Topological equivalence of complexes.

Von

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The "internal transformation" (simple subdivision of a cell), which is the basis of combinatory analysis situs as developed in the works of Dehn, Heegaard, Kneser[1]), and others, does not lend itself very readily to the direct investigation of topological invariants: owing to the great variety of ways in which a single n-cell may be subdivided it is necessary to interpolate discussions of the properties of spheres and their intersections not always germane to the real subject matter. It is the object of this paper to shew that a certain general transformation, of which "internal transformation" is a special case, can always be carried out by a series of "moves" of a much more specialised kind, whose effects are easily followed without the help of general theorems about spheres.

The moves employed are those that I defined in an earlier paper[2]), but the standpoint there adopted has been modified in one respect. The "n-arrays", discussed in §§ 1 and 2 below, are no longer regarded as being themselves the subject matter of combinatory topology, but are thought of rather as instruments for proving theorems about the n-*complexes* introduced in § 3. This does not necessitate any change in the definitions or results of FI and FII, save that the symbol "$\rightarrow$" is no longer read as "is topologically equivalent to", which is given a different sense.

§ 1 is a summary of the definitions and leading theorems of FI and FII, § 2 contains a new proof and generalisation of FII Theorem 10, § 3 gives its application to n-complexes.

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[1]) Dehn and Heegaard, Encykl. der Math. Wiss. III, AB 3 *Analysis Situs*; H. Kneser, Proc. Amsterdam 27 (1924), p. 601; E. Bilz, Math. Zeitschr. 18 (1923), S. 1.

[2]) The Foundations of Combinatory Analysis Situs, I and II, Proc. Amsterdam 29 (1926), p. 611 and 627, referred to as FI and FII. See also "Additions and Corrections" to these papers, Proc. Amsterdam 30 (1927), p. 670, cited as F. Add.

§ 1.

Arrays[3].

If n is a positive integer or zero, an *n-array* is formed from a finite or enumerable set of objects by choosing as the *units* of the array certain of the groups of $n+1$ objects contained in the set[4]. The choice is unrestricted save that every object must belong to at least one unit. The objects are called the *vertices* of the n-array, and if $0 \leqq k \leqq n$ any $k+1$ vertices all belonging to the same unit form a *k-component* of the array. An $(n-1)$-component is called a *face*; units with a face in common are *adjacent*. An *n-simplex* is an n-array with only one unit. The *logical sum*, $\Gamma \dotplus \Delta \dotplus \ldots$, of a number of n-arrays, $\Gamma, \Delta, \ldots$, is the n-array whose units are all the units of all the arrays[5]. — The *sum mod* 2 of $\Gamma, \Delta, \ldots$, denoted by $\Gamma + \Delta + \ldots$, is the n-array whose units are the n-simplexes contained in [6] an odd number of the arrays. When no two of $\Gamma, \Delta, \ldots$ have a common unit the logical sum and the sum mod 2 are the same and may be called the "sum", simply, and denoted by $\Gamma + \Delta + \ldots$. If the n-array Γ contains the n-array Δ, $\Gamma - \Delta$ is the sum of the n-simplexes contained in Γ but not in Δ.

If S and T are simplexes with no common vertex ST is the simplex containing all the vertices of both. If Γ and Δ are arrays with no common vertex $\Gamma\Delta$ is the sum of all products ST, where S is a unit of Γ and T of Δ.

An n-array is *regular* if each face belongs to at most two units and each vertex to a finite number. If is *connected* if every two units are the extreme members of a sequence of units such that adjacent members of the sequence are adjacent units of the array.

The sum of the faces of an n-array, Γ, that belong each to only one unit is the *boundary* of Γ, denoted by $\overline{\Gamma}$, (or, when that is incovenient, by B (Γ)). A component not contained in the boundary is *internal*. The sum mod 2 of the boundaries of the units of Γ is the *margin* of Γ, denoted by $\widehat{\Gamma}$.

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[3]) The definitions in § 1 are taken, with certain slight modifications, from *F* I and *F* II, where proofs of the theorems will be found. It should be noticed that Theorem 10 of *F* II is not assumed. The present paper replaces § 9 of *F* II.

[4]) P. Alexandroff pointed out almost simultaneously (Math. Annalen 96 (1926), p. 489) that an n-simplex may be regarded as being simply its $n+1$ vertices. See also the same author's paper, Math. Annalen 94 (1925), p. 296.

[5]) The corresponding notation for sets of points was introduced by Carathéodory, *Reelle Funktionen*, p. 28.

[6]) The array Γ^1 *contains* the array Γ^2 if every unit of Γ^2 is a unit or component of Γ^1.

It is easily proved that *if* Γ *and* Δ *have no common vertex and* $d(\Gamma\Delta) > 0$ [7]*,* $\overline{\Gamma\Delta}$ *is* $\Gamma\cdot\overline{\Delta} + \Delta\cdot\overline{\Gamma}$ (F I 1) *and* $\widehat{\Gamma\Delta}$ *is* $\Gamma\cdot\widehat{\Delta} + \Delta\cdot\widehat{\Gamma}$ (F I 1a). Here it is agreed that if Γ is a vertex $\overline{\Gamma}$ (*or* $\widehat{\Gamma}$) shall be omitted from terms containing it, and if Γ is unbounded (*or* has no margin) terms containing $\overline{\Gamma}$ (*or* $\widehat{\Gamma}$) are to be omitted. E. g., if α is a vertex $\overline{\alpha\Gamma}$ is $\Gamma + \alpha\cdot\overline{\Gamma}$ [8]).

(In the rest of this paper all arrays are supposed to contain only a finite number of vertices.)

S_n is said to have *regular contact* with Γ_n, not containing it, if it is the product of two components U and V, (S_n is UV), such that (I) U belongs to $\overline{\Gamma}_n$, (II) U is interior to $\Gamma_n + UV$, (III) V does not belong to Γ_n. (U and V must each contain at least one vertex.)

If Γ_n *is regular the common faces of* Γ_n *and* UV *form the array* $U\cdot\overline{V}$*, lying in* $\overline{\Gamma}_n$ (F I 4 and 5).

If Γ_n is bounded it is a *move of type* 1 to add to (Γ_n) a simplex, S_n, having regular contact with it, and a *move of type* 2 to remove T_n having regular contact with $\Gamma_n - T_n$.

If Γ_n contains $S_k\cdot\overline{T}_{n-k}$, but $\Gamma_n - S_k\cdot\overline{T}_{n-k}$ contains neither S_k nor T_{n-k}, it is a *move of type* 3 to substitute $\overline{S}_k\cdot T_{n-k}$ for $S_k\cdot\overline{T}_{n-k}$ in Γ_n. (The case $k = n$ is included).

If Δ can be obtained from Γ by a succession of moves of any of the three types we write $\Gamma \rightarrow \Delta$ [9]); if only moves of types p and q are required we write $\Gamma \underset{pq}{\rightarrow} \Delta$.

If $\Gamma_n \rightarrow S_n$, Γ_n is an *n-element* (E_n). The boundary of an $(n+1)$-element is an *n-sphere* (Σ_n). *The necessary and sufficient condition that* Γ_n *should be an n-sphere is that* $\Gamma_n \underset{3}{\rightarrow} \overline{S}_{n+1}$ (F I 9 and 14).

If S_k and Σ_{n-k-1} have no common vertex $S_k\Sigma_{n-k-1}$ is a *complete* $(n:k)$*-cluster*; if S_k and E_{n-k-1} have no common vertex $S_k E_{n-k-1}$ is an *incomplete* $(n:k)$*-cluster.* S_k is the *core,* Σ_{n-k-1} and E_{n-k-1} are the *shells.* An $(n:0)$-cluster is an *n-star.*

An *n-manifold* (M_n) is a connected n-array such that the sum of the units at each vertex is an n-star. If all the stars are complete the manifold is unbounded, otherwise it is bounded.

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[7]) $d(\Gamma)$ means "the dimension-number of Γ".

[8]) The letters Γ and Δ will be used for arrays of uncertain character; S, T, U, V, for simplexes; small Greek letters for vertices. If a lower index is present, it denotes the dimension number; upper indices are merely distinguishing marks.

[9]) This differs from the definition of F I in the case of unbounded arrays, but the two definitions are equivalent in the case of *manifolds* in view of F I 21 and F II Theorem 5. "$\Gamma \rightarrow \Delta$" is no longer to be read "Γ is topologically equivalent to Δ" but, *e. g.*, as "Γ leads to Δ". Cf. F. Add., under "Topological equivalence".

The units of M_n containing the component S_k form an $(n:k)$-cluster, complete or incomplete as S_k is internal to M_n or in $\overline{M}_n$. (F I 19 and 31.) This $(n:k)$-cluster is called the *S_k-cluster in M_n.*

The numerous small theorems — *$(n:k)$-clusters are n-elements, n-spheres a ren-manifolds*, etc. — required to set in motion the theory based on these definitions will be found in FI. The following results from FII may here be mentioned:

FII Theorems 1, 3a, 3b: *If $M^1 \rightarrow M^2$ then $M^1 \underset{12}{\rightarrow} M^2$, $M^1 \underset{23}{\rightarrow} M^2$, $M^1 \underset{13}{\rightarrow} M^2$.*

FII Theorem 2. *If $M^1 \underset{3}{\rightarrow} M^2$ and $\Gamma + M^1$ is a manifold, then $\Gamma + M^1 \underset{3}{\rightarrow} \Gamma + M^2$, provided Γ contains no internal component of M^2.*

FII Theorem 4. *If $\overline{E}^1$ is $\overline{E}^2$, then $E^1 \underset{3}{\rightarrow} E^2$.*

FII Theorem 5. *If the unbounded manifolds M^1 and M^2 contain units S and T, respectively, such that $M^1 - S \rightarrow M^2 - T$, then $M^1 \underset{3}{\rightarrow} M^2$*[10].

FII Theorem 6. *If $\overline{E}_n^1$ is $\overline{E}_n^2$ and M_n contains E_n^1, then $M_n \rightarrow (M_n - E_n^1) + E_n^2$, provided $M_n - E_n^1$ contains no internal component of E_n^2.*

(If M_n is unbounded $M_n \underset{3}{\rightarrow} (M_n - E_n^1) + E_n^2$.)

Corollary 1. *If $\overline{E}_n^1$ is $\overline{E}_n^2$, but E_n^1 and E_n^2 have no common internal component, $E_n^1 + E_n^2$ is an n-sphere.*

Corollary 2. *$\Sigma_n - E_n$ is an n-element.*

FII Theorem 7. *Suppose M_n contains E_n and $\overline{E}_n$ contains E_{n-1}, and let E_n^* be a second n-element whose boundary contains E_{n-1}. Then if all components of $M - E_n$ belonging to E_n or E_n^* belong to E_{n-1}, $M_n \rightarrow (M_n - E_n) + E_n^*$.*

FII Theorem 8a. *If the common part of M_n and E_n is an $(n-1)$-element in the boundary of each, $M_n \rightarrow M_n + E_n$.*

FII Theorem 8b. *If M_n contains E_n, and the part of E_n in $\overline{M}_n$ is an $(n-1)$-element, then $M_n \rightarrow M_n - E_n$.*

FII Lemma 7a. *If E and Π have no common vertex, Π being a sphere or an element, then $E\Pi$ is an element.*

FII Lemma 7b. *If Σ and Σ' have no common vertex, $\Sigma\Sigma'$ is a sphere.*

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[10]) Theorems 1, 3, 5, give all the general relations between the three moves. The only other plausible suggestion — "If $M^1 \rightarrow M^2$ and $\overline{M}^1$ is $\overline{M}^2$ then $M^1 \underset{3}{\rightarrow} M^2$" — is false. See F. Add. p. 671.

§ 2.

Assemblies of pieces.

An *n-dimensional assembly of pieces* is a collection of elements, (the "pieces" of the assembly), of all dimension numbers from 0 to n such that (denoting by G_i an i-dimensional piece):

$P(\mathrm{I})$ if G_i contains a unit or internal component of G_j, $\bar{G}_i$ contains G_j unless G_i and G_j are identical;

$P(\mathrm{II})$ if $i > 0$ every unit of $\bar{G}_i$ belongs to an $(i-1)$-dimensional piece;

$P(\mathrm{III})$ if $i < n$ G_i is contained in an $(i+1)$-dimensional piece.

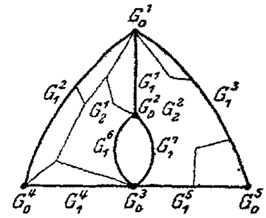

Fig. 1.
2-dimensional assembly with two 2-pieces.

Two assemblies $\mathsf{\Gamma}$ and $\mathsf{\Gamma}'$ have the *same structure* if a (1, 1) correspondence $\mathfrak{A}$ can be set up between their pieces so that (a) $d(\mathfrak{A}G_i) = i$; (b) G_i (of $\mathsf{\Gamma}$) contains G_j (of $\mathsf{\Gamma}$) if, and only if, $\mathfrak{A}G_i$ contains $\mathfrak{A}G_j$. $\mathfrak{A}$ is the *structural relation*.

If $\mathsf{\Gamma}$ denotes an n-assembly, Γ denotes the sum of the n-dimensional pieces of $\mathsf{\Gamma}$ [11]).

Lemma 1. *If all j-pieces of the assembly $\mathsf{\Gamma}_n$ for which $j > k$ are complete stars, and G_k is any k-piece, the units of Γ_n that contain a unit of G_k form an array $G_k \Delta_{n-k-1}$.*

It must be shewn that if U_k and V_k are units of G_k, and $U_k X_{n-k-1}$ is a unit of Γ_n, then $V_k X_{n-k-1}$ is a unit of Γ_n. Let G_n be the n-piece containing $U_k X_{n-k-1}$. The centre, α, of G_n does not, by $P(\mathrm{I})$, belong to G_k, and therefore $U_k X_{n-k-1}$ is $\alpha U_k Y_{n-k-2}$. The pieces of $\mathsf{\Gamma}_n$ contained in $\bar{G}_n$ constitute an $(n-1)$-assembly and so, using an inductive hypothesis, $\bar{G}_n$, which contains $U_k Y_{n-k-2}$, contains $V_k Y_{n-k-2}$. Hence G_n contains $V_k X_{n-k-1}$.

The array $G_k \Delta_{n-k-1}$ is called the *G_k-cluster in Γ_n*, Δ_{n-k-1} is its *shell*.

If Γ_n is a manifold Δ_{n-k-1} is a sphere or an element. (For if U_k is a unit of G_k, $U_k \Delta_{n-k-1}$ is the *U_k-cluster* in Γ_n.)

Lemma 2. *Let G_k^0 be a k-piece of the n-assembly M, (M being a manifold), and let H_k^0 be a k-element, whose boundary is identical with $\bar{G}_k^0$ but of which no unit or component not contained in G_k^0 belongs to M. Then there is an assembly $\mathsf{\Gamma}$ with these properties:*

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11) The conventions governing the use of letters may be extended to assemblies; *e. g.*, the use of M to denote an n-assembly implies that the sum of its n-pieces is a manifold.

(a) $M \rightarrow \Gamma$;

(b) *M and Γ have the same structure (with structural relation $\mathfrak{A}$, say)*;

(c) *if $j > k$ and G_j is a j-piece of M, $\mathfrak{A} G_j$ is a complete star*;

(d) *if $j \leqq k$ $\mathfrak{A} G_j$ is G_j, save that $\mathfrak{A} G_k^0$ is H_k^0*[12]).

The lemma being true when $k = n$, in virtue of FII, Theorems 4 and 2[13]), suppose it true when k is replaced by number greater than itself. By applying this hypothesis to all the $(k+1)$-dimensional pieces of M in turn, (taking H_{k+1}^0 itself to be a complete star), M may be transformed into an assembly M′ with the same structure, in which all pieces with $k+1$ or more dimensions are complete stars, while other pieces are unchanged, and $M \rightarrow M'$. We may then suppose, without loss of generality, that in M itself all pieces of more than k dimensions are complete stars.

The boundary of any piece, G_j, containing G_k^0 has, (by Lemma 1, and FII Lemma 7a and Corollary 2 to Theorem 6), the form $E_{j-1} + G_k^0 \Sigma_{j-k-2}$. *If G_h (with centre α) is contained $\overline{G}_j$ and contains G_k^0, then $\alpha \Sigma_{h-k-2}$ is contained in Σ_{j-k-2} and αE_{h-1} in E_{j-1}*. (Remark A.) For $\alpha \Sigma_{h-k-2} G_k^0$ must be contained in $\Sigma_{j-k-2} G_k^0$, the set of all units of $\overline{G}_j$ containing a unit of G_k^0; and if αE_{h-1} did not belong entirely to E_{j-1} it would contain an internal component of $G_k^0 \Sigma_{j-k-2}$, *i. e.* an internal component of G_k^0.

Consider the set of j-arrays, $\mathfrak{A} G_j$, defined as follows (G_j being a typical cell of M):

$\mathfrak{A} G_k^0$ is H_k^0;

if G_j is $\alpha^j (E_{j-1} + G_k^0 \Sigma_{j-k-2})$, $\mathfrak{A} G_j$ is $\alpha^j (E_{j-1} + H_k^0 \Sigma_{j-k-2})$;

if G_j does not contain G_k^0, $\mathfrak{A} G_j$ is G_j.

This set of arrays may be called $\mathfrak{A}$M. The following properties are evident: *All vertices of $\mathfrak{A} G_i$ not in H_k^0 belong to G_j* (Remark B). *The necessary and sufficient condition that $\overline{\mathfrak{A} G_h}$ should contain an array $H_k^0 \Gamma_q$ is that $\overline{G}_h$ should contain $G_k^0 \Gamma_q$*. (Remark C.) *If a unit U_j of G_j contains no unit of G_k^0, U_j belongs to $\mathfrak{A} G_j$; and conversely, a unit of $\mathfrak{A} G_j$ containing no unit of H_k^0 belongs to G_j*. (Remark D.)

It can now be shewn that $\mathfrak{A}$M *is an n-assembly having the properties required of* Γ.

(a) $\mathfrak{A} G_j$ is a j-element.

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[12]) The invariance of the property of being a manifold was proved by Weyl (Revista di Matem. Hisp.-Amer., 1923) by a process somewhat similar to that here adopted. Weyl's fundamental definitions are, however, of a radically different character.

[13]) Quoted on p. 402.

Since α^j does not (by definition) belong to E_{j-1} or Σ_{j-k-2}, nor (by hypothesis) to H_k^0, it is sufficient to shew that $E_{j-1}+H_k^0\Sigma_{j-k-2}$ is a $(j-1)$-sphere, (F I 12). $\mathrm{B}(H_k^0\Sigma_{j-k-2})$ is $\overline{H}_k^0\Sigma_{j-k-2}$, which is $\overline{G}_k^0\Sigma_{j-k-2}$, which is $\overline{E}_{j-1}$; units and internal components of $H_k^0\Sigma_{j-k-2}$ contain units or internal components of H_k^0, and so do not belong to E_{j-1}. Hence (F II, Theorem 6, Corollary 1), $E_{j-1}+H_k^0\Sigma_{j-k-2}$ is a $(j-1)$-sphere.

(b) Every unit of $\overline{\mathfrak{A}G_j}$ belongs to an $\mathfrak{A}G_{j-1}$.

A unit, U_{j-1}, of $\overline{\mathfrak{A}G_j}$ not containing a unit of H_k^0 belongs to $\overline{G}_j$, and therefore to some piece G_{j-1}, i. e. (by Remark D) to $\mathfrak{A}G_{j-1}$. A unit V_{j-1} that contains a unit of H_k^0 belongs to an array $H_k^0X_{j-k-2}$ contained in $\overline{\mathfrak{A}G_j}$; $G_k^0X_{j-k-2}$ is contained in $\overline{G}_j$, and therefore in a piece G_{j-1}; and $\mathfrak{A}G_{j-1}$ contains V_{j-1}.

(c) If $\overline{G}_j$ contains G_h, $\overline{\mathfrak{A}G_j}$ contains $\mathfrak{A}G_h$.

This follows at once from Remark A unless G_h contains G_k^0. In that case (with the notation of Remark A) αE_{h-1} is contained in E_{j-1}, $H_k^0\alpha\Sigma_{h-k-2}$ in $H_k^0\Sigma_{j-k-2}$, and therefore $\mathfrak{A}G_h$ in $\overline{\mathfrak{A}G_j}$.

(d) If $\mathfrak{A}G_h$ contains a unit or internal component of $\mathfrak{A}G_j$, $\overline{\mathfrak{A}G_h}$ contains $\mathfrak{A}G_j$.

By (c) it is sufficient to shew that $\overline{G}_h$ contains $\overline{G}_j$. If $j \leqq k$ the result follows from Remark D (cf. (b)). If $j > k$ $\mathfrak{A}G_j$ is a complete star, whose centre, β, is contained in $\mathfrak{A}G_h$. Since β is also the centre of G_j it is not interior to H_k^0 and therefore (Remark A) belongs to G_h. Hence, by P(II), $\overline{G}_h$ contains G_j.

From these results it follows that $\mathfrak{A}\mathsf{M}$ is an n-assembly. The complete symmetry of the relations between M and $\mathfrak{A}\mathsf{M}$ being thus established it may be inferred that

(c′) If $\overline{\mathfrak{A}G_j}$ contains $\mathfrak{A}G_h$, then $\overline{G}_j$ contains G_h, shewing that M and $\mathfrak{A}\mathsf{M}$ have the same structure. Finally,

(e) $M \to \mathfrak{A}M$.

If $G_k^0\Pi_{n-k-1}$, (say E_n^1), is the sum of all units of M containing a unit of G_k^0, the corresponding sum in $\mathfrak{A}M$ is, by Remark C, $H_k^0\Pi_{n-k-1}$, (say E_n^2). The arrays $M-E_n^1$ and $\mathfrak{A}M-E_n^2$ are identical, and so are the $(n-1)$-elements $\overline{G}_k^0.\,\Pi_{n-k-1}$ and $\overline{H}_k^0\cdot\Pi_{n-k-1}$, which are respectively the part of $\overline{E}_n^1$ interior to M and the part of $\overline{E}_n^2$ interior to $\mathfrak{A}M$. Hence (by F II Theorem 6, if Π_{n-k-1} is a sphere, by Theorem 7 if it is an element) $M \to (M-E_n^1)+E_n^2$, which is $\mathfrak{A}M$.

Theorem 1. *If the assemblies* M *and* Γ *have the same structure and* M *is a manifold, then* Γ *is a manifold and* $M \to \Gamma$.

The 1-dimensional pieces in M can, by Lemma 2, be changed one by one into their assigned correlates in Γ, while all other pieces are changed into complete stars. The right boundaries having thus been provided for the 2-pieces these may next be changed, without disturbing the 1-pieces; and so on. To ensure that this process can be continued until Γ is finally reached it is only necessary to shew that the final condition imposed on H_k^0 in Lemma 2 is fulfilled at every stage.

Suppose then that M has been successfully changed into $M^{(k)}$, in which some k-dimensional pieces, and all j-pieces for which $j < k$, are "regulated", (i. e. congruent[14]) to their assigned Γ-correlates), while the remaining "unregulated" pieces are complete stars. Let the operation to be justified be the replacement of the k-piece, G_k^0, of $M^{(k)}$ by a k-element, with boundary $\bar{G}_k^0$, congruent to G_k^* of Γ, ($\bar{G}_k^0$ and $\bar{G}_k^*$ being congruent by hypothesis). It can certainly be arranged that no internal *vertex* of the new k-piece belongs to $M^{(k)}$, for the choice of the internal vertices is unrestrained. If, on the other hand, there is an internal component, U^*, of G_k^*, with vertices lying in $\bar{G}_k^*$, such that the simplex with the corresponding vertices in $\bar{G}_k^0$ belongs to $M^{(k)}$, then U must be a unit or internal component of some piece, G_j, of $M^{(k)}$[15]). If G_j is not G_k^0 itself it is one of the pieces already regulated, for the unregulated pieces are complete stars whose centres do not belong to $\bar{G}_k^0$. Hence G_j is congruent to its Γ-correlate, G_j^*, and $j > k$. It follows that U^*, which is a unit or internal component of G_k^*, belongs to G_j^*; and this is incompatible with $P(I)$.

§ 3.

Complexes.

(In this section the objects so far called "elements", "spheres", "manifolds", "pieces", will be called "△ elements", "△ spheres", "△ manifolds", and "△ pieces", to distinguish them from the "○ elements", "○ spheres" etc. now to be introduced[16]). "Array", "unit", "component", having no new analogues, are still used, without prefix, in the same sense as before.)

A *cell* is simply an object associated with a positive integer, its *dimension number*. A collection of cells, of dimension numbers from 0 to n, becomes an *n-set of cells*, if certain pairs of cells with dimension numbers differing by 1 are associated together. The association of an i-cell,

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[14]) Two arrays are *congruent* if they are completely similar.

[15]) If Γ is any assembly every component of Γ is a unit or internal component of some piece.

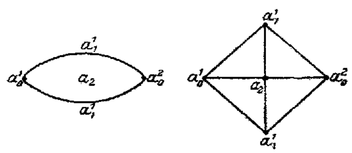
[16]) The connection between the △- and ○-entities having once been cleared up it should be possible to drop the prefixes in most contexts.

a_i, with an $(i+1)$-cell, a_{i+1}, is expressed in the statement "a_i *bounds* a_{i+1}", and more generally, if a_i bounds a_j and a_j bounds a_k, a_i is said to bound a_k.

A *proper* n-set is one in which if $i < n$ every i-cell bounds at least one $(i+1)$-cell, and if $j > 0$ every j-cell is bounded by at least one $(j-1)$-cell. From any proper n-set, ${}_{\bigcirc}\Gamma$, an n-array ${}_{\triangle}\Gamma$, called the *skeleton* of ${}_{\bigcirc}\Gamma$, is formed by taking as vertices the cells of ${}_{\bigcirc}\Gamma$ and as units all sets of $n+1$ cells, $a^{(0)}$, $a^{(1)}, \dots, a^{(n)}$, such that $a^{(i)}$ bounds $a^{(i+1)} (i = 0, \dots n)$. Evidently $a^{(i)}$ must be an i-cell[17].

If a_k is a cell of a proper n-set, the proper k-set formed by a_k and all the cells bounding it is denoted by ${}_{\bigcirc}\{a_k\}$. The array ${}_{\triangle}\{a_k\}$ is seen to be the sum of all k-components of ${}_{\triangle}\Gamma$ whose vertices form a sequence descending from a_k. Clearly if $a_n^1, a_n^2, \dots$, are the n-cells of ${}_{\bigcirc}\Gamma$, ${}_{\triangle}\Gamma$ is $\sum_{(i)} {}_{\triangle}\{a_n^i\}$.

The *boundary* $(n-1)$-*cells* of an n-set are those that bound only one n-cell; the *boundary* k-*cells*, $(k < n-1)$, are those that bound a boundary $(n-1)$-cell. Other cells, including all n-cells, are *internal*. The *boundary* of ${}_{\bigcirc}\Gamma$ (denoted by ${}_{\bigcirc}\overline{\Gamma}$ or $B({}_{\bigcirc}\Gamma)$) is the set of all boundary cells, (if any). If ${}_{\bigcirc}\Gamma$ is a proper n-set ${}_{\bigcirc}\overline{\Gamma}$ is evidently a proper $(n-1)$-set. If there is no boundary ${}_{\bigcirc}\Gamma$ is *unbounded*[18].



${}_{\bigcirc}\{a_2\}$ ${}_{\triangle}\{a_2\}$

Fig. 2.

Theorem 2. *If a_k is a boundary cell of the proper n-set ${}_{\bigcirc}\Gamma$, ${}_{\triangle}\{a_k\}$ belongs to $B({}_{\triangle}\Gamma)$.*

By hypothesis there exists in ${}_{\bigcirc}\Gamma$ a sequence of cells ascending from a_k to a cell a_{n-1} bounding only one n-cell, a_n. If then $a_k a_{k-1} \dots a_0$ is a unit of ${}_{\triangle}\{a_k\}$, $a_{n-1} a_{n-2} \dots a_k \dots a_0$ is a face of ${}_{\triangle}\Gamma$ containing it and belonging to only one unit of ${}_{\triangle}\Gamma$, *viz.* $a_n a_{n-1} \dots a_k \dots a_0$.

Corollary 1. *The skeleton of ${}_{\bigcirc}\overline{\Gamma}$ is contained in $B({}_{\triangle}\Gamma)$.*

(Set $k = n-1$ in Theorem 2).

Corollary 2. *If ${}_{\triangle}\Gamma$ is unbounded ${}_{\bigcirc}\Gamma$ is unbounded.*

(From Corollary 1).

An $n{\bigcirc}$*sphere* is a proper n-set whose skeleton is an $n{\triangle}$sphere.

From Corollary 2 to Theorem 2 it follows that *$n{\bigcirc}$ spheres are unbounded.*

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[17]) A sequence of cells $a_i, a_{i+1}, \dots, a_j$, in which each cell bounds its successor is called a sequence *ascending from a_i* or *descending* from a_j.

[18]) The *margin* of an n-set of cells may be defined analogously to the margin of an array (p. 400) and has properties similar to those of the boundary.

An *n-complex* is an *n*-set of cells (not necessarily "proper") such that the boundary of every *k*-cell[19]) is a $(k-1)$○sphere, $(k=1, 2, \ldots, n)$.

Theorem 3. *If* ${}_{\bigcirc}\Gamma$ *is a proper n-complex the skeleton of* ${}_{\bigcirc}\overline{\Gamma}$ *is* $B({}_{\triangle}\Gamma)$.

In view of Theorem 2, Corollary 1, it is sufficient to shew that every unit of $B({}_{\triangle}\Gamma)$ belongs to the skeleton of ${}_{\bigcirc}\overline{\Gamma}$.

In the first place *the vertices of any boundary face of* ${}_{\triangle}\Gamma$ *form a sequence descending from a boundary* $(n-1)$*-cell.* For if $a_n \ldots a_{j+1} a_{j-1} \ldots a_0$ were a bounding face there would be only one *j*-cell, a_j, such that a_{j+1}, a_j, a_{j-1} is a descending sequence. But $\bar{a}_{j+1}$ is unbounded and therefore there are at least two *j*-cells in $\bar{a}_{j+1}$ bounded by a_{j-1}. Again if $a_{n-1}\, a_{n-2} \ldots a_0$ is a bounding face of ${}_{\triangle}\Gamma$, a_{n-1} is a boundary cell, for if it bounded both a_n and a'_n, $a_n\, a_{n-1} \ldots a_0$ and $a'_n\, a_{n-1} \ldots a_0$ would be two units of ${}_{\triangle}\Gamma$ containing the given face.

From this Theorem 3 follows at once, for if a_{n-1} is a bounding $(n-1)$-cell all members of any sequence descending from it are by definition boundary cells, i. e. the members of any such sequence are the vertices of a unit of the skeleton of ${}_{\bigcirc}\Gamma$[20]).

Both $B({}_{\triangle}\Gamma)$ and the skeleton of ${}_{\bigcirc}\overline{\Gamma}$ may, then, be denoted by ${}_{\angle}\overline{\Gamma}$.

Corollary. *If* a_k *is a boundary vertex of* ${}_{\triangle}\Gamma$ *it is a boundary cell of* ${}_{\bigcirc}\Gamma$.

Theorem 4. *Every n*○*sphere is an n-complex.*

Let the theorem be assumed true of ○spheres of less than *n* dimensions.

If a_n is an *n*-cell of the *n*○sphere ${}_{\bigcirc}\Sigma$, the array ${}_{\triangle}\{a_n\}$, being the sum of all units of ${}_{\triangle}\Sigma$ containing the vertex a_n, is a complete star with centre a_n (*FI* 18). Hence the skeleton of $\bar{a}_n$, whose units are those of ${}_{\triangle}\{a_n\}$ with the vertex a_n left out, is an $(n-1)$△sphere: the boundaries of *n*-cells of ${}_{\bigcirc}\Sigma$ are $(n-1)$○spheres. It now follows from the inductive hypothesis that ${}_{\bigcirc}\Sigma$ is an *n*-complex.

An *n*○*element* is a proper *n*-complex whose skeleton is an *n*∠element[21]). By Theorem 3 its boundary is an $(n-1)$○sphere.

Theorem 5. *If* ${}_{\bigcirc}\Gamma$ *is a proper n-complex the arrays* ${}_{\triangle}\{a_k\}$ *formed from all the cells of* ${}_{\bigcirc}\Gamma$ *are the pieces of an n*△*assembly.*

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[19]) "The boundary of ${}_{\bigcirc}\{a_k\}$" may be abbreviated to "the boundary of a_k" and denoted by $\bar{a}_k$.

[20]) It will be noticed that the only property of ○spheres used in this proof is their unboundedness. For a systematic investigation of the properties that the boundaries of cells must be assumed to possess see the paper of Weyl already cited.

[21]) That an *n*○element cannot be defined to be any *n*-set whose skeleton is an *n*△element, and then *proved* to be an *n*-complex, is shewn by the simple example of a 1-cell bounded by a single 0-cell.

The arrays $_\triangle\{a_k\}$, being complete stars, are elements (FI 12). If a_j bounds a_k, $_\triangle\{a_j\}$ is contained in $_\triangle\{a_k\}$: P(III) is satisfied. If an internal component of $_\triangle\{a_j\}$ belongs to $_\triangle\{a_k\}$ the vertex a_j belongs to $_\triangle\{a_k\}$; i. e., in $_\circ\Gamma$, a_j bounds a_k. Hence $_\triangle\{a_j\}$, which does not contain a_k, belongs to $B\cdot{}_\triangle\{a_k\}$. This is P(I). The vertices of a boundary face, U_{k-1}, of $_\triangle\{a_k\}$ form a sequence descending from some $(k-1)$-cell, a_{k-1}, and U_{k-1} is a unit of $_\triangle\{a_{k-1}\}$. This is P(II).

Since all the pieces in this n-assembly are complete stars the "$_\triangle\{a_k\}$-cluster in $_\triangle\Gamma$" exists. (cf. Lemma 1).

Let $_\circ\Gamma$ be a complex and a_k one of its cells. If the dimension numbers of all cells bounded by a_k are diminished by $k+1$ while the bounding relations between the cells are maintained, the modified cells form an $(n-k-1)$-set called the *neighbour complex of* a_k in $_\circ\Gamma$, (NC(a_k) in $_\circ\Gamma$)[22]). (The name "complex" will be justified presently.)

Lemma 3. *The necessary and sufficient condition that a_k should belong to $_\circ\overline{\Gamma}$ is that* NC(a_k) *should be bounded.*

(Obvious.)

Lemma 4. *If $_\circ\Gamma$ is a proper complex, the shell of the $_\triangle\{a_k\}$-cluster in $_\triangle\Gamma$ is the skeleton of* NC(a_k) *in $_\circ\Gamma$.*

Let $_\triangle\Gamma_{n-k-1}$ be the shell of the $_\triangle\{a_k\}$-cluster in $_\triangle\Gamma$. A unit, U_{n-k-1}, of $_\triangle$NC(a_k) has for vertices a descending sequence of cells $a_n, a_{n-1}, \ldots, a_{k+1}$ all bounded by a_k. If then V_k is a unit of $_\triangle\{a_k\}$, i. e. if the vertices of V_k form a sequence descending from a_k, $U_{n-k-1}V_k$ is a unit of $_\triangle\Gamma$, and therefore U_{n-k-1} is a unit of $_\triangle\Gamma_{n-k-1}$.

Conversely the vertices of a unit, U_{n-k-1}, of $_\triangle\Gamma_{n-k-1}$ form a sequence descending from an n-cell a_n; for if V_k is a unit of $_\triangle\{a_k\}$ the vertices of $U_{n-k-1}V_k$ descend from a_n through a_k to a_0, and V_k exhausts the first k dimension numbers. Hence U_{n-k-1} is a unit of $_\triangle\{a_{n(-k-1)}\}$, where $a_{n(-k-1)}$ denotes a_n considered as an $(n-k-1)$-cell of NC(a_k).

An $n\circ$*manifold* is a connected n-complex in which the NC of every 0-cell is an $(n-1)\circ$sphere or $(n-1)\circ$element. From Lemma 3 is follows that an $n\circ$manifold is unbounded if, and only if, the NC of every 0-cell is an $n\circ$sphere. An $n\circ$manifold is clearly a proper set, and so Lemma 4 gives

Theorem 6a. *The skeleton of a $\circ$manifold is a $\triangle$manifold.*

Theorem 7a. *In an unbounded $n\circ$manifold* NC(a_k) *is an $(n-k-1)\circ$sphere.*

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[22]) cf. H. Kneser, Proc. Amsterdam 27 (1924), p. 601.

It is now clear that if ${}_{\circ}\Gamma$ is any complex $\mathrm{NC}(a_k)$ is an $(n-k-1)$-complex[23]). For if $\bar{a}_j$ contains a_k the boundary of $a_{j(-k-1)}$ is all the cells $a_{i(-k-1)}$, where a_i is any cell bounding a_j and bounded by a_k; i. e. $\mathrm{B}(a_{j(-k-1)})$ is $\mathrm{NC}(a_k)$ in $\bar{a}_j$. Since $\bar{a}_j$ is an unbounded manifold this NC is a $(j-k-2)\circ$sphere.

Hence (by Lemma 4):

Theorem 6b. *A proper complex ${}_{\circ}\Gamma$ is an $n\circ$manifold if ${}_{\triangle}\Gamma$ is an $n\triangle$manifold.*

Theorem 7b. *In a bounded manifold* $\mathrm{NC}(a_k)$ *is an* $(n-k-1)\circ$*sphere or* $(n-k-1)\circ$*element.*

$\circ$spheres and $\circ$elements are $\circ$manifolds.

The definitions of *$n\circ$assemblies* and of *structural similarity* are derived from those on p. 403 by substituting "$\circ$element" for "element", "$\circ$piece" for "piece", "internal cell" for "unit or internal component" in $P(\mathrm{I})$, and "$(i-1)$-cell" for "unit" in $P(\mathrm{II})$.

Theorem 8. *The necessary and sufficient condition that the collection of $\circ$elements $({}_{\circ}G_i)$ should be an $n\circ$assembly is that the collection $({}_{\triangle}G_i)$ should be an $n\triangle$assembly.*

Suppose $({}_{\circ}G_i)$ is known to be an $n\circ$assembly. Then the arrays ${}_{\triangle}G_i$ are $\triangle$elements.

$P(\mathrm{I})$: If ${}_{\triangle}G_i$ has a component with vertices descending, say, from a_k which is a unit or internal component of ${}_{\triangle}G_j$, a_k is an internal cell of ${}_{\circ}G_j$ (Theorem 2). Hence ${}_{\circ}\bar{G}_i$ contains ${}_{\circ}G_j$ and so ${}_{\triangle}\bar{G}_i$ contains ${}_{\triangle}G_j$.

$P(\mathrm{II})$: If $a_{i-1}\ldots a_0$ is a unit of ${}_{\triangle}\bar{G}_i$, where $i>0$, a_{i-1} is a boundary cell of ${}_{\circ}G_i$, (proof of Theorem 3), and so belongs to a piece ${}_{\circ}G_{j-1}$; and $a_{i-1}\ldots a_0$ belongs to ${}_{\triangle}G_{j-1}$.

$P(\mathrm{III})$: If ${}_{\triangle}G_i$ is any $i\triangle$piece and $i<n$, ${}_{\circ}G_i$ is contained in some ${}_{\circ}G_{i+1}$, and ${}_{\triangle}G_{i+1}$ contains ${}_{\triangle}G_i$.

The proof of the converse is precisely similar, using Theorem 3 (Corollary) and Theorem 2.

Theorem 9. *The necessary and sufficient condition that ${}_{\circ}\Gamma$ and ${}_{\circ}\Gamma'$ should have the same structure is that their skeleton$\triangle$assemblies, ${}_{\triangle}\Gamma$ and ${}_{\triangle}\Gamma'$, should have the same structure.*

(Now obvious.)

A generalised $n\circ$assembly is a collection of $\circ$elements of dimension numbers from 0 to n, satisfying $P(\mathrm{I})$ and $P(\mathrm{II})$, (modified as above), but not necessarily $P(\mathrm{III})$.

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[23] H. Kneser, *op. cit.*, pointed out that this need not be postulated in the definition.

If the generalised $n\circ$ assemblies ${}_{\circ}\Gamma^1$ and ${}_{\circ}\Gamma^2$ have the same structure, and all the pieces of ${}_{\circ}\Gamma^1$ are single cells, ${}_{\circ}\Gamma^2$ is said to be obtained from ${}_{\circ}\Gamma^1$ by *subdivision*. This is most conveniently denoted by using a small letter $(\mathfrak{a}, \mathfrak{b}, \ldots)$ for the structural relation. If ${}_{\circ}\mathfrak{a}\Gamma^1$ is ${}_{\circ}\Gamma^2$, and ${}_{\circ}\Gamma_k$ is any k-set in ${}_{\circ}\Gamma^1$, ${}_{\circ}\mathfrak{a}\Gamma_k$ denotes the sum of the corresponding k-pieces in ${}_{\circ}\Gamma^2$. In particular ${}_{\circ}\mathfrak{s}\Gamma^1$ denotes a complex "constituted similarly" to the skeleton of ${}_{\circ}\Gamma^1$[24]), and ${}_{\circ}\mathfrak{s}^r\Gamma^1$ is written for ${}_{\circ}\mathfrak{s}\mathfrak{s}^{r-1}\Gamma^1$[25]).

Topological Equivalence. Let ${}_{\circ}\Gamma$ and ${}_{\circ}\Gamma'$ be two n-complexes. If there exist n-complexes ${}_{\circ}\Gamma^1, {}_{\circ}\Gamma^2, \ldots, {}_{\circ}\Gamma^q$, (${}_{\circ}\Gamma^1$ is ${}_{\circ}\Gamma$, ${}_{\circ}\Gamma^q$ is ${}_{\circ}\Gamma'$) with the property that ${}_{\circ}\Gamma^i$ and ${}_{\circ}\Gamma^{i+1}$ can be organised into generalised $n\circ$ assemblies with the same structure, then ${}_{\circ}\Gamma$ and ${}_{\circ}\Gamma'$ are said to be *topologically equivalent*[26]).

Theorem 10. *If ${}_{\circ}M^1$ and ${}_{\circ}M^2$ are $n\circ$ manifolds, the necessary and sufficient condition for their topological equivalence is that ${}_{\triangle}M^1 \underset{123}{\rightleftarrows} {}_{\triangle}M^2$.*

That the condition is necessary follows from Theorem 1. For if a manifold is organised into an assembly satisfying $P(\mathrm{I})$ and $P(\mathrm{II})$, $P(\mathrm{III})$ is necessarily satisfied also.

If it is known that ${}_{\triangle}M^1 \rightarrow {}_{\triangle}M^2$, then $\triangle$subdivision-processes ${}_{\triangle}\mathfrak{a}$ and ${}_{\triangle}\mathfrak{b}$ exist such that ${}_{\triangle}\mathfrak{a}M^1$ is ${}_{\triangle}\mathfrak{b}M^2$[27]). If ${}_{\circ}M^3$ is a complex constituted similarly[28]) to ${}_{\triangle}\mathfrak{a}M^1$ the $\triangle$processes ${}_{\triangle}\mathfrak{a}$ and ${}_{\triangle}\mathfrak{b}$ serve in an obvious way as patterns for $\circ$processes for dividing ${}_{\circ}M^1$ and ${}_{\circ}M^2$ into ${}_{\circ}M^3$.

With the help of this theorem Theorems 6, 7, 8a, 8b, and Lemmas 7a and 7b of FII, quoted on p. 402, can be extended to complexes, provided that "is topologically equivalent to" is read for "$\rightarrow$"; also the following theorems from S:

Theorem 11[29]). *If ${}_{\circ}E^1_n$ and ${}_{\circ}E^2_n$ are $\circ$elements, and ${}_{\circ}\mathfrak{a}\overline{E}^1_n$ is ${}_{\circ}\mathfrak{s}^r\overline{E}^2_n$, there is a division process ${}_{\circ}\mathfrak{b}$ and an integer s such that ${}_{\circ}\mathfrak{b}E^1_n$ is ${}_{\circ}\mathfrak{s}^{r+s}E^2_n$, and ${}_{\circ}\mathfrak{b}$ is ${}_{\circ}\mathfrak{s}^s\mathfrak{a}$ in $\overline{E}^1_n$.*

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[24]) *i. e.*, to each k-component, U_k, of ${}_{\triangle}\Gamma^1$, there corresponds a k-cell, u_k, of ${}_{\circ}\mathfrak{s}\Gamma^1$, and if U_j is a component of U_k, u_j bounds u_k.

[25]) The corresponding notation for arrays was introduced in an earlier paper "On the superposition of n-dimensional manifolds", Journal Lond. Math. Soc. 2 (1927), p. 56—64, referred to as S. The only modification is that the statement "${}_{\triangle}\Gamma^1$ is ${}_{\triangle}\Gamma^2$" implies that all the pieces in ${}_{\triangle}\Gamma^1$ are *simplexes.*

[26]) cf. Weyl, *l. c.* The additional transformations allowed by Weyl (his axioms C and D) lead to no increased generality, with the present definitions, in view of FII Lemmas 7a and 7b (quoted on p. 402.).

[27]) This is Theorem 2 of S.

[28]) cf. f. n. [24]) above.

[29]) S., Theorem 2, Case 2.

Theorem 12[30]). *If* $_{\circ}M_n^1$ *and* $_{\circ}M_n^2$ *are topologically equivalent* $\circ$*manifolds, and if* $_{\circ}E_n^1$ *and* $_{\circ}E_n^2$ *are* $\circ$*elements, contained respectively in* $_{\circ}M_n^1$ *and* $_{\circ}M_n^2$ *but having no cell in their boundaries (if any), then* $_{\circ}M_n^1$ *can be superposed on* $_{\circ}M_n^2$ *so that* $_{\circ}E_n^1$ *falls on* $_{\circ}E_n^2$; *i. e. there exist a division process* $_{\circ}\mathfrak{a}$ *and an integer r such that* $_{\circ}\mathfrak{a}\,M_n^1$ *is* $_{\circ}\mathfrak{s}^r M_n^2$ *and* $_{\circ}\mathfrak{a}\,E_n^1$ *is* $_{\circ}\mathfrak{s}^r E_n^2$.

From Theorem 11 follows the general

Theorem on Superposition: *If* $_{\circ}\Gamma$ *and* $_{\circ}\Gamma'$ *are any two equivalent n-complexes there is a division process* $_{\circ}\mathfrak{a}$ *and an integer r such that* $_{\circ}\mathfrak{a}\,\Gamma$ *is* $_{\circ}\mathfrak{s}^r\Gamma'$.

For if $_{\circ}\Gamma$ and $_{\circ}\Gamma'$ can themselves be organised into assemblies with the same structure, it is only necessary to "superpose" corresponding 1-, 2-, ..., n-pieces successively, making the structural relation between k-pieces agree with that already set up between their boundaries.

If $_{\circ}\Gamma$ and $_{\circ}\Gamma'$ are the end members of the chain $_{\circ}\Gamma^1, {_{\circ}\Gamma^2}, \ldots, {_{\circ}\Gamma^q}$, where $_{\circ}\Gamma^i$ and $_{\circ}\Gamma^{i+1}$ can be organised as assemblies with the same structure, $_{\circ}\Gamma^{q-1}$ can first be superposed on $_{\circ}\Gamma'$, giving $_{\circ}\mathfrak{s}^{r_1}\Gamma'$; then $_{\circ}\Gamma^{q-2}$ may be superposed on $_{\circ}\mathfrak{s}^{r_1}\Gamma'$, giving $_{\circ}\mathfrak{s}^{r_2}\Gamma'$; and so on.

Corollary. *At most one intermediate complex is required to exhibit topological equivalence between two complexes.*

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[30]) *S.*, Theorem 3.

(Eingegangen am 4. 4. 1927.)