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The Scalar Curvature of Minimal Hypersurfaces in Spheres

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Introduction

Let M be a closed minimally immersed hypersurface of S^{n+1}, and h its second fundamental form. We denote the square of the length of h by S. It follows from the Gauss and Codazzi equations that the apparently extrinsic quantity S is in fact intrinsic and is given by

$$S=n(n-1)-R,$$

where R is the scalar curvature of M. If $0\leqq S(x)\leqq n$ for all $x\in M$, then either $S\equiv 0$ or $S\equiv n$ [6]. In the case that $S\equiv n$, Chern, do Carmo, Kobayashi; and Lawson independently proved that M is a Clifford torus [2, 4]. In a previous paper [5] we showed that there exists a positive real number $C(n)>0$ $\left[C(3)=3 \text{ and } C(n)\geqq \frac{1}{12n} \text{ in general}\right]$ such that if $S(x)$ is a constant bigger than n then it must be bigger than or equal to $n+C(n)$. We also learned that Naoya Doi has worked on the same problem [3]. His main results are for minimal hypersurface M of S^{n+1} with constant scalar curvature and whose sectional curvature is bounded above by 1, he proved that either $S=0$ or $S\geqq 2(2n-3)$. However, under his assumptions it is easily seen that either $S=0$; or $n=3$, $S=6$ and M is the classical isoparametric example. To see this, let $e_1, \ldots, e_n$ be a local orthonormal frame field of principal curvature directions of M, then the sectional curvature in the direction e_i, e_j, for $i\neq j$, is $1+\lambda_i\lambda_j$, where λ_i's are the principal curvatures. The conditions

$$\sum_1^n \lambda_i=0,$$

$$\sum_i \lambda_i^2=S, \text{ constant},$$

$$1+\lambda_i\lambda_j\leqq 1, \quad \text{for} \quad i\neq j,$$

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imply that there are at most one positive λ_i and one negative λ_i and the principal curvatures of M are constants. Hence either $\lambda_i = 0$ for all i, which gives $S=0$; or by Theorem 1 of [5], $n=3$, $S=6$ and M is the classical isoparametric example.

In this paper, we prove a pinching theorem for minimal hypersurfaces in S^{n+1} for $n \leqq 5$:

Theorem A. *Let M be a closed minimally immersed hypersuface in S^{n+1}, $n \leqq 5$, and S the square of the length of the second fundamental form of M. Then there exists $\delta(n) > 0$ such that if $n \leqq S(x) \leqq n + \delta(n)$ then $S(x) \equiv n$, hence M is a Clifford torus.*

The proof uses the equations for Δh, $\Delta(\nabla h)$, and some integral equations naturally associated to the second fundamental form h. Since we do not assume that S is a constant function, the estimate we have in [5] can not be applied here, and the technique of using integral equations seems essential.

In the end of this paper, we give a proof of the theorem we stated in [5], namely:

Theorem B. *Let M^3 be a closed minimal hypersurface with constant scalar curvature R in S^4. If all three principal curvatures of M are distinct at every point, then $R=0$.*

1. Formulas Involving the Second Fundamental Form of Minimal Hypersurfaces

In this section, we will review the notation used in our previous paper [5], derive several new formulas and give a proof of Theorem A. Let M be a minimal hypersurface of S^{n+1}, and $e_1, \ldots, e_n$ a local orthonormal frame field on M, $w_1, \ldots, w_n$ its dual coframe. Then we have

$$I = \sum_i w_i^2, \tag{1.1}$$

$$II = h = \sum_{i,j} h_{ij} w_i w_j, \qquad h_{ij} = h_{ji}, \tag{1.2}$$

$$\nabla h = \sum_{i,j,k} h_{ijk} w_i w_j w_k, \qquad h_{ijk} = h_{ikj}, \tag{1.3}$$

$$\nabla^2 h = \sum_{i,j,k,l} h_{ijkl} w_i w_j w_k w_l, \tag{1.4}$$

where

$$h_{ijkl} = h_{ijlk} + \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}.$$

$$\nabla^3 h = \sum_{\substack{i,j,k,l\\m}} h_{ijklm} w_i w_j w_k w_l w_m, \tag{1.5}$$

where

$$h_{ijklm} = h_{ijkml} + \sum_r h_{rjk} R_{rilm} + \sum_r h_{irk} R_{rjlm} + \sum_r h_{ijr} R_{rklm},$$

$$\Delta h_{ij} = (n-S) h_{ij}, \tag{1.6}$$

where

$$S=|h|^2.$$

Since we do not assume S is a constant function, the same computation as in [5] will lead to a slightly different formula for $\Delta(\nabla h)$:

$$\begin{aligned}\Delta h_{ijk}=&(2n+3-S)h_{ijk}+2\sum_{m,l}(h_{iml}h_{mk}h_{jl}+h_{jml}h_{mi}h_{lk}+h_{kml}h_{mi}h_{lj})\\&-\sum_{m,l}(h_{ijm}h_{ml}h_{lk}+h_{ikm}h_{ml}h_{lj}+h_{jkm}h_{ml}h_{li})\\&-(S_ih_{jk}+S_jh_{ik}+S_kh_{ij}).\end{aligned}\tag{1.7}$$

Then it follows easily from (1.6), (1.7) that we have:

Theorem 1. *Let M be an immersed minimal hypersurface of S^{n+1}, h its second fundamental form. Then*

$$\tfrac{1}{2}\Delta S=S(n-S)+|\nabla h|^2.\tag{1.8}$$

$$\tfrac{1}{2}\Delta(|\nabla h|^2)=(2n+3-S)|\nabla h|^2+3(2B-A)-\tfrac{3}{2}|\nabla S|^2+|\nabla^2 h|^2,\tag{1.9}$$

where

$$S=|h|^2,$$

$$A=\sum_{\substack{i,j,k\\l,m}}h_{ijk}h_{ijl}h_{km}h_{ml},$$

$$B=\sum_{\substack{i,j,k\\l,m}}h_{ijk}h_{klm}h_{jl}h_{im}.$$

Near any given point $p\in M$, we can choose a local frame field $e_1,\dots,e_n$ so that at p, we have

$$h_{ij}=\lambda_i\delta_{ij}.$$

Let f_k denote the smooth function on M given by $\operatorname{tr}(h^k)$, i.e., $f_k=\sum_i\lambda_i^k$. With such frame field, we have shown in [5] that

$$h_{ijij}=h_{jiji}+t_{ij},\tag{1.10}$$

where

$$t_{ij}=(\lambda_i-\lambda_j)(1+\lambda_i\lambda_j);$$

$$|\nabla^2 h|^2\geqq\frac{3}{2S}\sum_k(S+f_3\lambda_k-S\lambda_k^2)^2,\quad\text{at } p.\tag{1.11}$$

Hence we have

Proposition 2. *With the same assumption as in Theorem 1, we have*

$$|\nabla^2 h|^2\geqq\tfrac{3}{2}(Sf_4-f_3^2-S^2-S(S-n)).\tag{1.12}$$

Next we obtain a pointwise estimate for $A-2B$,

$$\begin{aligned}3(A-2B)&=\sum_{ijk} h_{ijk}^2(\lambda_i^2+\lambda_j^2+\lambda_k^2-2\lambda_i\lambda_j-2\lambda_j\lambda_k-2\lambda_i\lambda_k)\\ &=\sum_{\substack{ijk\\ \text{distinct}}} h_{ijk}^2(2(\lambda_i^2+\lambda_j^2+\lambda_k^2)-(\lambda_i+\lambda_j+\lambda_k)^2)\\ &\quad+3\sum_{i\neq k} h_{iik}^2(\lambda_k^2-4\lambda_i\lambda_k)+\sum_i h_{iii}^2(-3\lambda_i^2)\\ &<2S\sum_{\substack{ijk\\ \text{distinct}}} h_{ijk}^2+\frac{(\sqrt{17}+1)}{2}S\cdot 3\sum_{ik} h_{iik}^2,\end{aligned}$$

where $\lambda_k^2-4\lambda_i\lambda_k<\frac{\sqrt{17}+1}{2}S$ is obtained by standard Lagrange multiplier method with respect to the constraints $\sum\lambda_i=0$ and $\sum\lambda_i^2=S$. Hence we obtain

Proposition 3. *With the same assumptions as in Theorem 1, we have*

$$3(A-2B)\leqq\frac{\sqrt{17}+1}{2}S|\nabla h|^2. \tag{1.13}$$

Now we will derive some integral equations. Following Cheng and Yau [1], for a given symmetric two tensor $l=\sum_{ij} w_i w_j$ and a smooth function u on M, we consider the smooth function $\sum_{ij} l_{ij}u_{ij}$.

Then if M is a closed manifold, by Stokes' theorem we have

$$\int_M \sum_{ij} l_{ij}u_{ij}=-\int_M \sum_{ij} l_{ijj}u_i. \tag{1.14}$$

Moreover, if l satisfies the Codarzi equations, i.e., ∇l is a symmetric three tensor, and $\operatorname{tr} l$ is constant, then

$$\sum_j l_{ijj}=\sum_j l_{jji}=(\operatorname{tr} l)_i=0.$$

So in particular,

$$\int_M \sum_{ij} h_{ij}(f_3)_{ij}=0. \tag{1.15}$$

On the other hand, if we choose a local frame field $e_1,\ldots,e_n$ such that $h_{ij}(P)=\lambda_i\delta_{ij}$ at P, then

$$\begin{aligned}\tfrac{1}{3}\sum_{ij} h_{ij}(f_3)_{ij}&=\tfrac{1}{3}\sum_k \lambda_k(f_3)_{kk}\\ &=\sum_k \lambda_k\Big(\sum_i h_{iikk}\lambda_i^2+2\sum_{ij} h_{ijk}^2\lambda_i\Big)\\ &=\sum_{ik} h_{iikk}\lambda_k\lambda_i^2+2\sum_{ijk} h_{ijk}^2\lambda_i\lambda_k\\ &=\sum_{ik}(h_{kkii}+(\lambda_i-\lambda_k)(1+\lambda_i\lambda_k))\lambda_k\lambda_i^2+2B\end{aligned}$$

$$= \sum_i \left(\frac{S_{ii}}{2} - \sum_{jk} h_{ijk}^2 \right) \lambda_i^2 + \sum_{ik} \lambda_i^2 \lambda_k (\lambda_i - \lambda_k)(1 + \lambda_i \lambda_k) + 2B$$

$$= \sum_{ijk} \frac{h_{ik} h_{kj}}{2} S_{ij} + S f_4 - f_3^2 - S^2 - (A - 2B).$$

Integrating both sides and using (1.15), we obtain

$$\int_M A - 2B = \int_M S f_4 - f_3^2 - S^2 + \sum_{ijk} \frac{h_{ik} h_{kj}}{2} S_{ij},$$

$$= \int_M S f_4 - f_3^2 - S^2 - \sum_{ijk} (h_{ik} h_{kj})_j \frac{S_i}{2},$$

$$= \int_M S f_4 - f_3^2 - S^2 - \sum_{ijk} h_{ikj} h_{kj} \frac{S_i}{2},$$

$$= \int_M S f_4 - f_3^2 - S^2 - \frac{|\nabla S|^2}{4}.$$

Hence we have:

Theorem 4. *Let M be an immersed closed minimal hypersurface of S^{n+1}, h its second fundamental form, $S = |h|^2$ and $f_k = t_r(h^k)$. Then*

$$\int_M A - 2B = \int_M S f_4 - f_3^2 - S^2 - \frac{|\nabla S|^2}{4}, \tag{1.16}$$

where A and B are as in Theorem 1.

We can now give a proof of Theorem A.

Proof. Integrating (1.8) and (1.9), we get

$$\int_M \frac{f'(S)}{2} |\nabla S|^2 = \int_M S(S-n) f(S) - f(S) |\nabla h|^2, \tag{1.17}$$

$$\int_M |\nabla^2 h|^2 = \int_M (S - 2n - 3) |\nabla h|^2 + 3(A - 2B) + \tfrac{3}{2} |\nabla S|^2. \tag{1.18}$$

Using (1.12)

$$\int_M (S - 2n - 3)|\nabla h|^2 + 3(A - 2B) + \tfrac{3}{2}|\nabla S|^2 \geqq \int_M \tfrac{3}{2}(S f_4 - f_3^2 - S^2 - S(S-n))$$

$$= \int_M \tfrac{3}{2}(S f_4 - f_3^2 - S^2) - \tfrac{3}{2} |\nabla h|^2, \qquad \text{using Theorem 4}$$

$$= \int_M \tfrac{3}{2}(A - 2B) + \tfrac{3}{8} |\nabla S|^2 - \tfrac{3}{2} |\nabla h|^2.$$

So

$$\int_M (S - 2n - \tfrac{3}{2}) |\nabla h|^2 + \tfrac{3}{2}(A - 2B) + \tfrac{9}{8} |\nabla S|^2 \geqq 0.$$

Using Proposition 3 and (1.17) we have

$$0 \leqq \int_M (S - 2n - \tfrac{3}{2}) |\nabla h|^2 + \frac{\sqrt{17} + 1}{4} S |\nabla h|^2 + \tfrac{9}{4}(S^2(S-n) - S |\nabla h|^2).$$

Now suppose $n \leqq S(x) < n + \varepsilon$, then

$$\int_M S^2(S-n) \leqq (n+\varepsilon) \int_M S(S-n) = (n+\varepsilon) \int_M |\nabla h|^2$$

So we have

$$0 \leqq \int_M \left(\frac{\sqrt{17}-4}{4} S - \frac{6-n-9\varepsilon}{4} \right) |\nabla h|^2 .$$

Hence, if $n \leqq 5$ and $\varepsilon = \dfrac{6-1.13n}{5+\sqrt{17}}$, then

$$(\sqrt{17}-4)S - (6-n-9\varepsilon) < 0 .$$

Therefore, we have $|\nabla h| = 0$, i.e., $S \equiv n$. This proves Theorem A.

3. Proof of Theorem B

Since $R = 6 - S$, S is a constant function. We may assume $S \geqq 6$ [5]. We denote f_3 by f. If f is also a constant function, then all three principal curvatures are constants and $S = 6$, so we are done. If f is not a constant function, we will derive a contradiction. The derivation is rather long, so we divide it into four steps. Let $f(p) = \max_M f$. We may assume $f(p) \neq 0$, otherwise we use $\min f$. We choose a local frame e_1, e_2, e_3, e_4 such that at p

$$h_{ij}(p) = \lambda_i \delta_{ij} ,$$

with $\lambda_1 < \lambda_2 < \lambda_3$. Then

Step 1.

(i) $h_{ijk}(p) = 0$ for all i, j, k, except $h_{123}^2(p) = \dfrac{S(S-3)}{6}$.

(ii) $h_{iijk}(p) = h_{iiik}(p) = h_{kiii}(p) = 0$, if i, j, k are distinct.

(iii) $3 \sum_{i,l} h_{iill}^2(p) - 2 \sum_i h_{iiii}^2(p) = 3S(S-3)^2$.

To see this, we note that

$$\sum_i \lambda_i = 0, \text{ since } M \text{ is minimal}.$$

$$\sum_i h_{iik}(p)\lambda_i = \frac{S_k(p)}{2} = 0, \ S \text{ being constant}.$$

$$\sum_i h_{iik}(p)\lambda_i^2 = \frac{f_k(p)}{3} = 0, \ f(p) \text{ being } \max f .$$

By assumption, $\lambda_1, \lambda_2, \lambda_3$ are distinct, hence $h_{iik}(p) = 0$ for all i, k.

Using (1.8) and the assumption S is a constant, we have

$$y = \sum h_{ijk}^2(p) = S(S-3) = 6h_{123}^2 + 3 \sum_{i \neq k} h_{iik}^2 + \sum_i h_{iii}^2 .$$

Hence $h_{123}^2(p)=\frac{S(S-3)}{6}$. To obtain (ii), we take the derivative of the constant function $y=|\nabla h|^2$. We have

$$\sum_{ijk} h_{ijk}(p)h_{ijkl}(p)=0,$$

so $h_{iijk}(p)=0$, if i, j, k are distinct. For $j\neq k$, we also have

$$\sum_i h_{iijk}(p)=0,$$

$$\sum_i h_{iijk}(p)\lambda_i=\frac{S_{jk}(p)}{2}-\sum_{il} h_{ilj}(p)h_{ilk}(p)=0.$$

Since λ_1, λ_2, λ_3 are distinct, we have $h_{jjjk}(p)=h_{kkjk}(p)=0$.

Using (1.9), we have

$$\sum h_{ijkl}^2=S(S-3)(S-9)+3(A-2B).$$

Substitute (i) into

$$\begin{aligned}3(A-2B)(p)&=\sum h_{ijk}^2(p)(\lambda_i^2+\lambda_j^2+\lambda_k^2-2\lambda_i\lambda_j-2\lambda_i\lambda_k-2\lambda_j\lambda_k)\\&=6h_{123}^2(p)(\lambda_1^2+\lambda_2^2+\lambda_3^2-2\lambda_1\lambda_2-2\lambda_1\lambda_3-2\lambda_2\lambda_3)\\&=2S^2(S-3).\end{aligned}$$

This proves (iii).

Step 2. $f(p)>0$, $\frac{S}{6}<\lambda_1^2<\frac{S}{2}$, $0<\lambda_2^2<\frac{S}{6}$, and $\lambda_1<\lambda_2<0$.

Since $f(p)$ is $\max f$, we have

$$\begin{aligned}\tfrac{1}{3}\Delta f(p)&=(3-S)f(p)+6\sum_{ijk} h_{ijk}^2(p)\lambda_i, \quad \text{using (i) of Step 1}.\\&=(3-S)f(p)\leqq 0.\end{aligned}$$

Hence $f(p)>0$.

Using Lagrange multiplier to maximizing $\sum_{i=1}^{3} x_i^3$ with respect to the constraints $\sum_{i=1}^{3} x_i=0$ and $\sum_{i=1}^{3} x_i^2=S$, we see that $f^2\leqq\frac{S^3}{6}$ and equality holds if and only if two of the x_i's are equal to $\sqrt{\frac{S}{6}}$ or $-\sqrt{\frac{S}{6}}$. Hence by our assumption, $f^2<\frac{S^3}{6}$ on M, and $\frac{S}{6}<\lambda_1^2<\frac{S}{2}$, $0<\lambda_2^2<\frac{S}{6}$, $\lambda_1<\lambda_2<0$.

Step 3. $h_{iiil}(p)=-\frac{1}{3}(S-3)\lambda_i+g_i\lambda_l+Zg_ig_l$, where

$$g_i=\lambda_i^2-\frac{f}{S}\lambda_i-\frac{S}{3}.$$

To obtain this, we solve the following rank 5 linear system of six equations and six unknowns h_{iiil}, $i \leqq l$:

$$\sum_i h_{iiil}(p) = 0,$$

$$\sum_i h_{iiil}(p)\lambda_i = -\sum_{ij} h^2_{ijl}(p) + \frac{S_{ll}(p)}{2} = -\frac{S(S-3)}{3}, \tag{2.1}$$

where

$$h_{iijj} = h_{jjii} + (1+\lambda_i\lambda_j)(\lambda_i - \lambda_j).$$

Step 4. A Contradition

The real value Z in Step 3 is determined by substituting Step 3 into (iii) of Step 1, in fact,

$$x = Z\left(\frac{S^2}{6} - \frac{f^2(p)}{S}\right) = \left(\frac{1}{3} - \frac{1}{S}\right) f(p)$$
$$\pm \left[\frac{f^2(p)}{S^2}\left(\frac{19}{9}S^2 - \frac{8}{3}S + 1\right) + \frac{7}{9}S(S-6)\left(S - \frac{15}{7}\right)\right]^{1/2}. \tag{2.2}$$

Note that $S(S-6)(S-\frac{15}{7}) \geqq 0$ for $S \geqq 6$ and

$$\tfrac{19}{9}S^2 - \tfrac{8}{3}S + 1 \geqq (\tfrac{4}{3}S - 1)^2.$$

Hence we have

$$x > \left(\frac{5}{3} - \frac{2}{S}\right) f(p), \quad \text{if} \quad x \geqq 0; \tag{2.3}$$

or

$$x < -f(p), \quad \text{if} \quad x < 0. \tag{2.4}$$

On the other hand, $f(p)$ is $\max f$, so for each l, we have

$$\tfrac{1}{3} f_{ll}(p) = \sum_i h_{iiil}(p)\lambda_i^2 + 2\sum_{ij} h^2_{ijl}(p)\lambda_i \leqq 0. \tag{2.5}$$

Using the computation in Steps 1–3 and the fact that

$$f = f_3 = 3\lambda_l\left(\lambda_l^2 - \frac{S}{2}\right),$$

we have

$$-\frac{3}{S}x\left(\lambda_l^2 - \frac{S}{6}\right)\left(\lambda_l^2 - \frac{2S}{3}\right) \leqq \lambda_l\left(\lambda_l^2 - \frac{S}{6}\right)\left(\frac{9}{S}\lambda_l^4 - \frac{15}{2}\lambda_l^2 + 2S - 3\right) \tag{2.6}$$

for all $l = 1, 2, 3$. However, if $x > 0$, a direct computation shows that (2.6) with $l = 1$ and (2.3) can not hold simultaneously. Similarly if $x < 0$, then (2.6) with $l = 2$ and (2.4) can not be true simultaneously. This contradiction proves our theorem.

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