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On Automorphic Forms and Hodge Theory

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Introduction

The study of the periods of Γ-automorphic forms, for $\Gamma \subsetneq \mathrm{SL}_2(\mathbb{R})$ a Fuchsian group of the first kind, led Eichler [3] and Shimura [6] to establish an isomorphism

$$\delta : S_{k+2} \xrightarrow{\sim} H^1_P(\Gamma, V^k_{\mathbb{R}})$$

between the space S_{k+2} of cusp forms with respect to Γ of weight $k+2$, and the parabolic Eichler cohomology group $H^1_P(\Gamma, V^k_{\mathbb{R}})$, where $V^k_{\mathbb{R}}$ denotes the k-th symmetric power of the standard representation of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$.

As is well-known, this correspondence between cusp forms and cohomology classes is the starting point of the derivation of the Ramanujan-Petersson conjecture from the Weil conjectures. Deligne remarked in [1] that the Shimura isomorphism could be written in the form of a Hodge decomposition: the cusp forms can be interpreted as the global sections of a sheaf $\omega^k \otimes \Omega^1$ on the compact Riemann surface $\bar{S} = \Gamma \backslash \bar{\mathbb{H}}$ (see Sect. 1 for the definition of ω), and the Shimura isomorphism is essentially equivalent to

$$H^0(\bar{S}, \omega^k \otimes \Omega^1) \oplus \overline{H^0(\bar{S}, \omega^k \otimes \Omega^1)} \xrightarrow{\sim} H^1(\bar{S}, V^k), \qquad (*)$$

where V^k denotes here a sheaf on $\bar{S}$ whose restriction to $S = \Gamma \backslash \mathbb{H}$ is locally isomorphic to $\mathrm{Sym}^k(\mathbb{C}^2)$, but which has *degenerate* stalks at the points of $\bar{S} - S$. In fact, for $k = 0$, $(*)$ is just the well-known Hodge decomposition of $H^1(\bar{S}, \mathbb{C})$.

The aim of this paper is to give a direct proof of the preceeding isomorphism in the spirit of *Hodge theory with degenerating coefficients.*

A decomposition theorem of Hodge type for the cohomology of a complex non-singular projective variety with values in a local system of $\mathbb{C}$-vector spaces was proved by Deligne. In the special case of dimension 1, Zucker has generalized Deligne's theorem by allowing the system to have isolated degenerate stalks. However, in our particular case, we show that the exact knowledge of the sheaves involved (especially the explicit description of the degenerate stalks) makes it possible to avoid the technical difficulties in Zucker's work. We avoid any use of techniques such as L^2-cohomology, weight filtration or Laplace operators.

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Some of the classical computations in Eichler and Shimura's proof, such as the computation of dimensions on both sides of $(*)$, are circumvented in our proof by Serre duality. Others, such as those involving the Petersson scalar product, do occur in what seems to be a more natural context.

0. Some General Remarks on Sheaves

Let X be a topological space, $\Sigma \subseteq X$ a subset of isolated points, $Y = X - \Sigma$ and $j: Y \to X$ the inclusion. Let F_Y be a sheaf (of abelian groups, rings, modules) on Y. The sheaf $j_* F_Y$ is an extension of F_Y to X and its stalks are given by

$$(j_* F_Y)_x = \varinjlim_U F_Y(U \cap Y),$$

where U runs through the open neighborhoods of a point x in X. Thus an element of $(j_* F_Y)_x$, for $x \in \Sigma$, is the germ of a section f of F_Y defined on a punctured neighborhood $U - x$ of x.

In this paper we shall consider only such extensions of F_Y to X which are subsheaves of $j_* F_Y$. These sheaves are described by the

(0.1) Proposition. *Given any subgroup $A_x \subseteq (j_* F_Y)_x$ for each $x \in \Sigma$, there exists a unique extension $F_X \subseteq j_* F_Y$ of F_Y to X with*

$$F_{X,x} = A_x \quad \textit{for} \quad x \in \Sigma.$$

Proof. If U is an open neighborhood of x in X, then we have the canonical map $F_Y(U \cap Y) = (j_* F_Y)(U) \to (j_* F_Y)_x$, $f \mapsto f_x$. We define the extension F_X of F_Y by

$$F_X(U) = \{ f \in F_Y(U \cap Y) \mid f_x \in A_x \text{ for all } x \in U \cap \Sigma \}.$$

If $G_X \subseteq j_* F_Y$ is a further extension such that $G_{X,x} = A_x$ for $x \in \Sigma$, then clearly $G_X \subseteq F_X$ and this inclusion is an equality, since it is so on the stalks.

Now let Γ be a group acting continuously from the left on the topological space X. If F is a sheaf on X, then we say that Γ acts on F (from the right), or that F is a *Γ-sheaf*, if for every $\gamma \in \Gamma$ we have an isomorphism

$$F \to \gamma_* F$$

[i.e., an isomorphism $F(U) \to (\gamma_* F)(U) = F(\gamma^{-1} U)$ for each open set $U \subseteq X$], satisfying the obvious conditions with respect to the group structure of Γ. We say that a homomorphism $F \to G$ between sheaves with Γ-action is *equivariant*, if for each $\gamma \in \Gamma$ the diagram

$$\begin{array}{ccc} F & \longrightarrow & \gamma_* F \\ \downarrow & & \downarrow \\ G & \longrightarrow & \gamma_* G \end{array}$$

is commutative.

Consider now the quotient space $S = \Gamma \backslash X$ and the projection

$$\pi: X \to S.$$

For every Γ-sheaf F_X on X the *fixed sheaf*

$$F=(F_X)^\Gamma$$

on S is defined in the following way: if $U\subseteq S$ is open, then $\pi^{-1}(U)$ is an open and Γ-invariant subset of X and $F_X(\pi^{-1}(U))$ is a Γ-module. We set

$$F(U)=F_X(\pi^{-1}(U))^\Gamma.$$

For each point $x\in X$ we set $\Gamma_x=\{\gamma\in\Gamma|\gamma x=x\}$. In the situation in which we shall work, every point $x\in X$ has a basis of Γ_x-invariant open neighborhoods such that

$$U\cap\gamma U=\emptyset \quad\text{for}\quad \gamma\in\Gamma-\Gamma_x.$$

Under this assumption we show

(0.2) Proposition. *If* $F=(F_X)^\Gamma$ *and* $\bar{x}=\pi(x)$, *then*

$$F_{\bar{x}}=(F_{X,x})^{\Gamma_x}.$$

Proof. Let U be a Γ_x-invariant neighborhood of x such that $U\cap\gamma U=\emptyset$ for $\gamma\in\Gamma-\Gamma_x$. Then $\pi^{-1}(\pi(U))=\dot{\bigcup}_{\gamma\in\Gamma/\Gamma_x}\gamma U$ (disjoint union), and hence $F_X(\pi^{-1}(\pi(U)))$ $=\prod_{\gamma\in\Gamma/\Gamma_x}F_X(\gamma U)$ is the Γ-module induced by the Γ_x-module $F_X(U)$, i.e.,

$$F(\pi(U))=F_X(\pi^{-1}(\pi(U)))^\Gamma=F_X(U)^{\Gamma_x}.$$

Let $\mathscr{A}_X$ be a Γ-sheaf of rings and $\mathscr{A}$ its fixed sheaf on S. We call a $\Gamma-\mathscr{A}_X$-module $\mathscr{F}_X$ stalkwise free, if for every point $x\in X$ there exists a Γ_x-equivariant $\mathscr{A}_{X,x}$-isomorphism $\mathscr{F}_{X,x}\cong\mathscr{A}^n_{X,x}$. It is easy to prove the

(0.3) Proposition. *Let* $\mathscr{F}_X,\mathscr{G}_X$ *be* $\Gamma-\mathscr{A}_X$*-modules and* $\mathscr{F},\mathscr{G}$ *the associated fixed sheaves on* S. *If* $\mathscr{F}_X$ *or* $\mathscr{G}_X$ *is stalkwise free, then*

$$\left(\mathscr{F}_X\bigotimes_{\mathscr{A}_X}\mathscr{G}_X\right)^\Gamma=\mathscr{F}\bigotimes_{\mathscr{A}}\mathscr{G}.$$

1. Sheaf Theoretic Interpretation of Automorphic Forms

The group $SL_2(\mathbb{R})$ of real matrices $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with determinant 1 acts on the upper half plane $\mathbb{H}=\{z\in\mathbb{C}|\mathrm{Im}(z)>0\}$ from the left by

$$z\to\sigma z=\frac{az+b}{cz+d},$$

and on the sheaf $\mathcal{O}_{\mathbb{H}}$ of holomorphic functions on $\mathbb{H}$ (from the right) by $f(z)\to f^\sigma(z):=f(\sigma z)$.

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$. A point $s\in\mathbb{R}\cup\{\infty\}$ is a *cusp* of Γ if it is the only fixed point of an element $\gamma\in\Gamma$. We can write $s=\sigma\infty$ with $\sigma\in SL_2(\mathbb{R})$. If $\Gamma_s=\{\gamma\in\Gamma|\gamma s=s\}$, then the group $\Gamma_s^\sigma=\sigma^{-1}\Gamma_s\sigma$ is generated by a matrix $\tau=\begin{pmatrix}1 & h_s\\ 0 & 1\end{pmatrix}$

or by $-\tau$, or by τ and $-\tau$. The cusp is called *regular* if $\tau\in\Gamma_s^\sigma$, and *irregular* otherwise. We denote by Σ the set of cusps of Γ and we set

$$\bar{\mathbb{H}}=\mathbb{H}\cup\Sigma.$$

Taking the sets $U_N=\sigma\mathbb{H}_N\cup\{s\}$, where $\mathbb{H}_N=\{z\in\mathbb{C}|\mathrm{Im}(z)>N\}$, as a basis of neighborhoods of the cusps $s=\sigma\infty$, $\bar{\mathbb{H}}$ becomes a topological space with a continuous Γ-action. Let

$$j:\mathbb{H}\to\bar{\mathbb{H}}$$

be the inclusion. We make $\bar{\mathbb{H}}$ into a ringed space as follows:

(1.1) Definition. Let $\mathcal{O}_{\bar{\mathbb{H}}}\subseteq j_*\mathcal{O}_{\mathbb{H}}$ denote the extension of $\mathcal{O}_{\mathbb{H}}$ to $\bar{\mathbb{H}}$ given at the cusps $s=\sigma\infty$ by

$$\mathcal{O}_{\bar{\mathbb{H}},s}=\{f\in(j_*\mathcal{O}_{\mathbb{H}})_s|f^\sigma(z)=O(|z|^m),(\mathrm{Im}(z)\to\infty)\}.$$

That is, the germs in $\mathcal{O}_{\bar{\mathbb{H}},s}$ are given by the holomorphic functions f on the sets $\sigma\mathbb{H}_N$, such that $z^{-m}f^\sigma(z)$ is bounded on $\mathbb{H}_N$ for some integer $m=m(f)$. These are the functions meromorphic at the cusps:

$$f^\sigma(z)=\sum_{n\geqq -m} a_n z^{-n} \quad (|z|>N').$$

If $s=\sigma\infty$ is a cusp, then the function σz is an invertible global section of $\mathcal{O}_{\bar{\mathbb{H}}}$. When $\Sigma\neq\emptyset$, we shall assume, without restriction, that $\infty\in\Sigma$.

The left action of Γ on $\bar{\mathbb{H}}$ induces a right action on $\mathcal{O}_{\bar{\mathbb{H}}}$,

$$\mathcal{O}_{\bar{\mathbb{H}}}\to\gamma_*\mathcal{O}_{\bar{\mathbb{H}}}, \quad f(z)\to f^\gamma(z).$$

On the constant sheaf $V_{\bar{\mathbb{H}}}=\mathbb{C}^2$ we have the standard action of Γ: we let $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ act from the right by

$$(x,y)\gamma=(x,y)\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

We consider the Γ-sheaf

$$\mathcal{O}_{\bar{\mathbb{H}}}(V):=\mathcal{O}_{\bar{\mathbb{H}}}\bigotimes_{\mathbb{C}} V_{\bar{\mathbb{H}}}$$

of $\mathbb{C}^2$-valued meromorphic functions on $\bar{\mathbb{H}}$ which are holomorphic on $\mathbb{H}$. Throughout our considerations the global section

$$e_1-ze_2\in\mathcal{O}_{\bar{\mathbb{H}}}(V)(\bar{\mathbb{H}})=\mathcal{O}_{\bar{\mathbb{H}}}(\bar{\mathbb{H}})e_1\oplus\mathcal{O}_{\bar{\mathbb{H}}}(\bar{\mathbb{H}})e_2,$$

where $e_1=(1,0)$, $e_2=(0,1)$ plays a predominant role. An elementary calculation shows that, up to a constant, it is characterized by the transformation formula

$$(e_1-ze_2)\gamma=(cz+d)^{-1}(e_1-ze_2)$$

for all $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R})$.

(1.2) Definition. We define the Γ-invariant $\mathcal{O}_{\bar{\mathbb{H}}}$-submodule $\omega_{\bar{\mathbb{H}}}$ of $\mathcal{O}_{\bar{\mathbb{H}}}(V)$ by

$$\omega_{\bar{\mathbb{H}}}=\mathcal{O}_{\bar{\mathbb{H}}}(e_1-ze_2),$$

and denote by $\omega_{\bar{\mathbb{H}}}^k = \mathcal{O}_{\bar{\mathbb{H}}}(e_1 - ze_2)^k$ its k-th tensor power, $k \in \mathbb{Z}$.

$\omega_{\bar{\mathbb{H}}}$ can be interpreted as the "square root" of the $\mathcal{O}_{\bar{\mathbb{H}}}$-module

$$\Omega_{\bar{\mathbb{H}}}^1 := d\mathcal{O}_{\bar{\mathbb{H}}} = \mathcal{O}_{\bar{\mathbb{H}}} dz$$

of differentials $df = \frac{df}{dz} dz$ because of the isomorphism

$$\omega_{\bar{\mathbb{H}}}^2 \xrightarrow{\sim} \Omega_{\bar{\mathbb{H}}}^1, \quad (e_1 - ze_2)^2 \to dz,$$

which is Γ-equivariant since $(dz)^\gamma = d(\gamma z) = (cz+d)^{-2} dz$.

From now on we assume that $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group of the first kind. Then, as is well known, the quotient space $\bar{S} = \Gamma \backslash \bar{\mathbb{H}}$ has a natural structure of a compact Riemann surface. We consider the projection

$$\pi : \bar{\mathbb{H}} \to \bar{S}.$$

The images of the cusps are again called cusps. We set $S = \Gamma \backslash \mathbb{H} = \bar{S} - \{\text{cusps}\}$.

(1.3) Proposition. (i) $(\mathcal{O}_{\bar{\mathbb{H}}})^\Gamma$ *is the sheaf* $\mathcal{O}$ *of holomorphic functions on* $\bar{S}$.

(ii) $(\Omega_{\bar{\mathbb{H}}}^1)^\Gamma$ *is the sheaf* $\Omega^1(\Sigma)$ *of meromorphic* 1*-forms on* $\bar{S}$ *which are holomorphic on* S *and have a pole of order at most* 1 *at the cusps.*

Proof. (i) Let $U \subseteq \bar{S}$ be open and $f \in (\mathcal{O}_{\bar{\mathbb{H}}})^\Gamma(U) = \mathcal{O}_{\bar{\mathbb{H}}}(\pi^{-1}(U))^\Gamma$. f is a Γ-invariant function on $\pi^{-1}(U)$ and thus defines a function $\tilde{f}$ on U. Since f is holomorphic on $\pi^{-1}(U) \cap \mathbb{H}$, $\tilde{f}$ is holomorphic on $U \cap S$. f is, in particular, invariant under Γ_s for each cusp $s = \sigma\infty \in \pi^{-1}(U)$, so the function $f^\sigma(z)$ is invariant under the translation $\tau = \begin{pmatrix} 1 & h_s \\ 0 & 1 \end{pmatrix} \in \Gamma_s^\sigma$ and holomorphic on $\mathbb{H}_N$, for N large. It thus has a Fourier expansion

$$f^\sigma(z) = \sum_{n=-\infty}^{\infty} a_n q^n, \quad q = e^{2\pi i z/h_s},$$

in which the coefficients a_n must be 0 for $n < 0$, since $f^\sigma(z) = O(|z|^m)$, $q = O(1)$, but $q^{-1} \neq O(|z|^m)$ for all m. We therefore have

$$f(z) = \sum_{n=0}^{\infty} a_n q_s^n, \quad q_s = e^{2\pi i \sigma^{-1} z/h_s},$$

in the punctured neighborhood $\sigma\mathbb{H}_N$ of s. Since the analytic structure of $\bar{S}$ is defined in such a way that the Γ_s-invariant function q_s is a uniformizing parameter at s, $\tilde{f}$ is also holomorphic at $\bar{s} = \pi(s)$. Hence $(\mathcal{O}_{\bar{\mathbb{H}}})^\Gamma(U)$ is in fact the space of holomorphic functions on U.

(ii) For each cusp $s = \sigma\infty$ we have

$$d(\sigma^{-1} z) = \frac{h_s}{2\pi i} \frac{dq_s}{q_s}, \text{ where } \quad q_s = e^{2\pi i \sigma^{-1} z/h_s},$$

and hence

$$(\Omega_{\bar{\mathbb{H}}}^1)_{\bar{s}}^\Gamma = (\Omega_{\bar{\mathbb{H}},s}^1)^{\Gamma_s} = (\mathcal{O}_{\bar{\mathbb{H}},s} d(\sigma^{-1} z))^{\Gamma_s} = q_s^{-1} (\mathcal{O}_{\bar{\mathbb{H}},s})^{\Gamma_s} dq_s.$$

By what we have seen above the elements of this space can be interpreted as germs of meromorphic 1-forms at $\bar{s}$ having a pole of order at most 1. The elements of $(\Omega^1_{\bar{\mathbb{H}}})^\Gamma(U)=\Omega^1_{\bar{\mathbb{H}}}(\pi^{-1}(U))^\Gamma$ are thus in $1-1$-correspondence with the meromorphic 1-forms on U which are holomorphic on $U\cap S$ and have at most a simple pole at the cusps.

If f is a function on $U\subseteq\mathbb{H}$, then for each $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{R})$ and $k\in\mathbb{Z}$ we define the function $f\circ[\sigma]_k$ on $\sigma^{-1}U$ by

$$f\circ[\sigma]_k(z)=(cz+d)^{-k}f^\sigma(z).$$

The following theorem shows that the Γ-automorphic forms of weight k in the classical sense are the global sections of the sheaf

$$\omega^k:=(\omega^k_{\bar{\mathbb{H}}})^\Gamma.$$

(1.4) Theorem. *If $U\subseteq\bar{S}$ is open, then $\omega^k(U)$ is canonically isomorphic to the space of holomorphic functions $f(z)$ on $\pi^{-1}(U)\cap\mathbb{H}$ with the properties:*

(i) *$f\circ[\gamma]_k(z)=f(z)$ for all $\gamma\in\Gamma$, $z\in\pi^{-1}(U)\cap\mathbb{H}$.*

(ii) *For each cusp $s=\sigma\infty\in\pi^{-1}(U)$ we have a Fourier expansion*

$$f\circ[\sigma]_k(z)=q^\varepsilon\sum_{n=0}^{\infty}a_nq^n,\qquad q=e^{2\pi iz/h_s},\qquad z\in\mathbb{H}_N,$$

where $\varepsilon=0$ if k is even or s is regular and $\varepsilon=1/2$, otherwise.

Proof. $\omega^k(U)=\omega^k_{\bar{\mathbb{H}}}(\pi^{-1}(U))^\Gamma=[\mathcal{O}_{\bar{\mathbb{H}}}(\pi^{-1}(U))(e_1-ze_2)^k]^\Gamma$ consists of the Γ-invariant sections

$$\varphi=f(e_1-ze_2)^k,\qquad f\in\mathcal{O}_{\bar{\mathbb{H}}}(\pi^{-1}(U)),$$

and we associate to φ the function f on $\mathbb{H}\cap\pi^{-1}(U)$. The Γ-invariance means

$$f(e_1-ze_2)^k=(f(e_1-ze_2)^k)\gamma=f\circ[\gamma]_k(e_1-ze_2)^k,$$

i.e., $f=f\circ[\gamma]_k$. For the function $g=f\circ[\sigma]_k$ this means in particular $g\circ[\varrho]_k=g$ for all $\varrho\in\Gamma_s^\sigma=\sigma^{-1}\Gamma_s\sigma$. Since Γ_s^σ is generated by $\tau=\begin{pmatrix} 1 & h_s \\ 0 & 1\end{pmatrix}$ or by τ and $-\tau$ if s is regular (respectively by $-\tau$ if s is irregular) we have

$$g(z+h_s)=g(z)\qquad[\text{respectively } g(z+h_s)=(-1)^kg(z)].$$

Therefore $f\circ[\sigma]_k$ has a Fourier expansion

$$f\circ[\sigma]_k(z)=q^\varepsilon\sum_{n=-\infty}^{\infty}a_nq^n\quad\text{for}\quad z\in\mathbb{H}_N,$$

with ε defined as in (ii). The coefficients a_n must be zero for $n<0$ because of the growth condition $f\circ[\sigma]_k(z)=f^\sigma(z)(cz+d)^{-k}=O(|z|^m)$.

An automorphic form is a *cusp form* if the constant coefficient a_0 in its Fourier expansion at each cusp is zero. We denote by $S_k(\Gamma)$ the space of cusp forms of weight k with respect to Γ. By (1.4) this is the space of global sections of the sheaf

$$\omega_0^k:=(\omega^k_{\bar{\mathbb{H}}}\otimes\mathcal{O}_{\bar{\mathbb{H}}}(-\Sigma))^\Gamma,$$

where $\mathcal{O}_{\bar{\mathbb{H}}}(-\Sigma)$ is the sheaf of functions on $\bar{\mathbb{H}}$ which are holomorphic on $\mathbb{H}$ and zero at the cusps.

We obtain a very simple description of the sheaf ω_0^k if Γ has no elements of finite order and no irregular cusps. In this case the isotropy group Γ_s is trivial for each point $s \in \mathbb{H}$, and is generated by $\sigma\begin{pmatrix}1 & h_s\\0 & 1\end{pmatrix}\sigma^{-1}$ if $s=\sigma\infty$ is a cusp. It is easy to see the following

(1.5) Lemma. *Assume that Γ has no elements of finite order and no irregular cusps. Then for each point $s \in \bar{\mathbb{H}}$ we have*

$$\mathcal{O}_{\bar{\mathbb{H}}}(V)_s = \mathcal{O}_{\bar{\mathbb{H}},s} v_1 \oplus \mathcal{O}_{\bar{\mathbb{H}},s} v_2, \qquad \omega_{\bar{\mathbb{H}},s} = \mathcal{O}_{\bar{\mathbb{H}},s} v_1,$$

with the Γ_s-invariant germs $v_1 = (e_1 - ze_2)\sigma^{-1}$, $v_2 = e_2\sigma^{-1}$, where σ is given by $s=\sigma\infty$ if s is a cusp and is 1 if $s \in \mathbb{H}$.

In particular $\mathcal{O}_{\bar{\mathbb{H}}}(V)$ and $\omega_{\bar{\mathbb{H}}}$ are stalkwise free $\Gamma - \mathcal{O}_{\bar{\mathbb{H}}}$-modules.

Remark. Since the germs v_1, v_2 are Γ_s-invariant, they can also be considered as germs in $\mathcal{O}(V)_{\bar{s}}$, $\bar{s} = \pi(s)$, by (0.2), so that

$$\mathcal{O}(V)_{\bar{s}} = \mathcal{O}_{\bar{s}} v_1 \oplus \mathcal{O}_{\bar{s}} v_2 \quad \text{and} \quad \omega_{\bar{s}} = \mathcal{O}_{\bar{s}} v_1 .$$

We shall make use of this fact later.

(1.6) Corollary. *If Γ has no elements of finite order and no irregular cusps, then*

$$S_{k+2}(\Gamma) = H^0(\bar{S}, \omega^k \otimes \Omega^1).$$

Proof. By (1.3) the Γ-equivariant isomorphism $\omega_{\bar{\mathbb{H}}}^2 \cong \Omega_{\bar{\mathbb{H}}}^1$ yields an isomorphism

$$\omega^2 \cong \Omega^1(\Sigma).$$

Since $\omega_{\bar{\mathbb{H}}}$ is a stalkwise free $\Gamma - \mathcal{O}_{\bar{\mathbb{H}}}$-module we obtain by (0.3)

$$\omega_0^{k+2} = (\omega_{\bar{\mathbb{H}}}^k \otimes \omega_{\bar{\mathbb{H}}}^2 \otimes \mathcal{O}_{\bar{\mathbb{H}}}(-\Sigma))^\Gamma = \omega^k \otimes \Omega^1(\Sigma) \otimes \mathcal{O}(-\Sigma) = \omega^k \otimes \Omega^1 .$$

2. Hodge Structures

Although we shall make no use of the known results from Hodge theory, let us briefly recall the basic set up of this theory. The result we are aiming at, the *Shimura isomorphism*, turns out to be of a typical Hodge theoretical nature. We shall present a proof of it, which follows perfectly the formulation of Hodge theory given by Deligne and Griffiths (cf. [2, 9]), with the difference that we need not to make any use of harmonic forms, and thus can keep our arguments rather elementary.

Let V be a finite dimensional complex vector space, defined over $\mathbb{R}$, so that we have a complex conjugation $v \to \bar{v}$ on V.

(2.1) Definition. A *Hodge decomposition* of V of weight k is a decomposition

$$V = \bigoplus_{p+q=k} V^{pq} \quad \text{with} \quad V^{pq} = \bar{V}^{qp}.$$

To each Hodge decomposition one assigns the *Hodge filtration*

$$V=F^0 \supseteqq F^1 \supseteqq \dots \supseteqq F^k \supseteqq F^{k+1}=0, \tag{1}$$

where $F^p=\bigoplus_{j=p}^{k} V^{j,k-j}$. This filtration has the property

$$F^p \oplus \bar{F}^q = V, \quad p+q=k+1. \tag{2}$$

Conversely, every decreasing filtration (1) of V with the property (2) determines a Hodge decomposition of weight k, namely

$$V^{pq}=F^p \cap \bar{F}^q \cong F^p/F^{p+1} \quad \text{for} \quad p+q=k.$$

A *polarization* of a Hodge structure is a non-degenerate bilinear form $(,)$ on V, defined over $\mathbb{R}$, such that $\langle x,y\rangle := i^{p-q}(x,\bar{y})$ is a positive definite hermitian inner product on V^{pq}, and that the Hodge decomposition is orthogonal. The classical example of a polarized Hodge structure of weight k is the cohomology $V=H^k(X,\mathbb{C})$ of a non-singular projective variety X, or more generally of a compact Kähler manifold. In this case V^{pq} is the space of harmonic forms of type (p,q) on X.

Now let S be a complex non-singular quasi-projective variety. We denote by Ω^p, $\mathscr{E}^p$ the sheaves of holomorphic and of complex valued C^∞-differential forms of degree p on S, respectively; we set $\mathcal{O}=\Omega^0$, $\mathscr{E}=\mathscr{E}^0$.

Let V be a complex local system on S defined over $\mathbb{R}$, that is $V=V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ where $V_{\mathbb{R}}$ is a locally constant sheaf of finite dimensional real vector spaces. Let

$$\Omega^p(V):=\Omega^p \otimes_{\mathbb{C}} V, \qquad \mathscr{E}^p(V)=\mathscr{E}^p \otimes_{\mathbb{C}} V$$

be the sheaves of holomorphic, resp. $C^\infty-p$-forms "with values in V". If

$$f: X \to S$$

is a proper smooth morphism of quasi-projective complex varieties, then the sheaf

$$V=R^k f_* \mathbb{C}$$

is an example of such a local system. In this case the fibers of the $\mathcal{O}$-module

$$\mathcal{O}(V)=\mathcal{O} \otimes_{\mathbb{C}} R^k f_* \mathbb{C}$$

(i.e. the fibers of the associated vector bundle) are the cohomology groups $H^k(X_s,\mathbb{C})$ of the fibers X_s of f. Each of these cohomology groups carries a Hodge structure and these Hodge structures vary with s. The $\mathcal{O}$-module $\mathcal{O}(V)$ certainly has no decomposition into a direct sum of $\mathcal{O}$-modules analogous to (2.1), since holomorphic functions do not remain holomorphic under complex conjugation. However one can prove that there exists a decomposition of $\mathscr{E}$-modules

$$\mathscr{E}(V)=\bigoplus_{p+q=k} \mathscr{E}^{pq}(V), \tag{3}$$

which in the fibers induces the Hodge decomposition of $H^k(X_s,\mathbb{C})$.

Thus, the Hodge decomposition of $H^k(X_s, \mathbb{C})$ varies differentiably with s. But this "variation of Hodge structures" is induced by a holomorphic structure, namely, there is a filtration $(\mathscr{F}^p)$ of the $\mathcal{O}$-module $\mathcal{O}(V)$, which in the fibers induces the Hodge filtration of $H^k(X_s, \mathbb{C})$ and by which one gets the decomposition (3) by

$$\mathscr{E}^{pq}(V) = \mathscr{F}^p_{\mathscr{E}} \cap \bar{\mathscr{F}}^q_{\mathscr{E}} \cong \mathscr{F}^p_{\mathscr{E}}/\mathscr{F}^{p+1}_{\mathscr{E}} \quad \text{for} \quad p+q=k,$$

where $\mathscr{F}^p_{\mathscr{E}} = \mathscr{F}^p \bigotimes_{\mathcal{O}} \mathscr{E}$. By the "transversality theorem" of Griffiths we have for this filtration

$$\nabla \mathscr{F}^p \subseteq \mathscr{F}^{p-1} \bigotimes_{\mathcal{O}} \Omega^1 \subseteq \mathcal{O}(V) \bigotimes_{\mathcal{O}} \Omega^1 = \Omega^1(V),$$

where

$$\nabla = d \otimes 1 : \mathcal{O}(V) \to \Omega^1(V)$$

is induced by the complex differentiation $d : \mathcal{O} \to \Omega^1$. ∇ is called the *Gauß-Manin connection.*

These results now lead to the following abstract definition of a Hodge structure on a local system.

(2.2) Definition. Let V be a complex local system defined over $\mathbb{R}$ on the smooth, quasi-projective variety S. A *Hodge structure of* weight k on V (or a *variation of Hodge structures*) is a filtration $(\mathscr{F}^p)$ of the $\mathcal{O}$-module $\mathcal{O}(V)$ such that

(i) $\nabla \mathscr{F}^p \subseteq \mathscr{F}^{p-1} \otimes \Omega^1$.

(ii) $\mathscr{E}(V) = \bigoplus_{p+q=k} \mathscr{E}^{pq}(V)$, where $\mathscr{E}^{pq}(V) = \mathscr{F}^p_{\mathscr{E}} \cap \bar{\mathscr{F}}^q_{\mathscr{E}}$ for $p+q=k$.

There is also the notion of a *polarization* on V and the central result in this context is the theorem of Deligne (cf. [9]): a polarized Hodge structure of weight k on V induces a polarized Hodge structure of weight $k+q$ on the cohomology $H^q(S, V)$ of S with coefficients in the local system V, if S is projective.

3. The Sheaf V^k and its Polarized Hodge Structure

We consider the constant Γ-sheaf $V_{\bar{\mathbb{H}}} = \mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ on the extended upper half plane $\bar{\mathbb{H}}$ and its k-th symmetric power

$$V^k_{\bar{\mathbb{H}}} = \mathrm{Sym}^k_{\mathbb{C}}(\mathbb{C}^2) = \bigoplus_{j=0}^{k} \mathbb{C}e_1^j \cdot e_2^{k-j},$$

and we define the sheaf V^k on $\bar{S}$ by

$$V^k = (V^k_{\bar{\mathbb{H}}})^\Gamma.$$

For the next three paragraphs we assume that Γ has no elements of finite order and no irregular cusps. In this case the stalks of V^k are given by

$$V^k_{\bar{s}} = \mathrm{Sym}^k_{\mathbb{C}}(\mathbb{C}^2)^{\Gamma_s} = \begin{cases} \mathrm{Sym}^k_{\mathbb{C}}(\mathbb{C}^2) & \text{for} \quad \bar{s} = \pi(s) \in S, \\ \mathbb{C}(e_2 \sigma^{-1})^k & \text{for} \quad \bar{s} = \pi(\sigma\infty) \in \bar{S} \backslash S. \end{cases}$$

Thus V^k is a complex local system of dimension $k+1$ on S (defined over $\mathbb{R}$) and has degenerate stalks of dimension 1 at the cusps. We are going to show that V^k is in a natural way endowed with a polarized Hodge structure (in a modified sense) of weight k.

a) The connection ∇. Let $\mathcal{O}_{\bar{\mathbb{H}}}(V^k)=\mathcal{O}_{\bar{\mathbb{H}}}\bigotimes_{\mathbb{C}} V^k_{\bar{\mathbb{H}}}$ and $\Omega^1_{\bar{\mathbb{H}}}(V^k)=\Omega^1_{\bar{\mathbb{H}}}\bigotimes_{\mathbb{C}} V^k_{\bar{\mathbb{H}}}$. We set

$$\mathcal{O}(V^k):=\mathcal{O}_{\bar{\mathbb{H}}}(V^k)^\Gamma \quad \text{and} \quad \Omega^1(\Sigma)(V^k):=\Omega^1_{\bar{\mathbb{H}}}(V^k)^\Gamma=\Omega^1(\Sigma)\bigotimes_{\mathcal{O}}\mathcal{O}(V^k).$$

The Γ-equivariant, $\mathbb{C}$-linear, surjective map

$$d\otimes 1:\mathcal{O}_{\bar{\mathbb{H}}}(V^k)\to\Omega^1_{\bar{\mathbb{H}}}(V^k)$$

induces a $\mathbb{C}$-linear map

$$\nabla:\mathcal{O}(V^k)\to\Omega^1(\Sigma)(V^k),$$

which we call the *connection* of $\mathcal{O}(V^k)$. We denote by $\tilde{\Omega}^1(V^k)$ the image of ∇.

(3.1) Proposition. *The sequence*

$$0\to V^k\to\mathcal{O}(V^k)\xrightarrow{\nabla}\tilde{\Omega}^1(V^k)\to 0$$

is exact.

Proof. Tensoring the exact sequence $0\to\mathbb{C}\to\mathcal{O}_{\bar{\mathbb{H}}}\xrightarrow{d}\Omega^1_{\bar{\mathbb{H}}}\to 0$ with the constant sheaf $V^k_{\bar{\mathbb{H}}}$ we obtain the exact sequence

$$0\to V^k_{\bar{\mathbb{H}}}\to\mathcal{O}_{\bar{\mathbb{H}}}(V^k)\xrightarrow{d\otimes 1}\Omega^1_{\bar{\mathbb{H}}}(V^k)\to 0,$$

to which we apply the left exact functor $F\to F^\Gamma$.

b) The Hodge filtration. Given an $\mathcal{O}_{\bar{\mathbb{H}}}$-module or an $\mathcal{O}$-module $\mathcal{F}$, we denote by $\mathcal{F}^k$ its k-th symmetric power ($k\geqq 0$) over $\mathcal{O}_{\bar{\mathbb{H}}}$ or $\mathcal{O}$, respectively, and by $\cdot$ the product in the symmetric algebra $\bigoplus_{k=0}^{\infty}\mathcal{F}^k$.

Regarding that $\mathcal{O}_{\bar{\mathbb{H}}}(V^k)=\mathcal{O}_{\bar{\mathbb{H}}}(V)^k$, the filtration $\mathcal{O}_{\bar{\mathbb{H}}}(V)\supseteq\omega_{\bar{\mathbb{H}}}\supseteq 0$ induces a Γ-equivariant filtration

$$\mathcal{O}_{\bar{\mathbb{H}}}(V^k)=\mathcal{F}^0_{\bar{\mathbb{H}}}\supseteq\mathcal{F}^1_{\bar{\mathbb{H}}}\supseteq\ldots\supseteq\mathcal{F}^k_{\bar{\mathbb{H}}}\supseteq\mathcal{F}^{k+1}_{\bar{\mathbb{H}}}=0$$

given by $\mathcal{F}^p_{\bar{\mathbb{H}}}=\omega^p_{\bar{\mathbb{H}}}\cdot\mathcal{O}_{\bar{\mathbb{H}}}(V^{k-p})$. By (1.5) all these $\Gamma-\mathcal{O}_{\bar{\mathbb{H}}}$-modules are stalkwise free. Passing to the fixed sheaves we obtain the filtration

$$\mathcal{O}(V^k)=\mathcal{F}^0\supseteq\mathcal{F}^1\supseteq\ldots\supseteq\mathcal{F}^k\supseteq\mathcal{F}^{k+1}=0$$

where $\mathcal{F}^p=\omega^p\cdot\mathcal{O}(V^{k-p})$ by (0.3).

(3.2) Proposition. $\nabla\mathcal{F}^p\subseteq\mathcal{F}^{p-1}\bigotimes_{\mathcal{O}}\Omega^1(\Sigma)$.

Proof. On $\bar{\mathbb{H}}$ we have

$$\mathcal{F}^p_{\bar{\mathbb{H}}}=\omega^p_{\bar{\mathbb{H}}}\cdot\mathcal{O}_{\bar{\mathbb{H}}}(V^{k-p})=\mathcal{O}_{\bar{\mathbb{H}}}(e_1-ze_2)^p\cdot V^{k-p}_{\bar{\mathbb{H}}}.$$

If f is a section of $\mathcal{O}_{\bar{\mathbb{H}}}$ and x a section of $V_{\bar{\mathbb{H}}}^{k-p}$, then

$$(d\otimes 1)(f(e_1-ze_2)^p\cdot x)=(e_1-ze_2)^p\cdot x\otimes df-pf(e_1-ze_2)^{p-1}\cdot e_2x\otimes dz,$$

and this is a section of $\mathcal{F}_{\bar{\mathbb{H}}}^{p-1}\otimes\Omega_{\bar{\mathbb{H}}}^1$, since $e_2\cdot x$ is a section of V^{k-p+1}. Therefore $d\otimes 1$ maps $\mathcal{F}_{\bar{\mathbb{H}}}^p$ into $\mathcal{F}_{\bar{\mathbb{H}}}^{p-1}\otimes\Omega_{\bar{\mathbb{H}}}^1$. Hence

$$\nabla\mathcal{F}^p\subseteqq(\mathcal{F}_{\bar{\mathbb{H}}}^{p-1}\otimes\Omega_{\bar{\mathbb{H}}}^1)^\Gamma=\mathcal{F}^{p-1}\otimes\Omega^1(\Sigma)$$

by (1.5), (0.3) and (1.3).

c) The Hodge decomposition. In order to establish the Hodge decomposition it is not sufficient to tensor the above filtration with the sheaf of C^∞-functions on $\bar{S}$. We shall work with a larger sheaf which we define as follows. Let $\mathcal{E}_{\mathbb{H}}$ be the sheaf of complex valued C^∞-functions on $\mathbb{H}$. We define the extension $\mathcal{E}_{\bar{\mathbb{H}}}$ of $\mathcal{E}_{\mathbb{H}}$ by

$$\mathcal{E}_{\mathbb{H},s}=\left\{f\in(j_*\mathcal{E}_{\mathbb{H}})_s\,\Big|\,\frac{\partial^{\nu+\mu}f^\sigma}{\partial z^\nu\partial\bar{z}^\mu}=O(|z|^m)\text{ for all }\nu,\mu\geqq 0\right\}$$

for $s=\sigma\infty\in\Sigma$, where $m=m(f,\nu,\mu)$. We set

$$\mathcal{E}=(\mathcal{E}_{\bar{\mathbb{H}}})^\Gamma.$$

The restriction $\mathcal{E}_S$ of $\mathcal{E}$ to S is the sheaf of complex valued C^∞-functions on S. Now the sheaf

$$\mathcal{E}(V^k):=\Big(\mathcal{E}_{\bar{\mathbb{H}}}\bigotimes_{\mathbb{C}}V_{\bar{\mathbb{H}}}^k\Big)^\Gamma=\mathcal{E}\bigotimes_{\mathcal{O}}\mathcal{O}(V^k),$$

which obviously admits a complex conjugation, also admits a Hodge decomposition:

(3.3) Proposition. $\mathcal{E}(V^k)=\bigoplus_{p+q=k}\omega_{\mathcal{E}}^p\cdot\bar{\omega}_{\mathcal{E}}^q$ *and* $\omega_{\mathcal{E}}^p\cdot\bar{\omega}_{\mathcal{E}}^q=\mathcal{F}_{\mathcal{E}}^p\cap\bar{\mathcal{F}}_{\mathcal{E}}^q$, *where the subscript* $\mathcal{E}$ *indicates the tensor product with the* $\mathcal{O}$*-module* $\mathcal{E}$.

Proof. Since $\mathcal{E}(V^k)$ is the k-th symmetric power of $\mathcal{E}(V)$ over $\mathcal{E}$, it suffices to prove the proposition for $k=1$. Each section $v=fe_1+ge_2$ of

$$\mathcal{E}_{\bar{\mathbb{H}}}(V)=\mathcal{E}_{\bar{\mathbb{H}}}\bigotimes_{\mathbb{C}}V_{\bar{\mathbb{H}}}=\mathcal{E}_{\bar{\mathbb{H}}}e_1\oplus\mathcal{E}_{\bar{\mathbb{H}}}e_2$$

can be written in the form

$$v=\tilde{f}(e_1-ze_2)+\tilde{g}(e_1-\bar{z}e_2),$$

where $\tilde{f}=(\bar{z}-z)^{-1}(\bar{z}f+g)$, $\tilde{g}=(z-\bar{z})^{-1}(zf+g)$ are again sections of $\mathcal{E}_{\bar{\mathbb{H}}}$. This decomposition is unique, i.e.,

$$\mathcal{E}_{\bar{\mathbb{H}}}(V)=\Big(\mathcal{E}_{\bar{\mathbb{H}}}\bigotimes_{\mathcal{O}_{\bar{\mathbb{H}}}}\omega_{\bar{\mathbb{H}}}\Big)\oplus\Big(\mathcal{E}_{\bar{\mathbb{H}}}\bigotimes_{\mathcal{O}_{\bar{\mathbb{H}}}}\bar{\omega}_{\bar{\mathbb{H}}}\Big),$$

and thus

$$\mathcal{E}(V)=\omega_{\mathcal{E}}\oplus\bar{\omega}_{\mathcal{E}}.$$

The Propositions (3.2) and (3.3) show that the filtration $(\mathcal{F}^p)$ of $\mathcal{O}(V^k)$ satisfies the conditions of a Hodge structure of V^k in a modified sense, due to the presence of degenerate stalks.

d) The polarization. We now show that the Hodge structure of V^k is endowed with a natural polarization. We consider the bilinear form

$$B^1(x,y) = -\det(x,y)$$

on $\mathbb{C}^2$, and its k-th symmetric power

$$B^k(x_1\cdot\ldots\cdot x_k, y_1\cdot\ldots\cdot y_k) = \frac{1}{(k!)^2}\sum_{\sigma,\tau\in\mathfrak{S}_k}\prod_{i=1}^{k} B^1(x_{\sigma(i)}, y_{\tau(i)})$$

on $V^k_{\bar{\mathbb{H}}} = \mathrm{Sym}^k_{\mathbb{C}}(\mathbb{C}^2)$. This form is defined over $\mathbb{R}$, is Γ-invariant and its linear extension is a Γ-invariant $\mathcal{E}_{\bar{\mathbb{H}}}$-bilinear form on $\mathcal{E}_{\bar{\mathbb{H}}}(V^k)$

$$B^k : \mathcal{E}_{\bar{\mathbb{H}}}(V^k)\times\mathcal{E}_{\bar{\mathbb{H}}}(V^k)\to\mathcal{E}_{\bar{\mathbb{H}}}.$$

Taking fixed sheaves we obtain an $\mathcal{E}$-bilinear form

$$B^k : \mathcal{E}(V^k)\times\mathcal{E}(V^k)\to\mathcal{E}.$$

(3.4) Proposition. *B^k is a non-degenerate bilinear form on the $\mathcal{E}$-module $\mathcal{E}(V^k)$ with the properties*

(i) *$B^k(x,y) = (-1)^k B^k(y,x)$ for any sections x,y of $\mathcal{E}(V^k)$.*

(ii) *The Hodge decomposition $\mathcal{E}(V^k) = \bigoplus_{p+q=k}\omega^p_{\mathcal{E}}\cdot\bar{\omega}^q_{\mathcal{E}}$ is orthogonal with respect to the form $B^k(x,\bar{y})$.*

(iii) *The form $\langle x,y\rangle := i^{p-q}B^k(x,\bar{y})$ is positive definite on $\omega^p_{\mathcal{E}}\cdot\bar{\omega}^q_{\mathcal{E}}$, $p+q=k$, i.e., for a section x of $\omega^p_{\mathcal{E}}\cdot\bar{\omega}^q_{\mathcal{E}}$ we have*

$$\langle x,x\rangle \geqq 0 \quad \textit{and} \quad \langle x,x\rangle = 0 \quad \textit{iff} \quad x=0.$$

Proof. Let $s\in\bar{\mathbb{H}}$, $v_1 = (e_1 - ze_2)\sigma^{-1}$, $\bar{v}_1 = (e_1 - \bar{z}e_2)\sigma^{-1}$, where $\sigma = \mathrm{Id}$ if $s\in\mathbb{H}$ and $\sigma\infty = s$ if s is a cusp. Since by (1.5) v_1 is a Γ_s-invariant basis of $\omega_{\bar{\mathbb{H}},s}$ the stalk of $\mathcal{E}(V^k)$ at $\bar{s} = \pi(s)$ is given by

$$\mathcal{E}(V^k)_{\bar{s}} = \bigoplus_{p+q=k}\omega^p_{\mathcal{E},\bar{s}}\cdot\bar{\omega}^q_{\mathcal{E},\bar{s}} = \bigoplus_{p+q=k}\mathcal{E}_{\bar{s}}v_1^p\cdot\bar{v}_1^q.$$

An easy calculation shows that

$$\langle v_1^p\cdot\bar{v}_1^q, v_1^{p'}\cdot\bar{v}_1^{q'}\rangle = \begin{cases} 0 & \text{if } p\neq p' \\ \binom{k}{p}^{-1}(2\,\mathrm{Im}\,\sigma^{-1}z)^k & \text{if } p=p' \end{cases}$$

for $p+q = p'+q' = k$, and this immediately implies the proposition.

We now extend the sheaves $\mathcal{E}^i_{\mathbb{H}}$ of complex-valued C^∞-differential forms on $\mathbb{H}$ of degree i to $\bar{\mathbb{H}}$ by

$$\mathcal{E}^1_{\bar{\mathbb{H}}} := \mathcal{E}_{\bar{\mathbb{H}}}dz\oplus\mathcal{E}_{\bar{\mathbb{H}}}d\bar{z}\subseteq j_*\mathcal{E}^1_{\mathbb{H}}$$

and $\mathscr{E}^i_{\mathbb{H}} := \Lambda^i \mathscr{E}^1_{\mathbb{H}}$ (of course $\mathscr{E}^i_{\mathbb{H}}=0$ for $i>2$). We set $\mathscr{E}^i_{\mathbb{H}}(V^k)=\mathscr{E}^i_{\mathbb{H}} \bigotimes_{\mathbb{C}} V^k_{\mathbb{H}}$. The total differential $d: j_* \mathscr{E}^i_{\mathbb{H}} \to j_* \mathscr{E}^{i+1}_{\mathbb{H}}$ induces the Γ-invariant maps $d: \mathscr{E}^i_{\mathbb{H}} \to \mathscr{E}^{i+1}_{\mathbb{H}}$ and

$$d\otimes 1: \mathscr{E}^i_{\mathbb{H}}(V^k) \to \mathscr{E}^{i+1}_{\mathbb{H}}(V^k).$$

Hence, for the fixed sheaves

$$\mathscr{E}^i := (\mathscr{E}^i_{\mathbb{H}})^\Gamma \quad \text{and} \quad \mathscr{E}^i(V^k) := \mathscr{E}^i_{\mathbb{H}}(V^k)^\Gamma = \mathscr{E}^i \bigotimes_{\mathscr{E}} \mathscr{E}(V^k)$$

we obtain the maps $d: \mathscr{E}^i \to \mathscr{E}^{i+1}$ and

$$\nabla : \mathscr{E}^i(V^k) \to \mathscr{E}^{i+1}(V^k),$$

induced by $d\otimes 1$.

The bilinear form $B^k: \mathscr{E}(V^k) \times \mathscr{E}(V^k) \to \mathscr{E}$, together with the exterior product of differential forms $\mathscr{E}^i \otimes \mathscr{E}^j \xrightarrow{\wedge} \mathscr{E}^{i+j}$ defines a product

$$\mathscr{E}^i(V^k) \times \mathscr{E}^j(V^k) \xrightarrow{\wedge} \mathscr{E}^{i+j}$$

by the rule $(\alpha\otimes v, \beta\otimes w) \mapsto B^k(v,w)\alpha\wedge\beta$. It is easy to check the following

(3.5) Lemma. *If ξ, η are sections of $\mathscr{E}^i(V^k), \mathscr{E}^j(V^k)$, then*

(i) $\xi\wedge\eta = (-1)^{ij+k}\eta\wedge\xi$.

(ii) $d(\xi\wedge\eta) = (\nabla\xi)\wedge\eta + (-1)^i \xi\wedge(\nabla\eta)$.

4. The Hodge Filtration of $H^\cdot(\bar{S}, V^k)$

Again we assume that Γ has no elements of finite order and no irregular cusps. We now show that the filtration $(\mathscr{F}^p)$ of $\mathcal{O}(V^k)$ together with the connection ∇ induces a filtration of the $\mathbb{C}$-vector space $H^1(\bar{S}, V^k)$. We consider the complex

$$\Omega^\cdot(V^k): \mathcal{O}(V^k) \to \tilde{\Omega}^1(V^k) \to 0,$$

which by (3.1) is a resolution of the sheaf V^k. Because of (3.2) the filtration $(\mathscr{F}^p)$ of $\mathcal{O}(V^k)$ yields a filtration $(\mathfrak{F}^p)$ of the complex $\Omega^\cdot(V^k)$:

$$\begin{array}{rccc} \Omega^\cdot(V^k) = \mathfrak{F}^0: & \mathcal{O}(V^k) & \to & \tilde{\Omega}^1(V^k) \\ & \cup\!\shortparallel & & \cup\!\shortparallel \\ \mathfrak{F}^p: & \mathscr{F}^p & \to & \mathscr{F}^{p-1}\otimes\Omega^1(\Sigma)\cap\tilde{\Omega}^1(V^k). \end{array}$$

This filtration yields for every p a complex $\mathfrak{F}^p/\mathfrak{F}^{p+1}$ and we obtain a hypercohomology spectral sequence

$$E_1^{p,q} = \mathbb{H}^{p+q}(\bar{S}, \mathfrak{F}^p/\mathfrak{F}^{p+1}) \Rightarrow \mathbb{H}^{p+q}(\bar{S}, \Omega^\cdot(V^k)) = H^{p+q}(\bar{S}, V^k),$$

which converges to the cohomology $H^\cdot(\bar{S}, V^k)$ because $\Omega^\cdot(V^k)$ is a resolution of V^k.

(4.1) Proposition. *The terms $E_1^{p,q}$ are given by*

$$E_1^{p,q} = \mathbb{H}^{p+q}(\bar{S}, \mathfrak{F}^p/\mathfrak{F}^{p+1}) = \begin{cases} H^q(\bar{S}, \omega^{-k}) & \textit{for } p=0 \\ H^{k+q}(\bar{S}, \omega^k\otimes\Omega^1) & \textit{for } p=k+1 \\ 0 & \textit{for } p\neq 0, k+1. \end{cases}$$

Proof. Let $v_1=(e_1-ze_2)\sigma^{-1}$, $v_2=e_2\sigma^{-1}\in\mathcal{O}_{\bar{\mathbb{H}},s}$ where σ is chosen such that $s=\sigma\infty$ if s is a cusp and $\sigma=\mathrm{Id}$ if $s\in\mathbb{H}$. By (1.5) the germs $v_1^j\cdot v_2^{k-j}\in\mathcal{O}_{\bar{\mathbb{H}},s}(V^k)$ are Γ_s-invariant and form a basis of the $\mathcal{O}_{\bar{s}}$-module $\mathcal{O}(V^k)_{\bar{s}}$, $\bar{s}=\pi(s)$. Moreover we have

$$\mathscr{F}^p_{\bar{s}}=\bigoplus_{j=p}^{k}\mathcal{O}_{\bar{s}}v_1^j\cdot v_2^{k-j}.$$

The homomorphism $\mathcal{O}_{\bar{\mathbb{H}}}(V^k)\to\omega_{\bar{\mathbb{H}}}^{-k}$ given by $(e_1-ze_2)^j\cdot e_2^{k-j}\to 0$ for $j<k$, $e_2^k\to(e_1-ze_2)^{-k}$ is Γ-equivariant and has kernel $\omega_{\bar{\mathbb{H}}}\cdot\mathcal{O}_{\bar{\mathbb{H}}}(V^{k-1})$. Taking fixed sheaves we get a homomorphism $\mathcal{O}(V^k)\to\omega^{-k}$ with kernel $\omega\cdot\mathcal{O}(V^{k-1})$ which is surjective, since at every point $\bar{s}\in\bar{S}$ the germ v_2^k is mapped onto the germ v_1^{-k}. Therefore we have $\mathscr{F}^0/\mathscr{F}^1\cong\omega^{-k}$. This means that $\mathfrak{F}^0/\mathfrak{F}^1$ is the complex $\omega^{-k}\to 0$, and therefore

$$E_1^{0,q}=\mathbb{H}^q(\bar{S},\mathfrak{F}^0/\mathfrak{F}^1)=H^q(\bar{S},\omega^{-k}).$$

We next show that

$$(\mathscr{F}^k\otimes\Omega^1(\Sigma))\cap\tilde{\Omega}^1(V^k)=\omega^k\otimes\Omega^1.$$

This equation holds trivially on S because $\Omega^1(\Sigma)|_S=\Omega^1_S$. Let $\bar{s}=\pi(\sigma\infty)$ be a cusp. In order to compute the stalk

$$\tilde{\Omega}(V^k)_{\bar{s}}=\nabla(\mathcal{O}(V^k)_{\bar{s}})=(d\otimes 1)(\mathcal{O}_{\bar{\mathbb{H}}}(V^k)_s^{\Gamma_s})$$

we observe that $\nabla v_1=-v_2d(\sigma^{-1}z)$, $\nabla v_2=0$. For

$$v=\sum_{j=0}^{k}f_jv_1^j\cdot v_2^{k-j}\in\mathcal{O}(V^k)_{\bar{s}}$$

we thus have

$$\begin{aligned}\nabla v&=\sum_{j=0}^{k}v_1^j\cdot v_2^{k-j}\otimes df_j-\sum_{j=0}^{k}jf_jv_1^{j-1}\cdot v_2^{k-j+1}\otimes d(\sigma^{-1}z)\\&=\sum_{j=0}^{k-1}v_1^j\cdot v_2^{k-j}\otimes(df_j-(j+1)f_{j+1}d(\sigma^{-1}z))+v_1^k\otimes df_k.\end{aligned}$$

An element $v_1^k\otimes\alpha\in(\mathscr{F}^k\otimes\Omega^1(\Sigma))_{\bar{s}}=\mathcal{O}_{\bar{s}}v_1^k\otimes\Omega^1(\Sigma)_{\bar{s}}$ belongs therefore to $\tilde{\Omega}^1(V^k)_{\bar{s}}$ iff $v_1^k\otimes\alpha=v_1^k\otimes df$, where $f\in\mathcal{O}_{\bar{s}}$, and hence iff $v_1^k\otimes\alpha\in(\omega^k\otimes\Omega^1)_{\bar{s}}$. This gives the result that $\mathfrak{F}^{k+1}/\mathfrak{F}^{k+2}=\mathfrak{F}^{k+1}$ is the complex $0\to\omega^k\otimes\Omega^1$, so that

$$E_1^{k+1,q}=\mathbb{H}^{k+1+q}(\bar{S},\mathfrak{F}^{k+1})=H^{k+q}(\bar{S},\omega^k\otimes\Omega^1).$$

We now show that the complex $\mathfrak{F}^p/\mathfrak{F}^{p+1}$ is acyclic for $p\neq 0$, $k+1$. For each point $\bar{s}\in\bar{S}$ we have

$$\mathscr{F}^p_{\bar{s}}/\mathscr{F}^{p+1}_{\bar{s}}=\mathcal{O}_{\bar{s}}v_1^p\cdot v_2^{k-p}\bmod\mathscr{F}^{p+1}_{\bar{s}}$$

and by the above computation

$$\begin{aligned}\nabla(fv_1^p\cdot v_2^{k-p})&=v_1^p\cdot v_2^{k-p}\otimes df-pfv_1^{p-1}\cdot v_2^{k-p+1}\otimes d(\sigma^{-1}z)\\&\equiv-pfv_1^{p-1}\cdot v_2^{k-p+1}\otimes d(\sigma^{-1}z)\bmod(\mathscr{F}^p_{\bar{s}}\otimes\Omega^1(\Sigma)_{\bar{s}})\cap\tilde{\Omega}^1(V^k)_{\bar{s}}.\end{aligned}$$

This shows that the map

$$\mathcal{O}_{\bar{s}} v_1^p \cdot v_2^{k-p} \bmod \mathcal{F}_{\bar{s}}^{p+1} \xrightarrow{\nabla} \mathcal{O}_{\bar{s}} v_1^{p-1} \cdot v_2^{k-p+1} \otimes \Omega^1(\Sigma)_{\bar{s}} \bmod (\mathcal{F}_{\bar{s}}^p \otimes \Omega^1(\Sigma)_{\bar{s}}) \cap \tilde{\Omega}^1(V^k)_{\bar{s}}$$

is an isomorphism for $p=1, \ldots, k$. Therefore $\mathfrak{F}^p/\mathfrak{F}^{p+1}$ is acyclic and

$$E_1^{p,q} = \mathbb{H}^{p+q}(\bar{S}, \mathfrak{F}^p/\mathfrak{F}^{p+1}) = 0$$

for $p \neq 0, k+1$.

(4.2) Theorem. (i) *The spectral sequence* $\mathbb{H}^{p+q}(\bar{S}, \mathfrak{F}^p/\mathfrak{F}^{p+1}) \Rightarrow H^{p+q}(\bar{S}, V^k)$ *degenerates and yields a filtration*

$$H^1(\bar{S}, V^k) = F^0 \supseteqq F^1 \supseteqq \ldots \supseteqq F^{k+1} \supseteqq 0,$$

where $F^0/F^1 = H^1(\bar{S}, \omega^{-k})$ *and* $F^1 = \ldots = F^{k+1} = H^0(\bar{S}, \omega^k \otimes \Omega^1)$.

(ii) $H^i(\bar{S}, V^k) = 0$ *for* $i \neq 1$ *if* $k \neq 0$.

Proof. We have to show that all the differentials

$$d_r^{p,q}: E_r^{p,q} \to E_r^{p+r, q-r+1} \qquad (r \geqq 1)$$

are zero. We first assume $k>0$. Then $H^0(\bar{S}, \omega^{-k}) = 0$, because there are no automorphic forms of negative weight, and $H^q(\bar{S}, \omega^{-k}) = 0$ for $q>1$, since this group is dual to $H^{1-q}(\bar{S}, \omega^k \otimes \Omega^1)$ by Serre's duality theorem. Therefore $E_1^{0,q} = 0$ for $q \neq 1$. Since $E_1^{k+1,q} = H^{k+q}(\bar{S}, \omega^k \otimes \Omega^1)$ is also zero for $k+q \neq 0$ (again by Serre duality), we see that the only terms $E_1^{p,q}$ different from zero are $E_1^{0,1}$ and $E_1^{k+1,-k}$. This implies $d_r^{p,q} = 0$ for all $r \geqq 1$. If $k=0$, then $d = d_1^{0,0}$ sits in the exact sequence

$$0 \to H^0(\bar{S}, \mathbb{C}) \to H^0(\bar{S}, \mathcal{O}) \xrightarrow{d} H^0(\bar{S}, \Omega^1)$$

associated to the exact sequence $0 \to \mathbb{C} \to \mathcal{O} \to \Omega^1 \to 0$. Since $H^0(\bar{S}, \mathbb{C}) = H^0(\bar{S}, \mathcal{O}) = \mathbb{C}$ we have $d=0$ also in this case. Thus the spectral sequence degenerates, i.e., $E_1^{p,q} = E_\infty^{p,q}$ and therefore

$$F^p/F^{p+1} = E_\infty^{p,1-p} = E_1^{p,1-p}.$$

From (4.1) follows

$$F^0/F^1 = H^1(\bar{S}, \omega^{-k}) \quad \text{and} \quad F^1 = \ldots = F^{k+1} = E_1^{k+1,-k} = H^0(\bar{S}, \omega^k \otimes \Omega^1).$$

(4.3) Corollary. $\dim H^1(\bar{S}, V^k) = 2 \dim S_{k+2}(\Gamma)$.

Proof. From the theorem we obtain the exact sequence

$$0 \to H^0(\bar{S}, \omega^k \otimes \Omega^1) \to H^1(\bar{S}, V^k) \to H^1(\bar{S}, \omega^{-k}) \to 0,$$

in which the last group is dual to the first one by Serre duality.

5. The Polarized Hodge Decomposition of $H^1(\bar{S}, V^k)$

We are going to show in this paragraph that the filtration of $H^1(\bar{S}, V^k)$ obtained by (4.2) is in fact a Hodge structure of $H^1(\bar{S}, V^k)$.

We consider the sheaves $\mathscr{E}^i$, $\mathscr{E}^i(V^k)$ on $\bar{S}$, which we have defined in Sect. 3, with the connection $\nabla : \mathscr{E}^i(V^k) \xrightarrow{\wedge} \mathscr{E}^{i+1}(V^k)$ and the product $\mathscr{E}^i(V^k) \times \mathscr{E}^j(V^k) \longrightarrow \mathscr{E}^{i+j}$. We denote by $\tilde{\mathscr{E}}^1(V^k)$ the image of $\mathscr{E}(V^k)$ under the map $\nabla : \mathscr{E}(V^k) \to \mathscr{E}^1(V^k)$. In the same way as Proposition (3.1) we obtain the

(5.1) Proposition. *The sequence* $0 \to V^k \to \mathscr{E}(V^k) \to \tilde{\mathscr{E}}^1(V^k) \to 0$ *is exact.*

The sheaves $\omega^k \otimes \Omega^1 \subseteq \Omega^1(V^k)$ and $\overline{\omega^k \otimes \Omega^1} \subseteq \overline{\Omega^1(V^k)}$ are subsheaves of $\tilde{\mathscr{E}}^1(V^k)$ and we obtain from the exact sequence (5.1) a homomorphism

$$H^0(\bar{S}, \omega^k \otimes \Omega^1) \oplus H^0(\bar{S}, \overline{\omega^k \otimes \Omega^1}) \subseteq H^0(\bar{S}, \tilde{\mathscr{E}}^1(V^k)) \xrightarrow{\delta} H^1(\bar{S}, V^k).$$

We know already by (4.3) that the direct sum and $H^1(\bar{S}, V^k)$ have the same dimension and we now prove the

(5.2) Theorem. (i) *For* $H^1(\bar{S}, V^k)$ *we have the Hodge decomposition of type* $(k+1, 0), (0, k+1)$

$$H^1(\bar{S}, V^k) \xleftarrow[\sim]{\delta} H^0(\bar{S}, \omega^k \otimes \Omega^1) \oplus \overline{H^0(\bar{S}, \omega^k \otimes \Omega^1)}.$$

(ii) *The polarization on* $\mathscr{E}(V^k)$ *induces a polarization of this Hodge structure, which agrees with the Petersson scalar product on the vector space on the right.*

Proof. Let $\varphi \in S_{k+2} = H^0(\bar{S}, \omega^k \otimes \Omega^1)$ and $\psi = \bar{S}_{k+2}$. Considering φ and ψ as sections over S we have

$$\varphi \in H^0(S, \omega_S^k \otimes \Omega_S^1) = H^0(\mathbb{H}, \omega_{\mathbb{H}}^k \otimes \Omega_{\mathbb{H}}^1)^\Gamma$$

and we can write

$$\varphi = f(e_1 - z e_2)^k \otimes dz \quad \text{and} \quad \psi = \bar{g}(e_1 - \bar{z} e_2)^k \otimes d\bar{z},$$

where $f, g \in \mathcal{O}_{\mathbb{H}}(\mathbb{H})$ are cusp forms of weight $k+2$. We now form the product $\varphi \wedge \psi$ defined in Sect. 3. A straightforward calculation shows that

$$\varphi \wedge \psi = (-2i)^k f \cdot \bar{g} \operatorname{Im}(z)^k dz \wedge d\bar{z}.$$

It is very well known (and elementary to prove) that the integral $\int_S \varphi \wedge \psi$ is convergent, and we get a non-degenerate bilinear form on $S_{k+2} \oplus \bar{S}_{k+2}$ by $(\varphi, \psi) := \int_S \varphi \wedge \psi$.

In order to prove (i) we have only to show that δ is injective on $S_{k+2} \oplus \bar{S}_{k+2}$, i.e., that

$$\nabla H^0(\bar{S}, \mathscr{E}(V^k)) \cap (S_{k+2} \oplus \bar{S}_{k+2}) = (0).$$

This is now a computation of well known type. Let $h \in H^0(\bar{S}, \mathscr{E}(V^k))$ be such that $\nabla h \in S_{k+2} \oplus \bar{S}_{k+2}$, and let $\varphi = f(e_1 - z e_2)^k \otimes dz$ be an element in S_{k+2}. Since by (3.5) $\nabla h \wedge \varphi = d(h \wedge \varphi) - h \wedge \nabla \varphi = d(h \wedge \varphi)$, we have

$$\int_S \nabla h \wedge \varphi = \int_S d(h \wedge \varphi),$$

where $d(h \wedge \varphi) \in H^0(S, \mathscr{E}_S^2)$. Let $S_N = \bar{S} - \bigcup_{\bar{s} \in \pi(\Sigma)} U_{N,\bar{s}}$, where $U_{N,\bar{s}} = \pi(U_{N,s})$ is the neighborhood of the cusp $\bar{s} = \pi(\sigma\infty)$ given by $U_{N,s} = \sigma \mathbb{H}_N \cup \{s\}$. Since

$$\int_S \nabla h \wedge \varphi = \lim_{N \to \infty} \int_{S_N} d(h \wedge \varphi),$$

we can now use Stokes' theorem. Thus

$$\int_{S_N} d(h \wedge \varphi) = \int_{\partial S_N} h \wedge \varphi = \sum_{\bar{s}} \int_{\partial U_{N,\bar{s}}} h \wedge \varphi .$$

We rewrite the last integrals in the variable of $\mathbb{H}_N$. In the neighborhood $U_{N,s}$ of $s = \sigma\infty$ we can write $h = \sum_{j=0}^{k} h_j v_1^j \cdot v_2^{k-j}$, where $v_1 = (e_1 - z e_2)\sigma^{-1}$, $v_2 = e_2 \sigma^{-1}$ and $h_j \in \mathscr{E}_{\bar{\mathbb{H}}}(U_{N,s})^{\Gamma_s}$. The map $\sigma: \mathbb{H}_N \to U_{N,s} - \{s\}$ transforms φ into

$$\varphi\sigma = f^\sigma(z)((e_1 - z e_2)\sigma)^k \otimes d(\sigma z) = f \circ [\sigma]_{k+2}(z)(e_1 - z e_2)^k \otimes dz,$$

and therefore $h \wedge \varphi$ into

$$\begin{aligned}(h \wedge \varphi)\sigma &= \left(\sum_{j=0}^{k} h_j^\sigma(z)(e_1 - z e_2)^j \cdot e_2^{k-j} \right) \wedge (f \circ [\sigma]_{k+2}(z)(e_1 - z e_2)^k \otimes dz) \\ &= h_0^\sigma(z) \cdot f \circ [\sigma]_{k+2}(z) dz .\end{aligned}$$

Setting $z = x + iy$ we thus have

$$\int_{\partial U_{N,s}} h \wedge \varphi = \int_{\substack{y=N \\ |x| < h_s}} h_0^\sigma(z) f \circ [\sigma]_{k+2}(z) dz = \int_{|x| < h_s} h_0^\sigma(x+iN) f \circ [\sigma]_{k+2}(x+iN) dx.$$

Since f is a cusp form of weight $k+2$, we can write for large N

$$f \circ [\sigma]_{k+2}(z) = q g(q)$$

where $q = e^{2\pi i z/h_s}$, $z \in \mathbb{H}_N$, and g is a holomorphic function in q. Since g remains bounded on this neighborhood, and since $h_0 \in (\mathscr{E}_{\bar{\mathbb{H}},s})^{\Gamma_s}$, we can find a constant c and an integer m such that

$$|h_0^\sigma(z) \cdot f \circ [\sigma]_{k+2}(z)| < c(\operatorname{Im} z)^m \cdot |q| = c(\operatorname{Im} z)^m e^{-2\pi \operatorname{Im}(z)/h_s}$$

for $z \in \mathbb{H}_N$ and N sufficiently large. We get

$$\begin{aligned}\left| \int_{\partial U_{N,s}} h \wedge \varphi \right| &\leqq \int_{|x| < h_s} |h_0^\sigma(x+iN) \cdot f \circ [\sigma]_{k+2}(x+iN)| dx \\ &\leqq c \int_{|x| < h_s} N^m e^{-2\pi N/h_s} dx \leqq 2 h_s c N^m e^{-2\pi N/h_s},\end{aligned}$$

which tends to zero as $N \to \infty$. We have proved that

$$\int_S \nabla h \wedge \varphi = 0$$

for every $\varphi \in S_{k+2}$. Analogously one sees that $\int_S \nabla h \wedge \psi = 0$ for every $\psi \in \bar{S}_{k+2}$. That is, the element ∇h is orthogonal to $S_{k+2} \oplus \bar{S}_{k+2}$ under the non-degenerate pairing on this space. This proves that $\nabla h = 0$.

In order to prove (ii) we need only to observe that, if $\varphi, \psi \in S_{k+2}$, then $\langle \varphi, \psi \rangle := i^{k+1}(\varphi, \bar{\psi})$ is the Petersson scalar product of the two cusp forms.

6. The General Shimura Isomorphism

Throughout the last three paragraphs we have assumed that Γ has no elements of finite order and no irregular cusps. The reason for this was, that under this assumption the $\mathcal{O}_{\bar{\mathbb{H}}}$-modules which we have considered were stalkwise $\Gamma-\mathcal{O}_{\bar{\mathbb{H}}}$-free, so that the functor $F \to F^\Gamma$ preserved the tensor product. We can now drop this assumption because a Fuchsian group of the first kind contains always a normal subgroup Γ' of finite index without elements of finite order and without irregular cusps (cf. [8]).

Let $\chi: \Gamma \to \mathbb{C}^*$ be an abelian character. A cusp form of weight k *with character* χ is defined by replacing the condition $f \circ [\gamma]_k = f$ in the usual definition by $f \circ [\gamma]_k = \chi(\gamma) f$. We denote by $S_k(\Gamma, \chi)$ the space of cusp forms of weight k with character χ.

If $F_{\bar{\mathbb{H}}}$ is a Γ-sheaf on $\bar{\mathbb{H}}$, then we denote by $F_{\bar{\mathbb{H}}}(\chi)$ the Γ-sheaf which as a sheaf is $F_{\bar{\mathbb{H}}}$, but on which the action $f \mapsto f^\gamma$ of $\gamma \in \Gamma$ is changed into $f \mapsto \chi(\gamma) \cdot f^\gamma$. We set

$$F(\chi) = F_{\bar{\mathbb{H}}}(\chi)^\Gamma .$$

(6.1) Theorem (The Shimura isomorphism). *Let Γ be a Fuchsian group of the first kind, $\chi: \Gamma \to \mathbb{C}^*$ an abelian character and $k \geqq 0$. Then we have a canonical isomorphism*

$$S_{k+2}(\Gamma, \chi) \oplus \overline{S_{k+2}(\Gamma, \chi)} \cong H^1(\bar{S}, V^k(\chi)) .$$

Proof. Let $\Gamma' \subseteqq \Gamma$ be a normal subgroup of finite index without elements of finite order and without irregular cusps on which χ is trivial. Let $\bar{S}' = \Gamma' \backslash \bar{\mathbb{H}}$. Then $\bar{S}' \to \bar{S}$ is a ramified Galois covering with Galois group $\Delta = \Gamma/\Gamma'$. The sheaves

$$\omega^k(\chi)_{\bar{S}'} \otimes \Omega^1_{\bar{S}'} = (\omega^k_{\bar{\mathbb{H}}}(\chi) \otimes \Omega^1_{\bar{\mathbb{H}}}(-\Sigma))^{\Gamma'} \quad \text{and} \quad V^k(\chi)_{\bar{S}'} = V^k_{\bar{\mathbb{H}}}(\chi)^{\Gamma'}$$

are Δ-sheaves on $\bar{S}'$. By (5.2) we have an isomorphism

$$H^0(\bar{S}', \omega^k(\chi)_{\bar{S}'} \otimes \Omega^1_{\bar{S}'})^\Delta \oplus \overline{H^0(\bar{S}', \omega^k(\chi)_{\bar{S}'} \otimes \Omega^1_{\bar{S}'})^\Delta} \xrightarrow[\sim]{\delta'} H^1(\bar{S}', V^k(\chi)_{\bar{S}'})^\Delta .$$

Furthermore we have

$$S_{k+2}(\Gamma, \chi) \cong H^0(\bar{S}, (\omega^k_{\bar{\mathbb{H}}}(\chi) \otimes \Omega^1_{\bar{\mathbb{H}}}(-\Sigma))^\Gamma) = H^0(\bar{S}', \omega^k(\chi)_{\bar{S}'} \otimes \Omega^1_{\bar{S}'})^\Delta .$$

The first isomorphism was proved in Sect. 1 for $\chi = 1$, but the same proof holds for the general case. Hence we have only to show that

$$H^1(\bar{S}, V^k(\chi)) = H^1(\bar{S}', V^k(\chi)_{\bar{S}'})^\Delta .$$

Since Δ is finite, the functor $F \mapsto F^\Delta$ is exact on sheaves of $\mathbb{C}$-vector spaces. Therefore we have a spectral sequence (cf. [4], Proposition 5.2.3)

$$E_2^{i,j} = H^i(\Delta, H^j(\bar{S}', V^k(\chi)_{\bar{S}'})) \Rightarrow H^{i+j}(\bar{S}, V^k(\chi)) .$$

The finiteness of Δ implies $E_2^{i,j} = 0$ for $i \neq 0$, and thus the spectral sequence yields the required isomorphism.

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