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Skew-Symmetric Vanishing Lattices and Their Monodromy Groups

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1. Introduction

Let R denote the ring $\mathbb{Z}$ or the field $\mathbb{F}_2$, and let V be a free R-module of finite rank μ with an alternating bilinear form $\langle -, - \rangle : V \times V \to R$. This form determines a map j from V to its dual V^*, given by $x \to \langle x, - \rangle$. If V_0 denotes its kernel and V' its image, then we shall identify V' and V/V_0. For $a \in V$ we have a so-called symplectic transvection $T_a : V \to V$ given by $T_a(x) = x - \langle x, a \rangle a$. Notice that T_a is an element of the group $\mathrm{Sp}^{\#} V$ of automorphisms of $(V, \langle -, - \rangle)$ that act trivially on V^*/V' (hence on V_0 as well). Given any subset Δ of V we let $\Gamma_\Delta \subset \mathrm{Sp}^{\#} V$ denote the subgroup generated by the transvections $T_\delta, \delta \in \Delta$. We shall study triples $(V, \langle -, - \rangle, \Delta)$ satisfying:

(i) Δ is a Γ_Δ-orbit.

(ii) Δ generates V.

(iii) If rank $V > 1$, then there exist $\delta_1, \delta_2 \in \Delta$ such that $\langle \delta_1, \delta_2 \rangle = 1$.

Such a triple we call a *vanishing lattice* (over R), and Γ_Δ is called its *monodromy group*. In the sequel we exclude the (trivial) rank 1 case since it is exceptional. We conjecture that condition (iii) is already implied by (i) and (ii).

We prove that the monodromy group of a vanishing lattice over $\mathbb{Z}$ contains a certain congruence subgroup of $\mathrm{Sp}^{\#} V$ and that the problem of describing Δ reduces to the corresponding problem in the modulo 2 reduction of V (thereby generalizing [1, Theorem 1], [10, Theorem 1], and [5, Theorem 1]). Furthermore we generalize the results of Wajnryb's paper [10] concerning vanishing lattices over $\mathbb{F}_2$ and classify these structures. This yields information about the subgroup of Γ_Δ consisting of elements that act trivially on V' ($R = \mathbb{Z}$ or $\mathbb{F}_2$).

The interest of the notion of vanishing lattice is explained by the fact that it occurs in at least two places in algebraic geometry: the vanishing homology of an odd-dimensional isolated complete intersection singularity and the vanishing homology of many Lefschetz pencils on an even-dimensional projective variety both carry such a structure. In these cases our results give rather precise information on the vanishing cycles and monodromy groups (Sect. 6).

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2. Vanishing Lattices

We use the notations introduced in Sect. 1.

(2.1) **Lemma.** *Let $\Delta' \subset V$ be a set of generators such that all $T_{\delta'}, \delta' \in \Delta'$ are in a single conjugacy class in $\Gamma_{\Delta'}$. If there exist $\delta_1, \delta_2 \in \Delta'$ such that $\langle \delta_1, \delta_2 \rangle = 1$, then $(V, \langle -, - \rangle, \Delta)$ (where $\Delta = \Gamma_{\Delta'} \cdot \Delta'$) is a vanishing lattice, and $\Gamma_\Delta = \Gamma_{\Delta'}$.*

Proof. Let $\delta \in \Delta$. There exist $\delta' \in \Delta'$ and $g \in \Gamma_{\Delta'}$ such that $\delta = g(\delta')$. Now $T_\delta = g T_{\delta'} g^{-1}$, so $T_\delta \in \Gamma_{\Delta'}$. This proves $\Gamma_{\Delta'} = \Gamma_\Delta$.

It remains to prove that every $\delta' \in \Delta'$ is in $\Gamma_{\Delta'} \cdot \delta_1$. There is $g \in \Gamma_{\Delta'}$ such that $T_{\delta'} = g T_{\delta_1} g^{-1}$. Now $g(\delta_1) = g(T_{\delta_1}(\delta_2) - \delta_2) = g T_{\delta_1}(\delta_2) - g(\delta_2) = T_{\delta'} g(\delta_2) - g(\delta_2) \in R \cdot \delta'$. Hence $g(\delta_1) = \pm \delta'$.

Since $-\delta_1 = T_{\delta_2} T_{\delta_1}^2 T_{\delta_2}(\delta_1) \in \Gamma_{\Delta'} \cdot \delta_1$ it follows that $\delta' \in \Gamma_{\Delta'} \cdot \delta_1$.

(2.2) En passant we proved: if $(V, \langle -, - \rangle, \Delta)$ is a vanishing lattice then $-\Delta = \Delta$.

(2.3) Recall (see for example [3, p. 79]) that for an alternating bilinear form on V there is a basis $(e_1, f_1, \ldots, e_n, f_n, g_1, \ldots, g_p)$ such that with respect to this basis the matrix of $\langle -, - \rangle$ is given by

$$\begin{bmatrix} \begin{matrix} 0 & d_1 \\ -d_1 & 0 \end{matrix} & & & & & \\ & \begin{matrix} 0 & d_2 \\ -d_2 & 0 \end{matrix} & & & \Theta & \\ & & \ddots & & & \\ & \Theta & & \begin{matrix} 0 & d_n \\ -d_n & 0 \end{matrix} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

where in the $\mathbb{Z}$-case all d_i are natural numbers and d_i divides d_{i+1} $(i = 1, \ldots, n-1)$ and in the $\mathbb{F}_2$-case all d_i are 1. We call such a basis *symplectic*.

(2.4) Let $\langle -, - \rangle$ be an alternating bilinear form on the $\mathbb{Z}$-module V with d_1 (defined above) equal to 1. Chmutov [5] proves that G_V, the subgroup of $\mathrm{Sp}^{\#} V$ acting trivially on $V^*/2V'$ is generated by the elements T_a^2, $a \in V$.

It is easy to see that G_V contains the congruence subgroup modulo $2d_n$ of $\mathrm{Sp}^{\#} V$, which coincides with the congruence subgroup modulo 2^{r+1} where r is the number of 2's in the prime factorization of d_n.

Our first main result is:

(2.5) **Theorem.** *Let $(V, \langle -, - \rangle, \Delta)$ be a vanishing lattice over $\mathbb{Z}$. Then $\Gamma_\Delta \supset G_V$.*

Proof. We shall prove that the statement is true if Δ contains a finite set of generators $\{a_1, \ldots, a_v\}$ for V such that $T_{a_1}, \ldots, T_{a_v}$ generate Γ_Δ and $\langle a_1, a_i \rangle = 1$ for $i = 2, \ldots, v$.

Using this result and some lemmas we shall show that for any vanishing lattice such a set of generators exists.

This will complete the proof.

Suppose we have a set of generators as above.

If $\nu=\mu$, we are ready by [5, Proposition 3].

If $\nu>\mu$, we define a free $\mathbb{Z}$-module $\tilde{V}$ of rank ν with a basis $(\tilde{a}_1, \ldots, \tilde{a}_\nu)$ and with a bilinear form $\langle -, - \rangle$ defined by $\langle \tilde{a}_i, \tilde{a}_j \rangle = \langle a_i, a_j \rangle$

$$\tilde{\Gamma} := \Gamma_{\{\tilde{a}_1, \ldots, \tilde{a}_\nu\}} \qquad \tilde{\Delta} := \tilde{\Gamma} \cdot \tilde{a}_1 .$$

Now $\tilde{a}_1 = T_{\tilde{a}_i} T_{\tilde{a}_1}(\tilde{a}_i)$ for $i=2, \ldots, n$ since $\langle \tilde{a}_1, \tilde{a}_i \rangle = \langle a_1, a_i \rangle = 1$, and if $\tilde{\delta} = \tilde{g}(\tilde{a}_1)$ for $\tilde{g} \in \tilde{\Gamma}$ then $T_{\tilde{\delta}} = \tilde{g} T_{\tilde{a}_1} \tilde{g}^{-1} \in \tilde{\Gamma}$. So $\Gamma_{\tilde{\Delta}} = \tilde{\Gamma}$ and $\tilde{\Delta} = \Gamma_{\tilde{\Delta}} \cdot \tilde{a}_1$. Therefore $(\tilde{V}, \langle -, - \rangle, \tilde{\Delta})$ is a vanishing lattice.

Let $g \in G_V$. Now (2.4) tells us that there exist $r \in \mathbb{N}$, $x_j \in V$, $\varepsilon_j \in \{-1, 1\}$, $(j=1, \ldots, r)$ such that $g = \prod_{j=1}^{r} T_{x_j}^{2\varepsilon_j}$.

For every x_j there is $\tilde{x}_j \in \tilde{V}$ which maps to x_j under the surjection $\tilde{V} \to V$ given by $\tilde{a}_i \to a_i$.

Put $\tilde{g} = \prod_{j=1}^{r} T_{\tilde{x}_j}^{2\varepsilon_j} \in G_{\tilde{V}}$.

But $G_{\tilde{V}} \subset \Gamma_{\tilde{\Delta}} = \tilde{\Gamma}$, so $\tilde{g}$ is a product of elements $T_{\tilde{a}_i}, T_{\tilde{a}_i}^{-1}$ and hence g is a product of the corresponding T_{a_i} and $T_{a_i}^{-1}$. This proves $G_V \subset \Gamma_\Delta$.

(2.6) **Lemma.** *Let $(V, \langle -, - \rangle, \Delta)$ be a vanishing lattice and consider the equivalence relation on Δ generated by $\langle \delta_1, \delta_2 \rangle = 1$. Then Δ is an equivalence class.*

Proof (for $R=\mathbb{Z}$). $T_{\delta_2} T_{\delta_1}(\delta_2) = \delta_1$ if $\langle \delta_1, \delta_2 \rangle = 1$.

Hence any equivalence class $\Delta' \subset \Delta$ is a $\Gamma_{\Delta'}$-orbit. Γ_Δ permutes equivalence classes.

Let Δ' be an equivalence class and $\delta \in \Delta$. We show that there is $\delta' \in \Delta'$ such that $T_\delta(\delta') = \delta'$. (This is sufficient: it implies that Γ leaves Δ' invariant, so $\Delta' = \Gamma \cdot \Delta' = \Delta$.)

Take $\delta_1 \in \Delta'$. There is $\delta_2 \in \Delta$ such that $\langle \delta_1, \delta_2 \rangle = 1$. We have $\mathbb{Z} \cdot \delta_1 + \mathbb{Z} \cdot \delta_2 \to \mathbb{Z}$

$$\delta_i \to \langle \delta_i, \delta \rangle .$$

A generator δ' of the kernel of this mapping is indivisible in $\mathbb{Z} \cdot \delta_1 + \mathbb{Z} \cdot \delta_2$, hence $\delta' \in \Gamma_{\{\delta_1, \delta_2\}} \cdot \delta_1$ by [1, Theorem 2].

$T_\delta(\delta') = \delta' - \langle \delta', \delta \rangle \delta = \delta'$, and $\delta' \in \Delta'$ since $T_{\delta_1}(\Delta') = T_{\delta_2}(\Delta') = \Delta'$.

Proof (for $R = \mathbb{F}_2$). Any equivalence class is a Γ_Δ-orbit, hence equal to Δ.

(2.7) **Lemma.** *Let $\delta_0 \in \Delta$ and $\Delta_0 := \{\delta \in \Delta | \langle \delta_0, \delta \rangle = 1$ or $\delta = \delta_0\}$. Then $\Gamma_{\Delta_0} = \Gamma_\Delta$ and $V = R \cdot \Delta_0$.*

Proof. For any $\delta \in \Delta$ there exists a sequence $\delta_0, \delta_1, \ldots, \delta_l = \delta$ such that $\langle \delta_{i-1}, \delta_i \rangle = 1$ for $i=1, \ldots, l$. Let $l(\delta)$ denote the minimal length of such a sequence.

By induction on $l(\delta)$ we prove: $\delta \in \Gamma_{\Delta_0} \cdot \delta_0$ (hence $T_\delta \in \Gamma_{\Delta_0}$). If $l(\delta) = 0$ then $\delta = \delta_0$ so there is nothing to prove. If $l = l(\delta) > 0$ we have $\delta_0, \delta_1, \ldots, \delta_{l-1}, \delta_l = \delta$.

There exists $g\in\Gamma_{\Delta_0}$ such that $g(\delta_{l-1})=\delta_0$, from the induction hypothesis.

$$\langle\delta_0, g(\delta_l)\rangle=\langle g(\delta_{l-1}), g(\delta_l)\rangle=\langle\delta_{l-1},\delta_l\rangle=1$$

$$T_{g(\delta_l)}\in\Gamma_{\Delta_0},\quad \text{so}\quad T_{\delta_l}=g^{-1}T_{g(\delta_l)}g\in\Gamma_{\Delta_0}$$

$$\delta_l=T_{\delta_{l-1}}^{-1}T_{\delta_l}^{-1}(\delta_{l-1})\in\Gamma_{\Delta_0}\cdot\delta_0$$

$$V=R\cdot\Delta=R\cdot(\Gamma_{\Delta_0}\cdot\Delta_0)\subset R\cdot\Delta_0.$$

The proof of Theorem (2.5) is now completed by the following

(2.8) **Proposition.** *Let $(V,\langle -,-\rangle,\Delta)$ be a vanishing lattice. Then Δ contains a finite set of generators $\{a_1,\dots,a_v\}$ for V such that $T_{a_1},\dots,T_{a_v}$ generate Γ_Δ and $\langle a_1,a_i\rangle=1$ for $i=2,\dots,v$.*

Proof (for $R=\mathbb{Z}$; for $R=\mathbb{F}_2$: Take the finite set Δ_0). Take $\delta_0\in\Delta$. Now Δ_0 is countable so we can arrange its elements in a sequence $a_1=\delta_0, a_2,\dots$

$$V_l:=\mathbb{Z}\cdot\{a_1,\dots,a_l\}.\ \ \Gamma_l:=\text{subgroup of } \Gamma_\Delta \text{ generated by } T_{a_1},\dots,T_{a_l}.$$

We have $V=\bigcup_l V_l$, $\Gamma_\Delta=\bigcup_l \Gamma_l$.

There exists $l_0\in\mathbb{N}$ such that rank $V_{l_0}=$ rank V, so for such l_0 V/V_{l_0} is finite and

$$V_{l_0}/V_{l_0}\subset V_{l_0+1}/V_{l_0}\subset\dots\subset V/V_{l_0}.$$

Then there is $l_1\geqq l_0$ such that $V_l=V$ for $l\geqq l_1$.

Now $(V,\langle -,-\rangle,\Gamma_{l_1}\cdot\{a_1,\dots,a_{l_1}\})$ is a vanishing lattice.

We can apply Theorem (2.5) to it since it satisfies the extra hypothesis about a suitable set of generators.

Therefore $\Gamma_{l_1}\supset G_V$, and Γ_Δ/G_V is finite, so $\Gamma_\Delta/\Gamma_{l_1}$ is finite. Since $\Gamma_{l_1}/\Gamma_{l_1}\subset\Gamma_{l_1+1}/\Gamma_{l_1}\subset\dots\subset\Gamma_\Delta/\Gamma_{l_1}$, there exists $v\in\mathbb{N}$, $v\geqq l_1$ such that $\Gamma_\Delta=\Gamma_l$ for $l\geqq v$. This completes the proof of the proposition.

(2.9) **Theorem.** *Let $(V,\langle -,-\rangle,\Delta)$ be a vanishing lattice over $\mathbb{Z}$, and $x\in V$. Then: $x\in\Delta$ iff there are $y\in V$ and $\delta\in\Delta$ such that $\langle x,y\rangle=1$ and $x-\delta\in 2V$.*

Proof. Apply Proposition (2.8) and make $(\tilde{V},\langle -,-\rangle,\tilde{\Delta})$ as in the proof of Theorem (2.5). Apply Chmutov's corollary of Proposition 1 to $\tilde{V}$ [5].

(2.10) The group $\Gamma=\Gamma_\Delta$ acts on V' with its induced form $\langle -,-\rangle$. We denote the image of Γ in $\mathrm{Sp}^{\#}V'$ by Γ_s and the kernel of $\Gamma\to\Gamma_s$ by Γ_u. We refer to Γ_u, respectively, Γ_s as the *unipotent*, respectively, *simple* part of the monodromy group.

For $g\in\Gamma_u$ there exist $v_i\in V$, $w_i\in V_0$ such that $g(x)=x+\sum_i\langle x,v_i\rangle w_i$. So we can identify Γ_u with a subgroup of $V_0\otimes V'$. In particular Γ_u is abelian. The following is a first step in estimating Γ_u.

(2.11) **Lemma.** *Let $\delta\in\Delta$. $V_{00}:=\{v\in V_0\,|\,\delta+v\in\Delta\}$ does not depend on the choice of δ. It is a subgroup of V_0.*

If we identify Γ_u with a subgroup of $V_0\otimes V'$ as above, then Γ_u contains $V_{00}\otimes V'$.

Proof. Let $v\in V_{00}$. Then there is $g\in\Gamma$ such that $\delta+v=g(\delta)$. Now let $\delta'\in\Delta$. There is $h\in\Gamma$ such that $h(\delta)=\delta'$. So $\delta'+v=h(\delta)+v=h(\delta+v)=hg(\delta)\in\Delta$. This proves the first statement, from which the second immediately follows.

It remains to prove that for any $\delta \in \Delta$, $v \in V_{00}$, Γ_u contains the automorphism $x \to x + \langle x, \delta \rangle v$. ($\Delta$ generates V.) But this map is $T_{\delta+v}^{-1} T_\delta$, which is an element of Γ since $\delta + v \in \Delta$.

(2.12) A *quadratic function* associated to $(V, \langle -, - \rangle)$ is an $\mathbb{F}_2$-valued function on V satisfying

$$q(\lambda x + \mu y) = \lambda^2 q(x) + \mu^2 q(y) + \lambda\mu \overline{\langle x, y \rangle}$$

(where the bar above $\langle x, y \rangle$ denotes taking the class of $\langle x, y \rangle$ in $\mathbb{F}_2$, if $R = \mathbb{Z}$). From the above formula it is clear that q is a homomorphism of groups on any subgroup of $j^{-1}(2V^*)$, in particular on V_0.

It is also clear that q completely determines $\langle -, - \rangle$ if $R = \mathbb{F}_2$.

In this paper we shall call a triple $(V, \langle -, - \rangle, q)$ where q is a quadratic function associated to $(V, \langle -, - \rangle)$, a *quadratic form*.

(2.13) Let $R = \mathbb{F}_2$. Suppose that $q(V_0) = 0$ and that $(e_1, f_1, \ldots, e_n, f_n, g_1, \ldots, g_p)$ is a symplectic basis for $(V, \langle -, - \rangle)$ (2.3). We recall that then the *Arf invariant* of q can be defined as $\sum_{i=1}^{n} q(e_i) q(f_i)$ (see [6, 10, p. 151]. For fixed dimensions of V and V_0 there exist at most 3 quadratic forms, up to isomorphism [4, 6].

(i) $q(V_0) = 0$, $\mathrm{Arf}(q) = 1$, $\quad |q^{-1}(1)| = 2^{2n+p-1} + 2^{n+p-1}$

(ii) $q(V_0) = 0$, $\mathrm{Arf}(q) = 0$, $\quad |q^{-1}(1)| = 2^{2n+p-1} - 2^{n+p-1}$

(iii) $q(V_0) = \mathbb{F}_2$ (so $p \geqq 1$), $\quad |q^{-1}(1)| = |q^{-1}(0)| = 2^{2n+p-1}$.

[For $R = \mathbb{Z}$ all this is more complicated since $j^{-1}(2V^*)$ can be different from $V_0 + 2V$.]

(2.14) Let V be a free R-module of finite rank with an alternating bilinear form $\langle -, - \rangle$. To any basis B of V we can associate a unique quadratic form by requiring q_B to take the value 1 on all elements of B.

Now let $(V, \langle -, - \rangle, \Delta)$ be a vanishing lattice. We say a basis B is *weakly distinguished* if $\Gamma_B = \Gamma_\Delta$. In this case Γ respects q_B, so in particular $q_B(\delta) = 1$ for all $\delta \in \Delta$. Furthermore for $v \in V_{00}$, $\delta \in \Delta$ we have $1 = q_B(\delta + v) = q_B(\delta) + q_B(v) + \overline{\langle \delta, v \rangle} = 1 + q_B(v)$. So $q_B(v) = 0$, hence $V_{00} \subset \mathrm{Ker}(q|_{V_0})$.

3. Vanishing Lattices over $\mathbb{F}_2$

In this section R shall always be $\mathbb{F}_2$.

(3.1) Bases B and B' are said to be equivalent if $B' \subset \Gamma_B \cdot B$ and $B \subset \Gamma_{B'} \cdot B'$. (This is indeed an equivalence relation on the set of bases of V.) If B and B' are equivalent, then $\Gamma_B = \Gamma_{B'}$, $q_B = q_{B'}$, and if B is a weakly distinguished basis for $(V, \langle -, - \rangle, \Delta)$, then so is B'.

(3.2) Let $B = (a_1, \ldots, a_\mu)$ be a basis of V. We define $\mathrm{gr}(B)$ to be the graph which has B as its set of points and in which a_i, a_j are connected by an edge iff $\langle a_i, a_j \rangle = 1$.

For any $x = \sum_{i=1}^{\mu} x_i a_i \in V$ we define $\mathrm{gr}(B, x)$ to be the subgraph of $\mathrm{gr}(B)$ spanned by $\{a_i \in B \mid x_i = 1\}$. It is easy to see that $\mathrm{gr}(B)$ is connected if B is weakly distinguished.

A graph is said to have *radius one* if it has a point connected to any other point. Therefore we shall say that a basis $B=(a_1, \dots, a_\mu)$ of V has radius one if $\langle a_1, a_i\rangle = 1$ for each $i \neq 1$.

Wajnryb [10] proved that any basis B for which $\mathrm{gr}(B)$ is connected (e.g. weakly distinguished B) is equivalent to a basis of radius one.

(3.3) A basis is called *special* if it is equivalent to a basis $B=(a_1, \dots, a_\mu)$ such that for some k, $1 \leqq k \leqq \mu$ we have $\langle a_i, a_j\rangle = 0$ iff $i=j$ or $i, j \geqq k+1$.

This is a straightforward generalization of a notion introduced by Wajnryb [10] (he assumed $V_0 = 0$).

(3.4) **Lemma.** *If $B=(a_1, \dots, a_\mu)$ is a basis of radius one in V, and if $\mathrm{gr}(B, a_1+a_2+a_3+a_4+a_5)$ is a complete graph, then at least one of the following statements holds:*

(i) $a_1+a_2+a_3+a_4+a_5 \in \Delta_B$.

(ii) $a_1+a_2+a_3+a_4+a_5 \in V_0$.

(iii) *B is special.*

In order to prove this lemma we slightly modify an argument due to Wajnryb [10]. For convenience we give the complete proof instead of only indicating the modifications.

Proof. Let $J \subset B$ be a maximal set containing a_2, a_3, a_4, a_5, but not a_1, such that the subgraph of $\mathrm{gr}(B)$ spanned by J is complete. If $a_i, a_j, a_k, a_l \in J$ are distinct, then $a_1+a_i+a_j+a_k+a_l \in \Gamma_B(a_1+a_2+a_3+a_4+a_5)$. ($T_{a_2}T_{a_i}$ replaces a_2 by a_i in $a_1+a_2+a_3+a_4+a_5$.)

Let $a_p \in B$ be such that $\langle a_p, a_i\rangle = 0$ for all $a_i \in J$. In B we replace a_p by $a_p + a_1 = T_{a_1}(a_p)$.

Since $T_{a_p} = T_{a_1} T_{a_p+a_1} T_{a_1}$ the new basis is equivalent to the old one. Also we get a new J, having one extra point: $a_p + a_1$.

If $a_p \in B$ is such that $\langle a_p, a_i\rangle = 1$ for all $a_i \in J$ except one, we also replace a_p by $a_p + a_1$. J does not change, but the new base vector $b_p = a_p + a_1$ satisfies $\langle b_p, a_i\rangle = 1$ for exactly one $a_i \in J$.

If there exists $a_p \in B$ and $a_i, a_j, a_k, a_l \in J$ such that

$$\langle a_p, a_i\rangle = \langle a_p, a_j\rangle = 1, \qquad \langle a_p, a_k\rangle = \langle a_p, a_l\rangle = 0$$

then $T_{a_p}T_{a_k}T_{a_j}T_{a_l}T_{a_i}T_{a_p}(a_1+a_i+a_j+a_k+a_l) = a_1$. So (i) holds.

Let $J_1 := B \backslash (\{a_1\} \cup J)$.

We may assume that for any $a_p \in J_1$ there exists exactly one $a_{p'} \in J$ such that $\langle a_p, a_{p'}\rangle = 1$.

If J has exactly 4 elements, then $J = \{a_2, a_3, a_4, a_5\}$, so $\langle a_1+a_2+a_3+a_4+a_5, x\rangle = 0$ for all $x \in V$. So in this case (ii) holds. If not J contains at least 5 elements.

Case A. There exist a_p, $a_q \in J_1$ such that $\langle a_p, a_q\rangle = 0$, $p \neq q$, $p' \neq q'$. So we may assume a_2, a_3, a_4, $a_{p'}$, $a_{q'}$ are distinct elements of J. Then:

$$T_{a_3}T_{a_p}T_{a_q}T_{a_1}T_{a_4}T_{a_{q'}}T_{a_q}T_{a_2}T_{a_1}T_{a_p}(a_1+a_2+a_3+a_4+a_{q'}) = a_1 .$$

This proves that $a_1+a_2+a_3+a_4+a_{q'} \in \Delta_B$, hence $a_1+a_2+a_3+a_4+a_5 \in \Delta_B$. So (i) holds.

Case B. There exist $a_p, a_q \in J_1$ such that $\langle a_p, a_q\rangle = 1$ and $p' = q'$. Let $a_2, a_3, a_4, a_{p'}, a_r$ be distinct elements of J.

Then $T_{a_r}T_{a_q}T_{a_{p'}}T_{a_p}T_{a_4}T_{a_{p'}}T_{a_3}T_{a_1}T_{a_q}T_{a_2}T_{a_1}T_{a_p}(a_1+a_2+a_3+a_4+a_r)=a_1$. Hence (i) holds.

Case C. There exist $a_p, a_q \in J_1$ such that $\langle a_p, a_q\rangle = 1$ and $p' \neq q'$. We replace the basis vector a_p by $a_p + a_{p'} + a_{q'} = T_{a_{q'}}T_{a_{p'}}a_p$. J does not change.

Since $T_{a_p} = T_{a_{p'}}T_{a_{q'}}T_{a_p+a_{p'}+a_{q'}}T_{a_{q'}}T_{a_{p'}}$ the new basis is equivalent to the old one.

For $a_i \in J$ we have:

$$\langle a_p + a_{p'} + a_{q'}, a_i\rangle = 1 \quad \text{iff} \quad a_i = a_{q'}$$
$$\langle a_p + a_{p'} + a_{q'}, a_q\rangle = 0.$$

So we reduced case C to:

Case D. For all $a_p, a_q \in J_1$ we have: $\langle a_p, a_q\rangle = 0$ and $p' = q'$. We replace all $a_p \in J_1$ by $a_p + a_1$.

The new basis is equivalent to B. So B is special and (iii) holds.

(3.5) **Theorem.** *For any weakly distinguished basis B of V which is not special, Δ_B is the set of $x \in V - V_0$ with $q_B(x) = 1$.*

Proof. We may assume that B is a basis with radius one (3.2). Throughout the proof, we omit the index B from Δ_B, Γ_B, q_B. Suppose first $x \in \Delta$.

If $x \in V_0$, then $x = a_1$ since Γ acts trivially on V_0. Hence $\Delta = \{a_1\}$, and B is special which contradicts the hypothesis.

So $x \notin V_0$ and $q(x) = q(a_1) = 1$ since Γ respects q.

Conversely, let $x \in V \setminus V_0$ with $q(x) = 1$. Write $x = \sum_{i \in I} a_i$ ($I \subset \{1, \ldots, \mu\}$). We show: If $|I| > 1$, then $x \in \Gamma \cdot \left(\sum_{i \in I'} a_i\right)$ for $I' \subset \{1, \ldots, \mu\}$ with $|I'| < |I|$. This is sufficient.

If there is an $i_0 \in I$ such that $\langle x, a_{i_0}\rangle = 1$, then $T_{a_{i_0}}(x) = \sum_{i \in I \setminus \{i_0\}} a_i$ and we are done.

Hence we may assume: $\langle x, a_{i_0}\rangle = 0$ for all $i_0 \in I$.

We consider the two cases: $1 \in I$, $1 \notin I$ separately.

Case 1. $1 \in I$. Then $|I|$ is odd, since $\langle x, a_1\rangle = 0$.

There exists $j \in \{1, \ldots, \mu\} \setminus I$ such that $\langle x, a_j\rangle = 1$ (for $x \notin V_0$). If there is an $i_0 \in I$ such that $\langle a_{i_0}, a_j\rangle = 0$, then $T_{a_{i_0}+a_1}T_{a_j}(x) = x + a_j + a_{i_0} + a_1 = \sum_{i \in I'} a_i$, where $I' = (I \cup \{j\}) \setminus \{1, i_0\}$. [$T_{a_{i_0}+a_1} \in \Gamma$ since $a_{i_0} + a_1 = T_{a_{i_0}}(a_1) \in \Delta$.]

Clearly $|I'| < |I|$. So we may assume that $\langle a_{i_0}, a_j\rangle = 1$ for all $i_0 \in I$. We distinguish two subcases:

(a) If there exist distinct $i_1, i_2 \in I$ with $\langle a_{i_1}, a_{i_2}\rangle = 0$, then

$$T_{a_{i_2}+a_1}T_{a_{i_1}}T_{a_j}(x) = x + a_j + a_{i_1} + a_{i_2} + a_1 = \sum_{i \in I'} a_i.$$

$I' = (I \cup \{j\}) \setminus \{1, i_1, i_2\}$. Notice that $a_{i_2} + a_1 \in \Delta$ and that $|I'| < |I|$.

(b) For all i_1, $i_2 \in I$ with $i_1 \neq i_2$ we have $\langle a_{i_1}, a_{i_2}\rangle = 1$. If $|I|=3$, then $q(x)=0$ which contradicts the hypothesis. So $|I| \geqq 5$. Let $1, i_1, i_2, i_3, i_4$ be distinct elements of I.

$\langle a_1 + a_{i_1} + a_{i_2} + a_{i_3} + a_{i_4}, a_j\rangle = 1$ and B is not special, hence $a_1 + a_{i_1} + a_{i_2} + a_{i_3} + a_{i_4} \in \Delta$ (Lemma 3.4).

$$T_{a_1 + a_{i_1} + a_{i_2} + a_{i_3} + a_{i_4}} T_{a_j}(x) = x + a_j + a_1 + a_{i_1} + a_{i_2} + a_{i_3} + a_{i_4} = \sum_{i \in I'} a_i,$$

where

$$I' = (I \cup \{j\}) \backslash \{1, i_1, i_2, i_3, i_4\} \quad (\text{hence } |I'| < |I|).$$

Case 2. $1 \notin I$. If $\langle a_1, x\rangle = 1$ we have for any $i_0 \in I$:

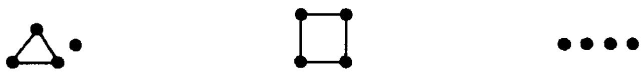
$$T_{a_{i_0}} T_{a_1}(x) = x + a_1 + a_{i_0} = \sum_{i \in (I \backslash \{i_0\}) \cup \{1\}} a_i .$$

Although $|(I\backslash\{i_0\} \cup \{1\}| = |I|$, we are in case 1 now, so this has already been dealt with, and we may assume $\langle a_1, x\rangle = 0$. Hence $|I|$ is even.

If $|I|=2$, $x = a_{i_1} + a_{i_2}$.

Since $q(x) = q(a_{i_1}) + q(a_{i_2}) + \langle a_{i_1}, a_{i_2}\rangle = 1+1+0=0$, this case is excluded ($\langle a_{i_1}, a_{i_2}\rangle = \langle x, a_{i_2}\rangle$, and we treated the case of this being 1 in the beginning of the proof).

If $|I|=4$ every vertex of $\mathrm{gr}(B,x)$ has even degree, hence degree 2 or degree 0. This leaves us with three possible graphs:



Since $q(x)=1$ only the first of these graphs can be $\mathrm{gr}(B,x)$, so

$$x = a_{i_1} + a_{i_2} + a_{i_3} + a_{i_4} \quad \text{with} \quad \langle a_{i_j}, a_{i_k}\rangle = 1 \quad \text{if } j \neq k \text{ and } 1 \notin \{j,k\}$$
$$= 0 \quad \text{in all other cases.}$$

Now we replace in our basis B the vector a_{i_1} by $a_{i_1} + a_1$. We get a basis B' which satisfies the hypothesis of the theorem. B' is equivalent to B, hence $\Delta_{B'} = \Delta$, $\Gamma_{B'} = \Gamma$, $q_{B'} = q$.

Since $\mathrm{gr}(B',x)$ is a complete graph on 5 points, it follows that $x \in \Delta$ (Lemma 3.4).

If $|I| \geqq 6$ we use the following

(3.6) **Assertion.** $\mathrm{gr}(B,x)$ *has a full subgraph* $\mathrm{gr}(B,y)$ *which is of one of the following types:*

and for such y: $y \in \Delta$.

We postpone the proof of this assertion and finish the proof of the theorem:

There exists $j \in \{1, \ldots, \mu\} \backslash I$ such that $\langle x, a_j\rangle = 1$.

Take y as in the assertion. If $\langle y, a_j\rangle = 1$, then

$$T_y T_{a_j}(x) = x + a_j + y = \sum_{i \in I'} a_i \quad \text{with} \quad |I'| \leqq |I| - 2.$$

If $\langle y, a_j\rangle = 0$, then $T_y T_{a_1} T_{a_j}(x) = x + a_1 + a_j + y = \sum_{i \in I'} a_i$ with $|I'| \leqq |I| - 1$.

This completes the proof.

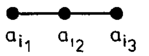
Now we prove the assertion (3.6): If (i) does not occur, $\mathrm{gr}(B,x)$ has at most one point of degree 0. Moreover if such a point occurs all other points are connected by edges, so (iii) occurs. Hence we may assume all points have degree $\geqq 2$ (every point has even degree since $\langle x, a_i\rangle = 0$ for all $i \in I$). So every point is connected to at least 2 other points.

If (ii) does not occur, all points connected to a certain fixed point are connected among themselves. Hence (iii) occurs if there is a point with degree $\geqq 4$. So if (iii) does not occur $\mathrm{gr}(B,x)$ has only connected components of the form

This contradicts the hypothesis $q(x)=1$.

For the various subgraphs we have:

(i) $y = a_{i_1} + a_{i_2} + a_{i_3}$ and $T_{a_{i_1}} T_{a_{i_2}} T_{a_{i_3}} T_{a_1}(y) = a_1$.

(ii) $y = a_{i_1} + a_{i_2} + a_{i_3}$ 

$T_{a_{i_2}} T_{a_{i_3}}(y) = a_{i_1}$.

(iii) $y = a_{i_1} + a_{i_2} + a_{i_3} + a_{i_4} + a_{i_5}$.

Apply Lemma (3.4) to $T_{a_{i_1}} T_{a_1}(y) = y + a_{i_1} + a_1$.

This proves the assertion.

(3.7) **Corollary.** *Under the conditions of* (3.5) $V_{00} = \mathrm{Ker}(q|_{V_0})$.

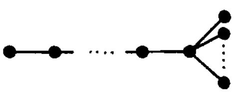
(3.8) **Theorem.** *Let* $(V, \langle -, - \rangle, \Delta)$ *be a vanishing lattice admitting a weakly distinguished basis* B *which is not special.*

Then Γ *is the subgroup* $O^{\#}(q_B)$ *of* $\mathrm{Sp}^{\#} V$ *consisting of those automorphisms which respect* q_B.

Proof. If $V_0 = V_{00}$, q_B induces a quadratic function q' on V', which takes the value 1 on any element of the image of Δ. Hence $\Gamma_s = O(q')$ [10, Theorem 4]. Since $\Gamma_u = V_0 \otimes V'$ the desired result follows. If $V_0 \neq V_{00}$, there exists $v_0 \in V_0$ such that $q_B(v_0) = 1$ [Corollary (3.7)]. Making use of this v_0 we can lift any element of $V' \backslash \{0\}$ to an element of $\Delta = q_B^{-1}(1) \backslash V_0$. Now $\Gamma_s = \mathrm{Sp}(V')$ ([3, Sect. 5, Exercise 11] or [10, Theorem 4]). But any element of $\mathrm{Sp}(V')$ can be lifted to an element of $\mathrm{Sp}^{\#} V$ respecting q_B, uniquely up to an element of $V_{00} \otimes V' \cong \Gamma_u$. So in this case we also have the result.

(3.9) The orders of $O^{\#}(q_B)$ and $\mathrm{Sp}^{\#} V$ can easily be computed from the orders of $O(q)$ for nondegenerate quadratic forms and of $\mathrm{Sp}(V)$ for nondegenerate alternating forms. For these we refer to [4, 5].

(3.10) Now we turn our attention to special bases. It is easy to see that any special basis is equivalent to a basis B for which $\mathrm{gr}(B)$ is a tree as indicated in the following picture (examples are the Dynkin diagrams A_μ, D_μ):



Indeed, an equivalent basis satisfying the condition of (3.3) is obtained from B by taking elements corresponding to chains containing the left end point of the graph.

(3.11) **Lemma.** *If B is a special weakly distinguished basis of the above type, then: $x \in \Delta$ iff $\mathrm{gr}(B,x)$ is connected or $\mathrm{gr}(B,x)$ has an odd number of components and is obtained by deleting from a connected subgraph another connected subgraph containing a point that has degree $\geqq 3$ in $\mathrm{gr}(B)$. (If such a point exists, it is unique.) Furthermore, $x \in V_{00}$ iff $\mathrm{gr}(B,x)$ consists of an even number of end points of $\mathrm{gr}(B)$, all of them connected in $\mathrm{gr}(B)$ to a point of degree $\geqq 3$ (degree 2 if $\mu = 3$).*

Proof. Exercise.

4. Classification of Vanishing Lattices Over $\mathbb{F}_2$

In this section R is $\mathbb{F}_2$.

(4.1) Let $(V, \langle -,- \rangle, \Delta)$ be a vanishing lattice, $\dim V = \mu$. We can choose $a_1, \ldots, a_\nu \in \Delta$ such that $T_{a_1}, \ldots, T_{a_\nu}$ generate $\Gamma = \Gamma_\Delta$ (hence $a_1, \ldots, a_\nu$ generate V). We suppose ν is the minimal number for which we can do this. We may assume that $\langle a_1, a_i \rangle = 1$ for all $i \neq 1$ (3.2).

Now let us consider a ν-dimensional vector space $\tilde{V}$ over $\mathbb{F}_2$ with a basis $B = (\tilde{a}_1, \ldots, \tilde{a}_\nu)$. We define an alternating bilinear form $\langle -,- \rangle$ on $\tilde{V}$ by putting $\langle \tilde{a}_i, \tilde{a}_j \rangle = \langle a_i, a_j \rangle$. Let $\tilde{\Delta} := \Gamma_B \cdot B$.

$(\tilde{V}, \langle -,- \rangle, \tilde{\Delta})$ is a vanishing lattice and B is a weakly distinguished basis of radius 1 for it.

Let $\pi: \tilde{V} \to V$ be the surjection defined by $\pi(\tilde{a}_i) = a_i$, $i = 1, \ldots, \nu$. Clearly, $\mathrm{Ker}(\pi) \subset \tilde{V}_0$. It is also easy to see that $\tilde{\Gamma}$ respectively $\tilde{\Delta}$ maps onto Γ, respectively Δ.

We distinguish two cases.

(4.2) *Case 1.* B is not special. Then $\tilde{\Delta} = q_B^{-1}(1) \backslash \tilde{V}_0$ (Theorem 3.5). There are two subcases:

(a) q_B vanishes on $\mathrm{Ker}\,\pi$. Then q_B induces a quadratic function q on V, and Γ respects q.

So $q(\delta) = 1$ for $\delta \in \Delta$.

On the other hand for any $v \in q^{-1}(1) \backslash V_0$, there exists $\tilde{v} \in q_B^{-1}(1) \backslash \tilde{V}_0$ such that $\pi(\tilde{v}) = v$. Hence $v \in \pi(\tilde{\Delta}) = \Delta$.

This proves $\Delta = q^{-1}(1) \backslash V_0$.

Furthermore $\Gamma = O^{\#}(q)$, since $\Gamma_B = O^{\#}(q_B)$ (Theorem 3.8).

So these vanishing lattices correspond to quadratic forms.

For fixed $p = \dim V_0$, $2n + p = \dim V$, we have at most three vanishing lattices of this kind which we denote by their monodromy groups:

$$O_1^{\#}(2n, p; \mathbb{F}_2) \quad \text{if} \quad \mathrm{Arf}(q) = 1$$

$$O_0^{\#}(2n, p; \mathbb{F}_2) \quad \text{if} \quad \mathrm{Arf}(q) = 0$$

$$O^{\#}(2n, p; \mathbb{F}_2) \quad \text{if} \quad q(V_0) = \mathbb{F}_2 \quad (p \geqq 1).$$

The number of elements of Δ is given by $2^{2n+p-1} + 2^{n+p-1}$, respectively, $2^{2n+p-1} - 2^{n+p-1}$, respectively, $2^{2n+p-1} - 2^{p-1}$.

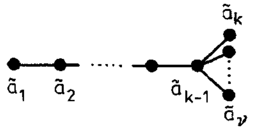
(b) There is a $\tilde{v}_0 \in \mathrm{Ker}(\pi)$ with $q_B(\tilde{v}_0)=1$. Then for any $v \in V - V_0$ there is $\tilde{v} \in \tilde{V} - \tilde{V}_0$ such that $\pi(\tilde{v}) = v$ and $q_B(\tilde{v}) = 1$. Hence $\Delta = V - V_0$ and $\Gamma = \mathrm{Sp}^{\#} V$, since $\Gamma_B = O^{\#}(q_B)$.

For fixed $p = \dim V_0$ and $2n + p = \dim V$ this gives us only one vanishing lattice (up to isomorphism).

We denote it by its monodromy group $\mathrm{Sp}^{\#}(2n, p; \mathbb{F}_2)$.

Δ has $2^{2n+p} - 2^p$ elements.

(4.3) *Case 2.* B is special. Then B is equivalent to a basis B' such that $\mathrm{gr}(B')$ is given by



Let k be the length of the longest chain in this graph.

$\dim \tilde{V}_{00} = v - 2$, if $k = 3$, and $v - k$ if $k \neq 3$.

If $v = k$, $|\tilde{\Delta}| = \frac{1}{2}k(k+1)$. Hence in general $|\tilde{\Delta}| = 2^{v-k-1}k(k+1)$ (including $k = 3$!).

Suppose $\tilde{v} \in \tilde{V}_{00} \cap \mathrm{Ker}\,\pi$. Then $\tilde{v} = \sum_{i \in I} \tilde{a}_i$ for some

$$I \subset \{k, k+1, \ldots, v\}, \quad |I| \text{ even} \quad (k \neq 3)$$
$$I \subset \{1, 3, 4, \ldots, v\}, \quad |I| \text{ even} \quad (k = 3).$$

If $i_0 \in I$ we can delete a_{i_0} from our set of generators, since $a_{i_0} = \sum_{i \in I \setminus \{i_0\}} a_i \in \pi(\Gamma_{B''} \cdot B'')$ where $B'' = B' \setminus \{i_0\}$. This contradicts our minimality hypothesis, so $I = \emptyset$, hence $\tilde{V}_{00} \cap \mathrm{Ker}(\pi) = 0$. Since $\dim \tilde{V}_0 - \dim \tilde{V}_{00} \leqq 1$, it follows that $\dim \mathrm{Ker}(\pi) \leqq 1$, so we may assume

$$\mathrm{Ker}(\pi) = 0 \quad \text{or} \quad \mathrm{Ker}(\pi) = \{0, \tilde{a}_1 + \tilde{a}_3 + \ldots + \tilde{a}_k\} \qquad (k \geqq 5 \text{ odd})$$

(add an even number of the elements $\tilde{a}_k, \ldots, \tilde{a}_v$ to $\tilde{a}_k$ if necessary). If $\mathrm{Ker}(\pi) = 0$ we find at most 2 vanishing lattices for fixed $p = \dim V_0$, $2n + p = \dim V$ (up to isomorphism)

namely one with $k = 2n$	which we shall call	$A^{\mathrm{ev}}(2n, p; \mathbb{F}_2)$	
and one with $k = 2n+1$	called	$A^{\mathrm{odd}}(2n, p; \mathbb{F}_2)$	$(p \geqq 1)$.

If $\mathrm{Ker}(\pi) = \{0, \tilde{a}_1 + \tilde{a}_3 + \ldots + \tilde{a}_k\}$, then $\tilde{\Delta}$, $\tilde{\Gamma}$ project bijectively to Δ, respectively, Γ, since $\tilde{a}_1 + \tilde{a}_3 + \ldots + \tilde{a}_k \notin \tilde{V}_{00}$. We call this vanishing lattice $A'(2n, p; \mathbb{F}_2)$. ($p = \dim V_0$, $2n + p = \dim \mathrm{V}$.)

For these three vanishing lattices Γ is determined by the splitting exact sequence

$$0 \to V_{00} \otimes V' \to \Gamma \to \Gamma_s \to 1,$$

where $\Gamma_s \cong \mathscr{S}_{k+1}(k \neq 3)$, $\Gamma_s \cong \mathscr{S}_3(k = 3)$ (symmetric groups).

In the sequel we shall write $O_1^{\#}(2n, p)$ instead of $O_1^{\#}(2n, p; \mathbb{F}_2)$ etc.

(4.4) So for fixed dimensions $2n$ and p of V, V_0, respectively, there exist at most 7 vanishing lattices up to isomorphism, namely

$$
\begin{array}{ll}
O_1^{\#}(2n,p) & \\
O_0^{\#}(2n,p) & \\
O^{\#}(2n,p) & (p\geqq 1)\\
\mathrm{Sp}^{\#}(2n,p) & \\
A^{\mathrm{ev}}(2n,p) & \\
A^{\mathrm{odd}}(2n,p) & (p\geqq 1)\\
A'(2n,p) & (n\geqq 2).
\end{array}
$$

For the quadratic form q with $\mathrm{Arf}\,q=0$ and $\dim V-\dim V_0=2, q^{-1}(1)$ does not generate V and if $\dim V-\dim V_0=4, q^{-1}(1)$ consists of 2 orbits of $\Gamma_{q^{-1}(1)}$.

So we only find vanishing lattices $O_0^{\#}(2n,p)$ for $n\geqq 3$.

In the same way we only find vanishing lattices $O^{\#}(2n,p)$ for $n\geqq 2$. $A^{\mathrm{ev}}(2,p)$ only occurs for $p=0$, since the length of the longest chain in the graph has to be 2. Furthermore we have to check whether or not any two of these 7 vanishing lattices are isomorphic. We do this by comparing the number of elements of the respective Δ's.

Only for small n some of these numbers are equal, and for those cases we actually find isomorphisms. The result is the following:

(4.5)
$$
\begin{array}{ll}
O_1^{\#}(2,0)\cong \mathrm{Sp}^{\#}(2,0)\cong A^{\mathrm{ev}}(2,0) & \\
O_1^{\#}(2,p)\cong \mathrm{Sp}^{\#}(2,p)\cong A^{\mathrm{odd}}(2,p) & (p\geqq 1)\\
O_1^{\#}(4,p)\cong A^{\mathrm{ev}}(4,p) & \\
\mathrm{Sp}^{\#}(4,p)\cong A'(4,p) & \\
O^{\#}(4,p)\cong A^{\mathrm{odd}}(4,p) & (p\geqq 1)\\
O_0^{\#}(6,p)\cong A'(6,p). &
\end{array}
$$

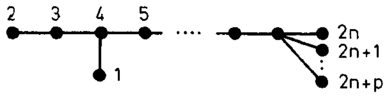
Existence of the vanishing lattices of type A and A' is clear. This also proves existence of

$$
\begin{array}{ll}
O_1^{\#}(2n,p) & \text{for } n\leqq 2\\
O_0^{\#}(6,p) & \\
O^{\#}(4,p) & \\
\mathrm{Sp}^{\#}(2n,p) & \text{for } n\leqq 2.
\end{array}
$$

We shall prove existence of $O_1^{\#}(2n,p)$ and $O^{\#}(2n,p)$ for $n\geqq 3$ and of $O_0^{\#}(2n,p)$ for $n\geqq 4$ by giving weakly distinguished bases.

It will follow that vanishing lattices of type $\mathrm{Sp}^{\#}(2n,p)$ exist [form a quotient of $O^{\#}(2n,p+1)$].

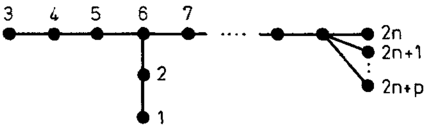
(4.6) Consider a basis B with $\mathrm{gr}(B)$ given by:



$(n \geqq 3)$

$$\begin{aligned} \mathrm{Arf}(q_B) &= 1 \quad \text{if} \quad n \equiv 2, 3 \bmod 4 \\ &= 0 \quad \text{if} \quad n \equiv 0, 1 \bmod 4. \end{aligned}$$

Also consider B' with $\mathrm{gr}(B')$ given by:

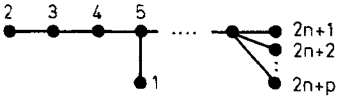


$(n \geqq 4)$

$$\begin{aligned} \mathrm{Arf}(q_{B'}) &= 1 \quad \text{if} \quad n \equiv 0, 1 \bmod 4 \\ &= 0 \quad \text{if} \quad n \equiv 2, 3 \bmod 4. \end{aligned}$$

It is not difficult to see that these bases are not special ($\Delta = q_{B'}^{-1}(1)$). It follows from Theorems (3.5), (3.8) that these bases are weakly distinguished bases for $O_1^{\#}(2n, p)$, $O_0^{\#}(2n, p)$.

For $O^{\#}(2n, p)$ we use



$(n \geqq 3)$

(4.7) For $\mathrm{Sp}^{\#}(2n, p)$ $(n \neq 1)$ and $A'(2n, p)$ $(n \neq 3)$ weakly distinguished bases do not exist [(3.8), (3.11), and (4.5)].

For the A-type lattices we have weakly distinguished bases by construction.

For $A'(6, p)$ a weakly distinguished basis might exist a priori, since $A'(6, p) \cong O_0^{\#}(6, p)$.

We shall prove that it does not exist (first for $p = 0$). We may assume a weakly distinguished basis has radius one (3.2). If there would be one it would consist of 6 elements from

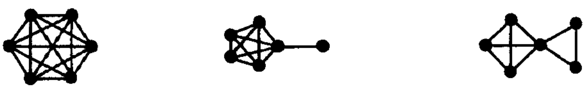
$$\{\delta' \in \Delta \mid \langle \delta', a_1 \rangle = 1\} \cup \{a_1\} \quad \text{for some} \quad a_1 \in \Delta$$

but we can write this set as

$$\{a_1\} \cup \{b_1, \ldots, b_6\} \cup \{a_1 + b_1, \ldots, a_1 + b_6\}$$

with $\langle b_i, b_j \rangle = 1$ for all $i \neq j$.

If we take 6 elements from this set that form a basis B, $\mathrm{gr}(B)$ is one of the following:



But all these correspond to weakly distinguished bases for $A^{\mathrm{ev}}(6, 0)$, not for $A'(6, 0)$.

Furthermore, if a weakly distinguished basis would exist for some $A'(6, p)$, $p \geqq 1$, we could use an argument similar to (4.3), to prove that there would exist one for $A'(6, 0)$.

Now we have proved:

(4.8) **Theorem.** (i) *Up to isomorphism, the vanishing lattices over $\mathbb{F}_2$ are given by the following list:*

$$\begin{array}{lll} O_1^{\#}(2n,p) & \text{for } n\geqq 1, & p\geqq 0 \\ O_0^{\#}(2n,p) & \text{for } n\geqq 3, & p\geqq 0 \\ O^{\#}(2n,p) & \text{for } n\geqq 2, & p\geqq 1 \\ \mathrm{Sp}^{\#}(2n,p) & \text{for } n\geqq 1, & p\geqq 0 \\ A^{\mathrm{ev}}(2n,p) & \text{for } n=1, & p=0 \quad \text{or} \quad n\geqq 2, \quad p\geqq 0 \\ A^{\mathrm{odd}}(2n,p) & \text{for } n\geqq 1, & p\geqq 1 \\ A'(2n,p) & \text{for } n\geqq 2, & p\geqq 0. \end{array}$$

(ii) *Among those we only have the following isomorphisms:*

$$\begin{array}{l} O_1^{\#}(2,0)\cong \mathrm{Sp}(2,0)=A^{\mathrm{ev}}(2,0) \\ O_1^{\#}(2,p)\cong \mathrm{Sp}(2,p)=A^{\mathrm{odd}}(2,p) \quad (p\geqq 1) \\ O_1^{\#}(4,p)\cong A^{\mathrm{ev}}(4,p) \\ O^{\#}(4,p)\cong A^{\mathrm{odd}}(4,p) \quad (p\geqq 1) \\ \mathrm{Sp}^{\#}(4,p)\cong A'(4,p) \\ O_0^{\#}(6,p)\cong A'(6,p). \end{array}$$

(iii) *Weakly distinguished bases exist for the members of the $O_1^{\#}$, $O_0^{\#}$, $O^{\#}$-series and the two A-series with the exception of $O_0^{\#}(6,p)$.*

Weakly distinguished bases do not exist for other vanishing lattices (the A' and $\mathrm{Sp}^{\#}$-series, with the exception of $\mathrm{Sp}^{\#}(2,p)\cong O_1^{\#}(2,p)$).

(4.9) From the discussions (4.2) and (4.3) it follows that vanishing lattices not admitting a weakly distinguished basis do admit a set of generators $\{a_1,\ldots,a_{\dim V+1}\}$ such that the T_{a_i} generate Γ.

(4.10) $V_{00}=V_0$ in all cases except

$$O^{\#}(2n,p) \quad \text{and} \quad A^{\mathrm{odd}}(2n,p) \quad (n\geqq 2).$$

In these exceptional cases V_{00} is of codimension 1 in V_0.
[$V_{00}=\mathrm{Ker}(q|_{V_0})$ for $O^{\#}(2n,p)$.]

(4.11) To conclude this section we shall characterize vanishing lattices of the types A and A' by the property that they do not contain a vanishing sublattice of type $O_1^{\#}(6,0)$ (in a sense to be made precise later). A first step is the following lemma which is a generalization of [10, Lemma 5].

(4.12) **Lemma.** *Let $(V,\langle -,-\rangle,\Delta)$ be a vanishing lattice of one of the types A or A'. Let $a,b,c\in\Delta$, such that $\langle a,b\rangle=\langle a,c\rangle=\langle b,c\rangle=0$ and $a+b$, $a+c$, $b+c\notin V_0$. Then: $a+b+c\in\Delta$ iff $(V,\langle -,-\rangle,\Delta)$ is of type $A'(6,p)$.*

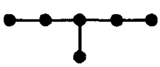
Proof. Suppose $a+b+c\in\Delta$. We divide out V_0 and consider the vanishing lattice V'. Then a', b', and c' are distinct elements of Δ' since $a+b$, $a+c$, $b+c\notin V_0$. Now we are in the situation of Wajnryb's Lemma 5. Its proof shows that $\dim V'=6$. So V' is of type $A^{ev}(6,0)$ or $A'(6,0)$. It is easily seen that the first case is excluded [Wajnryb's Lemma 3 or our Lemma (3.11)]. If V is not of type $A'(6,p)$ we can divide out a subspace of V_0 to obtain type $A^{odd}(6,1)$ still satisfying our condition. But this is excluded by the same argument as above.

The converse is an easy consequence of the isomorphism between $A'(6,p)$ and $O_0^{\#}(6,p)$.

(4.13) **Proposition.** *Let $(V,\langle -,-\rangle,\Delta)$ be a vanishing lattice. Then the following statements are equivalent:*

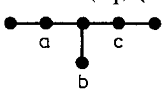
(i) *$(V,\langle -,-\rangle,\Delta)$ is not of one of the types A^{ev}, A^{odd}, A'.*

(ii) *There exist $V_1\subset V$, $\Delta_1\subset\Delta$ such that $(V_1,\langle -,-\rangle|_{V_1\times V_1},\Delta_1)$ is a vanishing lattice of type $O_1^{\#}(6,0)$.*

Proof. Suppose that $(V,\langle -,-\rangle,\Delta)$ is not of one of the types A^{ev}, A^{odd}, A'. Then it either admits a weakly distinguished basis as in (4.6) or it is of type $\mathrm{Sp}^{\#}$, hence a quotient of a lattice of type $O^{\#}(2n,p)$ with $n\geqq 4$. The latter has a weakly distinguished basis as in (4.6). In both cases the corresponding graph has a full subgraph  which corresponds to a weakly distinguished basis of a vanishing lattice of type $O_1^{\#}(6,0)$.

Let $(V_1,\langle -,-\rangle|_{V_1\times V_1},\Delta_1)$ be the sublattice of V corresponding to this graph [in the second case we take its image in $\mathrm{Sp}^{\#}(2n,p-1)$; if $n=3$, then $V=V_1$, but $\Delta\neq\Delta_1$].

It follows from Theorem (3.5) that V_1 is as desired.

On the other hand let V_1 be a subspace such that $(V_1,\langle -,-\rangle|_{V_1\times V_1},\Delta_1)$ is a vanishing lattice of type $O_1^{\#}(6,0)$. In the graph 

corresponding to a weakly distinguished basis, the points a,b,c represent elements of Δ satisfying the condition of the previous lemma. It follows that V is either of type $A'(6,p)$ or not of any of the types A^{ev}, A^{odd}, A'. By dividing out V_0 we see that the first case cannot occur since $A'(6,0)$ and $O_1^{\#}(6,0)$ are not isomorphic.

(4.14) We have the following table:

	Possess a weakly distinguished basis	Do not possess a weakly distinguished basis
Contain a sublattice $O_1^{\#}(6,0,\mathbb{F}_2)$	$O_1^{\#}(2n,p;\mathbb{F}_2)$ $n\geqq 3$ $O_0^{\#}(2n,p;\mathbb{F}_2)$ $n\geqq 4$ $O^{\#}(2n,p;\mathbb{F}_2)$ $n\geqq 3, p\geqq 1$	$\mathrm{Sp}^{\#}(2n,p;\mathbb{F}_2)$, $n\geqq 3$
Do not contain a sublattice $O_1^{\#}(6,0;\mathbb{F}_2)$	$A^{ev}(2n,p;\mathbb{F}_2)$ $A^{odd}(2n,p;\mathbb{F}_2)$ $p\geqq 1$	$A'(2n,p;\mathbb{F}_2)$

5. The Unipotent Part of the Monodromy Group

In this section we lift our results about V_{00} to integral lattices. Also we give a description of Γ_u (over $\mathbb{Z}$ or $\mathbb{F}_2$). Over $\mathbb{Z}$ this description is complete only if $j^{-1}(2V^*)=V_0+2V$.

(5.1) **Lemma.** *Let $(V, \langle -, - \rangle, \Delta)$ be a vanishing lattice over $\mathbb{Z}$. Then V_{00} is the inverse image of $\bar{V}_{00}$ under the natural projection $V_0 \to \bar{V}_0$. ($\bar{V}$ is the vanishing lattice $V/2V$ over $\mathbb{F}_2$.)*

In particular $V_{00} \supset 2V_0$.

Proof. It is clear that any element of V_{00} projects into $\bar{V}_{00}$. Conversely, let $v \in V_0$, such that v projects into $\bar{V}_{00}$, and let $\delta \in \Delta$.

Now $\delta + v \in \Delta$ iff there exists $y \in V$ such that $\langle \delta + v, y \rangle = 1$ and $\delta + v$ projects into $\bar{\Delta}$. But $\langle \delta + v, y \rangle = \langle \delta, y \rangle = 1$ for some $y \in V$ and δ projects into $\bar{\Delta}$, v into $\bar{V}_{00}$, so $\delta + v$ projects into $\bar{\Delta}$. Hence $v \in V_{00}$.

(5.2) If B is a weakly distinguished basis of V that projects to a nonspecial basis of $\bar{V}$, it follows that $V_{00} = \mathrm{Ker}(q_B|_{V_0})$.

(5.3) **Theorem.** *Identifying Γ_u with a subgroup of $V_0 \otimes V'$ as in* (2.10), *we have: $V_{00} \otimes V' \subset \Gamma_u \subset V_{00} \otimes V' + V_0 \otimes jj^{-1}(2V^*)$.*

In particular, if $j^{-1}(2V^*) = V_0 + 2V$ ($R = \mathbb{F}_2$ and $R = \mathbb{Z}$ with d_n, as defined in (2.3), *odd), then $\Gamma_u = V_{00} \otimes V'$.*

Proof. The first inclusion has been proven in (2.11).

Now choose a basis $(v_1, \ldots, v_p)$ for V_0 such that $v_2, \ldots, v_p \in V_{00}$, and let $g \in \Gamma_u$. There exist $w_1, \ldots, w_p \in V$ such that $g(x) = x + \sum_{i=1}^{p} \langle x, w_i \rangle v_i$ for all $x \in V$. For $\delta \in \Delta$ we have: $g(\delta) \in \Delta$ so $\langle \delta, w_1 \rangle v_1 \in V_{00}$.

Hence $v_1 \in V_{00}$ or $\langle \delta, w_1 \rangle \in 2R$ for all $\delta \in \Delta$. In the first case we are done, as well as in the second, for it follows that $j(w_1) \in 2V^*$ since Δ generates V.

6. Singularities and Lefschetz Pencils

(6.1) Let V be the middle homology group of the Milnor fibre of a complex isolated complete intersection singularity of odd dimension. On V we have the intersection form $\langle -, - \rangle$, which is skew symmetric. The monodromy group of the singularity is generated by transvections T_δ with respect to the vanishing cycles δ. V is generated by the set Δ of vanishing cycles and unless the singularity is an ordinary double point there are two vanishing cycles with intersection number one. (For these well known facts we refer to the lecture notes of Looijenga [9].)

It follows that $(V, \langle -, - \rangle, \Delta)$ is a vanishing lattice. We shall apply the results of the preceding sections to it. From (2.5) and (2.9) we immediately get the following theorem which is a generalization of [1] (A_μ-singularities), [10] (plane curve singularities), and [5] (hypersurface singularities).

(6.2) **Theorem.** *The monodromy group of an isolated complete intersection singularity contains the subgroup G of $g \in \mathrm{Aut}(V)$ that act trivially on $V^*/2V'$. G contains* *the congruence subgroup modulo* 2^{r+1} *of the group* $\mathrm{Sp}^{\#} V$ (*r and* $\mathrm{Sp}^{\#} V$ *are defined in Sect.* 2).

The set of vanishing cycles consists of those $x \in V$ *that satisfy* $\langle x, V\rangle = \mathbb{Z}$ *and project to elements of* $\bar{\Delta}$ *in the modulo* 2 *reduction* $\bar{V}$ *of* V.

(6.3) It is obvious that G is contained in the congruence subgroup modulo 2 of the monodromy group. Equality however does not hold in general. An example is provided by the singularity $\tilde{E}_6$, given by $x_1^3 + x_2^3 + x_3^3 + \lambda x_1 x_2 x_3 + x_4^2 = 0$. The third power of its classical monodromy operator is in the congruence subgroup, but not in G.

(6.4) **Theorem.** $V_{00} \otimes V' \subset \Gamma_u \subset V_{00} \otimes V' + V_0 \otimes jj^{-1}(2V^*)$.

If $j^{-1}(2V^*) = 2V + V_0$, *as is for example the case for a curve singularity then* $\Gamma_u = V_{00} \otimes V'$.

(6.5) We can apply the results of Sect. 4 to the modulo 2 reduction of the vanishing lattice of a singularity. We get the following results: If a singularity admits a weakly distinguished basis for the vanishing homology, then the modulo 2 reduction cannot be of type $\mathrm{Sp}^{\#}(2n, p; \mathbb{F}_2)$ with $n \geqq 3$ or of type $A'(2n, p; \mathbb{F}_2)$ (4.8 iii). (It is not known whether or not weakly distinguished bases exist for all isolated complete intersection singularities.)

If a singularity is adjacent to the E_6-singularity, then the modulo 2 reduction of the vanishing homology contains a $O_1^{\#}(6, 0; \mathbb{F}_2)$ lattice corresponding to this E_6, so we do not have type A or A' (4.13). In both cases the condition can be weakened to the corresponding condition on the modulo 2 reduction of the vanishing lattice.

If we restrict ourselves to isolated hypersurface singularities we get the following precise result:

(6.6) **Theorem.** *The modulo 2 reduction of the vanishing lattice associated to an isolated hypersurface singularity is of type:*

$A^{\mathrm{ev}}(2n, 0; \mathbb{F}_2)$ *for the* A_{2n}*-singularity*

$A^{\mathrm{odd}}(2n, 0; \mathbb{F}_2)$ *for the* A_{2n+1}*-singularity*

$A^{\mathrm{odd}}(2n-2, 2; \mathbb{F}_2)$ *for the* D_{2n}*-singularity*

$A^{\mathrm{ev}}(2n, 1; \mathbb{F}_2)$ *for the* D_{2n+1}*-singularity*

$O_1^{\#}(2n, p; \mathbb{F}_2)$, $O_0^{\#}(2n, p; \mathbb{F}_2)$ *or* $O^{\#}(2n, p; \mathbb{F}_2)$ *for other singularities.*

Proof. Isolated hypersurface singularities admit a (weakly) distinguished basis [7]. This excludes the $\mathrm{Sp}^{\#}$ and A'-types.

From the classification of hypersurface singularities [2] we know that only the A_k and D_k singularities are not adjacent to E_6. So for other singularities we have $O_1^{\#}$, $O_0^{\#}$ or $O^{\#}$.

The theorem now follows. (Look at the intersection diagrams for the A_k and D_k singularities.)

(6.7) We shall now describe another situation where vanishing lattices occur. A good reference for this is [8].

A *pencil* of hyperplanes in a complex projective space $\mathbb{P}^N$ consists of all hyperplanes containing a fixed $(N-2)$-dimensional projective subspace A, called the *axis* of the pencil. Hyperplanes of $\mathbb{P}^N$ correspond to points of the dual projective space $\check{\mathbb{P}}^N$, and a set of hyperplanes is a pencil if it corresponds to a line L in $\check{\mathbb{P}}^N$. We shall consider the intersections of hyperplanes in the pencil with a nonsingular subvariety X of dimension n in $\mathbb{P}^N$. If H is a hyperplane, then $H \cap X$ will be nonsingular iff H is not tangent to X. The hyperplanes that are tangent to X correspond to the points of a subvariety $\check{X}$ of $\check{\mathbb{P}}^N$, called the *dual variety*. In general $\check{X}$ is a hypersurface with singularities. A generic line L intersects $\check{X}$ transversally in r regular points, where $r = \deg \check{X}$ if $\check{X}$ is a hypersurface, $r = 0$ if $\check{X}$ is of lower dimension. Then the axis A of the corresponding pencil intersects X transversally, and if we blow up X along $X \cap A$, the resulting space Y will be nonsingular. The fibres of the associated map $f: Y \to L$ are the hyperplane sections $H_t \cap X$, and the critical values of f are the points of $\check{X} \cap L$. The singular fibres contain exactly one singular point, which is non-degenerate, i.e. an ordinary double point. Removing the critical points and the singular fibres we get a locally trivial fibre bundle $f: Y^* \to L^*$. The fundamental group of L^* acts on the homology of the fibre. This *monodromy action* is trivial except in the middle dimension $n-1$. The monodromy group Γ in dimension $n-1$ is generated by the transformations $T_{\delta_i}: x \to x - \langle x, \delta_i \rangle \delta_i$ where $\langle -, - \rangle$ denotes the intersection form and the δ_i are the so-called vanishing cycles corresponding to the ordinary double points mentioned above. The T_{δ_i} are conjugated in Γ.

The vanishing cycles δ_i generate a subgroup V of the middle homology group, called the *vanishing homology*.

If n is odd, the intersection form is symmetric, if n is even it is skew symmetric. We assume n even and consider the triple $(V, \langle -, - \rangle, \Gamma \cdot \{\delta_1, \ldots, \delta_r\})$. It follows from Lemma (2.1) that this is a vanishing lattice if there are $\delta, \delta' \in \Gamma \cdot \{\delta_1, \ldots, \delta_r\}$ such that $\langle \delta, \delta' \rangle = 1$. Such δ, δ' exist if there is a hyperplane section of X having an isolated singularity other than an ordinary double point. We conjecture that this condition can always be met. If X is a hypersurface this can at least be achieved in some multiple embedding of $\mathbb{P}^N$. Then the theory of the preceding sections applies.

Added to revised version. For any vanishing lattice V, the group Γ/G can be identified with a certain canonical central extension of the monodromy group of the mod 2 reduction.

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