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On Conformal Minimal Immersions of S^2 into $\mathbb{C}P^n$

John Bolton[1], Gary R. Jensen[2], Marco Rigoli[3], and Lyndon M. Woodward[1]

[1] Department of Mathematical Sciences, University of Durham, Durham DHI 3 LE, UK
[2] Department of Mathematics, Washington University, MO 63130, USA
[3] Institute of Mathematics, ICTP, I-34100 Trieste, Italy

1. Introduction

In this paper we consider conformal minimal immersions of the Riemann sphere S^2 into $\mathbb{C}P^n$, where this latter space is given the Fubini-Study metric of constant holomorphic sectional curvature 4. As is well known, particular examples of such immersions are given by holomorphic or antiholomorphic immersions of S^2 into $\mathbb{C}P^n$ [13]. Also, since $\mathbb{R}P^n$ is a totally real, totally geodesic submanifold of $\mathbb{C}P^n$ whose double cover is S^n, it follows that every conformal minimal immersion of S^2 into S^n determines a totally real conformal minimal immersion of S^2 into $\mathbb{C}P^n$. However these examples are rather special in that they have constant Kähler angle.

Before stating our main results we should remark that any immersion of S^2 can be made conformal by first applying a suitable diffeomorphism of S^2 to itself.

Our main results are as follows.

The Veronese Sequence (Theorem 5.2.). For each $p=0,\ldots,n$, let $\psi_p: S^2 \to \mathbb{C}P^n$ be given by

$$\psi_p[z_0, z_1] = [g_{p,0}(z_0/z_1), \ldots, g_{p,n}(z_0/z_1)] \ ,$$

where $[z_0, z_1] \in \mathbb{C}P^1 = S^2$, and for $j=0,\ldots,n$, $g_{p,j}(z)$ is given by

$$g_{p,j}(z) = \frac{p!}{(1+z\bar{z})^p} \sqrt{\binom{n}{j}}\, z^{j-p} \sum_k (-1)^k \binom{j}{p-k} \binom{n-j}{k} (z\bar{z})^k \ .$$

Then ψ_p is a conformal minimal immersion with constant curvature $4(n+2p(n-p))^{-1}$ and constant Kähler angle θ_p given by

$$(\tan \tfrac{1}{2}\theta_p)^2 = \frac{p(n-p+1)}{(p+1)(n-p)} \ .$$

Each ψ_p is an embedding unless $n=2p$, in which case ψ_p is a totally real immersion, and is essentially the Veronese immersion of S^2 into $\mathbb{R}P^n$ given by Boruvka [2]. We call $\psi_0, \ldots, \psi_n$ the *Veronese sequence*.

Characterisation of the Veronese Sequence (Theorem 5.4). Let $\psi : S^2 \to \mathbb{C}P^n$ be a conformal minimal immersion with constant curvature and assume that $\psi(S^2)$ is *linearly full* (i.e. not contained in any hyperplane of $\mathbb{C}P^n$). Then, up to a holomorphic isometry of $\mathbb{C}P^n$, the immersion ψ is an element of the Veronese sequence.

Rigidity Theorem (Theorem 6.1). Let $\psi, \psi' : S^2 \to \mathbb{C}P^n$ be conformal minimal immersions. Then ψ, ψ' differ by a holomorphic isometry of $\mathbb{C}P^n$ if and only if they have the same Kähler angles and induced metrics at each point.

In the case of holomorphic immersions this reduces to the rigidity theorem of Calabi [3] for complex submanifolds of a Kähler manifold. It also generalises the rigidity theorem of Barbosa [1] for minimal immersions of S^2 into S^n.

Totally Real Minimal Immersions (Theorem 7.3). If $\psi : S^2 \to \mathbb{C}P^n$ is a totally real minimal immersion (i.e. has constant Kähler angle $\pi/2$) then, up to a holomorphic isometry of $\mathbb{C}P^n$, $\psi(S^2)$ is contained in $\mathbb{R}P^n$.

Pinching Theorems for Curvature (Sect. 8). We prove pinching theorems for curvature which generalise results of Calabi [5], Lawson [12], Rigoli [14], and Jensen-Rigoli [10, 11], and go some way towards proving a conjecture of Simon [15].

Minimal Immersions with Constant Kähler Angle (Sect. 9). We conjecture that if $\psi : S^2 \to \mathbb{C}P^n$ is a conformal minimal immersion with constant Kähler angle which is not holomorphic, anti-holomorphic or totally real then ψ also has constant curvature and so essentially is an element of a Veronese sequence. We prove this conjecture (assuming that ψ is linearly full) for many values of n. We also prove pinching theorems for Kähler angle.

Conformal minimal immersions are harmonic maps, and much progress has been made recently in understanding harmonic maps into $\mathbb{C}P^n$. Specifically, Wolfson [16] has introduced the notion of the *harmonic sequence* associated to such a harmonic map, and this is the main tool we use in this paper. We consider conformal minimal immersions of S^2 because in this case the harmonic sequence terminates and is essentially the Frenet frame of an associated holomorphic curve. In the case of a minimal immersion of S^2 into S^n, this curve is just the directrix curve described in [6].

The construction of the harmonic sequence and a discussion of its local properties are outlined in Sect. 2, while in Sect. 3 we describe its topological properties. Formulae for Kähler angle and curvature are given in Sect. 4, and our results are presented in Sects. 5–9.

This is a slightly modified version of a paper originally submitted to Mathematische Annalen by Bolton and Woodward. However, similar results had been obtained independently and somewhat earlier by Jensen and Rigoli in a paper also submitted to Mathematische Annalen. It was therefore agreed that the present paper should be submitted bearing all four names. The first and last authors would like to thank Philip Higgins, Tony Scholl and Steve Wilson for some helpful remarks; the other two authors would similarly like to thank Simon Salamon and McKenzie Wang.

2. Minimal Immersions and Harmonic Sequences

We will first consider minimal immersions of S^2 into $\mathbb{C}P^n$ in the context of harmonic maps. In [16] Wolfson has developed the notion of the harmonic sequence determined by a harmonic map of a Riemann surface M into a complex Grassmannian. If M is homeomorphic to S^2, and the Grassmannian is $\mathbb{C}P^n$, this is essentially the Frenet frame of a certain associated holomorphic curve. We now describe these ideas briefly and show how they can be adapted to deal with conformal minimal immersions of S^2 into $\mathbb{C}P^n$.

Let M be a smooth manifold and let V be a complex vector subbundle of the trivial bundle $M\times\mathbb{C}^{n+1}$ over M. Let $\mathbb{C}^{n+1}$ be endowed with the Hermitian inner product defined by

$$\langle (x_0,\dots,x_n),(y_0,\dots,y_n)\rangle = x_0\bar{y}_0+\dots+x_n\bar{y}_n ,$$

where $(x_0,\dots,x_n)$, $(y_0,\dots,y_n)$ are elements of $\mathbb{C}^{n+1}$. Then V has a connection ∇, induced from the trivial connection on $M\times\mathbb{C}^{n+1}$, given by

$$\nabla s=\pi_V ds ,$$

where s is a section of V and $\pi_V : M\times\mathbb{C}^{n+1}\to V$ denotes orthogonal projection onto V.

If L denotes the canonical line bundle over $\mathbb{C}P^n$ defined by

$$L=(\{(p,v)\in\mathbb{C}P^n\times\mathbb{C}^{n+1} : v\in p\} ,$$

then L and its orthogonal complement $L^\perp$ both have induced connections and Hermitian metrics. Thus we have a Hermitian metric and connection on $\mathrm{Hom}(L,L^\perp)$ and there is a canonical isomorphism

$$h: T\mathbb{C}P^n\to \mathrm{Hom}(L,L^\perp) ,$$

given by

$$h(x)(s)=\pi_{L^\perp}ds(x) ,$$

where $x\in T\mathbb{C}P^n$ and s is a local section of L. Under this isomorphism, the complex structure, metric and connection on $\mathrm{Hom}(L,L^x)$ correspond to the complex structure, Fubini Study metric and connection (also denoted by ∇) on $\mathbb{C}P^n$ with constant holomorphic sectional curvature 4.

If M is a smooth manifold then there is a bijective correspondence between (smooth) complex line subbundles of $M\times\mathbb{C}^{n+1}$ and smooth maps $\psi: M\to\mathbb{C}P^n$. This correspondence is given by $\psi\leftrightarrow\psi^*L$. If $d\psi: TM\to T\mathbb{C}P^n$ is the derivative of ψ, then $h\circ d\psi$ is a bundle map covering ψ and the corresponding section δ of $\mathrm{Hom}(TM\otimes\psi^*L,\psi^*L^\perp)$ is given by

$$\delta(x\otimes s)=\pi_{L^\perp}ds(x) ,$$

where a section s of ψ^*L is considered as a $\mathbb{C}^{n+1}$-valued map defined on M. If M is a Riemann surface, the holomorphic part

$$\partial : T^{(1,0)}M\otimes\psi^*L\to\psi^*L^\perp$$

of δ is given in terms of a local complex co-ordinate z on M by

$$\partial(\partial/\partial z \otimes s) = (h \circ d\psi(\partial/\partial z))(s) = \pi_{L^\perp}(\partial s/\partial z) \ , \tag{2.1}$$

while the antiholomorphic part

$$\bar{\partial}: T^{(0,1)}M \otimes \psi^*L \to \psi^*L^\perp$$

of δ is given by

$$\bar{\partial}(\partial/\partial \bar{z} \otimes s) = (h \circ d\psi(\partial/\partial \bar{z}))(s) = \pi_{L^\perp}(\partial s/\partial \bar{z}) \ . \tag{2.2}$$

If V is any complex vector bundle over the Riemann surface M, then by the Koszul-Malgrange theorem each connection on V determines a holomorphic structure on V. Thus we have holomorphic structures on ψ^*L and $\psi^*L^\perp$, and Wolfson shows that ψ is harmonic if and only if ∂ is a holomorphic bundle map. Using these ideas Wolfson goes on to construct inductively an associated sequence

$$\dots, L_{-2}. L_{-1}, L_0, L_1, L_2 \dots$$

of complex line subbundles of the trivial bundle $M \times \mathbb{C}^{n+1}$ and bundle maps

$$\partial_p: T^{(1,0)}M \otimes L_p \to L_{p+1}$$

and

$$\bar{\partial}_p: T^{(0,1)}M \otimes L_p \to L_{p-1} \ .$$

Here $L_p = \psi_p^* L$ for a suitable harmonic map $\psi_p: M \to \mathbb{C}P^n$ and ∂_p (resp. $\bar{\partial}_p$) is essentially the map ∂ (resp. $\bar{\partial}$) defined above for the map ψ_p. Then ∂_p (resp. $\bar{\partial}_p$) is a holomorphic (resp. antiholomorphic) bundle map. If $\partial_p \equiv 0$ but $\partial_{p-1} \not\equiv 0$ (resp. $\bar{\partial}_p \equiv 0$ but $\bar{\partial}_{p+1} \not\equiv 0$) then the sequence terminates with L_p at the right (resp. left) hand end. Otherwise the set Z_p (resp. Y_p) of points of M over which ∂_p (resp. $\bar{\partial}_p$) is singular is a set of isolated points and, except at these points we have that L_{p+1} (resp. L_{p-1}) is the image of ∂_p (resp. $\bar{\partial}_p$). See [16] for details.

If ψ is a conformal immersion then ψ is minimal if and only if it is harmonic. Thus we may assign to a conformal minimal immersion a corresponding associated harmonic sequence.

If M is homeomorphic to S^2 then Wolfson shows that for $p \neq q$, L_p is orthogonal to L_q. In particular, L_{p-1} is orthogonal to L_{p+1}, i.e. the image of ∂_p is orthogonal to the image of $\bar{\partial}_p$. It now follows that each ψ_p is conformal, so is itself a minimal immersion of M (perhaps with isolated singularities). Clearly the harmonic sequence has length at most $n+1$, and it is easy to see that the sequence has length exactly $n+1$ if and only if ψ is *linearly full*, i.e. ψ is not contained in any hyperplane of $\mathbb{C}P^n$.

If $\psi: S^2 \to \mathbb{C}P^n$ is a linearly full conformal minimal immersion we will relabel the associated sequence $L_0, \dots, L_n$ and we will denote by ψ_p the conformal minimal immersion (perhaps with isolated singularities) corresponding to the line bundle L_p. We will call the sequence

$$\psi_0, \dots, \psi_n$$

the *harmonic sequence* determined by ψ, with $\psi = \psi_p$ for some $p = 0, \dots, n$. Note that $L_1, \dots, L_n$ is essentially the analogue for the holomorphic curve ψ_0 of the Frenet frame of a real space curve.

To help with later calculations, we give a local description of the above. Let $\psi: S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion and let z be a local complex co-ordinate on S^2. We define a sequence $f_0, \dots, f_n$ of local sections of $L_0, \dots, L_n$ inductively such that f_0 is a nowhere zero local section of L_0 and

$$f_{p+1} = (h \circ d\psi_p(\partial/\partial z))f_p = \partial_p(\partial/\partial z \otimes f_p) , \quad p = 0, \dots, n-1 .$$

If $Z = Z_0 \cup \dots \cup Z_{n-1}$ then, on the complement of Z we have

$$\frac{\partial f_p}{\partial z} = f_{p+1} + \frac{1}{|f_p|^2} \left\langle \frac{\partial f_p}{\partial z}, f_p \right\rangle f_p , \quad p = 0, \dots, n , \tag{2.3}$$

and since the image of $\bar{\partial}_p$ is in L_{p-1} we have that for suitable complex valued functions a_p, b_p

$$\frac{\partial f_p}{\partial \bar{z}} = a_p f_p + b_p f_{p-1} , \quad p = 1, \dots, n . \tag{2.4}$$

In fact, taking the inner product of (2.4) with f_{p-1} we see that

$$b_p = -\frac{|f_p|^2}{|f_{p-1}|^2} , \quad p = 1, \dots, n . \tag{2.5}$$

We now assume (without loss of generality) that $\partial f_0/\partial \bar{z} \equiv 0$. Then an easy induction argument using (2.3), (2.4) and the fact that

$$\partial^2 f_p/\partial z \partial \bar{z} = \partial^2 f_p/\partial \bar{z} \partial z$$

shows that $a_p = 0$ for $p = 0, \dots, n$. Thus

$$\frac{\partial f_p}{\partial \bar{z}} = -\frac{|f_p|^2}{|f_{p-1}|^2} f_{p-1} , \quad p = 1, \dots, n . \tag{2.6}$$

It also follows that

$$\left.\begin{aligned} &\frac{\partial^2}{\partial z \partial \bar{z}} \log |f_p|^2 = \frac{|f_{p+1}|^2}{|f_p|^2} - \frac{|f_p|^2}{|f_{p-1}|^2} , \quad p = 1, \dots, n . \\ &\frac{\partial^2}{\partial z \partial \bar{z}} \log |f_0|^2 = \frac{|f_1|^2}{|f_0|^2} . \end{aligned}\right\} \tag{2.7}$$

Note that (2.6) shows for each $p = 0, \dots, n$ that f_p is a holomorphic section of L_p.

If we define γ_p by putting

$$\left.\begin{aligned} &\gamma_p = \frac{|f_{p+1}|^2}{|f_p|^2} , \quad p = 0, \dots, n-1 , \\ &\gamma_{-1} = \gamma_n = 0 , \end{aligned}\right\} \tag{2.8}$$

then for $p = 1, \dots, n$,

$$|d\psi_p(\partial/\partial z)|^2 = \gamma_p = \frac{|b_{p+1} f_p|^2}{|f_{p+1}|^2} = |d\psi_{p+1}(\partial/\partial \bar{z})|^2 . \tag{2.9}$$

Also, using (2.7) and (2.8) one may derive the unintegrated Plücker formulae for the γ_p's, namely

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \gamma_p = \gamma_{p+1} - 2\gamma_p + \gamma_{p-1} \ , \quad p = 0, \dots, n-1 \ . \tag{2.10}$$

Remark. If ∂_p is never zero, then for $r = 0, \dots, n-1$ we have that γ_r/γ_p is a globally defined function on S^2. If ψ_0 is a non-singular holomorphic curve in $\mathbb{C}P^n$ then the functions γ_r/γ_0 are the higher order curvatures considered by Calabi [4], Lawson [12], and Rigoli [14].

3. Ramification Indices, Chern Classes and Degrees

Let $\psi_0, \dots, \psi_n : S^2 \to \mathbb{C}P^n$ be a harmonic sequence and let

$$\partial_p : T^{(1,0)} S^2 \otimes L_p \to L_{p+1} \ , \quad p = 0, \dots, n-1 \ ,$$

be the associated bundle homomorphisms described in Sect. 2. Since ∂_p is holomorphic if follows that the singularities are isolated zeros and we define the *ramification index* $r(\partial_p)$ of ∂_p by:

$$r(\partial_p) = \text{sum of the indices of the singularities of } \partial_p \ . \tag{3.1}$$

Thus $r(\partial_p)$ is a non-negative integer, and if $c_1(L_p)$ denotes the first Chern class of the line bundle L_p it follows from the Hopf Index Theorem that

$$r(\partial_p) = c_1(L_{p+1}) - c_1(L_p) - 2 \ . \tag{3.2}$$

Thus we have immediately that

$$r(\partial_0) + \dots + r(\partial_{n-1}) = c_1(L_n) - c_1(L_0) - 2n \tag{3.3}$$

and

$$\sum_{p=0}^{n-1} (n-p) r(\partial_p) = (n+1)(-c_1(L_0) - n) \ . \tag{3.4}$$

Recall that the first Chern class of the canonical line bundle L over $\mathbb{C}P^n$ is equal to minus the generator of $H^2(\mathbb{C}P^n)$ so that if d_0 (resp. d_n) denotes the degree of the holomorphic curve ψ_0 (resp. $\bar{\psi}_n$) then

$$c_1(L_0) = -d_0 \ , \quad c_1(L_n) = d_n \ . \tag{3.5}$$

Hence (3.3) and (3.4) may be rewritten as

$$\sum_{p=0}^{n-1} r(\partial_p) = d_0 + d_n - 2n \ , \tag{3.6}$$

and

$$\sum_{p=0}^{n-1} (n-p) r(\partial_p) = (n+1)(d_0 - n) \ . \tag{3.7}$$

The harmonic sequence is said to be *totally unramified* if $r(\partial_p) = 0$ for $p = 0, \dots, n-1$, and we see immediately from (3.7) that this is the case if and only if the degree d_0 of ψ_0 is n. If $\psi : S^2 \to \mathbb{C}P^n$ is a linearly full conformal minimal immersion with corresponding harmonic sequence $\psi_0, \ldots, \psi_n$ we say that ψ is a *totally unramified minimal immersion* if $\psi_0, \ldots, \psi_n$ is totally unramified.

Next observe that in terms of a local complex coordinate z on S^2 we have for $p=0, \ldots, n$,

$$c_1(L_p)=\frac{1}{2\pi i}\int_{S^2} -\frac{\partial^2}{\partial z \partial \bar{z}} \log |f_p|^2 d\bar{z} \wedge dz \ , \tag{3.8}$$

so that from (3.2), for $p=0, \ldots, n-1$,

$$r(\partial_p)+2=\frac{1}{2\pi i}\int_{S^2} -\frac{\partial^2}{\partial z \partial \bar{z}} \log \gamma_p d\bar{z} \wedge dz \ . \tag{3.9}$$

Thus using (2.7), (2.8), and (2.10) we have

$$c_1(L_p)=\frac{1}{2\pi i}\int_{S^2} (\gamma_{p-1}-\gamma_p) d\bar{z} \wedge dz \ , \tag{3.10}$$

and

$$r(\partial_p)+2=\frac{-1}{2\pi i}\int_{S^2} (\gamma_{p-1}-2\gamma_p+\gamma_{p+1}) d\bar{z} \wedge dz \ . \tag{3.11}$$

Now consider the p-th osculating curve [9]

$$\sigma_p : S^2 \to \mathbb{C}P^{\binom{n+1}{p+1}-1}$$

which has a local lift $\hat{\sigma}_p$ into $\mathbb{C}^{\binom{n+1}{p+1}}$ given by

$$\hat{\sigma}_p=f_0 \wedge \ldots \wedge \left(\frac{\partial}{\partial z}\right)^p f_0 \ , \quad p=0, \ldots, n \ . \tag{3.12}$$

We write $\hat{\sigma}_p=\varphi(z)\tilde{\sigma}_p$, where $\varphi(z)$ is the greatest common divisor of the $\binom{n+1}{p+1}$ components of $\hat{\sigma}_p$. Then $\tilde{\sigma}_p$ is a nowhere zero holomorphic curve. If

$$\beta_p=|\tilde{\sigma}_p|^2 \ , \tag{3.13}$$

then

$$|\varphi_p|^2 \beta_p=|f_0|^2 \cdots |f_p|^2 \ , \tag{3.14}$$

and the degree δ_p of σ_p is given by

$$\delta_p=\frac{1}{2\pi i}\int_{S^2} \frac{\partial^2}{\partial z \partial \bar{z}} \log \beta_p d\bar{z} \wedge dz \ , \tag{3.15}$$

which is equal to the degree of the polynomial function $\tilde{\sigma}_p$. But from (2.7), (2.8), and (3.14) we have

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \beta_p=\gamma_p \ , \tag{3.16}$$

so that

$$\delta_p=\frac{1}{2\pi i}\int_{S^2} \gamma_p d\bar{z} \wedge dz \ . \tag{3.17}$$

Thus it follows from (3.10), (3.11), and (3.17) that

$$c_1(L_p)=\delta_{p-1}-\delta_p\ , \tag{3.18}$$

$$r(\partial_p)+2=-(\delta_{p-1}-2\delta_p+\delta_{p+1})\ , \tag{3.19}$$

where we have put $\delta_{-1}=0$. This latter equation is the *global Plücker formula* (see [9]). We also remark in passing that the energy $E(L_p)$ of the map ψ_p defined by

$$E(L_p)=\frac{1}{2\pi i}\int_{S^2}(\gamma_{p-1}+\gamma_p)d\bar{z}\wedge dz\ , \tag{3.20}$$

is also an integer namely

$$E(L_p)=\delta_{p-1}+\delta_p\ . \tag{3.21}$$

In fact [see (4.4) and (4.6) below], if ψ_p is an immersion then its induced area form dA_p is given by

$$dA_p=(\gamma_{p-1}+\gamma_p)\frac{d\bar{z}\wedge dz}{2i}\ , \tag{3.22}$$

so that the area $A(\psi_p)$ of S^2 with the metric induced by ψ_p is given by

$$A(\psi_p)=\pi(\delta_{p-1}+\delta_p)\ . \tag{3.23}$$

It is an immediate consequence of the global Plücker formula (3.19) that

$$\delta_p=(p+1)(d_0-p)-\sum_{k=0}^{p-1}(p-k)r(\partial_k)\ ,$$

which, using (3.7) gives

$$\delta_p=(p+1)(n-p)+\frac{n-p}{n+1}\sum_{k=0}^{p-1}(k+1)r(\partial_k)+\frac{p+1}{n+1}\sum_{k=p}^{n-1}(n-k)r(\partial_k)\ . \tag{3.24}$$

In particular for a totally unramified harmonic sequence $\psi_0,\ldots,\psi_n:S^2\to\mathbb{C}P^n$ we have

$$\delta_p=(p+1)(n-p)\ . \tag{3.25}$$

4. Kähler Angles and Curvature

If $\psi:M\to\mathbb{C}P^n$ is a conformal immersion of a Riemann surface M we define the *Kähler angle* of ψ to be the function $\theta:M\to[0,\pi]$ given in terms of a complex coordinate z on M by

$$\left(\tan\frac{\theta(m)}{2}\right)=\frac{|d\psi(m)(\partial/\partial\bar{z})|}{|d\psi(m)(\partial/\partial z)|}\ ,\qquad m\in M\ . \tag{4.1}$$

It is clear that θ is globally defined and is smooth at m unless $\theta(m)=0$ or π. Observe that if $z=x+iy$, and J denotes the complex structure on $\mathbb{C}P^n$, then θ is the angle between $Jd\psi(\partial/\partial x)$ and $d\psi(\partial/\partial y)$. The Kähler angle, whose importance in the theory of minimal immersions of surfaces into Kähler manifolds was pointed out by Chern and Wolfson [7], gives a measure of the failure of ψ to be a holomorphic map. Indeed ψ is holomorphic if and only if $\theta(m)=0$ for all $m \in M$, while ψ is anti-holomorphic if and only if $\theta(m)=\pi$ for all $m \in M$.

Now suppose that $\psi: S^2 \to \mathbb{C}P^n$ is a linearly full conformal minimal immersion and let $\psi_0, \ldots, \psi_n$ be the associated harmonic sequence described in Sect. 2. Then each $\psi_p: S^2 \to \mathbb{C}P^n$ is a conformal minimal immersion on the complement of the finite set $Y_p \cap Z_p$ of points of S^2 (see Sect. 2). Thus the Kähler angle

$$\theta_p: S^2 - (Y_p \cap Z_p) \to [0, \pi]$$

is well-defined, and is smooth on $S^2 - (Y_p \cup Z_p)$. If we write

$$t_p = (\tan \tfrac{1}{2}\theta_p)^2 \ , \tag{4.2}$$

then in terms of a local complex co-ordinate z,

$$t_p = \frac{|d\psi_p(\partial/\partial \bar{z})|^2}{|d\psi_p(\partial/\partial z)|^2} = \frac{\gamma_{p-1}}{\gamma_p} \ . \tag{4.3}$$

Now suppose that ds_p^2 is the metric on $S^2 - (Y_p \cap Z_p)$ induced by ψ_p, so that

$$ds_p^2 = 2F_p d\bar{z} dz \ , \quad 2F_p = \gamma_{p-1} + \gamma_p \ . \tag{4.4}$$

Then the Laplacian Δ_p of ds_p^2 is given by

$$\Delta_p = \frac{2}{F_p} \frac{\partial^2}{\partial z \partial \bar{z}} \ , \tag{4.5}$$

the area form dA_p by

$$dA_p = F_p \frac{d\bar{z} \wedge dz}{i} \ , \tag{4.6}$$

and the curvature $K_p = K(\psi_p)$ by

$$K_p = -\frac{1}{F_p} \frac{\partial^2}{\partial z \partial \bar{z}} \log F_p \ . \tag{4.7}$$

In the following sections we will be studying conformal minimal immersions with constant Kähler angle. We note that if ψ_p is such an immersion then from (3.17) we have that

$$t_p = \frac{\delta_{p-1}}{\delta_p} \ , \tag{4.8}$$

while from (4.7), (4.4), and (4.3) it follows that

$$K_p = -\frac{1}{F_p} \frac{\partial^2}{\partial z \partial \bar{z}} \log \gamma_{p-1} \ . \tag{4.9}$$

5. The Veronese Sequence

Let $\psi: S^2 \to \mathbb{C}P^n$ be the holomorphic immersion (indeed embedding) defined by

$$\psi([z_0, z_1]) = \left[z_0^n, \sqrt{\binom{n}{1}} z_0^{n-1} z_1, \ldots, \sqrt{\binom{n}{r}} z_0^{n-r} z_1^r, \ldots, z_1^n\right], \tag{5.1}$$

where $[z_0, z_1] \in \mathbb{C}P^1 = S^2$. Alternatively, in terms of the holomorphic coordinate $z = z_1/z_0$ on S^2 we may write

$$\psi(z) = \left[1, \sqrt{\binom{n}{1}} z, \ldots, \sqrt{\binom{n}{r}} z^r, \ldots, z^n\right]. \tag{5.2}$$

As is well-known [14], this is of constant curvature and, up to holomorphic isometries of $\mathbb{C}P^n$, is the only such linearly full holomorphic curve. Let $\psi_0, \ldots, \psi_n$ be the harmonic sequence of $\psi = \psi_0$. We call $\psi_0, \ldots, \psi_n$ the *Veronese sequence*. Since ψ_0 is a rational curve of degree n it follows from (3.7) that each ψ_p is an immersion.

Recall from Sect. 2 that if $f_0: \mathbb{C} \to \mathbb{C}^{n+1}$ is the holomorphic lift of ψ_0 given by

$$f_0(z) = \left[1, \sqrt{\binom{n}{1}} z, \ldots, \sqrt{\binom{n}{r}} z^r, \ldots, z^n\right], \tag{5.3}$$

and $f_1, \ldots, f_n$ are defined inductively by

$$f_{p+1} = \frac{\partial f_p}{\partial z} - \frac{1}{|f_p|^2}\left\langle \frac{\partial f_p}{\partial z}, f_p\right\rangle f_p, \quad p = 0, \ldots, n-1, \tag{5.4}$$

then $\psi_p = [f_p]$. We proceed to establish an explicit formula for ψ_p and to prove that it has constant Kähler angle and constant curvature. To this end let $g_p: \mathbb{C} \to \mathbb{C}^{n+1}$ be the map defined by

$$g_p = (g_{p,0}, \ldots, g_{p,r}, \ldots, g_{p,n}),$$

where

$$g_{p,r}(z) = \frac{p!}{(1+z\bar{z})^p} \sqrt{\binom{n}{r}} z^{r-p} \sum_k (-1)^k \binom{r}{p-k}\binom{n-r}{k} (z\bar{z})^k. \tag{5.5}$$

Lemma 5.1. *If g_p is defined by* (5.5) *then*

$$|g_p|^2 = \frac{n!\,p!}{(n-p)!}(1+z\bar{z})^{n-2p}.$$

Proof.

$$|g_p|^2 = \frac{(p!)^2}{(1+z\bar{z})^{2p}} \sum_{r=0}^n \left\{\binom{n}{r} (z\bar{z})^{r-p} \left(\sum_k (-1)^k \binom{r}{p-k}\binom{n-r}{k} (z\bar{z})^k\right)^2\right\}. \tag{5.6}$$

But if x, y, t are indeterminates, then equating powers of y^{2p} in the n-th power of the identity

$$x(1+yt)(1+yt^{-1}) + (1-xyt)(1-xyt^{-1}) \equiv (1+x)(1+xy^2)$$

reveals that

$$\sum_{r=0}^{n}\binom{n}{r}x^r\left(\sum_k(-1)^k\binom{r}{p-k}\binom{n-r}{k}x^k\right)^2=\binom{n}{p}(1+x)^n x^p . \tag{5.7}$$

The result is now immediate from (5.6) and (5.7).

Now observe that $f_0=g_0$ [see (5.3) and (5.5)], so suppose that for some $p\geqq 0$ we have

$$g_k=f_k \quad \text{for} \quad 0\leqq k\leqq p . \tag{5.8}$$

Since

$$\left\langle\frac{\partial g_p}{\partial z},g_p\right\rangle=\frac{\partial}{\partial z}|g_p|^2-\left\langle g_p,\frac{\partial g_p}{\partial\bar z}\right\rangle , \tag{5.9}$$

it follows from (5.8) and the fact that $\langle f_p,f_{p-1}\rangle=0$ that $\langle g_p,\partial g_p/\partial\bar z\rangle=0$. Thus from (5.9) and Lemma 5.1,

$$\frac{1}{|g_p|^2}\left\langle\frac{\partial g_p}{\partial z},g_p\right\rangle=\frac{\partial}{\partial z}\log|g_p|^2=\frac{(n-2p)\bar z}{1+z\bar z} .$$

But then a straightforward calculation using (5.8) shows that,

$$g_{p+1}=\frac{\partial g_p}{\partial z}-\frac{1}{|g_p|^2}\left\langle\frac{\partial g_p}{\partial z},g_p\right\rangle g_p ,$$

so that, by (5.4), $f_{p+1}=g_{p+1}$. Thus by induction we have $f_p=g_p$ for $p=0,\dots,n$.

Theorem 5.2. *The Veronese sequence $\psi_0,\dots,\psi_n:S^2\to\mathbb{C}P^n$ is given by*

$$\psi_p=[f_{p,0},\dots,f_{p,n}] ,$$

where, for $z\in S^2$,

$$f_{p,r}(z)=\frac{p!}{(1+z\bar z)^p}\sqrt{\binom{n}{r}}\,z^{r-p}\sum_k(-1)^k\binom{r}{p-k}\binom{n-r}{k}(z\bar z)^k .$$

Furthermore ψ_p is a minimal immersion with induced metric

$$ds_p^2=\frac{n+2p(n-p)}{(1+z\bar z)^2}\,dzd\bar z$$

and hence is of constant curvature $K_p=4/(n+2p(n-p))$ and constant Kähler angle θ_p given by

$$(\tan\tfrac{1}{2}\theta_p)^2=\frac{p(n-p+1)}{(p+1)(n-p)} .$$

Finally, if $T:S^2\to S^2$ denotes the antipodal map, (well-defined for the metric ds_p^2 on S^2 since this is of constant curvature) then

$$\psi_{n-p}=\psi_p\circ T .$$

Proof. The first statement has already been established, while

$$\gamma_p = \frac{|f_{p+1}|^2}{|f_p|^2} = \frac{(p+1)(n-p)}{(1+z\bar{z})^2}$$

by Lemma 5.1, so that

$$ds_p^2 = (\gamma_{p-1} + \gamma_p)dzd\bar{z} = \frac{n+2p(n-p)}{(1+z\bar{z})^2}\, dzd\bar{z} \ .$$

Finally we observe that

$$[f_p(z)] = \left[f_{n-p}\left(-\frac{1}{\bar{z}}\right)\right] ,$$

and hence as $T: S^2 \to S^2$ is given by $z \to -1/\bar{z}$, this establishes that $\psi_{n-p} = \psi_p \circ T$.

Remarks. Using Lemma 5.1 it is easy to see that if $2p \neq n$ then f_p is injective and it readily follows that in this case ψ_p is an embedding; while if $n = 2m$ then ψ_m is essentially the Veronese immersion. More precisely, following Boruvka [2], let us suppose $n = 2m$ and write

$$X_{-s} = f_{m,m+s} = \frac{m!}{(1+z\bar{z})^m} \sqrt{\binom{2m}{m+s}}\, z^s \sum_k (-1)^k \binom{m-s}{k} \binom{m+s}{m-k} (z\bar{z})^k$$

$$X_s = f_{m,m-s} = \frac{m!}{(1+z\bar{z})^m} \sqrt{\binom{2m}{m-s}}\, z^{-s} \sum_k (-1)^k \binom{m-s}{m-k} \binom{m+s}{k} (z\bar{z})^k \ .$$

Some elementary manipulation then shows that

$$X_s = (-1)^s \bar{X}_{-s} \ ,$$

so that $i^s X_s = \overline{i^s X_{-s}}$ for $s = 0, \ldots, m$.

Thus writing

$$\left.\begin{aligned} Y_s &= \frac{i^s}{\sqrt{2}}(X_s + X_{-s}) \ . \\ Y_{-s} &= \frac{i^{s+1}}{\sqrt{2}}(X_s - X_{-s}) \ . \\ Y_0 &= X_0 \end{aligned}\right\} \qquad s = 1, \ldots, m \ ,$$

we have $Y_s = \bar{Y}_s$ for $s = -m, \ldots, 0, \ldots, m$. In particular, it follows that there is a holomorphic isometry of $\mathbb{C}P^n$ which maps the Veronese sequence of Theorem 5.2 into the harmonic sequence for the holomorphic curve $\psi: S^2 \to \mathbb{C}P^{2m}$ given by

$$z \mapsto [h_0(z), \ldots, h_{2m}(z)] \ ,$$

where

$$h_r(z)=\begin{cases} i^{m-r+1}\sqrt{\binom{2m}{r}}(z^{2m-r}-z^r) & \text{if} \quad 0\leqq r\leqq m-1 \\ \sqrt{2}\sqrt{\binom{2m}{m}}z^m & \text{if} \quad m=m \\ i^{r-m}\sqrt{\binom{2m}{r}}(z^r+z^{2m-r}) & \text{if} \quad m+1\leqq r\leqq 2m \, . \end{cases}$$

Furthermore the image of the m-th term of the harmonic sequence for this ψ is contained in $\mathbb{R}P^{2m}$ and is the Veronese immersion. For further details see Boruvka [2].

We now show that the above example essentially contains all linearly full conformal minimal immersions of S^2 into $\mathbb{C}P^n$ with constant curvature.

Lemma 5.3. *Let $\psi: S^2\to\mathbb{C}P^n$ be a linearly full conformal minimal immersion with constant curvature and constant Kähler angle. Then, up to a holomorphic isometry of $\mathbb{C}P^n$, the harmonic sequence determined by ψ is the Veronese sequence.*

Proof. Let $\psi_0,\dots,\psi_n$ be the harmonic sequence determined by ψ, and assume that $\psi=\psi_p$. If $p=0$ or n then ψ is a holomorphic or antiholomorphic curve of constant curvature on $\mathbb{C}P^n$, so the result follows from the rigidity theorem of Calabi [3]. Now assume $1\leqq p\leqq n-1$.

From (4.9), (4.4), and (4.3) we see that

$$K_p=-\frac{2t_p}{\gamma_{p-1}(1+t_p)}\frac{\partial^2}{\partial z\partial\bar{z}}\log\gamma_{p-1}\ ,$$

so from (2.10) we obtain

$$K_p=-\frac{2t_p}{1+t_p}\left\{t_{p-1}-2+\frac{1}{t_p}\right\}\, . \tag{5.10}$$

This shows that t_{p-1} is also constant. Now ψ_p is an immersion with constant non-zero Kähler angle so $r(\partial_{p-1})=0$. Thus ψ_{p-1} is also an immersion. If we put $\mu_p=F_p/F_{p-1}$ it follows from (4.4) and (4.3) that μ_p is constant, and from (4.7) we have

$$K_{p-1}=\mu_pK_p\ .$$

Thus K_{p-1} is also constant, so by inducting downwards we see that K_0 is constant. We may now apply Calabi's rigidity theorem [3] to complete the proof.

Theorem 5.4. *Let $\psi: S^2\to\mathbb{C}P^n$ be a linearly full conformal minimal immersion of constant curvature. Then, up to a holomorphic isometry of $\mathbb{C}P^n$, the harmonic sequence determined by ψ is the Veronese sequence.*

Proof. Suppose that ψ is the p-th element ψ_p of a harmonic sequence $\psi_0,\dots,\psi_n$. It follows from (3.22), (3.17) and the Gauss-Bonnet theorem that $K_p=4/(\delta_{p-1}+\delta_p)$, so from (4.7) and (4.4) we obtain

$$-\frac{2}{\gamma_{p-1}+\gamma_p}\frac{\partial^2}{\partial z\partial\bar{z}}\log(\gamma_{p-1}+\gamma_p)=\frac{4}{\delta_{p-1}+\delta_p} . \tag{5.11}$$

If we now choose a complex co-ordinate z on $S^2-\{pt\}$ so that the metric $ds_p^2=(\gamma_{p-1}+\gamma_p)d\bar{z}dz$ is given by

$$ds_p^2=\frac{\delta_{p-1}+\delta_p}{(1+z\bar{z})^2}d\bar{z}dz , \tag{5.12}$$

it follows from (5.11), (5.12), and (3.16) that

$$\frac{\partial^2}{\partial z\partial\bar{z}}\log\frac{\beta_{p-1}\beta_p}{(1+z\bar{z})^\alpha}=0 , \tag{5.13}$$

where

$$\alpha=\delta_{p-1}+\delta_p .$$

Now choose a nowhere zero holomorphic $\mathbb{C}^{n+1}$-valued function $f_0(z)$ such that $\psi_0=[f_0]$ (i.e. f_0 is a holomorphic section of L_0 over $\mathbb{C}=S^2-\{pt\}$). The map $\tilde{\sigma}_j$ defined following (3.12) is a polynomial function on $\mathbb{C}$ of degree δ_j and thus $\beta_{p-1}\beta_p(1+z\bar{z})^{-\alpha}$ is globally defined on $\mathbb{C}$ and has a non-zero constant limit c, say, as $z\to\infty$. Thus from (5.13)

$$\beta_{p-1}\beta_p=c(1+z\bar{z})^\alpha .$$

But then

$$\beta_{p-1}=c_{p-1}(1+z\bar{z})^{\delta_{p-1}} , \qquad \beta_p=c_p(1+z\bar{z})^{\delta_p} ,$$

where c_{p-1}, c_p are constants, and hence

$$\gamma_{p-1}=\frac{\delta_{p-1}}{(1+z\bar{z})^2} , \qquad \gamma_p=\frac{\delta_p}{(1+z\bar{z})^2} .$$

Hence ψ_p is of constant curvature and constant Kähler angle and the result follows from Lemma 5.3.

Remark 5.5. It is of interest to see what conditions one must place on a harmonic sequence in order that it be the Veronese sequence. For instance, one sees immediately from (5.10) that if a harmonic sequence has two consecutive Kähler angles constant then, up to a holomorphic isometry of $\mathbb{C}P^n$, it must be the Veronese sequence. We return to this question in later sections.

6. Rigidity

A holomorphic curve has Kähler angle identically zero, and Calabi [3] has shown that such a curve is determined by its induced metric. As we indicate below, a conformal minimal immersion of S^2 into S^n determines a conformal minimal immersion of S^2 into $\mathbb{C}P^n$ which has constant Kähler angle $\pi/2$, and Barbosa [1] has shown that such immersions are also determined by the induced metric. Thus Theorem 6.1, proved below, is a generalisation of both of these rigidity results.

Recall that if $\pi: S^n \to \mathbb{R}P^n$ is the standard double covering then every (conformal) immersion $\tilde{\psi}: S^2 \to S^n$ defines a (conformal) immersion $\psi = \pi\tilde{\psi}: S^2 \to \mathbb{R}P^n$, and conversely every immersion of S^2 into $\mathbb{R}P^n$ arises in this way. Next recall that $\mathbb{R}P^n$ is a totally geodesic submanifold of $\mathbb{C}P^n$ of constant curvature 1, so that if $i: \mathbb{R}P^n \to \mathbb{C}P^n$ denotes the inclusion map, then a conformal immersion $\psi: S^2 \to \mathbb{R}P^n$ is minimal if and only if $i\psi$ is minimal.

Theorem 6.1. *Let ψ, ψ' be conformal minimal immersions of S^2 into $\mathbb{C}P^n$. Then ψ, ψ' differ by a holomorphic isometry of $\mathbb{C}P^n$ if and only if they have the same Kähler angle and induced metric at each point.*

Proof. Let $\psi_0, \dots, \psi_m$ be the harmonic sequence determined by ψ. Since we have not assumed that ψ is linearly full we only know that $m \leqq n$. Assume that $\psi = \psi_p$ for some $p = 0, \dots, m$. We will apply our standard notation to the harmonic sequence determined by ψ, and denote the corresponding objects associated to the harmonic sequence determined by ψ' by the addition of the superscript $'$. Assume that $\psi' = \psi'_q$ and that $q \geqq p$. If follows from (4.3) and (4.4) that

$$\gamma_p = \gamma'_q , \qquad \gamma_{p-1} = \gamma'_{q-1} ,$$

so an easy induction argument using (2.10) shows that

$$\gamma_{p-j} = \gamma'_{q-j} , \qquad j = 1, \dots, p-1 .$$

In particular, $\gamma'_{q-p-1} = 0$, so that $p = q$ and also $\gamma_0 = \gamma'_0$. Thus ψ_0 and ψ'_0 are holomorphic curves which induce the same metric on S^2, so by Calabi's rigidity result, there is a holomorphic isometry g of $\mathbb{C}P^n$ such that $g\psi_0 = \psi'_0$. The construction of the harmonic sequence now shows that

$$g\psi = g\psi_p = \psi'_p = \psi' ,$$

so the theorem is proved.

7. Totally Real Conformal Minimal Immersions

An immersion of S^2 in $\mathbb{C}P^n$ is *totally real* if its Kähler angle is constant and equal to $\pi/2$. An immersion whose image lies in $\mathbb{R}P^n \subset \mathbb{C}P^n$ is of course totally real, and in this section we show that if $\psi: S^2 \to \mathbb{C}P^n$ is a totally real conformal minimal immersion then there is a holomorphic isometry g of $\mathbb{C}P^n$ such that the image of $g\psi$ is contained in $\mathbb{R}P^n$.

To fix notation, we let ψ be a linearly full conformal minimal immersion of S^2 into $\mathbb{C}P^n$ with associated harmonic sequence $\psi_0, \dots, \psi_n$, and let $\psi = \psi_p$.

Lemma 7.1. *If ψ_p is totally real, then $n = 2p$ and $\gamma_j = \gamma_{n-j-1}$ for $j = 0, \dots, n-1$.*

Proof. Since $\theta_p = \pi/2$, (4.2) and (4.3) show that $\gamma_{p-1} = \gamma_p$. An induction argument using (2.10) now shows that

$$\gamma_{p-r-1} = \gamma_{p+r} , \qquad r = 0, \dots \tag{7.1}$$

Thus $p + r \geqq n$ if and only if $p - r - 1 < 0$, from which it follows that $n = 2p$. Taking $r = p - j - 1$ in (7.1) we obtain the desired result.

Theorem 7.2. *Let $\psi_0 : S^2 \to \mathbb{C}P^n$ be a linearly full holomorphic curve with n even. Then $\gamma_0 = \gamma_{n-1}$ if and only if there is a holomorphic isometry g of $\mathbb{C}P^n$ such that $g\psi_j = \overline{g\psi_{n-j}}$ for all $j = 0, \ldots, n$.*

Proof. ($\Leftarrow$) The existence of such an isometry g implies that the metrics induced on S^2 by ψ_0 and by ψ_n are equal. Thus (4.4) shows that $\gamma_0 = \gamma_{n-1}$.

($\Rightarrow$) Since γ_0, γ_{n-1} are the induced metrics on the holomorphic curves $\psi_0, \bar{\psi}_n$, it follows from Calabi's rigidity theorem [3] that $A\psi_0 = \bar{\psi}_n$ for some $A \in U(n+1)$. However, the construction of the harmonic sequence shows that the harmonic sequence determined by $\bar{\psi}_n$ is $\bar{\psi}_n, \bar{\psi}_{n-1}, \ldots, \bar{\psi}_0$. Thus

$$A\psi_j = \bar{\psi}_{n-j} , \quad j = 0, \ldots, n . \tag{7.2}$$

Taking $j = 0$ and then $j = n$ in (7.2) we see that $\bar{A}A\psi_0 = \psi_0$. Since ψ_0 is linearly full it follows that $\bar{A}A = \lambda I$ for some $\lambda \in \mathbb{C}$. If we conjugate by A we see that A and $\bar{A}$ commute, so that λ is real. Also, $\lambda^{n+1} = 1$ and n is even so that $\lambda = 1$. Thus A is a unitary symmetric matrix so there is a real orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal. If we put $Q^{-1}AQ = D\,D$ for a unitary diagonal matrix D and then let $P = DQ^{-1}$ we see that $P \in U(n+1)$ and $A = {}^tPP$. It now follows from (7.2) that

$$P\psi_j = \overline{P\psi_{n-j}} , \quad j = 0, \ldots, n ,$$

so Theorem 7.2 is proved.

If g is a holomorphic isometry of $\mathbb{C}P^n$, then ψ and $g\psi$ have equal Kähler angles. The following theorem is thus an immediate consequence of Lemma 7.1 and Theorem 7.2.

Theorem 7.3. *Let ψ be a linearly full conformal minimal immersion of S^2 into $\mathbb{C}P^n$ with associated harmonic sequence $\psi_0, \ldots, \psi_n$ with $\psi = \psi_p$. Then the following three statements are equivalent:*

(i) *ψ is totally real.*

(ii) *$n = 2p$ and $\gamma_j = \gamma_{n-j-1}$ for $j = 0, \ldots, n-1$.*

(iii) *There is a holomorphic isometry g of $\mathbb{C}P^n$ such that the image of $g\psi$ lies in $\mathbb{R}P^n \subset \mathbb{C}P^n$.*

Remark 7.4. (i) A linearly full holomorphic curve φ_0 in $\mathbb{C}P^n$ with the property that $\varphi_0 = \bar{\varphi}_n$ has been called *totally isotropic* by Eells and Wood [8]. They show that n must be even for such a curve.

(ii) Let $\tilde{\psi} : S^2 \to S^n$ be a conformal minimal immersion whose image is not contained in any hyperplane section of S^n. As indicated earlier, $\tilde{\psi}$ determines a totally real conformal minimal immersion ψ into $\mathbb{C}P^n$, and it is clear that ψ is also linearly full. If $\psi_0, \ldots, \psi_n$ is the associated harmonic sequence, then Theorem 7.3 shows that n is even and if $n = 2p$ then $\psi = \psi_p$ and $r(\partial_j) = r(\partial_{n-j-1})$ for $j = 0, \ldots, n-1$. Using (3.24) one can now deduce that

$$\delta_{p-1} = \delta_p = p(p+1) + \sum_{k=0}^{p-1} (k+1) r(\partial_k) .$$

Thus (3.23) enables one to obtain some results of Calabi [5], namely that the area $A(\psi_p)$ is an integer multiple of 2π, and $A(\psi_p) \geqq 2\pi p(p+1)$ with equality if and only if ψ_p has no higher order singularities.

8. Pinching Theorems for Curvature

In this section we prove some pinching theorems for curvature of linearly full conformal minimal immersions of S^2 into $\mathbb{C}P^n$, and then show how these are relevant to the Simon conjecture [15].

Let $\psi: S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion and let $\psi_0, \ldots, \psi_n$ be its associated harmonic sequence with $\psi = \psi_p$. Recall that we have given $\mathbb{C}P^n$ the Fubini-Study metric of constant holomorphic sectional curvature 4.

Lemma 8.1. *Suppose that the curvature K_p of ψ_p satisfies either $K_p \geqq 4(\delta_{p-1}+\delta_p)^{-1}$ or $K_p \leqq 4(\delta_{p-1}+\delta_p)^{-1}$. Then $K_p = 4(\delta_{p-1}+\delta_p)^{-1}$.*

Proof. It follows from the Gauss Bonnet theorem and from (3.23) that

$$\int_{S^2} \left(K_p - \frac{4}{\delta_{p-1}+\delta_p}\right) dA_p = 0 \ .$$

The result is now immediate.

Theorem 8.2. *Let $\psi: S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion and suppose that ψ is the p-th element ψ_p of its harmonic sequence.*

(i) *If*

$$K(\psi) \geqq \frac{4}{n+2p(n-p)} \quad \textit{then} \quad K(\psi) = \frac{4}{n+2p(n-p)}$$

and ψ is totally unramified.

(ii) *If*

$$K(\psi) \leqq \frac{4}{n+2p(n-p)}$$

and if ψ is totally unramified then

$$K(\psi) = \frac{4}{n+2p(n-p)} \ .$$

(Recall also that when $K(\psi)$ is constant then, up to a holomorphic isometry of $\mathbb{C}P^n$, ψ belongs to the Veronese sequence.)

Proof. From (3.24) we see that

$$\delta_{p-1} + \delta_p \geqq n + 2p(n-p) \ ,$$

with equality if and only if ψ is totally unramified. The result is now immediate from Lemma 8.1 and Theorem 5.4.

Theorem 8.2 has several corollaries as we now show. The first is a strengthened version of a theorem of Lawson [12].

Theorem 8.3. *Let $\psi: S^2 \to \mathbb{C}P^n$ be a totally unramified holomorphic immersion and suppose that $K(\psi) \leqq 4/n$. Then $K(\psi) = 4/n$ and, up to a holomorphic isometry of $\mathbb{C}P^n$, ψ is the holomorphic curve ψ_0 in the Veronese sequence.*

Similarly, we have the following theorem of Rigoli [14].

Theorem 8.4. *Let $\psi : S^2 \to \mathbb{C}P^n$ be a linearly full holomorphic immersion and suppose that $K(\psi) \geqq 4/n$. Then $K(\psi) = 4/n$ and, up to a holomorphic isometry of $\mathbb{C}P^n$, ψ is the holomorphic curve ψ_0 in the Veronese sequence.*

Remark. Let K_q denote the curvature of the q^{th} element ψ_q of the Veronese sequence. Then $K_q = K_{n-q}$, and K_q is a decreasing function of q on the closed interval $[0, n/2]$. One may thus use Theorem 8.2 to prove pinching theorems for curvature of the following type:

Let $\psi : S^2 \to \mathbb{C}P^n$ be a totally unramified minimal immersion. If $K_q \leqq K(\psi) \leqq K_{q-1}$ for some integer q with $0 \leqq q \leqq n/2$ then $K(\psi) = K_{q-1}$ or K_q (so that ψ is the $(q-1)^{st}$, q^{th}, $(n-q)^{th}$ or $(n-q+1)^{st}$ element of the Veronese sequence).

We now consider the application of Theorem 8.2 to minimal immersions of S^2 into S^N. Recall that Simon [15] has made the following conjecture.

Conjecture. Let $\psi : S^2 \to S^N(1)$ be a conformal minimal immersion and suppose, for some integer $s \geqq 1$, that

$$\frac{2}{(s+1)(s+2)} \leqq K(\psi) \leqq \frac{2}{s(s+1)} .$$

Then

$$K(\psi) = \frac{2}{s(s+1)} \quad \text{or} \quad K(\psi) = \frac{2}{(s+1)(s+2)}$$

and, up to isometries of S^N, ψ is the unique linearly full conformal minimal immersion into the totally geodesic subspace S^{2s} or $S^{2(s+1)}$ of S^N, namely the Veronese immersion.

We now show how Theorem 8.2 may be used to go some way towards resolving the conjecture. In the process we give a new proof that the conjecture is true for $s \leqq 2$. Our theorem verifies the conjecture for all cases cited in [15], plus some more.

Recall first, from Remark 7.4 (ii), that each conformal minimal immersion $\tilde{\psi} : S^2 \to S^n$ whose image is not contained in any hyperplane section of S^n determines a linearly full conformal minimal immersion $\psi : S^2 \to \mathbb{C}P^n$, so we have the associated harmonic sequence $\psi_0, \ldots, \psi_n : S^2 \to \mathbb{C}P^n$. Furthermore by Theorem 7.3 we have $n = 2m$ for some m, and $\psi = \psi_m$. The following theorem is then an immediate consequence of Theorem 8.2.

Theorem 8.5. *Let $\tilde{\psi} : S^2 \to S^{2m}(1)$ be a linearly full conformal minimal immersion. If either*

(i) $K(\tilde{\psi}) \geqq \dfrac{2}{m(m+1)}$,

or

(ii) $K(\tilde{\psi}) \leqq \dfrac{2}{m(m+1)}$ *and the harmonic sequence associated to ψ is totally unramified,*

then

$$K(\tilde{\psi}) = \frac{2}{m(m+1)} ,$$

and up to isometries of S^{2m}, $\tilde{\psi}$ is the Veronese immersion.

Now suppose we are given a linearly full conformal minimal immersion $\tilde{\psi}: S^2 \to S^{2m}$ and suppose for some integer $s \geqq 1$ that

$$K(\tilde{\psi}) \geqq \frac{2}{(s+1)(s+2)} . \tag{8.1}$$

If $m \geqq s+1$ then

$$K(\tilde{\psi}) \geqq \frac{2}{(s+1)(s+2)} \geqq \frac{2}{m(m+1)} ,$$

so that by Theorem 8.5 we have $m = s+1$ and

$$K(\tilde{\psi}) = \frac{2}{(s+1)(s+2)} .$$

Thus condition (8.1) implies that $m \leqq s+1$. If in addition we have

$$K(\tilde{\psi}) \leqq \frac{2}{s(s+1)} ,$$

and m≦s then

$$K(\tilde{\psi}) \leqq \frac{2}{s(s+1)} \leqq \frac{2}{m(m+1)} ,$$

and hence by Theorem 8.5 if the associated harmonic sequence of ψ is totally unramified then

$$K(\tilde{\psi}) = \frac{2}{m(m+1)} ,$$

and $m = s$.

Theorem 8.6. *The Simon conjecture is true for linearly full conformal minimal immersions $\tilde{\psi}: S^2 \to S^{2m}$ if either i) ψ has totally unramified associated harmonic sequence; or (ii) $m > s \geqq 1$; or iii) $m = 2$, $s \geqq 1$. In particular it is true for $s \leqq 2$. If a counterexample exists for $s \geqq 3$ then the holomorphic curve ψ_0 in the harmonic sequence associated to ψ has degree $d_0 \geqq 2m+1$. (The "first" open case is $s = m = 3$.)*

Proof. The last statement in the theorem follows from (3.7) so it only remains to show that if

$$K(\tilde{\psi}) \leqq \frac{2}{s(s+1)}$$

and if $m \leqq 2 \leqq s$ then the harmonic sequence associated to ψ is totally unramified. Since ψ is a totally real immersion the only non-trivial case is $m = 2 \leqq s$ in which case $r(\partial_1) = r(\partial_2) = 0$. But then (5.10) shows that

$$t_1 = 1 - K_2 \geqq 1 - 2/s(s+1) ,$$

so it follows from (4.3) that $r(\partial_0) = 0$ and hence $r(\partial_3) = 0$ by Theorem 7.3.

9. Minimal Immersions with Constant Kähler Angle

In this section we will give support to the following.

Conjecture. Let $\psi: S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion with constant Kähler angle, and suppose that ψ is neither holomorphic, antiholomorphic nor totally real. Then, up to holomorphic isometries of $\mathbb{C}P^n$, ψ belongs to the Veronese sequence.

Theorem 9.1. *The conjecture is true if* $n \leqq 4$.

Proof. If $n \leqq 3$ then the harmonic sequence determined by ψ must have two consecutive Kähler angles constant, so Remark 5.5 shows that the conjecture holds in this case. Similar reasoning shows that if $n=4$ then it is enough to consider the case in which $\psi=\psi_2$. In this case (2.10) and (4.3) show that

$$\gamma_0-\gamma_3=3\gamma_2(t_2-1) \ . \tag{9.1}$$

By considering the harmonic sequence determined by $\bar{\psi}$ if necessary we may assume $t_2 \geqq 1$. However, if $t_2>1$ then (9.1) shows that $r(\partial_0)=0$, so by (4.8) and (3.24)

$$t_2=\frac{2(d_0-1)}{3(d_0-2)}>1$$

and hence $d_0<4$. But ψ is linearly full and $n=4$, so $d_0 \geqq 4$ by (3.7) and we have a contradiction.

Theorem 9.2. *Let $\psi: S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion with constant Kähler angle, and let $\psi_0, \ldots, \psi_n$ be the harmonic sequence determined by ψ, with $\psi=\psi_p$. If the degrees δ_{p-1}, δ_p are coprime, then ψ belongs to the Veronese sequence.*

Proof. Let $f_0(z)$ be a holomorphic $\mathbb{C}^{n+1}$-valued function defined on $\mathbb{C}=S^2-\{pt\}$ such that $\psi_0=[\mathrm{f}_0]$. For each $j=0,\ldots,p$, the map $\tilde{\sigma}_j$ defined following (3.12) is a polynomial function on $\mathbb{C}$ of degree δ_j and hence the function

$$\beta_{p-1}^{-\delta_p}\beta_p^{\delta_{p-1}}$$

is defined on the whole of $\mathbb{C}$, is never zero, and has a constant limit c as $z \to \infty$. Furthermore by (3.16), (4.8), and (4.3) we have

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \beta_{p-1}^{-\delta_p}\beta_p^{\delta_{p-1}}=0$$

so that

$$\beta_{p-1}^{-\delta_p}\beta_p^{\delta_{p-1}}=c \ .$$

Hence, as elements of $\mathbb{C}[z,\bar{z}]$, β_{p-1} and β_p have the same prime factors. Suppose then that π is a prime factor and that the highest powers of π dividing β_{p-1} and β_p are r, s respectively. Then $r\delta_p=s\delta_{p-1}$ and since $(\delta_{p-1},\delta_p)=1$ it follows that $\delta_{p-1}|r$, $\delta_p|s$. But if π is of degree d in z we have $rd \leqq \delta_{p-1}$, $sd \leqq \delta_p$ so that $d=1$ and

$$\beta_{p-1}=\pi^{\delta_{p-1}} \ , \qquad \beta_p=\pi^{\delta_p} \ .$$

Furthermore we may assume without loss of generality that π is self-conjugate so that $\pi=\alpha+\bar{\beta}z+\beta\bar{z}+\delta z\bar{z}$ for some complex numbers α, β, δ with α, δ real, and

$\alpha\delta-\beta\bar{\beta}>0$. But then, using (3.16),

$$\gamma_{p-1}+\gamma_p=(\delta_{p-1}+\delta_p)\frac{\partial}{\partial z\partial\bar{z}}\log\pi=(\delta_{p-1}+\delta_p)\cdot\frac{\alpha\delta-\beta\bar{\beta}}{\pi^2}$$

and so $ds_p^2=(\gamma_{p-1}+\gamma_p)dz\,d\bar{z}$ is of constant curvature.

Corollary 9.3. *The conjecture is true for totally unramified minimal immersions $\psi:S^2\to\mathbb{C}P^n$ when n and $n+2$ are consecutive prime integers.*

Proof. We observe that $(p(n-p+1),(p+1)(n-p))=(p(p+1),n-2p)$ and hence since n is prime it follows from (3.25) that

$$(\delta_{p-1},\delta_p)=(p+1,n-2p)=(p+1,n+2)=1\ .$$

Remark. There is a conjecture known as the Twin Primes Conjecture that there are infinitely many pairs of primes of the form $n,n+2$.

We now prove some pinching theorems for Kähler angles. Let $\psi:S^2\to\mathbb{C}P^n$ be a linearly full conformal minimal immersion and let $\psi_0,\dots,\psi_n$ be the associated harmonic sequence. We assume that $\psi=\psi_p$.

Lemma 9.4. *If the Kähler angle t_p of ψ_p satisfies either $t_p\geqq\delta_{p-1}/\delta_p$ or $t_p\leqq\delta_{p-1}/\delta_p$ then $t_p=\delta_{p-1}/\delta_p$.*

Proof. Using (3.17) we see that

$$\int_{S^2}(\delta_{p-1}\gamma_p-\delta_p\gamma_{p-1})d\bar{z}\wedge dz=0\ ,$$

so the result follows from (4.3).

Theorem 9.5. *Assume that ψ_p is totally unramified. If either $t_p\leqq N_p$ or $t_p\geqq N_p$, where*

$$N_p=\frac{p(n-p+1)}{(p+1)(n-p)}\ ,$$

then $t_p=N_p$.

The proof of the above theorem is immediate from (3.25).

Remark. One may use Theorem 9.5 to deduce pinching theorems for Kähler angle of the following type.

Let $\psi:S^2\to\mathbb{C}P^n$ be a totally unramified minimal immersion. If the Kähler angle $t(\psi)$ of ψ is such that $N_{q-1}\leqq t(\psi)\leqq N_q$ for some integer $q\in[0,n]$, then $t(\psi)=N_{q-1}$ or N_q.

It seems rather more difficult to deal with the case in which ψ_p is not totally unramified. However, using (3.24) we can deduce the following.

Theorem 9.6. *If $t_p\leqq\frac{1}{2}$ (resp. $t_p\geqq 2$), then $t_p=0$ (resp. $t_p=\infty$), i.e. ψ is a holomorphic (resp. antiholomorphic) curve.*

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