=0,\quad \Big(B_{(\varrho)n}+\eta^m\frac{\partial}{\partial x^m}B_{(\varrho)n}\Big)\Big(\lambda^n+\frac{d\eta^n}{ds}\Big)=0,$$

which give

$$(6.41)\qquad B_{(\varrho)n}\bar{\eta}^n+B_{(\varrho)nm}\eta^m\lambda^n=0,$$

or

$$(6.42)\qquad B_{(\varrho)n}\tilde{\eta}^n+(B_{(\varrho)nm}-B_{(\varrho)s}C^s_{.mn})\eta^m\lambda^n=0.$$

To investigate the stability of the constrained geodesic C, it is necessary to solve (6.37) and consider only those solutions which satisfy (6.42).

We shall now express (6.37) in another form. We have, analogous to (6.36),

$$(6.5)\qquad \bar{\bar{\eta}}^r = \frac{d^2\eta^r}{ds^2} + \eta^m \lambda^n \lambda^s \frac{\partial}{\partial x^s}\left\{ {r \atop mn} \right\} + \left\{ {r \atop mn} \right\} \lambda^n \left(\bar{\eta}^m - \left\{ {m \atop ab} \right\} \eta^a \lambda^b \right)$$

$$+ \left\{ {r \atop mn} \right\} \eta^m \left(\bar{\lambda}^n - \left\{ {n \atop ab} \right\} \lambda^a \lambda^b \right) + \left\{ {r \atop mn} \right\} \bar{\eta}^m \lambda^n,$$

or, by virtue of (5.31),

$$(6.51)\qquad \bar{\bar{\eta}}^r = \frac{d^2\eta^r}{ds^2} + 2\left\{ {r \atop mn} \right\} \bar{\eta}^m \lambda^n$$

$$+ \eta^m \lambda^n \lambda^s \left(\frac{\partial}{\partial x^s}\left\{ {r \atop mn} \right\} - \left\{ {r \atop an} \right\}\left\{ {a \atop ms} \right\} - \left\{ {r \atop ma} \right\}\left\{ {a \atop ns} \right\} \right) - \eta^m \lambda^s \lambda^t \left\{ {r \atop mn} \right\} C^n_{.st}.$$

Substitution for $d^2\eta^r/ds^2$ in (6.34) gives

$$(6.52)\qquad \boxed{\bar{\bar{\eta}}^r + (C^r_{.mn} + C^r_{.nm})\bar{\eta}^m \lambda^n + (G^r_{.msn} + C^r_{.nms})\lambda^m \eta^s \lambda^n - f''(s)\lambda^r = 0,}$$

where

$$(6.53)\qquad G^r_{.msn} = \frac{\partial}{\partial x^s}\left\{ {r \atop mn} \right\} - \frac{\partial}{\partial x^n}\left\{ {r \atop sm} \right\} + \left\{ {a \atop nm} \right\}\left\{ {r \atop sa} \right\} - \left\{ {a \atop sm} \right\}\left\{ {r \atop na} \right\},$$

the curvature tensor of the manifold, and

$$(6.54)\qquad C^r_{.nms} = \frac{\partial}{\partial x^s} C^r_{.nm} - C^r_{.am}\left\{ {a \atop ns} \right\} - C^r_{.na}\left\{ {a \atop ms} \right\} + C^a_{.nm}\left\{ {r \atop sa} \right\},$$

the covariant derivative of $C^r_{.nm}$.

Equation (6.52) is the *second tensorial equation for the disturbance vector* η^r. Its left hand side is precisely the left hand side of (6.37) expressed in a different manner. We note that, in the case of no constraint, (6.52) reduces immediately to the equation of Levi-Civita [16]).

We shall now obtain from (6.52) an invariant equation for the magnitude of the disturbance vector. Let μ^r be a unit vector codirectional with η^r, so that

$$(6.6)\qquad \eta^r = \eta\mu^r, \qquad g_{mn}\mu^m\mu^n = 1.$$

Then

$$(6.61)\qquad \bar{\eta}^r = \frac{d\eta}{ds}\mu^r + \eta\bar{\mu}^r,$$

and [17])

$$(6.62)\qquad \bar{\bar{\eta}}^r \mu_r = \frac{d^2\eta}{ds^2} - \eta\bar{\mu}^2.$$

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[16]) Math. Annalen 97 (1926), p. 315, equation (42).

[17]) Cf. GD., equation (9.21).

Hence, multiplying (6.52) by μ_r and summing as indicated, we obtain

$$(6.63)\qquad \frac{d^2\eta}{ds^2} - \eta\,\overline{\mu}^2 + (C^r_{.mn} + C^r_{.nm})\left(\frac{d\eta}{ds}\mu^m + \eta\,\overline{\mu}^m\right)\lambda^n\mu_r + (G^r_{.msn} + C^r_{.nms})\,\eta\,\lambda^m\mu^s\lambda^n\mu_r - f''(s)\,\lambda^r\mu_r = 0,$$

or

$$(6.64)\qquad \boxed{\begin{aligned} &\frac{d^2\eta}{ds^2} + \frac{d\eta}{ds}\mu^r\lambda^m\mu^n(C_{rmn} + C_{rnm}) \\ &+ \eta\left[(G_{rmsn} + C_{rnms})\mu^r\lambda^m\mu^s\lambda^n + (C_{rmn} + C_{rnm})\mu^r\lambda^m\overline{\mu}^n - \overline{\mu}^2\right] \\ &- f''(s)\,\lambda^r\mu_r = 0. \end{aligned}}$$

This is the *invariant equation for the magnitude of the disturbance vector.*

7. Isometric and normal correspondences.

So far we have been considering a general correspondence, given by the function $f(s)$, between the points of C and C^*. When there are no constraints, C and C^* being then free geodesics, there exists a correspondence of obvious simplicity. This correspondence is obtained by putting $f(s) = 0$; it makes $s^* = s$, and also makes PP^* normal to C, provided that OO^* has been chosen normal to C. In the case of paths (or constrained geodesics) it is no longer possible to obtain a correspondence which combines these two features, and we are led to consider two types of correspondence:

1. The *isometric correspondence,* defined by $s^* = s$. This gives

$$(7.1)\qquad f(s) = 0.$$

2. The *normal correspondence,* defined by the condition that PP^* should be normal to C. The analytical expression for this condition is

$$(7.2)\qquad g_{mn}\,\eta^m\lambda^n = 0.$$

The reductions in (6.37), (6.52) and (6.64) in the case of the isometric correspondence are very simple. The last term is to be omitted from each of these equations. We also know, by (6.22), that the particular first integral

$$(7.3)\qquad g_{mn}\,\overline{\eta}^m\lambda^n = 0$$

exists.

In the case of the normal correspondence, differentiation of (7.2) gives

$$(7.4)\qquad g_{mn}\,\overline{\eta}^m\lambda^n + g_{mn}\,\eta^m\overline{\lambda}^n = 0,$$

or, by (5.31),

$$g_{mn}\bar{\eta}^m \lambda^n - g_{mn} C^n_{.ab} \eta^m \lambda^a \lambda^b = 0. \tag{7.41}$$

Differentiating again, we obtain

$$g_{mn}\bar{\bar{\eta}}^m \lambda^n + g_{mn}\bar{\eta}^m \bar{\lambda}^n - g_{mn} C^n_{.abc} \eta^m \lambda^a \lambda^b \lambda^c - g_{mn} C^n_{.ab} \eta^m (- C^a_{.cd} \lambda^b \lambda^c \lambda^d - C^b_{.cd} \lambda^a \lambda^c \lambda^d) = 0. \tag{7.42}$$

Hence, by (6.23),

$$f''(s) = \eta^m \lambda^a \lambda^b \lambda^c (C_{mabc} - C_{mpb} C^p_{.ca} - C_{map} C^p_{.cb}). \tag{7.5}$$

This is the value for $f''(s)$ which must be substituted in (6.37) and (6.52) in the case of the normal correspondence. The reduction in (6.64) consists in the omission of the last term, since $\lambda^r \mu_r = 0$.

8. Dynamical significance of the paper.

Being given a conservative dynamical system with N coordinates x^r, subject to M constraints given by

$$B_{(\varrho)m}\, dx^m = 0, \tag{8.1}$$

where $B^r_{(\varrho)}$ are mutually orthogonal unit vectors, the equations of motion may be written[18])

$$\frac{d^2 x^r}{ds^2} + \left\{ {r \atop mn} \right\} \frac{dx^m}{ds}\frac{dx^n}{ds} = - B^r_{(\varrho)} B_{(\varrho)mn} \frac{dx^m}{ds}\frac{dx^n}{ds}, \tag{8.2}$$

where ds is the action line-element, given by

$$ds^2 = g_{mn}\, dx^m dx^n = (h - V)\, a_{mn}\, dx^m dx^n, \tag{8.21}$$

h being the constant total energy of the motion, V the potential energy, and $\frac{1}{2} a_{mn} \dot{x}^m \dot{x}^n$ the kinetic energy.

Since (8.2) is identical with (3.5), we have the result:

Theorem V. *The constrained geodesics of the present paper are the curves of motion of a conservative dynamical system, subject to non-holonomic constraints, the line-element being the action line-element.*

Thus the geometrical considerations as to stability apply to conservative dynamical systems subject to non-holonomic constraints, only those disturbances being considered which do not change the total energy.

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[18]) GD., p. 58, Theorem XVII (A).

(Eingegangen am 25. 8. 1927.)