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Ribbon Concordance of Knots in the 3-Sphere

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1. Introduction

Given a (smooth) concordance $C \subset S^3 \times I$ between knots $K_i \subset S^3 \times \{i\}$, $i=0,1$, C may be adjusted by a small isotopy so that the restriction to C of the projection $S^3 \times I \to I$ is a Morse function. This will typically have critical points of index 0 (local minima), 1 (saddle-points), and 2 (local maxima). If there are no local maxima, we say that C is a *ribbon concordance from* K_1 *to* K_0. The terminology is suggested by the fact that K is a ribbon knot [6, p. 172] if and only if there is a ribbon concordance from K to the unknot.

Write $K_1 \geqq K_0$ if there exists a ribbon concordance from K_1 to K_0. The relation $\geqq$ is clearly reflexive and transitive, and we conjecture that it is also anti-symmetric, that is, $K_1 \geqq K_0$ and $K_0 \geqq K_1$ implies $K_0 = K_1$. [Knots are always oriented, and $K_0 = K_1$ means that there exists an orientation-preserving diffeomorphism of pairs $(S^3, K_0) \to (S^3, K_1)$.] In other words

Conjecture 1.1. $\geqq$ *is a partial ordering on the set of knots in* S^3.

This would actually give a partial ordering on the semigroup of knots under connected sum $(+)$, since clearly $K_1 \geqq K_0$ implies $K_1 + K \geqq K_0 + K$ for any K.

Let G be a group. Recall that the lower central series of G is defined as follows: $G_0 = G$, $G_{\alpha+1} = [G, G_\alpha]$, and $G_\beta = \bigcap_{\alpha<\beta} G_\alpha$ if β is a limit ordinal. We say that G is *transfinitely nilpotent* if $G_\alpha = 1$ for some α, and that a knot K is *transfinitely nilpotent* if the commutator subgroup $[\pi(K), \pi(K)]$ is, where $\pi(K)$ is the group of K. Examples of such K are those in the class generated by fibred knots and 2-bridge knots under the operations of connected sum and cabling (see Corollary 5.4).

As evidence for the conjecture just stated, we have

Theorem 1.2. *If* $K_1 \geqq K_0$ *and* $K_0 \geqq K_1$, *and* K_1 *(say) is transfinitely nilpotent, then* $K_0 = K_1$.

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Let $\tilde{X}$ be the infinite cyclic covering of the exterior of a knot K, and τ the automorphism of $H^1(\tilde{X};\mathbb{Q})$ induced by the canonical generator of the group of covering transformations. We say that K is *$\mathbb{Q}$-anisotropic* if $H^1(\tilde{X};\mathbb{Q})$ contains no non-trivial τ-invariant subspace which is self-annihilating with respect to the non-singular skew-symmetric duality pairing $H^1(\tilde{X};\mathbb{Q})\times H^1(\tilde{X};\mathbb{Q})\to\mathbb{Q}$ [15].

The next theorem gives a sufficient condition for a knot to be minimal with respect to $\geqq$. We remark that the class of knots which satisfy its hypotheses includes all connected sums of coherently oriented torus knots (see Proposition 4.9).

Theorem 1.3. *Let K_1 be $\mathbb{Q}$-anistropic and transfinitely nilpotent. Then $K_1 \geqq K_0$ implies $K_0 = K_1$.*

Finally, we have

Theorem 1.4. *Let K be transfinitely nilpotent, and let C be a ribbon concordance from K to K. Then the exterior of C is a relative s-cobordism between the two copies of the exterior of K.*

Thus, modulo the 4-dimensional relative s-cobordism theorem, $(S^3\times I, C)\cong(S^3,K)\times I$.

Combining Theorems 1.3 and 1.4 gives the following, which suggests that the only ribbon concordance from a knot which is $\mathbb{Q}$-anisotropic and transfinitely nilpotent is the product concordance.

Corollary 1.5. *Let K_1 be $\mathbb{Q}$-anisotropic and transfinitely nilpotent, and let C be a ribbon concordance from K_1 to K_0. Then $K_0 = K_1$, and the exterior of C is a relative s-cobordism between the two copies of the exterior of K_1.*

Theorems 1.2–1.4 are proved in Sect. 3. Our whole approach is based on Stallings' results on homology and central series of groups [20]. We also need the fact, recently proved by Thurston (unpublished), that knot groups are residually finite, which we use in conjunction with a result of Gerstenhaber and Rothaus [7]. Anisotropy of the duality pairing is used in the same way as in Scharlemann's study [19] of concordances of torus knots.

In Sect. 4 we discuss the question of anisotropy, over $\mathbb{R}$ and $\mathbb{Q}$, relating it to the standard knot invariants. For example, $\mathbb{R}$-anisotropy can sometimes be detected by the Alexander polynomial (Corollary 4.7). Applied to torus knots, this strengthens and simplifies the results of [19, Sect. 2]. We also show that any connected sum of coherently oriented torus knots is $\mathbb{Q}$-anisotropic. With algebraic knots in mind, we consider the question of anisotropy for iterated torus knots. In particular, using results of Litherland [10], we show that most 2-stage iterated torus knots are $\mathbb{Q}$-anisotropic (Proposition 4.10). However, there do exist (other) 2-stage iterated torus knots K, indeed algebraic knots, such that at least $4K$ is $\mathbb{Q}$-isotropic. This may be contrasted with Rudolph's result [18] that non-trivial positive braids have positive signature.

Section 5 contains some remarks on transfinite nilpotency, and in Sect. 6 we close with some questions.

2. Terminology

We work in the smooth category.

Homology will be with integer coefficients unless otherwise specified.

Unlabelled maps are induced by inclusion.

$\langle\ \rangle$ denotes the normal closure of a set of elements in a group.

If X is a homology circle, that is, $H_*(X)\cong H_*(S^1)$, $\tilde{X}$ will denote the (unique) infinite cyclic covering of X.

Note that $\pi_1(\tilde{X})$ is naturally identified, via the covering projection, with the commutator subgroup of $\pi_1(X)$. Hence if $X\subset Y$ are homology circles such that $H_1(X)\to H_1(Y)$ is an isomorphism, then $\pi_1(X)\to\pi_1(Y)$ is injective (surjective) if and only if $\pi_1(\tilde{X})\to\pi_1(\tilde{Y})$ is injective (surjective).

The *exterior* of a submanifold M of N (we always assume $M\cap\partial N=\partial M$) is the closure of the complement of a tubular neighbourhood of M. It is of the same homotopy type as the complement $N-M$.

X will usually stand for the exterior of a knot K in S^3.

A *concordance* between knots K_0 and K_1 is a submanifold C of $S^3\times I$, with $C\cong S^1\times I$, such that $C\cap S^3\times\{i\}=K_i$, $i=0,1$. We shall always denote the exterior of K_i by X_i, and the exterior of C by Y.

3. Ribbon Concordance

Lemma 3.1. *If C is a ribbon concordance from K_1 to K_0, then $\pi_1(X_1)\to\pi_1(Y)$ is surjective and $\pi_1(X_0)\to\pi_1(Y)$ is injective.*

Proof. As we pass up through a level $S^3\times\{t\}$ corresponding to a critical point of index k, a k-handle is added to C, and a $(k+1)$-handle is added to Y. Hence, if C is ribbon from K_1 to K_0, then

$$Y=X_0\times I\cup 1\text{-handles}\cup 2\text{-handles},$$

and, dually,

$$Y=X_1\times I\cup 2\text{-handles}\cup 3\text{-handles}.$$

Therefore $\pi_1(X_1)\to\pi_1(Y)$ is surjective.

Since $H_*(X_0)\to H_*(Y)$ is an isomorphism, in the expression $Y=X_0\times I\cup 1$-handles $\cup 2$-handles there must be equal numbers of 1- and 2-handles, and the 2-handles must cancel the 1-handles homologically. Therefore

$$\pi_1(Y)\cong(\pi_1(X_0)*F)/\langle r_1,\ldots,r_n\rangle,$$

where F is the free group on $x_1,\ldots,x_n$, say, and the relators $r_j\in\pi_1(X_0)*F$ satisfy the condition that if $\varepsilon_i(r_j)$ is the exponent sum of x_i in r_j, then the $n\times n$ matrix $(\varepsilon_i(r_j))$ has determinant ± 1. [In fact, if $a\in\pi_1(X_0)$ is represented by a meridian of K_0, then r_j may be taken to be of the form $x_j^{-1}w_j^{-1}aw_j$, for some $w_j\in\pi_1(X_0)*F$, $j=1,\ldots,n$.]

Suppose $z\in\ker(\pi_1(X_0)\to\pi_1(Y))$. By a recent result of Thurston (unpublished), $\pi_1(X_0)$ is residually finite. Therefore, if $z\neq 1$, $\pi_1(X_0)$ has a finite quotient G such that $\bar{z}\neq 1$, where $^-:\pi_1(X_0)\to G$ is the quotient map. Hence $\pi_1(Y)$ has quotient

$$H=(G*F)/\langle\bar{r}_1,\ldots,\bar{r}_n\rangle,$$

where $^{-}$ now denotes the quotient map $\pi_1(X_0)*F \to G*F$ induced by $^{-}: \pi_1(X_0) \to G$ and id: $F \to F$. Since $\varepsilon_i(\bar{r}_j) = \varepsilon_i(r_j)$, the conditions of [7, Theorem 2] are satisfied, allowing us to conclude that the natural map $G \to H$ is injective. But since $z \mapsto 1 \in \pi_1(Y)$ by hypothesis, whereas $\bar{z} \neq 1$, this is impossible. Hence $\pi_1(X_0) \to \pi_1(Y)$ is injective, as asserted. □

Lemma 3.2. *Let C be a ribbon concordance from K_1 to K_0, where K_1 is transfinitely nilpotent and $H_1(\tilde{X}_1; \mathbb{Q}) \to H_1(\tilde{Y}; \mathbb{Q})$ is injective. Then $K_0 = K_1$, and $\pi_1(X_i) \to \pi_1(Y)$ is an isomorphism, $i = 0, 1$.*

Proof. By Lemma 3.1, $\pi_1(X_1) \to \pi_1(Y)$ is surjective; therefore $\pi_1(\tilde{X}_1) \to \pi_1(\tilde{Y})$ is surjective. This in turn implies the surjectivity of $H_1(\tilde{X}_1; R) \to H_1(\tilde{Y}; R)$, and hence of $H_1(\partial \tilde{Y}; R) \to H_1(\tilde{Y}; R)$, for any coefficients R. From the homology exact sequence of the pair $(\tilde{Y}, \partial \tilde{Y})$ we then have $H_1(\tilde{Y}, \partial \tilde{Y}; R) = 0$. Now let R be a field. By Milnor duality [15] for the infinite cyclic covering $\tilde{Y}$, $H_1(\tilde{Y}, \partial \tilde{Y}; R) \cong H_2(\tilde{Y}; R)$; therefore $H_2(\tilde{Y}; R) = 0$. Hence $H_2(\tilde{Y}) \otimes R = 0$, by the universal coefficient theorem. In particular, $H_2(\tilde{Y}) \otimes \mathbb{Q} = 0$, showing that $H_2(\tilde{Y})$ is $\mathbb{Z}$-torsion. Again, for any prime p, $H_2(\tilde{Y}) \otimes \mathbb{Z}_p = 0$, showing that multiplication by $p: H_2(\tilde{Y}) \to H_2(\tilde{Y})$ is surjective. But a surjective endomorphism of a finitely-generated module over a Noetherian ring is injective, and $H_2(\tilde{Y})$ is a finitely-generated $\mathbb{Z}[t, t^{-1}]$-module. Thus $H_2(\tilde{Y})$ has no p-torsion, for any prime p. Therefore $H_2(\tilde{Y}) = 0$.

Now $H_1(\tilde{X}_1)$ is torsion-free [5], hence $H_1(\tilde{X}_1) \to H_1(\tilde{X}_1) \otimes \mathbb{Q} = H_1(\tilde{X}_1; \mathbb{Q})$ is injective. Since $H_1(\tilde{X}_1; \mathbb{Q}) \to H_1(\tilde{Y}; \mathbb{Q})$ is injective by hypothesis, we must have $H_1(\tilde{X}_1) \to H_1(\tilde{Y})$ injective.

Thus $H_1(\tilde{X}_1) \to H_1(\tilde{Y})$ is an isomorphism and $H_2(\tilde{Y}) = 0$, and hence by [20, Theorem 3.4], $\pi_1(\tilde{X}_1) \to \pi_1(\tilde{Y})$ induces an injection $\pi_1(\tilde{X}_1)/\pi_1(\tilde{X}_1)_\alpha \to \pi_1(\tilde{Y})/\pi_1(\tilde{Y})_\alpha$ for all α. In particular, if $\pi_1(\tilde{X}_1)$ is transfinitely nilpotent, $\pi_1(\tilde{X}_1) \to \pi_1(\tilde{Y})$ is injective. It follows that $\pi_1(X_1) \to \pi_1(Y)$ is injective, and therefore an isomorphism.

By Lemma 3.1, we have an injection $\pi_1(X_0) \to \pi_1(Y) \cong \pi_1(X_1)$. Let $g: \partial X_0 \to \partial X_1$ be the obvious orientation-preserving homeomorphism which sends a longitude-meridian pair for K_0 to a longitude-meridian pair for K_1. Since the cobordism Y between X_0 and X_1 is a product from ∂X_0 to ∂X_1, the above injection $\pi_1(X_0) \to \pi_1(X_1)$ restricts to $g_*: \pi_1(\partial X_0) \to \pi_1(\partial X_1)$. Hence, by [24, Corollary 6.4], it is induced by a covering map $f: X_0 \to X_1$ such that $f | \partial X_0 = g$. Since g is a homeomorphism, f must be also. Then f extends to an orientation-preserving homeomorphism $(S^3, K_0) \to (S^3, K_1)$, showing that $K_0 = K_1$. We have also shown in the process that $\pi_1(X_i) \to \pi_1(Y)$ is an isomorphism, $i = 0, 1$. □

Let $\tilde{X}$ be the infinite cyclic covering of the exterior X of a knot K, and let F be a field. Then according to Milnor [15], the cup product pairing

$$\cup: H^1(\tilde{X}; F) \times H^1(\tilde{X}, \partial \tilde{X}; F) \to H^2(\tilde{X}, \partial \tilde{X}; F) \cong F$$

is non-singular. Since $k^*: H^1(\tilde{X}, \partial \tilde{X}; F) \to H^1(\tilde{X}; F)$, induced by inclusion, is an isomorphism, one obtains a non-singular skew-symmetric bilinear form

$$\cdot: H^1(\tilde{X}; F) \times H^1(\tilde{X}; F) \to F$$

by setting

$$x \cdot y = x \cup k^{*-1}(y).$$

The automorphism $\tau: H^1(\tilde{X}; F) \to H^1(\tilde{X}; F)$ induced by the canonical generator of the group of covering transformations of $\tilde{X}$ is an isometry of $\cdot$, that is, $\tau x \cdot \tau y = x \cdot y$.

Let us say that K is *F-isotropic* if $H^1(\tilde{X}; F)$ contains a non-zero τ-invariant subspace on which $\cdot$ is identically zero, and otherwise, *F-anisotropic*.

Now let C be a concordance with K at one end, and let Y be the exterior of C. The following is a well-known consequence of Milnor duality in the infinite cyclic covering $(\tilde{Y}, \partial\tilde{Y})$.

Lemma 3.3. *If K is F-anisotropic, then $H_1(\tilde{X}; F) \to H_1(\tilde{Y}; F)$ is injective.*

Proof. (Compare [19, proof of Theorem 2.4].) F coefficients are to be understood throughout.

We shall prove the equivalent assertion that $H^1(\tilde{Y}) \to H^1(\tilde{X})$ is surjective.

Let $X_0 = \partial Y - X$.

We have inclusions $i: \tilde{X} \to \partial\tilde{Y}$; $j: \partial\tilde{Y} \to \tilde{Y}$; $k: \tilde{X} \to (\tilde{X}, \partial\tilde{X})$; $l: \partial\tilde{Y} \to (\partial\tilde{Y}, \tilde{X}_0)$; and e: $(\tilde{X}, \partial\tilde{X}) \to (\partial\tilde{Y}, \tilde{X}_0)$. Note that e^* is an isomorphism, by excision.

Let $W = j^* H^1(\tilde{Y}) \subset H^1(\partial\tilde{Y})$, and $U = i^* W \subset H^1(\tilde{X})$. Let $U^\perp \subset H^1(\tilde{X})$ be the annihilator of U with respect to $\cdot$. Clearly $\tau(U) = U$, which implies $\tau(U^\perp) = U^\perp$.

Suppose $x \in U^\perp$. Then

$$i^*(w) \cup k^{*-1}(x) = 0 \in H^2(\tilde{X}, \partial\tilde{X}), \quad \text{for all} \quad w \in W.$$

By the naturality of the cup product, this gives

$$w \cup e^{*-1} k^{*-1}(x) = 0 \in H^2(\partial\tilde{Y}, \tilde{X}_0),$$

and thence

$$w \cup l^* e^{*-1} k^{*-1}(x) = 0 \in H^2(\partial\tilde{Y}),$$

for all $w \in W$. But by duality in $(\tilde{Y}, \partial\tilde{Y})$, W is its own annihilator under $\cup: H^1(\partial\tilde{Y}) \times H^1(\partial\tilde{Y}) \to H^2(\partial\tilde{Y})$ (see [15, p. 130]). Therefore $l^* e^{*-1} k^{*-1}(x) \in W$, so that $x = i^* l^* e^{*-1} k^{*-1}(x) \in U$. Thus $U^\perp \subset U$, and therefore $U^\perp$ is a τ-invariant subspace of $H^1(\tilde{X})$ which is self-annihilating with respect to $\cdot$. By hypothesis, we must have $U^\perp = \{0\}$, giving $U = H^1(\tilde{X})$ as desired. □

Proof of Theorem 1.3. This follows from Lemmas 3.2 and 3.3. □

Let $d(K)$ denote the degree of the Alexander polynomial of K; equivalently, $d(K) = \dim H_1(\tilde{X}; \mathbb{Q})$.

Lemma 3.4. (i) *$K_1 \geqq K_0$ implies $d(K_1) \geqq d(K_0)$.*

(ii) *Suppose $K_1 \geqq K_0$, $d(K_1) = d(K_0)$, and K_1 is transfinitely nilpotent. Then $K_0 = K_1$.*

Proof. Let C be a ribbon concordance from K_1 to K_0, and let $d(C) = \dim H_1(\tilde{Y}; \mathbb{Q})$.

(i) By Lemma 3.1, $\pi_1(X_1) \to \pi_1(Y)$ is surjective; hence $H_1(\tilde{X}_1; \mathbb{Q}) \to H_1(\tilde{Y}; \mathbb{Q})$ is surjective. This implies that $d(C) \leqq d(K_1)$, and also that $H_1(\partial\tilde{Y}; \mathbb{Q}) \to H_1(\tilde{Y}; \mathbb{Q})$ is surjective. Therefore, since $\dim \ker(H_1(\partial\tilde{Y}; \mathbb{Q}) \to H_1(\tilde{Y}; \mathbb{Q})) = \frac{1}{2} \dim H_1(\partial\tilde{Y}; \mathbb{Q})$ by Milnor duality (see [15, p. 130]), we have $d(C) = \frac{1}{2} \dim H_1(\partial\tilde{Y}; \mathbb{Q})$. But

$H_1(\partial\tilde{Y};\ \mathbb{Q})\cong H_1(\tilde{X}_0;\ \mathbb{Q})\oplus H_1(X_1;\ \mathbb{Q})$, giving $d(C)=\frac{1}{2}(d(K_0)+d(K_1))$. Hence $d(K_1)\geqq d(K_0)$.

(ii) If $d(K_1)=d(K_0)$, then the argument just given shows that $d(C)=d(K_1)$. Hence $H_1(\tilde{X}_1;\mathbb{Q})\to H_1(\tilde{Y};\mathbb{Q})$ is an isomorphism. The fact that $K_0=K_1$ now follows from Lemma 3.2. □

Proof of Theorem 1.2. By Lemma 3.4(i), $d(K_1)=d(K_0)$. Hence by Lemma 3.4(ii), $K_0=K_1$. □

Proof of Theorem 1.4. Let Y be the exterior of C, and write X, X' for the exterior of K in $S^3\times\{0\}$ and $S^3\times\{1\}$ respectively.

As in the proof of Lemma 3.4(ii), $H_1(\tilde{X}';\mathbb{Q})\to H_1(\tilde{Y};\mathbb{Q})$ is injective. Therefore, by Lemma 3.2, $\pi_1(X)\to\pi_1(Y)$ and $\pi_1(X')\to\pi_1(Y)$ are isomorphisms.

Now X is homotopy equivalent to a finite 2-complex, and

$$Y=X\times I\cup 1\text{-handles}\cup 2\text{-handles};$$

hence Y is also homotopy equivalent to a finite 2-complex. Moreover, $\pi_1(X)\cong\pi_1(Y)$, $\pi_2(X)=0$, and $H_2(X)\cong H_2(Y)=0$. Hence $\pi_2(Y)=0$, by [4, Theorem 2]. Therefore Y is a $K(\pi,1)$, and the inclusions $X\to Y$, $X'\to Y$ are homotopy equivalences. Finally, they are simple homotopy equivalences since $Wh(\pi_1(X))=0$ by [25, Theorem 17.5]. □

4. Anisotropy

Throughout this section, F will denote either $\mathbb{Q}$ or $\mathbb{R}$. Note that $\mathbb{R}$-anisotropy implies $\mathbb{Q}$-anisotropy.

As usual, we regard $H^1(\tilde{X};F)$ as a module over the Laurent polynomial ring $\Lambda=F[t,t^{-1}]$ by defining $tx=\tau(x)$ for all $x\in H^1(\tilde{X};F)$.

First, ignoring the form $\cdot$, we consider the question of the existence of a proper τ-invariant subspace of $H^1(\tilde{X};F)$, i.e. a proper Λ-submodule. Since Λ is a principal ideal domain, $H^1(\tilde{X};F)$ is isomorphic to a finite direct sum of cyclic Λ-modules, and its order is $(\Delta(t))$, where $\Delta(t)$ is the Alexander polynomial of K. Hence

Proposition 4.1. *$H^1(\tilde{X};F)$ has no proper $F[t,t^{-1}]$-submodule if and only if $\Delta(t)$ is irreducible in $F[t]$.* □

Corollary 4.2. *If $\Delta(t)$ is irreducible in $\mathbb{Q}[t]$, (or equivalently $\mathbb{Z}[t]$), then K is $\mathbb{Q}$-anisotropic.* □

For irreducible $\lambda\in\Lambda$, let V_λ be the λ-primary summand of $V=H^1(\tilde{X};F)$. Since $\tau(x)\cdot y=x\cdot\tau^{-1}(y)$, we have $(\lambda x)\cdot y=x\cdot(\bar{\lambda}y)$, where $^-:\Lambda\to\Lambda$ is the conjugation induced by $t\mapsto t^{-1}$. It follows that if $(\lambda_1)\neq(\bar{\lambda}_2)$, then V_{λ_1} and V_{λ_2} are orthogonal with respect to $\cdot$. In particular, if $(\lambda)\neq(\bar{\lambda})$, $\cdot$ vanishes on V_λ. Now $(\lambda)=(\bar{\lambda})$ if and only if $\lambda=\pm t^n\bar{\lambda}$, and the $-$ sign is impossible since $\Delta(1)\neq 0$. Hence

Proposition 4.3. *If $\Delta(t)$ has a non-symmetric irreducible factor in $F[t]$, then K is F-isotropic.* □

Let $V_\lambda \cong \bigoplus_{i=1}^{k} \Lambda/(\lambda^{r_i})$, with $r = \max\{r_i : 1 \leqq i \leqq k\}$. If $r > 1$, then $\lambda^{r-1} V_\lambda$ is a non-zero submodule of $H^1(\tilde{X}; F)$ on which $\cdot$ vanishes (compare [9, Lemma 12]). Since the minimum polynomial $\mu(t)$ of τ is just the product $\prod \lambda^r$ over the irreducible factors λ of Δ, we see

Proposition 4.4. *If $\mu(t)$ has a repeated root, then K is $\mathbb{Q}$-isotropic.* □

Following Milnor [15], one defines a symmetric form

$$\langle\, , \rangle : H^1(\tilde{X}; F) \times H^1(\tilde{X}; F) \to F$$

by

$$\langle x, y \rangle = \tau(x) \cdot y + \tau(y) \cdot x = (\tau - \tau^{-1})(x) \cdot y .$$

Clearly τ is an isometry of $\langle\, , \rangle$.

Since $\Delta(1)\Delta(-1) \neq 0$, $\tau - \tau^{-1}$ is invertible, and hence $\langle\, , \rangle$ is non-singular. Also, $\cdot$ can be recovered from $\langle\, , \rangle$ by the formula

$$x \cdot y = \langle (\tau - \tau^{-1})^{-1}(x), y \rangle .$$

Hence, with the obvious terminology,

Proposition 4.5. *K is F-isotropic if and only if $\langle\, , \rangle$ is F-isotropic.* □

Consider the case $F = \mathbb{R}$. Then any irreducible factor λ of Δ has degree 1 or 2, and those of degree 1 are necessarily non-symmetric, since $\Delta(-1) \neq 0$. So if K is to be $\mathbb{R}$-anisotropic, then by Proposition 4.3 all the irreducible factors of Δ must be of the form $\lambda(t) = t^2 - 2t\cos\theta + 1 = (t - e^{i\theta})(t - e^{-i\theta})$. In general, for such a λ write $V_\lambda = V_\theta$, and, as in [15], let $\sigma_\theta(K)$ be the signature of $\langle\, , \rangle | V_\theta$. The minimum polynomial of $\tau | V_\theta$ is of the form λ^r. If $r > 1$, then K is $\mathbb{Q}$-isotropic by Proposition 4.4. If $r = 1$ and $|\sigma_\theta(K)| < \dim V_\theta$, then there exists a non-zero $x \in V_\theta$ such that $\langle x, x \rangle = 0$, and hence, since the minimum polynomial λ has degree 2, a non-zero Λ-submodule of V_θ on which $\langle\, , \rangle$ vanishes (see [9, Proposition 14]). If $|\sigma_\theta(K)| = \dim V_\theta$, then $\langle x, x \rangle \neq 0$ for all non-zero $x \in V_\theta$. Finally, $\langle\, , \rangle$ is F-isotropic if and only if $\langle\, , \rangle | V_\lambda$ is F-isotropic for all λ. Hence

Proposition 4.6. *K is $\mathbb{R}$-anisotropic if and only if all the roots of $\Delta(t)$ lie on S^1 and, for each such root $e^{i\theta}$, $|\sigma_\theta(K)| = \dim V_\theta$.* □

Corollary 4.7. *If all the roots of $\Delta(t)$ are distinct and lie on S^1, then nK is $\mathbb{R}$-anisotropic for all n.*

Proof. Let $\lambda(t) = t^2 - 2t\cos\theta + 1$ be a factor of $\Delta(t)$. Since $\Delta(t)$ has distinct roots, for K we have $V_\lambda \cong \Lambda/(\lambda)$. Hence $|\sigma_\theta(K)| = 2$ (see [15, pp. 128–129]). Therefore $|\sigma_\theta(nK)| = |n\sigma_\theta(K)| = 2|n|$, and the result follows from Proposition 4.6. □

Let $[p, q]$ denote the torus knot of type p, q. In [19], Scharlemann proves that $[p, q]$ is $\mathbb{Q}$-anisotropic, by an explicit analysis of $\cdot : H^1(\tilde{X}; \mathbb{Q}) \times H^1(\tilde{X}; \mathbb{Q}) \to \mathbb{Q}$. However, the Alexander polynomial $\Delta(t)$ of $[p, q]$ is $\dfrac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}$, so Corollary 4.7 implies

Corollary 4.8. *$n[p, q]$ is $\mathbb{R}$-anisotropic, for all n.* □

Note that $\Delta(t) = \prod_{\substack{d|pq \\ d\nmid p,q}} \varphi_d(t)$, where $\varphi_d(t)$ is the cyclotomic polynomial whose roots are the primitive d-th roots of 1. In particular, if p and q are prime, then the $\mathbb{Q}$-anisotropy of $[p,q]$ actually follows from Corollary 4.2.

Let us say that $\sum_{i=1}^{n} [p_i, q_i]$ is a connected sum of *coherently oriented* torus knots if either $p_i q_i > 0$ for all i or $p_i q_i < 0$ for all i.

Proposition 4.9. *Any connected sum of coherently oriented torus knots is $\mathbb{Q}$-anisotropic.*

As a preliminary to the proof of Proposition 4.9, we recall the connection between the Milnor signatures $\sigma_\theta(K)$ and the Tristram-Levine signature function $\sigma_K : S^1 \to \mathbb{R}$ (see [23, 8]) defined by

$$\sigma_K(\xi) = \text{signature of } (1-\bar{\xi})A + (1-\xi)A^T,$$

where A is a Seifert matrix for K. Let J_K be the "jump function" associated with σ_K:

$$J_K(\theta) = \lim_{\varphi \to \theta^+} \sigma_K(e^{i\varphi}) - \lim_{\varphi \to \theta^-} \sigma_K(e^{i\varphi}).$$

Then (see [13]), for $0 < \theta < \pi$,

$$\sigma_\theta(K) = J_K(\theta).$$

Proof of Proposition 4.9. A formula for $J_{[p,q]}(\theta)$ is given by Litherland in [10] (see the proof of Lemma 4.11 below), from which it follows easily that if $p, q > 0$ and $\theta = \frac{2\pi}{d}$, then

$$\sigma_\theta([p,q]) = \begin{cases} -2, & \text{if} \quad d|pq, d\nmid p,q \\ 0, & \text{otherwise}. \end{cases}$$

Now let $K = \sum_{i=1}^{n} [p_i, q_i]$, and assume without loss of generality that $p_i, q_i > 0$, $1 \leqq i \leqq n$. Let $V = H^1(\tilde{X}; \mathbb{Q})$, $\theta = \frac{2\pi}{d}$, and m the number of indices i such that $d|p_i q_i$, $d\nmid p_i, q_i$. Then, by the preceding paragraph, $|\sigma_\theta(K)| = 2m$.

Let $\varphi = \varphi_d$ be the d-th cyclotomic polynomial. Then

$$V_\varphi \cong \bigoplus_1^m \Lambda/(\varphi), \quad \text{where} \quad \Lambda = \mathbb{Q}[t, t^{-1}].$$

Suppose $\langle\ ,\ \rangle | V_\varphi$ were $\mathbb{Q}$-isotropic. Then it would be Witt equivalent to a form on a proper quotient Λ-module of V_φ, which is necessarily of the form $\bigoplus_1^l \Lambda/(\varphi)$, for some $l < m$. But then, interpreting this over $\mathbb{R}[t, t^{-1}]$, we would have

$$|\sigma_\theta(K)| \leqq 2l < 2m,$$

a contradiction.

This shows that $\langle\ ,\ \rangle|V_\varphi$ is $\mathbb{Q}$-anisotropic for all irreducible factors φ of Δ, and hence that $\langle\ ,\ \rangle$ is $\mathbb{Q}$-anisotropic. $\square$

We now make a few remarks about the iterated torus knots $[p_1, q_1; p_2, q_2; \ldots; p_n, q_n]$ (the (p_1,q_1)-cable of the (p_2, q_2)-cable of ... the p_n, q_n torus knot). These include the *algebraic knots* [2].

First, the Alexander polynomial of the (p,q)-cable of a knot K is $\Delta_{p,q}(t)\Delta(t^q)$, where $\Delta_{p,q}$ is the polynomial of $[p,q]$ and Δ that of K. It follows that all the roots of the polynomial of $[p_1,q_1;\ldots;p_n, q_n]$ lie on S^1, and it is not hard to work out the condition on the p_i, q_i which is necessary and sufficient for these roots to be distinct. When this occurs, the knot will be $\mathbb{R}$-anisotropic by Corollary 4.7.

It is possible to obtain more delicate information by considering the signatures σ_θ, using the results of Litherland [10]. For example, one can show that most 2-stage iterated torus knots are $\mathbb{Q}$-anisotropic. In particular, we have

Proposition 4.10. *If p_1q_1 is odd, then $[p_1, q_1; p_2, q_2]$ is $\mathbb{Q}$-anisotropic.*

This is not the strongest possible statement; what we shall actually show is that $[p_1,q_1; p_2,q_2]$ is $\mathbb{Q}$-anisotropic if there do not exist distinct prime factors m,n of p_1 and q of q_1 such that either $\{m,n,q\}=\{2,3,5\}$ or $q=2$.

Adopting the notation of [10], write $f_{p,q}(x)=\frac{1}{2}J_{[p,q]}(2\pi x)$. So, for $0<x<1/2$, $\sigma_{2\pi x}([p,q])=2f_{p,q}(x)$.

Recall that $f_{P,Q}(x)\neq 0$ if and only if $x=k/pq$, say, where $(k,pq)=1$, $p|P$, $q|Q$, and $p,q\neq \pm 1$. Thus we may write $P=pp'$, $Q=qq'$.

Lemma 4.11. *Suppose $p,q\neq\pm 1$, $p',q'\geqq 1$, $(pp',qq')=1$, and $(k,pq)=1$. Then $f_{pp',qq'}(k/pq)$ is independent of p' and q'.*

Proof. According to Litherland [10], $f_{pp',qq'}(k/pq)$ is ± 1 according as

$$\left[\frac{a}{qq'}\right]+\left[\frac{b}{pp'}\right]+\left[\frac{kp'q'}{pp'\cdot qq'}\right] \qquad (*)$$

is even or odd, where a and b are given by expressing $pp'\cdot qq'(k/pq)=kp'q'$ as

$$kp'q'=a\cdot pp'+b\cdot qq'.$$

But suppose

$$k=\alpha p+\beta q.$$

Then

$$kp'q'=\alpha q'\cdot pp'+\beta p'\cdot qq',$$

so that $(*)=\left[\frac{\alpha}{q}\right]+\left[\frac{\beta}{p}\right]+\left[\frac{k}{pq}\right]$ is independent of p' and q'. $\square$

Proof of Proposition 4.10. The Alexander polynomial $\Delta(t)$ of $K=[p_1,q_1; p_2,q_2]$ is $\Delta_{p_1,q_1}(t)\Delta_{p_2,q_2}(t^{q_1})$. Let $\Lambda=\mathbb{Q}[t,t^{-1}]$, and $V=H^1(\tilde{X};\mathbb{Q})$. If φ is an irreducible factor of Δ over Λ, (so $\varphi=\varphi_d$, say, the d-th cyclotomic polynomial), then V_φ is isomorphic to either $\Lambda/(\varphi)$ or $\Lambda/(\varphi)\oplus\Lambda/(\varphi)$. Hence if $\langle\ ,\ \rangle|V_\varphi$ is $\mathbb{Q}$-isotropic, then

$V_\varphi \cong \Lambda/(\varphi) \oplus \Lambda/(\varphi)$ and $\langle\ ,\ \rangle | V_\varphi$ is Witt trivial, which implies that $\sigma_\theta(K)=0$ for all θ of the form $\frac{2\pi k}{d}$, $(k,d)=1$.

Now φ_d is a repeated factor of Δ if and only if $d=pq$, $p|p_1$, $q|q_1$, $p>1$, $q>1$, and $e^{\frac{2\pi i q_1}{pq}}$ is a primitive c-th root of 1, where $c|p_2q_2$, $c\nmid p_2, q_2$. Let $r=q_1/q$. Then $q_1/pq=r/p$, and $(r,p)=1$. Hence we must have $p=c=mn$, say, where $m|p_2$, $n|q_2$, $m>1$, $n>1$. By [10], for $\theta=\frac{2\pi k}{d}$, $(k,d)=1$, and $0<\theta<\pi \pmod{2\pi}$,

$$\frac{1}{2}\sigma_\theta(K)=f_{p_1,q_1}\left(\frac{k}{mnq}\right)+f_{p_2,q_2}\left(\frac{rk}{mn}\right).$$

Hence, appealing to Lemma 4.11 and remembering that $f_{P,Q}(x)=-f_{P,Q}(-x)$, $f_{-P,Q}=-f_{P,Q}$, the vanishing of all these $\sigma_\theta(K)$ implies that for some r such that $(r,mn)=1$,

$$f_{mn,q}\left(\frac{k}{mnq}\right)+f_{m,n}\left(\frac{rk}{mn}\right)=0 \qquad (**)$$

for all k such that $(k,mn)=1$.

The following lemma will therefore complete the proof of Proposition 4.10.

Lemma 4.12. *If an equation of the form* (**) *holds, with* m,n,q *coprime and* >1, *then either* $\{m,n,q\}=\{2,3,5\}$, *or* $q=2$.

Proof. (**) implies that if $(k_i,mnq)=1$, $i=1,2$, and $k_1\equiv k_2 \pmod{mn}$, then $f_{mn,q}\left(\frac{k_1}{mnq}\right)=f_{mn,q}\left(\frac{k_2}{mnq}\right)$. But it is implicitly proved in [10, Appendix] (see particularly Lemma A3), that this implies $(q,mn)=(3,4)$, $(5,6)$, or $(3,10)$, or $q=2$. Since m,n are coprime and >1, $(3,4)$ is impossible, and the proof is complete. □

For $\{m,n,q\}=\{2,3,5\}$, equations of type (**) do exist; namely, for $(m,n,q;r)=(2,3,5;5)$, $(2,5,3;7)$, and $(3,5,2;8)$. There is some evidence that the last is in fact the only example with $q=2$, in which case the phrase "or $q=2$" in Lemma 4.12 could be dropped.

Using Lemma 4.11 again, it follows for example that if $K=[6x,25;\ 3,2]$, $[10x,21;5,2]$, or $[15x,6;5,3]$, with $x\geqq 1$ and coprime to 5, 21, and 2 respectively, then the signatures $\sigma_\theta(K)$ corresponding to the primitive 30-th roots of 1 are all zero. We can therefore at least conclude that $4K$ is $\mathbb{Q}$-isotropic (see [9]). These can be realized by algebraic knots, the smallest examples being $[156,25;3,2]$, $[220,21;5,2]$, and $[255,16;5,3]$. We do not know whether or not any of these, or related examples, is itself $\mathbb{Q}$-isotropic, but at any rate the limitation of signature arguments is apparent. Also, as n increases, the class of n-stage iterated torus knots K for which $\mathbb{Q}$-isotropy is not precluded by the $\sigma_\theta(K)$, becomes increasingly large.

5. Transfinite Nilpotency

If K is a knot with exterior X, we shall throughout this section write $\pi(K)=\pi_1(X)$, and $\gamma(K)=\pi_1(\tilde{X})=[\pi(K),\pi(K)]$. We shall also assume that $\gamma(K)\neq 1$, i.e. that K is non-trivial.

First note that $\gamma(K)$ is never nilpotent, as it contains a free subgroup of rank $2g, g \geqq 1$, namely, the image of the fundamental group of a minimal Seifert surface. So the least possible ordinal α such that $\gamma(K)_\alpha = 1$ is $\alpha = \omega$. (Clearly $G_\omega = 1$ is equivalent to G being residually nilpotent.)

Question 5.1. *Does $\gamma(K)_\alpha = 1$ for some α imply $\gamma(K)_\omega = 1$?*

One obvious remark is that the least ordinal α for which $\gamma(K)_\alpha = 1$ is necessarily a limit ordinal, for otherwise $\gamma(K)$ would have a non-trivial centre, namely $\gamma(K)_{\alpha-1}$. But [16] $\gamma(K)$ is either free of rank $2g, g \geqq 1$, or a non-trivial free product with amalgamation $A *_F B$, where F is free of rank $2g, g \geqq 1$, and is therefore centreless.

The fibred knots are precisely those with $\gamma(K)$ free, and a free group is residually nilpotent [11, Sect. 5.5], in fact, residually torsion-free nilpotent [11, Sect. 5.7].

Other examples of K with $\gamma(K)$ residually nilpotent are provided by the 2-bridge knots. For Mayland [14] has shown that for such a knot, $\gamma(K)$ can be expressed as a union of parafree groups in such a way that [1, Proposition 2.1] applies. It follows that $\gamma(K)$ is residually torsion-free nilpotent.

If $\Delta(0) = \pm 1$, then it follows easily from [20, Theorem 3.4] that $\gamma(K)_n/\gamma(K)_{n+1} \cong F_n/F_{n+1}$, $1 \leqq n < \omega$, where F is free of rank equal to that of $H_1(\tilde{X})$. In particular, $\gamma(K)_n/\gamma(K)_{n+1}$ is torsion-free. Recently, Strebel [21] has shown that in fact $\gamma(K)_n/\gamma(K)_{n+1}$ is torsion-free for any K, so that $\gamma(K)$ is residually nilpotent if and only if it is residually torsion-free nilpotent.

Finally, $\gamma(K_1 + K_2) \cong \gamma(K_1) * \gamma(K_2)$, and by [12] a free product of residually torsion-free nilpotent groups is residually torsion-free nilpotent.

It would be interesting to know more about which knots are transfinitely nilpotent. [Of course not all are; if $\Delta(t) = 1$, then $\gamma(K)$ is perfect.] Here is a specific question.

Question 5.2. *Are alternating knots transfinitely nilpotent?*

We now make some remarks on satellite knots. Let K be a knot, and $h: S^1 \times D^2 \to S^3$ an embedding such that $h(S^1 \times 0) = K$, and, for $x \in \partial D^2$, the linking number $Lk(h(S^1 \times x), K) = 0$. Let J be a simple closed curve in $S^1 \times D^2$ which does not lie in a ball in $S^1 \times D^2$, and write $J(K)$ for $h(J)$. [If $K \neq 0$, the unknot, and J is not a core of $S^1 \times D^2$, then $J(K)$ is a *satellite*, of K.]

Suppose $K \neq 0$, and write $\pi(J) = \pi_1(S^1 \times D^2 - J)$. Then $\pi(J(K))$ is a free product with amalgamation $\pi(J) *_{\mathbb{Z} \times \mathbb{Z}} \pi(K)$. In particular, $\gamma(K) \subset \gamma(J(K))$, so $\gamma(J(K))_\alpha = 1$ implies $\gamma(K)_\alpha = 1$, and $\gamma(J(K))$ free implies $\gamma(K)$ free.

Let $q \geqq 0$ be the (homological) winding number of J in $S^1 \times D^2$. If $q = 0$, then it is easy to see that in the above free product with amalgamation decomposition, $\pi(K) \subset \gamma(J(K))$. Since $\pi(K)_\alpha = \gamma(K)$, $\alpha \geqq 1$, it follows that $J(K)$ is not transfinitely nilpotent.

Consider the standard embedding of $S^1 \times D^2$ in S^3, and let C be the core of the complementary solid torus. Suppose $J(0)$ can be fibred in such a way that C meets each fibre transversely, (necessarily) in q points. This is equivalent to the existence of a fibring of J in $S^1 \times D^2$ such that the intersection of each fibre with $\partial(S^1 \times D^2)$ consists of q longitudes. If this holds, let us say that J is *fibred*. An example is the curve J such that $J(K)$ is the (p,q)-cable of K. (Another, trivial, example is the curve obtained by locally tying a fibred knot in a core of $S^1 \times D^2$.)

Let J be fibred, and let S be the corresponding fibre of the exterior of J in $S^1 \times D^2$, a surface of genus g, say, with $q+1$ boundary components. If X is the exterior of K, then the infinite cyclic covering of the exterior of $J(K)$ is $S \times R^1 \cup \coprod_1^q \tilde{X}$, identified in the obvious way along the q copies of $S^1 \times R^1$. Hence $\gamma(J(K)) \cong F * \mathop{*}_1^q \gamma(K)$, where F is free of rank $2g$. Therefore by [12], $\gamma(J(K))$ is residually nilpotent if and only if $\gamma(K)$ is residually nilpotent. Also, $\gamma(J(K))$ is free if and only if $\gamma(K)$ is free.

Finally, the surjection $\pi(K) \to \mathbb{Z}$ induces a surjection $\pi(J(K)) \cong \pi(J) *_{\mathbb{Z}\times\mathbb{Z}} \pi(K) \to \pi(J(0))$. Hence $\gamma(J(0))$ is a quotient of $\gamma(J(K))$, so that $\gamma(J(K))$ finitely generated implies $\gamma(J(0))$ finitely generated. Hence [16] $\gamma(J(K))$ free implies $\gamma(J(0))$ free.

Collecting the above observations [and calling a knot K residually nilpotent if $\gamma(K)$ is], we have

Proposition 5.3. (1) *If $q=0$, then $J(K)$ is not transfinitely nilpotent.*

(2) *If $q \neq 0$, then $J(K)$ transfinitely nilpotent implies K transfinitely nilpotent, and $J(K)$ fibred implies K and $J(0)$ fibred.*

(3) *If J is fibred, then $J(K)$ is residually nilpotent if and only if K is residually nilpotent, and $J(K)$ is fibred if and only if K is fibred.* □

Corollary 5.4. *If K is in the class of knots generated by fibred knots and 2-bridge knots under the operations of connected sum and cabling, then K is residually nilpotent.* □

6. Some Questions

The notion of concordance can be expressed in terms of $\geqq$. For it is well-known that, ordering critical points by their values, any concordance can be isotoped so that its critical points have non-decreasing index, and those of a given index occur in any desired order (see [17], for example). Hence (as is equally well-known), any concordance C from K_0 to K_1, say, can be adjusted so that $C \cap S^3 \times \{1/2\}$ is a knot K', $C \cap S^3 \times [0,1/2]$ is ribbon from K' to K_0, and $C \cap S^3 \times [1/2,1]$ is ribbon from K' to K_1. Thus two knots K_0 and K_1 are concordant if and only if these exists K' such that $K' \geqq K_0$ and $K' \geqq K_1$. (In other words, if Conjecture 1.1 is true, a concordance class is a directed set with respect to $\geqq$.)

Question 6.1. *Let K_0 be minimal with respect to $\geqq$. Does K concordant to K_0 imply $K \geqq K_0$? Equivalently, if $K' \geqq K$ and $K' \geqq K_0$, is $K \geqq K_0$?*

If $K_0 = 0$, the unknot, then Question 6.1 is just the question as to whether all slice knots are ribbon knots [6, Problem 25].

Of course there definitely exist concordant knots K_0, K_1 such that neither $K_1 \geqq K_0$ nor $K_0 \geqq K_1$. For example, let $K_0 = T + (-T)$, where T is the trefoil, and $K_1 = E + (-E)$, where E is the figure eight knot. Then $K_0 \geqq 0$, $K_1 \geqq 0$, so K_0 and K_1

are concordant. But both are transfinitely nilpotent (in fact, fibred), and $d(K_0)=d(K_1)(=4)$, and therefore by Lemma 3.4(ii), either $K_1 \geqq K_0$ or $K_0 \geqq K_1$ would imply $K_0 = K_1$.

Question 6.2. *If* $K_1 \geqq K_2 \geqq \ldots$, *does there exist some* m *such that* $K_n = K_m$ *for all* $n \geqq m$?

An affirmative answer to Question 6.2 would imply that every concordance class contains a minimal representative.

Question 6.3. *Does every concordance class contain a unique minimal representative with respect to* $\geqq$?

This would follow from affirmative answers to Questions 6.1 and 6.2 and Conjecture 1.1.

Recall [22, Sect. 6] the Gromov-Milnor-Thurston notion of the *volume* $v(M)$ of a compact 3-manifold M, and define $v(K)=v(X)$.

Question 6.4. *Does* $K_1 \geqq K_0$ *imply* $v(K_1) \geqq v(K_0)$?

In the proof of Theorem 1.4, we had to show that $\pi_2(Y)=0$. The following seems reasonable.

Conjecture 6.5. *The exterior of a ribbon concordance is aspherical.*

The exterior Y of a ribbon concordance C is homotopy equivalent to a 2-complex, and therefore the asphericity of Y is equivalent to $\pi_2(Y)=0$. Note also that if E is a meridian 2-disc of C, then $Y \cup E$ is contractible. Hence Conjecture 6.5 is implied by the Whitehead conjecture [26] that a subcomplex of an aspherical 2-complex is aspherical.

Since the latter is known to be true for a subcomplex with a single 2-cell [3], it follows that Conjecture 6.5 is true for any ribbon concordance from K, say, to the unknot, with a single saddle-point. Equivalently, the exterior of any ribbon disc in B^4 with only one saddle point is aspherical.

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Notes added in proof. (1) In the proof of Lemma 3.1, instead of using [7] and the fact that knot groups are residually finite, we may use the main result of Howie, J.: On pairs of 2-complexes and systems of equations over groups. J. reine angew. Math. **324**, 165–174 (1981), and the fact that knot groups are locally indicable [this follows from Scott, G.P.: Compact submanifolds of 3-manifolds. J. London Math. Soc. **7**, 246–250 (1973)].

(2) Regarding Question 5.2, Theorem B of Mayland, E.J., Jr. and Murasugi, K.: On a structural property of the groups of alternating links. Can. J. Math. **28**, 568–588 (1976) asserts that if K is an alternating knot with $\Delta(0)=p^r$, p prime, then $\gamma(K)$ is residually a finite q-group for all primes $q \neq p$, and hence residually nilpotent. It follows that such knots may be included with fibred and 2-bridge knots in statement of Corollary 5.4.