# The Logarithm of the Dedekind $\eta$-Function

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*Dedicated to Friedrich Hirzebruch*

## 1. Introduction

The Dedekind $\eta$-function defined in the upper-half plane $H$ by

(1.1)
$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

is one of the most famous and well-studied functions in mathematics, particularly in relation to elliptic curves and modular forms. Its 24th power $\eta^{24}$ is a modular form of weight 12 for the group $SL(2, Z)$ or equivalently the expression

$$\eta(\tau)^{24} (d\tau)^6$$

is invariant under the action of $SL(2, Z)$. This gives the transformation properties of $\log \eta(\tau)$ under $SL(2, Z)$, up to the addition of an integer multiple of $\pi i/12$. In [12] Dedekind investigated this integer ambiguity and its dependence on a general element

(1.2)
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $SL(2, Z)$, expressing the answer in terms of the Dedekind sums (for $c \neq 0$)

$$s(a, c) = \sum_{k=1}^{c-1} \left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{ak}{c}\right)\right),$$

where for any real number $x$

$$((x)) = \begin{cases} 0 & \text{for integral } x \\ x - [x] - \frac{1}{2} & \text{otherwise}. \end{cases}$$

Dedekind's formula (for $c \neq 0$) is

$$\log \eta \left( \frac{a\tau + b}{c\tau + d} \right) = \log \eta(\tau) + \tfrac{1}{4}\log\{-(c\tau + d)^2\}$$

(1.3)
$$+ \pi i \left( \frac{a+d}{12c} - \operatorname{sign} c \cdot s(a, c) \right).$$

Here $\log\{-(c\tau + d)^2\}$ is that branch whose imaginary part has absolute value less than $\pi$, while $\log \eta$ is any fixed branch defined on $H$: for example that branch for which

$$\log \eta(\tau) - \frac{\pi i \tau}{12} \to 0 \quad \text{as} \quad \operatorname{Im} \tau \to \infty.$$

For $c = 0$, $A$ acts by an integer translation on $\tau$ and the effect on $\log \eta(\tau)$ is trivially read off from the definition of $\eta(\tau)$.

It should be noted that the problem which Dedekind posed and solved with the above formula is essentially a topological one. Using the cohomological properties of $SL(2, Z)$ it is not hard to show that, up to equivariant homotopy, there is a unique section

$$s = f(\tau)\, d\tau^6$$

of $T^{-6}$ ($T$ being the tangent bundle of $H$) which is nowhere zero and $SL(2, Z)$-invariant. The transformation properties of $\log f$ under $SL(2, Z)$ are then the same as those of $\log \eta^{24}$. Thus the analyticity of $\eta$ is not directly involved in Dedekind's formula.

In the past hundred years these ideas have been pursued in different directions by many people including Rademacher [31], C. Meyer [23, 24], Siegel [34], Hirzebruch [16, 17], W. Meyer [25], Atiyah et al. [6], Atiyah et al. [5], and Müller [27]. Recently ideas emerging from the physics of gauge theories and developed by Witten [35], Quillen [29, 30], and Bismut and Freed [9, 10] have cast new light on these problems. It seems therefore timely to attempt to survey the whole development of the theory of $\log \eta$, putting results in their natural order and in the appropriate general context. This is the aim of the present paper, in which the emphasis will be strongly geometrical. In a sense we shall show that the latest ideas from physics provide the key to a proper understanding of Dedekind's original results.

From a mathematical point of view the general context is that of the index theory of elliptic operators as applied to the signature of manifolds. We shall use many versions and generalizations of the original Hirzebruch signature theorem, involving families of operators, manifolds with boundary and group actions. All are involved in the story.

We shall now review the history of the subject in a little more detail. Rademacher [31] put the emphasis on an integer-valued class function $\psi$, defined on $SL(2, Z)$ by the formula

$$\psi(A) = \frac{a+d}{c} - 12 \operatorname{sign} c \cdot s(a \cdot c) - 3 \operatorname{sign}(c(a+d)) \quad \text{for} \quad c \neq 0$$

(1.4)
$$= \frac{b}{d} \quad \text{for} \quad c = 0$$

which is closely related to the Dedekind formula. In particular Rademacher gave a simple formula for $\psi(A)$ when $A$ is expressed explicitly in terms of the standard generators of SL $(2, Z)$. Meyer [24] and Siegel [34], following methods introduced by Hecke, computed the values $L_A(0)$ of certain $L$-functions $L_A(s)$ attached to real quadratic fields, in terms of the modular behaviour of $\log \eta(\tau)$ and hence in terms of Dedekind sums or the Rademacher invariant $\psi$.

Hirzebruch [16] made an extensive study of the "cusps" of Hilbert modular surfaces for real quadratic fields and found in particular an explicit resolution for them. He also attached a "singular defect" $\delta(A)$ to each such cusp, this being the correction term due to a cusp in the general Hirzebruch formula for the signature of a 4-manifold. Using his explicit resolution he was able to compute $\delta(A)$, finding the formula

$$\delta(A) = -\tfrac{1}{3}\psi(A),$$

where $\psi(A)$ appeared explicitly through the Rademacher formula in terms of the generators.

Comparing this formula for $\delta(A)$ with Meyer's evaluation of $L_A(0)$ Hirzebruch found that $\delta(A) = L_A(0)$. He then conjectured that this formula might continue to hold for totally real number fields of any degree, since both $\delta(A)$ and $L_A(0)$ could be defined quite generally. Hirzebruch also drew attention to another formula of Rademacher, giving an alternative expression for Dedekind sums, namely

$$(1.5) \qquad\qquad 4\,|c|\,s(a, c) = \sum_{k=1}^{|c|-1} \cot \frac{\pi k}{c} \cot \frac{\pi a k}{c}$$

(this formula is obtained by taking "Fourier transforms" over the finite abelian group of integers modulo $c$). This new expression appears in the equivariant signature theorem [2, 3] as the "signature defect" due to a cyclic singularity (i.e arising from an isolated fixed point for the action of a cyclic group). It seemed somewhat mysterious that the Dedekind sums should appear in connection with modular forms and also in topology.

Motivated in part by Hirzebruch's conjecture on totally real fields Atiyah, Patodi and Singer introduced in [6] a differential geometric $L$-function defined in terms of eigenvalues $\lambda$ of a certain first order differential operator (on odd-dimensional manifolds) by the series

$$\sum_{\lambda \neq 0} (\operatorname{sign} \lambda)\,|\lambda|^{-s}.$$

To avoid confusion with the Hirzebruch $L$-polynomials, which appear in the general formula for the signature of a $4k$-dimensional manifold, this series was not denoted by $L(s)$ but by $\eta(s)$ instead [unsigned series were traditionally denoted by $\zeta(s)$]. This will now lead to an unfortunate but by now inescapable clash of notation, since our main concern here is the Dedekind $\eta$-function $\eta(\tau)$! The main result of [6] was to identify $\eta(0)$, computed for a manifold $Y$, as the boundary correction to the Hirzebruch signature formula for a manifold $X$ with boundary $Y$. This was in a differential-geometric context, characteristic classes being represented by the appropriate Pontrjagin forms.

In [5] the results of [6] were applied to manifolds which occur as boundaries of neighbourhoods of the cusps of Hilbert modular surfaces. In this way the Hirzebruch conjecture was established, both $\delta(A)$ and $L_4(0)$ being identified with $\eta_Y(0)$. An independent but related proof was given by Müller in [27].

Recently Witten [35] has argued that "global anomalies" in gauge theories can be expressed in terms of the invariants $\eta(0)$ of [6]. Witten's arguments have been put on a rigorous footing and established quite generally by Bismut and Freed [9, 10], on the basis of earlier ideas of Quillen [29, 30]. The situation is roughly as follows.

If $M$ is an even-dimensional compact Riemannian manifold with a spin-structure one can define the Dirac operator $D$ acting from positive to negative spinors and the index-theorem [3] gives a topological formula for index $D$ in terms of Pontrjagin classes. More generally one can consider the Dirac-type operators, including the operator whose index is the signature, and there are corresponding formulae. The index theorem for families of elliptic operators [4] generalizes this situation to include "parameters", i.e. for a fibre bundle $Z \to X$ with fibre $M$. The index theorem now becomes a formula in $K(X)$ or, on passing to cohomology, a formula in $H^*(X)$. This theorem can then be "localized" to a formula in terms of differential forms on $X$, the main point being to define an appropriate local form of the index of families. This programme has been carried out by Bismut [8] following ideas of Quillen [30] and motivated by the ideas of local anomalies in the physics of gauge theories. The 2-dimensional component of the index of a family is of special interest as the first Chern class of the "determinant line-bundle" $\mathscr{L}$ over $X$.

Physicists need to introduce determinants of operators, and there is no difficulty in "regularizing" such determinants for positive self-adjoint elliptic operators (e.g. of Laplace type). The most elegant way is the $\zeta$-function regularization introduced by Ray and Singer [32], in which one puts

$$\zeta(s) = \Sigma \lambda^{-s}, \ \det = \exp(-\zeta'(0)).$$

For an operator $D$ of Dirac type this procedure, applied to $D^*D$ enables one to define $|\det D|$, but since $D$ is naturally complex there remains a problem of phase. It turns out that there is an essential difficulty (anomaly) in attempting to define this phase in an invariant way. However one can define its logarithmic derivative and this endows the line-bundle $\mathscr{L}$ above with a unitary *connection*. The curvature of this connection is then a measure of the (local) anomaly.

Even if $\mathscr{L}$ is flat, so that the local anomaly is zero, there may still be "global anomalies" arising from non-trivial monodromy of $\mathscr{L}$ round non-contractible closed loops in $X$. To each such loop one gets an odd-dimensional manifold $W$ fibred over the circle with fibre $M$. Witten has argued that the global anomaly is given essentially by the "adiabatic limit" of $\exp \pi i \eta_W(0)$, the limit in which the metrics on $M$ are shrunk by a factor $\varepsilon$ with $\varepsilon \to 0$. Bismut and Freed [9] using the earlier work of Bismut [8] have proved this result even when $\mathscr{L}$ is not flat, although in that case both sides depend on the actual loop (not just on its homotopy class).

All this discussion applies to any operator of Dirac-type. Note also that adding an even integer to $\eta_W(0)$ leaves the monodromy of $\mathscr{L}$ unaffected. This is related to the fact that, as an eigenvalue changes sign, $\eta(0)$ jumps by 2, so $\eta(0) \bmod 2Z$ is a continuous function of the parameters. In general we cannot control the

appearance of zero-eigenvalues, leading to such jumps. However, for the special case of the signature operator the zero-eigenspace represents harmonic forms so that, by Hodge theory, it is determined by the cohomology and cannot jump. Thus, for the signature, $\eta(0)$ is continuous and enables us to define a natural logarithm for the monodromy of $\mathscr{L}$. It is this extra refinement of the general theory which will specially concern us.

The relevance of these general ideas involving anomalies and index theory to our particular situation is fairly clear. From its very definition as an infinite product it is clear that the Dedekind $\eta$-function should be some regularized determinant, the manifold $M$ of the general theory will be a 2-torus with flat metric and SL $(2, Z)$ will appear as the group of components of the diffeomorphisms of $M$ (preserving orientation).

The topological properties of the signature are also important for us. It is well-known that the signature is multiplicative for products of manifolds. More generally [11] the same is true for fibrations provided the fundamental group of the base acts trivially on the cohomology of the fibre. However, multiplicativity fails for general fibrations and this can be understood in terms of the index theorem for families. This failure already takes place in the lowest dimension when base and fibre are both Riemann surfaces, as explained in [1] and [19].

The more general situation of the signature for a local coefficient system (not necessarily arising from a fibration) was investigated by Lusztig [21], while Meyer [25] studied surface fibrations over a surface with boundary. In particular, when the fibre $M$ is a torus, Meyer introduced a rational-valued class-function $\phi$ on SL $(2, Z)$ with the property that $\phi(A)$ measures the signature defect (in the category of torus-bundles) for the 3-manifold $W(A)$ associated to $A$: $W(A) \to S^1$ is the torus bundle with monodromy $A$. Meyer showed that his invariant $\phi$ was closely related to the Rademacher invariant $\psi$ and that, for hyperbolic elements $A$ of SL $(2, Z)$,

$$(1.6) \qquad\qquad \phi(A) = -\tfrac{1}{3}\psi(A).$$

Our presentation will start with this topological situation studied by Meyer and Lusztig. The main result is the signature theorem (2.13) for a local coefficient system on a surface with boundary. The theorem expresses the signature as the *relative* Chern class of a certain line-bundle $L$, with a trivialization $\sigma$ on the boundary. This trivialization is given by (and essentially equivalent to) a certain 2-cocycle for the group $G$ (indefinite unitary) of the local system. Equivalently $\sigma$ can be viewed as a section of a certain group extension $G_2$ of $G$ (with kernel $Z$). For semi-simple elements $A$ of $G$ we give an explicit computation of $\sigma(A)$. In particular when $A$ is a hyperbolic element of SL $(2, R)$ we show (2.15) that $\sigma = \sigma_0$, where $\sigma_0$ is the natural section defined by the one-parameter group through $\pm A$. The proofs in Sect. 2 use the theory of group extensions of Lie groups and are essentially geometric versions of Meyer's results. In Sect. 3 we give an alternative analytic approach based on the index of elliptic operators and generalizing Lusztig's proof.

In Sect. 4 we consider fibrations over a surface $X$ (with fibre $M$ of dimension $4k - 2$). The cohomology of the fibres then gives a local coefficient system of the type studied in Sect. 2 and Sect. 3. We describe the Quillen determinant line-bundle $\mathscr{L}$ and the results of Bismut and Freed [9] for the *signature operator*. The conclusion

is that $\mathcal{L}$, restricted to the boundary of $X$, has a canonical trivialization determined by the $\eta$-invariant, that $\mathcal{L} = L^*$ (where $L$ is the line-bundle of Sect. 2) and that the $\eta$-trivialization of $\mathcal{L}^2$ coincides with the (dual of the) $\sigma$-trivialization of $L^2$ if $\dim M = 2$, or more generally if $\mathrm{Hom}\,(\mathrm{Diff}^+(M), Z) = 0$. This can be viewed as the appropriate refinement, for the signature operator, of the general Bismut-Freed result. In the last part of Sect. 4 we recall various versions of the equivariant signature theorem and show in particular how Dedekind sums enter from isolated fixed-points. We derive formulae relating $\eta$-invariants for finite coverings with such fixed-point contributions. This material is essentially a summary of [7, Sect. 2].

In Sect. 5 we specialize to the case when the fibre $M$ is a 2-torus with flat (normalized) metric, so that the natural parameter space is the upper-half plane $H$ modulo $\mathrm{SL}(2, Z)$. The signature cocycle $\sigma$ of Sect. 2, for the group $\mathrm{SL}(2, Z)$, is now a coboundary and this leads to Meyer's invariant $\phi$:

$$-\sigma = \delta\phi \, .$$

Using the main result of Atiyah et al. [6] we prove (5.12) that $\phi(A) = \eta^0(A)$ where $\eta^0(A)$ is the *adiabatic* limit of $\eta_W(0)$ with $W = W(A)$ being the 3-manifold, fibred over $S^1$, associated to $A$. Moreover the results of Sect. 2 enable us to identify $\phi(A)$, for $A$ hyperbolic, with another topological invariant $\chi(A)$ which is essentially the quantity Dedekind studied. In fact if $\omega_A$ is the natural $A$-invariant differential (i.e. invariant under the corresponding one-parameter group) then

$$f(\tau) = \frac{\eta(\tau)^4 \, d\tau}{\omega_A}$$

is (up to 6-th roots of unity) $A$-invariant, and so we can define $\chi(A)$ by

$$\log f(A\tau) - \log f(\tau) = -\pi i \chi(A) \, .$$

Since it is easy to deal with the explicit term $\omega_A$ it is clear that determining $\chi(A)$ is effectively equivalent to Dedekind's problem.

We then move on to study Quillen's determinant line-bundle $\mathcal{L}$ and we prove, by several methods, that $\mathcal{L}$ is *flat* or, as physicists say, the local anomaly vanishes. Moreover

$$\omega = \eta(\tau)^4 \, d\tau$$

gives a covariant constant section of $\mathcal{L}$. If now $A \in \mathrm{SL}(2, Z)$ is hyperbolic and, if $H_A$ is the quotient of $H$ by the group generated by $A$, we can descend $\mathcal{L}$ to $H_A$. Since $\mathcal{L}$ is flat it will have a well-defined monodromy round the fundamental loop of $H_A$. Using the standard trivialization $\sigma_0$ of $\mathcal{L}$ (given by the one-parameter group through $\pm A$) we can take the logarithm of this monodromy. Dividing by $\pi i$ then defines an invariant $\mu(A)$. Topological considerations (or the use of $\omega$) then show that $\mu(A) = \chi(A)$.

A direct analytical computation of $\mu(A)$ on classical lines (as in [34]) shows that $\mu(A) = L_A(0)$ where $L_A(s)$ is the $L$-series defined in (5.49). Together with the previous results this then proves that, for hyperbolic $A$, we have the equalities:

$$L_A(0) = \mu(A) = \chi(A) = \phi(A) = \eta^0(A) \, .$$

By a simple argument, based on conformal invariance, we then prove that $\eta(A)$ $= \eta^0(A)$, provided we use the natural metric on $W(A)$ given by the $A$-invariant geodesic in $H$. We also show fairly directly that Hirzebruch's signature defect $\delta(A)$ coincides with Meyer's invariant $\phi(A)$.

These results, based on the Bismut-Freed theorem, give therefore a new analytical proof of the main result of [5] (for quadratic fields), namely the equalities

$$L_A(0) = \delta(A) = \eta(A).$$

In fact the analysis in [5] has common features with the Bismut-Freed approach, notably the use of the adiabatic limit. The main results of Sect. 5 are summarized in the final Theorem (5.60).

Having thus identified $\phi(A)$, by general methods, with various other invariants we proceed in Sect. 6 to the question of explicit computation. Again our approach will be based on geometrical methods, and in particular on the use of the fixed-point formula in the equivariant signature theorem. We use this first, in an obvious way, to compute $\phi(A)$ for elliptic elements $A$ (i.e. of finite order). We then move on to consider parabolic elements. These, together with the elliptic elements, occur naturally as the monodromy round exceptional fibres in algebraic families of elliptic curves. We prove a simple general formula (6.3) for $\phi(A)$ in terms of the structure of the associated exceptional fibre.

We then turn to the main case when $A$ is hyperbolic beginning with the simple case when $c = -1$, $a + d > 0$, $A$ being as usual

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Following Hirzebruch [16] we show that the 3-manifold $W(A)$ is in this case the boundary of the neighbourhood of a nodal rational curve $\Gamma$ embedded with normal degree $-(a + d)$. This leads, as in [16], to the simple formula (6.9) for the signature defect $\delta(A) = \phi(A)$.

The general case is then related to this special case by a covering argument and a further application of the fixed-point formula for the signature operator. The manifold $W(A)$ has $W(B)$ as a finite covering of order $|c|$, where (for $c < 0$)

$$(1.7) \qquad\qquad B = \begin{pmatrix} a & -bc \\ -1 & d \end{pmatrix}.$$

The difference

$$(1.8) \qquad\qquad \phi(A) - \frac{1}{|c|}\,\phi(B) = \eta(A) - \frac{1}{|c|}\,\eta(B)$$

is then an example of the invariant for finite coverings (independent of all metrics) described in Sect. 4. As explained in Sect. 4 it can be computed by using the manifold $Z(B)$ with boundary $W(B)$. The covering $W(B) \to W(A)$ extends to $Z(B)$ with a single singular or fixed-point. The Dedekind sum arising from this fixed-point then gives the difference (1.8). Together with the simple formula (6.3) for $\phi(B)$ this then leads to the general formula (6.14) for $\phi(A)$; in view of (1.6) this

essentially becomes the Rademacher formula (1.4). Moreover the equality $\phi(A)$ $= \chi(A)$ then leads (for hyperbolic $A$) to Dedekind's original formula (1.3).

For non-hyperbolic elements we can still (for $c \neq 0$) use the same covering method to compare $\phi(A)$ and $\phi(B)$, and hence derive Dedekind's formula (1.3) in general.

In earlier treatments Dedekind sums appear at the beginning of the story and are made the basis for subsequent developments. Here we have deliberately postponed them until the computational stage. The general theoretical ideas are best understood without reference to Dedekind sums. These are then easily seen to enter via fixed-point ideas related to finite coverings. In the introduction to [5], describing the contribution to the signature of Hilbert modular surfaces due to the various singularities, it was stated that elliptic fixed-points were easy to understand (via the $G$-signature theorem) but that cusps were much more subtle. Although the cusp story is indeed more subtle, and involves the ideas explained in the present paper, the computation of the cusp contribution to the signature is again related to the $G$-signature theorem in a natural way.

In [17] Hirzebruch notes the appearance of Dedekind sums in number theory (via the Dedekind $\eta$-function) and in topology (through the $G$-signature theorem). He asks whether there is perhaps some deep explanation for this fact. Hopefully this paper, following Hirzebruch's work on cusps, provides an answer by showing that the real connection between number theory and topology, in this context, hinges on fundamental ideas from the physics of gauge theories!

There are a number of generalizations and further problems which remain to be investigated. As explained in Sect. 3 a full generalization of [6] to deal with local coefficient systems would be desirable. Also there are obvious generalizations of many aspects of Sect. 5 in which we twist by a character of the torus. Such questions are treated classically in [34] and from a topological view-point in [26]. For surfaces of genus $\geq 2$ detailed formulae involving the Selberg zeta function might be interesting to derive. The computation of local anomalies in this case is of interest in string theory and is being actively investigated by many authors at the present time. The general identification of the $\sigma$ and $\eta$ trivializations of the Quillen line-bundle remains an open question in higher dimensions, as explained in Sect. 4. Moreover the question of providing some geometric interpretation of the results of [5, 27] for number fields of higher degree is rather intriguing. What should replace the monodromy of the determinant line-bundle when the base circle is replaced by a torus?

It will be clear by now that many of the ideas and problems investigated here have their origin in the work of Hirzebruch. I am also indebted to D. Quillen and D. Freed for their help in understanding the key role of the determinant line-bundle. Finally I should record the great stimulus provided in this whole area by the penetrating insight of E. Witten.

Because this paper attempts to present many points of view and integrate many previous results there are inevitably severe problems of notation and compatibility. In particular there are innumerable sign conventions and, while I have attempted to achieve consistency, it would be a minor miracle if I have completely succeeded.

Finally, a small comment on the normalization of the various invariants. In order to eliminate small numerical differences in the final theorem I have chosen to

normalize the various definitions appropriately. However, the notations remain consistent with the literature.

The material in this paper was presented in my Rademacher Lectures given at the University of Pennsylvania in January 1987.

## 2. The Signature of Local Coefficient Systems

In this section we shall review and extend results of Meyer [25] concerning the signature of local coefficient systems. We consider the following situation: a compact oriented surface $X$ with boundary $Y$ (consisting of disjoint circles $S_1, \ldots, S_r$) and a local coefficient system $E$ over $X$ of flat hermitian vector spaces. The hermitian form on $E$ is (possibly) indefinite and such a coefficient system corresponds to a homomorphism

$$\alpha: \quad \pi_1(X) \to U(p, q),$$

where $U(p, q)$ is the unitary group of the hermitian form $\sum_{i=1}^{p} |z_i|^2 - \sum_{j=1}^{q} |z_j|^2$. We then have a skew-hermitian form $A$ on $H^1(X, Y; E)$ induced by the cup-product and the hermitian form on $E$. The signature of the hermitian form $iA$ will be denoted by $\text{sign}(X, E)$ or $\text{sign}(X, \alpha)$.

Consider in particular the surfaces $X_n$ obtained by deleting $n$ discs from the 2-sphere. Since $\pi_1(X_n)$ is free on $(n-1)$ generators, a homomorphism $\alpha$: $\pi_1(X_n) \to U(p, q)$ is just given by $n-1$ elements $A_1, A_2, \ldots, A_{n-1}$ of $U(p, q)$ so that we can write

$$\text{sign}(X_n, \alpha) = \text{sign}(A_1, A_2, \ldots, A_{n-1})$$

indicating that we have a function of $(n-1)$ elements of $U(p, q)$. Note that it is a *symmetric* function of the $A_i$.

The additivity property of the usual signature extends to local coefficients. Decomposing $X_n$ as the connected sum of $X_3$ and $X_{n-1}$ leads to the formula

(2.1) $\quad \text{sign}(A_1, A_2) + \text{sign}(A_1 A_2, A_3, \ldots, A_{n-1}) = \text{sign}(A_1, A_2, \ldots, A_{n-1})$.

In particular taking $n = 4$ and using the symmetry we deduce

(2.2) $\quad \text{sign}(A_1, A_2) + \text{sign}(A_1 A_2, A_3) = \text{sign}(A_1, A_2 A_3) + \text{sign}(A_2, A_3)$.

This identity expresses the fact that $\text{sign}(A_1, A_2)$ *is a 2-cocycle for the group* $U(p, q)$. We shall call it the *signature cocycle* and its properties will be our main concern. Note that

(2.3) $\qquad \text{sign}(A, B) = 0 \quad$ if $\quad A = 1, \ B = 1 \quad$ or $\quad AB = 1$,

(2.4) $\qquad\qquad \text{sign}(A^{-1}, B^{-1}) = -\text{sign}(A, B)$.

*Note.* The signature cocycle was introduced along these lines by Meyer [25], though he considered only the symplectic groups $Sp(2p, R)$, and the associated real quadratic form on $H^1(X, Y; E)$. Of course $Sp(2p, R) \subset U(p, p)$ and our cocycle on $U(p, p)$ restricts to Meyer's cocycle on $Sp(2p, R)$. The signature for hermitian local

coefficient systems was introduced quite generally by Lusztig [21] and we shall return shortly to his results.

The signature cocycle is not continuous in its arguments but it is measurable. It therefore defines a signature cohomology class $\text{sign} \in H^2(U(p, q), Z)$. By general theory [18; 22] this cohomology group classifies central extensions of $U(p, q)$ by $Z$ and these in turn correspond to homomorphisms

$$\pi_1(U(p, q)) \to Z.$$

Our first task is to identify the *signature homomorphism* arising from the signature class. Consider as basic case $p = 1$, $q = 0$ so that $U(p, q) = U(1)$ and $\pi_1(U(1)) \cong Z$. The signature homomorphism $Z \to Z$ is therefore determined by some integer $M$ which we shall compute later. Switching to $p = 0$, $q = 1$ (i.e. changing the sign of the hermitian form) changes $M$ to $-M$. The naturality of the signature with respect to changes in $p$ and $q$ then shows that for $pq \neq 0$

$$\text{sign}: \pi_1(U(p, q)) \to Z$$

is given by $(M, -M)$, relative to the canonical generators coming from $\pi_1(U(p))$ and $\pi_1(U(q))$. If $p$ or $q = 0$ then the corresponding factor is omitted.

To compute the integer $M$ we will explicitly identify the signature cocycle for $U(1)$. From (2.3) we may assume none of $A$, $B$, $AB$ equal to 1. For $U(1)$ this implies that each bounding circle of $X = X_3$ is acyclic for our local coefficient system, so that $H^1(X, Y, E) \cong H^1(X, E)$. This makes the hermitian form non-degenerate. Since $H^1(X, E)$ is one-dimensional we must have

$$\text{sign}(A, B) = \pm 1.$$

Putting $A = \exp 2\pi i\theta$, $B = \exp 2\pi i\phi$ with $0 < \theta < 1$, $0 < \phi < 1$ consider the $(\theta, \phi)$ plane. $\text{sign}(\theta, \phi) = \text{sign}(A, B)$ must be constant in each triangle. From (2.4) it follows that

$$\text{sign}(1 - \theta, 1 - \phi) = -\text{sign}(\theta, \phi).$$
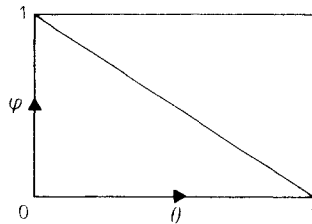
Hence $\text{sign}(\theta, \phi)$ is $+1$ on one triangle and $-1$ on the other. The correct sign depends on a careful check of our sign connections. We shall see in Sect. 3 (by other methods) that the correct sign is $+1$ on the upper triangle.

The canonical generator $g$ of $H^2(U(1), Z)$ corresponds to the universal covering

$$Z \to R \to U(1).$$

For $n > 1$, the element $ng$ corresponds to the extension

(2.5)
$$Z \xrightarrow{\ i\ } R \times Z_n \xrightarrow{\ p\ } U(1),$$

where $Z_n$ are the integers modulo $n$ and

$$j(1) = \left(\frac{1}{n}, 1\right), \quad p(x, r) = \exp 2\pi i \left(x - \frac{r}{n}\right).$$

In particular, taking $n = 2$, define the section

(2.6)
$$\sigma \colon U(1) \to R \times Z_2$$

by
$$\sigma(1) = (0, 0)$$

$$\sigma(\exp 2\pi i\theta) = (\theta - \tfrac{1}{2}, 1) \qquad 0 < \theta < 1.$$

The associated 2-cocycle $\tau$ of $U(1)$ is defined by

$$j\tau(\lambda, \mu) = \sigma(\lambda) + \sigma(\mu) - \sigma(\lambda\mu).$$

A direct computation using (2.6) then shows that $\tau$ is our signature cocycle. Thus we have established

(2.7) **Proposition.** *The signature cohomology class of $U(1)$ is twice the generator, and the signature cocycle is defined by the section (2.6).*

As noted earlier this then implies

(2.8) **Corollary.** *The signature cohomology class of $U(p, q)$ is $(2, -2)$ relative to the basis of $\pi_1 U(p) \times \pi_1 U(q)$: if $p$ or $q = 0$ the corresponding term is omitted.*

It is useful to have explicit models of the central extensions of $G = U(p, q)$. Consider first the universal central extension, i.e. the universal covering

(2.9)
$$Z^2 \to \tilde{G} \to G.$$

Given a homomorphism $\lambda \colon Z^2 \to Z$ we form the associated extension as

(2.10)
$$G_\lambda = \tilde{G} \times_\lambda Z,$$

where $(\tilde{g}u, n) \sim (\tilde{g}, \lambda(u) + n)$ for $u \in Z^2$, $n \in Z$, $\tilde{g} \in \tilde{G}$. To construct $\tilde{G}$ explicitly we introduce the homogeneous space $H(p, q) = U(p, q)/U(p) \times U(q)$ and the two homogeneous vector bundles $F^+$, $F^-$ of dimensions $p$, $q$ respectively. Let $L^\pm = \det F^\pm$ be the corresponding line-bundles and denote by $\log L^\pm$ the $C$-bundles which are the universal coverings of $L^\pm_* = L^\pm - (0\text{-section})$. Let

$$N = \log L^+ \oplus \log L^- \qquad \text{(fibre-wise product)}$$

be the $C \oplus C$-bundle over $H(p, q)$. Then $\tilde{G}$ is the subgroup of diffeomorphisms of $N$ which commutes with the natural action of $Z^2$ and induces the action of $G$ on the $C^* \times C^*$-bundle $L^+_* \times L^-_*$.

If $\lambda \colon Z^2 \to Z$ is given by the pair of integers $(m, n)$ it follows that $G_\lambda$ acts on $\log L^\lambda$, where $L^\lambda = (L^+)^m \otimes (L^-)^n$, commuting with the natural $Z$-action and

inducing the action of $G$ on $L_*^\lambda$. *An element of $G_\lambda$ is therefore an element $g$ of $G$ together with a lift of* the action of $g$ to $\log L^\lambda$. Equivalently, associating to $g$ (*in the* standard way) the line-bundle $L^\lambda(g)$ over the circle, *an element of $G_\lambda$ consists of $g$ together with a homotopy class of trivializations of $L^\lambda(g)$.*

*Remarks.* 1) The central $U(1)$ of $G = U(p, q)$ acts trivially on $H(p, q)$ and acts via $\xi \to \xi^p, \xi \to \xi^q$ on $L^+$ and $L^-$ respectively. Hence it acts via $\xi \to \xi^{mp + nq}$ on $L^\lambda$, where $\lambda = (m, n)$. Thus the action is trivial if $mp + nq = 0$. In particular this holds for $(m, n) = (2, -2)$ and $p = q$.

2) $G_\lambda$ is connected if $m$ and $n$ are coprime. In general it has $k$ components where $k$ is the H.C.F of $m$ and $n$.

In view of Corollary (2.8) we are especially interested in the case $\lambda = (2, -2)$ in which case $L^\lambda = (L^+)^2 \otimes (L^-)^{-2}$. For brevity we shall put $G_\lambda = G_2$. Since the signature cocycle defines this extension it follows that there is a section $\sigma: G \to G_2$ such that

$$(2.11) \qquad\qquad \sigma(A)\sigma(B)[\sigma(AB)]^{-1} = \mathrm{sign}(A, B)$$

for all $A, B \in G$. Since there is no homomorphism $G \to Z$ (other than 0) $\sigma$ is necessarily unique. Moreover, the naturality properties of the signature imply that sign $(A, B)$ and hence $\sigma$ *is invariant under inner automorphisms.*

From its definition $\sigma$ is clearly additive for direct sums. Hence formula (2.6), extended to direct sums, essentially identifies $\sigma$ on $U(p) \times U(q)$, i.e. on all *elliptic elements* of $G$. Later we shall discuss the identification of $\sigma$ on other classes of elements.

We now return to the question of computing $\mathrm{sign}(X, E)$ for a general local coefficient system $E$ over a surface $X$ with boundary $Y$. Following [21] we first fix a *splitting* of $E$, i.e. a vector bundle decomposition

$$E = E^+ \oplus E^-$$

which is orthogonal relative to the hermitian form and such that the form is positive on $E^+$ and negative on $E^-$. Such a splitting is equivalent to reducing the structure group of $E$ from $U(p, q)$ to $U(p) \times U(q)$ and corresponds to an equivariant map

$$f: \tilde{X} \to H(p, q) = \frac{U(p, q)}{U(p) \times U(q)}.$$

Here $\tilde{X}$ is the universal covering of $X$ and equivariance means that

$$f(gx) = \alpha(g) f(x),$$

where $g \in \pi_1(X)$ and $\alpha: \pi_1(X) \to U(p, q)$ is the representation defining the local coefficient system $E$. Since $H(p, q)$ is contractible such splittings exist and are unique up to homotopy.

The universal vector bundles $F^\pm$ on $H(p, q)$ pull back via $f$ to $\pi_1(X)$-equivariant vector bundles on $\tilde{X}$ and these descend to give $E^\pm$ on $X$. The same then follows for the corresponding determinant line-bundles. Consider in particular the line-bundle on $X$

$$L = (\det E^+) \otimes (\det E^-)^{-1}.$$

Our interpretation of the extension $G_2$ of $G$ shows that an element of $G_2$ over $\alpha(g)$, for $g \in \pi_1(X)$, defines a homotopy class of trivializations for the restriction of $L^2$ to the closed loop $S_g$ on $X$. In particular the signature section $\sigma: G \to G_2$ gives rise to a (class of) trivializations of $L^2$ on each closed loop in $X$: note that this is base-point independent (because conjugation acts trivially on line-bundles). Applying this to each boundary component of $X$ we see that we can define a relative first Chern class $c_1(L^2, \sigma)$ as an element of $H^2(X, Y) \cong Z$. Note that $\sigma$ can also be viewed as an isomorphism $(L^+)^2 \to (L^-)^2$ on $Y = \partial X$, so we can form a line-bundle on the double of $X$ and its Chern class (as an integer) coicides with $c_1(L, \sigma)$.

Taking $X = X_3$, the 2-sphere with 3 discs deleted, it is then a routine matter of reinterpretation to see that

$$(2.12) \qquad c_1(L, \sigma) = \sigma(A)\sigma(B)[\sigma(AB)]^{-1} = \text{sign}(A, B),$$

by definition of $\sigma$. Here $A, B$ are as before the elements of $G$ associated to two generating loops on $X_3$.

Since both the signature and the relative Chern class are additive for connected sums it follows that (2.12) holds for all $X$. Thus we have established

(2.13) **Theorem.** *Let $E$ be a flat hermitian vector bundle over the surface $X$, with a splitting $E = E^+ \oplus E^-$. Let $G_2$ be the extension of the appropriate unitary group $G = U(p, q)$ with class $(2, -2)$. Then there is a unique section $\sigma: G \to G_2$ such that*

$$\text{sign}(X, E) = c_1(L^2, \sigma),$$

*where $L^2 = (\det E^+)^2 \otimes (\det E^-)^{-2}$ is trivialized over $\partial X$ by the section $\sigma$. The section $\sigma$ gives rise to the signature cocycle of $G$ by (2.11).*

If $X$ is closed then Theorem (2.13) yields the following:

(2.14) **Corollary.** *If $E$ is a flat hermitian vector bundle over the closed surface $X$ then*

$$\text{sign}(X, E) = 2[c_1(L^+) - c_1(L^-)],$$

*where $L^\pm = \det E^\pm$, and $E = E^+ \oplus E^-$ is a splitting of $E$.*

This corollary was given a different (analytic) proof by Lusztig [21] and we shall return in Sect. 3 to his viewpoint. Note however that the correct sign in (2.7) is determined by the correct sign in (2.14), and this follows from Lusztig's proof.

Theorem (2.13) needs to be complemented by a more explicit determination of the section $\sigma$. For elliptic elements we have already seen how $\sigma$ is determined [extending (2.6) by additivity]. We now turn our attention to the general semi-simple element of $U(p, q)$. Such an element is conjugate to a direct sum:

$$A = A_1 \oplus A_2 \oplus \ldots \oplus A_r \oplus B,$$

where $B$ is an elliptic element of $U(p - r, q - r)$ and each $A_i$ is a *hyperbolic* element of $U(1, 1)$, i.e. whose two eigenvalues $\lambda, \mu$ have a real ratio $\lambda/\mu$. By additivity of $\sigma$ it is then sufficient to consider hyperbolic elements $A$ of $U(1, 1)$. Such an element lies on a one-parameter group $\exp(t\alpha)$, with $t = 1$ giving $A$. The extension $G_2 \to G = U(p, q)$ pulled back to $R$ (i.e. restricted to the one-parameter subgroup $R \to G$) splits canonically, and this canonical splitting $\sigma_0$ corresponds to the

natural trivialization of the bundle $L^\lambda$ over the circle given by contracting $A$ to 1 along $\exp t\alpha$. We shall prove

(2.15) **Proposition.** *For a hyperbolic element of* $U(1, 1)$ $\sigma$ *is the canonical splitting* $\sigma_0$.

*Proof.* The key fact about a hyperbolic element $A$ of $U(1, 1)$ is that, modulo the centre, it is conjugate to its own inverse [the adjoint group is PSL $(2, R)$ and here we have real eigenvalues $\lambda, \lambda^{-1}$]. On the other hand the section $\sigma$ is compatible with inverses (2.4). These two facts together will give the proof. Formally let us enlarge the one-parameter group by adding the $U(1)$-centre to give a homomorphism $R^2 \to U(1, 1)$. The extension $G_2 \to G = U(1, 1)$ when pulled back to $R^2$ again splits canonically. Let $\sigma_0$ denote this canonical splitting and put $\sigma = \sigma_0 \beta$ so that $\beta: R^2 \to Z$ measures the difference. Then $\beta$ (written additively) has the following properties

$$\beta(-u, -v) = -\beta(u, v) \quad \text{(compatibility with inverses)}$$

$$\beta(u, -v) \quad = \beta(u, v) \quad \text{(conjugacy invariance)}$$

$$\beta(u, v) \quad\quad\quad\quad\quad \text{independent of } u.$$

Here $u$ generates the centre and $v$ lies in SU $(1, 1)$. The independence of $u$ follows from the trivial action of the centre on the line-bundle $L^\lambda$. Clearly these properties of $\beta$ together imply $\beta = 0$ and so $\sigma = \sigma_0$ is the canonical section as required.

We have thus identified the signature section $\sigma$ on all semi-simple elements of $U(p, q)$. One should go further and examine in particular the unipotent elements but we shall not pursue this line.

## 3. The Analytic Approach

In this section we present briefly an alternative approach to Theorem (2.13) based on the analysis of elliptic operators. We begin by recalling the result of Lusztig [21] for a closed Riemann surface $X$ with a flat hermitian vector bundle $E$. From a Riemannian metric on $X$ and a splitting $E = E^+ \oplus E^-$ Lusztig defines an involution $\tau$ on $\Omega^*(X, E)$, the differential forms on $X$ with coefficients in $E$. He also defines an operator $D: \Omega^+(X, E) \to \Omega^-(X, E)$ where $\Omega^\pm$ are the $\pm 1$-eigenspaces of $\tau$, and shows that

$$\text{index } D = \text{sign}(X, E).$$

This construction works for all even-dimensional $X$ and is a generalization of that in [3]. Moreover the general index theorem for elliptic operators [3] then leads to the formula in Corollary (2.14).

Since the index theorem of [3] has been generalized in [6] to deal with manifolds with boundary it would be natural to attempt to extend (2.14) to the case of surfaces with boundary, and so prove (2.13). Whilst such an approach should be possible it cannot be carried out immediately because Lusztig's operator $D$ is not, near the boundary, of the type assumed in [6]. We recall that in [6] the elliptic operator near the boundary was assumed to be of the form

$$\sigma\left(\frac{\partial}{\partial u} + A\right),$$

where $u$ is the normal variable and $A$ is a self-adjoint operator on the boundary. For Lusztig's operator $A$ has *a self-adjoint symbol*, but is not itself self-adjoint. Thus the results of [6] would first have to be extended to such situations. In fact this is a quite natural extension and it is currently under investigation in various contexts.

Another possible approach would be to exploit the complex structure of $X$. The flat bundle $E$ can then be viewed as holomorphic and endowed with a meromorphic connection (on a compactification of $X$). One could then attempt to use the theory of differential equations with regular singular points in its modern form as developed by Deligne [13].

Instead of pursuing either of these interesting but somewhat lengthy programmes we shall adopt a hybrid method which will rapidly reproduce the results of Sect. 2.

We first extend Lusztig's result to the case of a surface $X$ with boundary $Y$, under the assumption that each monodromy element of the flat bundle $E$ on each of the components of $Y$ is *elliptic*. This assumption is just what is needed to bring Lusztig's operator into the standard form considered in [6]. We can therefore apply the main results of [6]. In [6, (4.14)] only $4k$-dimensional manifolds were considered and there was no auxiliary flat bundle $E$. However the appropriate modifications are easily made and, as in Lusztig [21], we have to use a homotopy connecting $D$ to the standard signature operator with coefficients in a bundle used in [3]: the important point is that the homotopy can be chosen trivial near the boundary, so that the boundary condition and so the index is unaltered. The conclusion is that

$$(3.1) \qquad \text{sign}(X, E) = \int_X 2[c_1(E^+) - c_1(E^-)] - \sum_j \eta(A_j),$$

where in the integrand $c_1$ stands for the first Chern form (using product connections near the boundary) and $\eta(A_j)$ is the $\eta$-invariant on the $j$th-bounding circle (with $A_j$ being the monodromy).

To put (3.1) into a more topological form we proceed as follows. By hypothesis each $A_j$ is elliptic and so lies in a maximal compact torus $T^p \times T^q$ of $U(p, q)$. Inside this torus we now deform $A_j$ by leaving fixed the 1-eigenspace and deforming all other eigenvalues to the value $-1$ (without crossing the value 1 in the process). Let $B_j$ be the resulting element. Then $B_j = B_j^{-1}$ and so $\eta(B_j) = 0$. Also this deformation leads to a deformation of our elliptic boundary value problem in which the index is unaltered (because no new eigenvalues equal to 1 are introduced). Thus, applying this for each $j$, (3.1) gets deformed to the formula

$$(3.2) \qquad \text{sign}(X, E) = \int_X 2[c_1(E^+) - c_1(E^-)],$$

where $c_1$ now stands for the Chern form defined by the product connections coming from the $B_j$. Since $B_j^2 = 1$ these are just trivializations of the line-bundles $(\det E^+)^2$ and $(\det E^-)^2$. Thus (3.2) is a topological formula and is easily seen to coincide with that given by Theorem (2.13).

Having derived the elliptic case of (2.13) from Lusztig's formula we now proceed to the general case. Let $A \in U(p, q)$, then we can write $A = BC$ where $B \in SU(p, q)$ and $C$ is in the central $U(1)$. Since $SU(p, q)$ is the commutator subgroup of $U(p, q)$ it follows that we can find a surface $X_0$ with one bounding circle

and a flat bundle on $X_0$ with monodromy $B$ on $\partial X_0$. Hence attaching a sphere with 3 holes we get a surface $X$ with two bounding circles having monodromy $A$ and $C^{-1}$ respectively.

(3.3)



Using the trivialization of $L = (\det E^+)^2 \otimes (\det E^-)^{-2}$ over the $C$-boundary (which is elliptic) given above there is then a unique trivialization of $L$ over the $A$-boundary which will give the correct signature formula in (2.13). To see that this is independent of the choice of $B$, $C$, $X_0$ we take another choice and then double up along the $A$-boundary. Additivity and formula (3.2) (for elliptic boundaries) proves the independence. Thus we have defined the trivialization of $L$ for all $A$, or equivalently a section $\sigma$ of $G_2 \to G = U(p, q)$ (see Sect. 2). Finally if $X$ has boundaries $S_1, \ldots, S_k$ and $E$ is a flat bundle with monodromies $A_1, \ldots, A_k$ we attach a figure like (3.3) to each boundary. This reduces us to elliptic monodromies and additivity then completes the proof of the general case of Theorem (2.12).

The one merit of the above proof is that it avoids recourse to the theory of group extensions of Lie groups used in Sect. 2.

## 4. The Signature of Fibrations

We shall now consider fibrations $Z \xrightarrow{M} X$, where $X$ as before is an oriented surface with boundary $Y$ and the fibre $M$ is a compact oriented manifold of dimension $4k - 2$. We further assume that the total space $Z$ is oriented (this is equivalent to assuming that $\pi_1(X)$ acts trivially on $H^{4k-2}(M)$). In particular the signature on $H^{2k}(Z)$ denoted by $\text{sign}(Z)$ is defined.

We shall use Riemannian metrics on $Z$ adapted to this fibration. More precisely we assume that $X$ is given a metric and that the projection $Z \to X$ is a Riemannian submersion. We also assume that the metrics on $X$ and $Z$ are products near their boundaries.

The main theorem of [6] then gives a formula for the signature of $Z$, namely

(4.1) $$\text{sign}\, Z = \int_Z L_k(p) - \eta(\partial Z),$$

where $L_k$ is the Hirzebruch polynomial in the Pontrjagin forms $p_j$ of $Z$ and $\eta(\partial Z)$ is the spectral invariant introduced in [6]. We recall that on $\partial Z$ there is a self-adjoint operator $A$ defined on even differential forms by $\phi \to (-1)^{k+p+1}(*d - d*)\phi$ for $\phi \in \Omega^{2p}$, and that the function $\eta(s)$ is then defined in terms of the eigenvalues $\lambda$ of $A$ by

$$\eta(s) = \sum_{\lambda \neq 0} \frac{\text{sign}\,\lambda}{|\lambda|^s}.$$

This function is holomorphic for $\text{Re}(s) > 0$ and its value at $s = 0$ is the $\eta$-invariant in (4.1).

Of course if $\partial X$ has components $S_1, \ldots, S_r$ then $\partial Z$ has corresponding components $W_1, \ldots, W_r$, fibred over the $S_j$, and

$$(4.2) \qquad\qquad \eta(\partial Z) = \sum_j \eta(W_j).$$

Following ideas of Witten [35], Bismut and Freed [9] have studied the "adiabatic limit" of (4.1) in which the metric $g_X$ of $X$ is rescaled to $g_X/\varepsilon^2$ and $\varepsilon \to 0$. Each term in (4.1) and (4.2) converges to a limit. Moreover the integral in (4.1), when written as a double integral $\int_X \int_M L_k(p)$, converges locally on $X$, so that (4.1) gives

$$(4.3) \qquad\qquad \operatorname{sign} Z = \int_X \left[ \lim_{t \to 0} \int_M L_k^i(p) \right] - \sum_j \eta^0(W_j),$$

where $\eta^0 = \lim_{t \to 0} \eta^i$ and $\eta^i$ refers to the rescaled metric. Note that the Pontrjagin forms and hence $\eta$ are conformally invariant, so that the adiabatic limit is equivalently desribed by shrinking the metric on the fibre.

Formula (4.3), when reduced modulo integers, has an interesting interpretation in terms of Quillen's "determinant line-bundle" which we shall now briefly recall.

Consider the signature operator $D$ of the manifold $M$. This is defined as $d + d^* : \Omega^+ \to \Omega^-$ where $\Omega^{\pm}$ are the $\pm 1$-eigenspaces of the involution

$$\phi \to i^{p(p-1)+2k-1} * \phi \qquad \text{for} \quad \phi \in \Omega^p.$$

Let $H^+$ and $H^-$ denote the solution spaces of $Du = 0$ and $D^*v = 0$ respectively, i.e. they consist of the harmonic forms in $\Omega^+$ and $\Omega^-$. If we now vary $M$ over the fibres of $Z \to X$ we get a family $D_x$ of operators and corresponding spaces $H_x^+$ and $H_x^-$ of harmonic forms which define vector bundles $H^+$ and $H^-$ over $X$. The Quillen line-bundle $\mathscr{L}$ is the bundle $\det H^- \otimes (\det H^+)^{-1}$ over $X$ endowed with a natural unitary connection defined in [9] to which we shall return shortly. Bismut and Freed prove that the *first Chern form* of $\mathscr{L}$ is given by:

$$(4.4) \qquad\qquad c_1(\mathscr{L}) = -\frac{1}{2} \lim_{t \to 0} \int_M L_k^i(p).$$

*Note.* Actually Bismut and Freed work with Dirac operators which requires a spin hypothesis, but this should be irrelevant for the signature operator. The minus sign arises because of Quillen's sign convention, which is dictated by holomorphic considerations (see below). The factor $\frac{1}{2}$ arises because we are dealing with the $L$-polynomials rather than the $\hat{A}$-polynomials as in [9].

If we apply (4.3) modulo integers when $X$ is a disc then we see that the *monodromy* of $\mathscr{L}$ around the bounding circle of $X$ is just $\exp(-\pi i \eta^0(W))$. Bismut and Freed establish this quite generally. In other words $-\pi i \eta^0(W)$ *is a distinguished choice for the logarithm of the monodromy of $\mathscr{L}$ around a circle*. Moreover (4.3) then links these distinguished choices on $\partial X$ with $\operatorname{sign} Z$.

Before proceeding further we should spell out the data on which $\eta^0(W)$ depends. Lifting the fibration $W \to S^1$ to a fibration $\tilde{W} \to R$ the metric on $\tilde{W}$ defines a diffeomorphism $\tilde{W} \cong M \times R$, and the metric then takes the form

$$ds^2 = \varrho_x + dx^2$$

where $\varrho_x$ is a one-parameter family of metrics on $M$. This path $\varrho\colon R \to \mathscr{M}$ in the space $\mathscr{M}$ of metrics on $M$ must be periodic under the diffeomorphism $g$ of $M$ [identifying $M \times (x)$ with $M \times (x+1)$]. Then $\eta^0(W)$ depends only on the pair $(g, \varrho)$ and is also invariant under the natural action of $\mathrm{Diff}^+(M)$ (orientation preserving diffeomorphisms) on such pairs. If $\mathrm{Diff}^+(M)$ acts freely on $M$ then we could replace this data by a closed path in $\mathscr{M}/\mathrm{Diff}^+(M)$ representing the conjugacy class of $[g] \in \pi_0(\mathrm{Diff}^+(M)) = \pi_1(\mathscr{M}/\mathrm{Diff}^+(M))$.

The work of Bismut-Freed identifies $\frac{1}{2}\eta^0$ modulo integers in terms of the monodromy of Quillen's line bundle $\mathscr{L}$. The question we now raise is that of extending this identification to take account of the integer ambiguity. At this point we should remark that, in the general framework of Dirac operators studied by Bismut and Freed, there is no way of fixing the integer ambiguity. This is because we cannot in general control the 0-eigenvalue of Dirac operators. However, for the special case of the signature operator, Hodge theory identifies harmonic forms with cohomology and this is what fixes the integer ambiguity and leads to the formula (4.3), with the topologically defined term $\mathrm{sign}\,Z$ on the left-hand side.

At this point we should make a comment on the connection in the Quillen line-bundle $\mathscr{L}$. This is defined quite generally (not just for the signature operator) but because of jumps in the 0-eigenvalue the connection is defined using all eigenvalues and it involves regularized infinite determinants. For the signature operator $D$ this implies the following. Let $D'$ be the restriction of $D$ to the orthogonal complement of the harmonic spaces $H^\pm$. The $\zeta$-function definition of determinants [as exp $-\zeta'(0)$] applied to $(D')^*D'$ enables one to define $|\det D'|$ in a natural fashion, but there is an "anomaly" in trying to define a complex-valued determinant. However, one can define *the logarithmic variation* along any path. This leads to Quillen's point of view in which $\det D'$ is interpreted not as a function on $X$ but as a section of a line-bundle $\mathscr{L}'$ with unitary connection over $X$. Since $\det D'$ is nowhere zero this section trivializes $\mathscr{L}'$ but not in a unitary or covariant constant manner.

To pass from $\mathscr{L}'$ to the Quillen line-bundle $\mathscr{L}$ one has to consider the 0-eigenvalues, i.e. the harmonic bundles $H^\pm$. These have natural metrics and connections inherited (by orthogonal projection) from the Hilbert space bundles of all forms and so

$$(4.5) \qquad \qquad \mathscr{H} = \det H^- \otimes (\det H^+)^{-1}$$

is a line-bundle with unitary connection. We then take

$$(4.6) \qquad \qquad \mathscr{L} = \mathscr{L}' \otimes \mathscr{H}$$

with its induced unitary connection. Note that, as $\mathscr{L}'$ is trivialized by $\det D'$, $\mathscr{L} \cong \mathscr{H}$ as a line-bundle but *the isomorphism does not preserve metric or connection.*

We return now to the re-interpretation of (4.3), rewriting it, using (4.4) as

$$(4.7) \qquad \qquad \mathrm{sign}\,Z = -2\int_X c_1(\mathscr{L}) - \sum_j \eta^0(W_j).$$

As we remarked earlier Bismut and Freed have shown, for each bounding circle $S_j$ of $X$, that $-\pi i \eta^0(W)$ is a (distinguished) choice for the logarithm of the monodromy of $\mathscr{L}$ around $S_j$. This means that we can use this choice to trivialize $\mathscr{L}$ (up to homotopy) on each $S_j$ and hence obtain a *relative* Chern class which we shall

denote by $c_1(\mathscr{L}, \eta)$ to indicate that it arises from using $\eta$-invariants. Then (4.5) simply becomes

(4.8)                                    $\mathrm{sign}\, Z = -2\, c_1(\mathscr{L}, \eta).$

We now compare (4.8) with Theorem (2.13) for the signature of local coefficient systems over $X$. The fibration $Z \to X$ gives a local coefficient system corresponding to the representation of $\pi_1(X)$ on $H^*(M)$. In particular the middle cohomology $H^{2k-1}(M)$ gives a flat bundle with a skew-form. Complexifying coefficients and multiplying by $i$ gives a flat bundle with a hermitian form [of type $(p, p)$ where $2p = \dim H^{2k-1}(M)$]. It is easy to see (as in [11]) that the signature of $X$ with coefficients in this flat bundle is equal to the signature of the total space $Z$, i.e.

(4.9)                                $\mathrm{sign}\,(X, H^{2k-1}(M)) = \mathrm{sign}\, Z.$

Thus (4.8) and (2.13) both give a formula for $\mathrm{sign}\, Z$ in terms of a relative first Chern class. Moreover, the line bundle $\mathscr{L}$ in (4.6) is topologically just the dual of the line-bundle $L$ in (2.13). This follows from (4.5), (4.6), the triviality of $\mathscr{L}'$, the definition of $L$ and the fact that contributions to $H^{\pm}$ from $H^j(M)$ and $H^{2k-1-j}(M)$, for $j \neq 2k-1$, cancel. The same cancellation is actually involved in establishing (4.9), and an alternative approach would have been to replace $H^{2k-1}(M)$ in (4.9) by $H^*(M)$, with appropriate factors of $\pm i$ to define the hermitian form.

It is now natural to ask whether the (homotopy) trivializations of $\mathscr{L}^2$ given by $\eta$ and $L^{-2}$ given by $\sigma$ coincide. The difference between these two trivializations would assign an integer $N_j$ to each component $S_j$ of $X$. These integers would be homotopy invariants depending only on the diffeomorphisms $g_j$ of $M$, and subtracting (4.8) from (2.13) shows that

$$\sum_j N(g_j) = 0.$$

In particular, taking $X$ to be a sphere with three holes, this shows that

(4.10)                                    $N \colon \mathrm{Diff}^+(M) \to Z$

is a homomorphism. Of course $N$ depends only on the group of components of $\mathrm{Diff}^+(M)$. If there are no such homomorphisms (except zero) then it follows that the $\eta$ and $\sigma$ trivializations agree. This can then be viewed as giving a formula for the variation $\delta$ of $\log \det D'$ around closed paths in $\mathscr{M}/\mathrm{Diff}^+(M)$ (or more precisely for pairs $(g, \varrho)$ as before). More precisely we have

(4.11)                    $\delta \log \det D' = -\pi i \eta^0 - \tfrac{1}{2} \log_\sigma \mathrm{Mon}(\mathscr{H}^2),$

where the symbols have the following meaning. The pair $(g, \varrho)$ define a bundle $W$ over $S^1$ with fibre $M$ and $\eta^0 = \eta^0(g, \varrho) = \eta^0(W)$ is the adiabatic limit of $\eta(W)$. The line bundle $\mathscr{H}$ is defined by (4.5) and is endowed with its natural connection coming from the spaces of harmonic forms, $\mathrm{Mon}(\mathscr{H}^2)$ denotes the monodromy along a period of the path $\varrho$ of metrics and $\log_\sigma$ denotes the branch of the logarithm defined by the signature section $\sigma$ of Sect. 2. As a formula in $R/2\pi i Z$ (4.11) is just a special case of the Bismut-Freed-Witten formula. The main point of (4.11) is therefore the identification of the integer ambiguity.

We should emphasize that (4.11) has only been established under the assumption that $\text{Diff}^+ M$ has no non-zero homomorphisms to the integers. It would be interesting to know if this restriction is superfluous. Put another way the difference of the $\eta$ and $\sigma$ trivializations gave the homomorphism $N$ of (4.10) and this is then an invariant of the manifold $M$. Perhaps it is always zero?

We may now ask for examples of manifolds $M$ having no non-trivial homomorphisms $\text{Diff}^+ M \to Z$. Consider the case of $\dim M = 2$. For the 2-sphere $S^2$ the group $\text{Diff}^+ (S^2)$ is connected. If $M = T^2$ is a 2-torus then the group of components of $\text{Diff}^+ (M)$ is $\text{SL} (2, Z)$ whose abelianization is finite. If $M$ is a surface of genus $g \geqq 2$ the group of components of $\text{Diff}^+ (M)$ is the Teichmuller group $\Gamma_g$ and Mumford [28] has shown that its abelianization is also finite. Hence our discussion and the conclusion (4.11) holds for $\dim M = 2$.

If we use the complex structure of the Riemann surface $M$ then the signature operator $D$ is equivalent to the operator

$$\bar\partial: \ \Omega^{0,0} \oplus \Omega^{1,0} \to \Omega^{0,1} \oplus \Omega^{1,1}$$

and the fibre of the line-bundle $\mathscr{H}$ can be identified with

$$(\det H^{0,1} \otimes \det H^{1,1}) \otimes (\det H^{0,0} \otimes \det H^{1,0})^{-1}$$

where $H^{p,q} = H^q(M, \Omega^p)$ with $\Omega^p$ here standing for holomorphic $p$-forms. The terms $H^{0,0}$ and $H^{1,1}$ cancel by duality while $H^{0,1}$ and $H^{1,0}$ are Serre duals. Hence the fibre of $\mathscr{L}$ can be identified with

$$(4.12) \qquad\qquad [\det H^0(M, \Omega^1)]^{-2}.$$

If $Z \to X$ is a holomorphic family of Riemann surfaces then (4.12) shows that $\mathscr{H}$ is naturally a holomorphic line-bundle. Moreover if $Z$ has a Kähler metric then the Bismut-Freed connection on $\mathscr{H}$ coincides with that determined by its metric and holomorphic structure, as shown by Freed [15] (see also Donaldson [15a]). The same applies to the Quillen line-bundle $\mathscr{L}$.

In the next section we shall investigate in detail the case when $M$ has genus 1, the original motivation for this paper. The case when $M$ has genus $\geqq 2$ also merits further treatment using the explicit formula for $\det \bar\partial$ in terms of the Selberg zeta function [33], but we shall not pursue this here.

In Sect. 6 we shall carry out some computations which rely on the equivariant version of (4.1). It will therefore be convenient to summarize here the general results in this direction: for a fuller account we refer to [7, Sect. 2]. In the situation of (4.1) assume that $g: Z \to Z$ is an isometry *having no fixed points on* $\partial Z$. Then we can define $\text{sign}(Z, g)$ as in [3] by considering the induced action of $g$ on the cohomology of $Z$ and (4.1) generalizes to:

$$(4.13) \qquad\qquad \text{sign}(Z, g) = L_k(p, g) - \eta(\partial Z, g),$$

where $L_k(p, g)$ is the sum over the fixed point set components of $g$ which occurs in the $G$-signature theorem for closed manifolds [3] and the boundary term $\eta(\partial Z, g)$ is defined by evaluating at $s = 0$ the function

$$(4.14) \qquad\qquad \eta(g, s) = \sum_{\lambda \neq 0} \frac{\text{sign}\,\lambda}{|\lambda|^s} (\text{Tr}\,g_\lambda),$$

where the eigenvalues $\lambda$ are as before those of the operator $A$ on $\partial Z$ and $g_\lambda$ is the induced action of $g$ on the $\lambda$-eigenspace (the sum is now over distinct $\lambda$). Thus (4.13) is a common generalization of (4.1) and the $G$-signature theorem of [3]. The proof is essentially the same as that in [6] except that the fixed-point contribution is identified as in [14]. Note that, since $g$ has no fixed-points on $\partial Z$, the term $L_k(p, g)$ is actually independent of the metric on $Z$: it is a topological invariant of the action of $g$ on $Z$.

If $g$ generates a finite (cyclic) group $G$ acting freely on $W = \partial Z$, then we can form the quotient manifold $W' = W/G$ and we can relate the term $\eta(\partial Z, g)$ in (4.13) with $\eta(W')$. Elementary character theory shows that

$$(4.15) \qquad \eta(W') = \frac{1}{|G|} \sum_{g \in G} \eta(W, g).$$

Thus the deviation from multiplicativity

$$(4.16) \qquad \eta(W') - \frac{1}{|G|} \eta(W) = \frac{1}{|G|} \sum_{g \neq 1} \eta(W, g)$$
$$= \frac{1}{|G|} \sum_{g \neq 1} \{L_k(p, g) - \operatorname{sign}(Z, g)\}$$

[using (4.13)], is *independent of the metric*.

If the fixed points of $g$ are *isolated*, then at each fixed point $P$ the differential of $g$ is a rotation on the tangent space at $P$ and is given by angles $\alpha_j$ ($j = 1, \ldots, 2k$) in orthogonal planes (the $\alpha_j$ are determined up to an even number of sign changes). The fixed point contribution $L_k(p, g)$ is then given by

$$(4.17) \qquad L_k(p, g) = i^{-n} \prod_{j=1}^{n} \cot \frac{\alpha_j}{2}, \qquad n = 2k.$$

If $|G| = c$ each $\alpha_j = \frac{2\pi a_j}{c}$, with $a_j$ an integer prime to $c$. Hence

$$(4.18) \qquad \sum_{g \neq 1} L_k(p, g) = i^{-n} \sum_{q=1}^{c-1} \prod_{j=1}^{n} \cot\left(\frac{\pi q a_j}{c}\right),$$

where the $a_j$ arise from a generator of $G$. In particular if $k = 1$ (so that dim $Z = 4$) and $G$ acts with integers $(1, a)$, we recognize on the right the Rademacher version (1.5) of the Dedekind sum, i.e.

$$(4.19) \qquad \frac{1}{|G|} \sum_{g \neq 1} L_1(p, g) = -4s(a, c).$$

Returning to (4.16), for $k = 1$, and assuming for simplicity that $G$ has just one fixed point, of type $(1, a)$, we get

$$(4.20) \qquad \eta(W') - \frac{1}{c} \eta(W) = -4s(a, c) - \frac{1}{c} \sum_{q=1}^{c-1} \operatorname{sign}(Z, g^q),$$

where $g$ generates $G$. Note that $Z' = Z/G$ is a rational homology manifold (with just one "cyclic" singularity) and the last term in (4.20) can be rewritten as

(4.21)         $$\frac{1}{c}\sum_{q=1}^{c-1} \text{sign}\,(Z, g^q) = \text{sign}\,(Z') - \frac{1}{c}\,\text{sign}\,(Z)$$

so that (4.20) becomes

(4.22)         $$\eta(W') - \frac{1}{c}\,\eta(W) = -4s(a, c) - \left\{ \text{sign}\,(Z') - \frac{1}{c}\,\text{sign}\,(Z) \right\}.$$

Notice now that none of the terms in (4.22) depends on the choice of a generator of $G$. The formula would therefore continue to make sense if the $c$-fold covering $W \to W'$ and its singular extension to $Z \to Z'$ are not given by a group action. This means $W \to W'$ is not a Galois covering and the singular point of $Z'$ is locally the quotient by a finite cyclic group. It is easy to extend (4.22) to this non-Galois situation as follows. Excising a small ball $B$ around the singular point we get a $c$-fold covering $Z_0 \to Z_0'$, where $Z_0 = Z - B$ and

$$\partial Z_0 = W - S, \quad \partial Z_0' = W' - S',$$

$S = \partial B$ being the 3-sphere and $S'$ the quotient lens space. Applying (4.1) to $Z_0$ and $Z_0'$ and observing that the Pontrjagin form expression behaves multiplicatively for any covering we deduce

(4.23)         $$\eta(W') - \frac{1}{c}\,\eta(W) = \left[ \eta(S') - \frac{1}{c}\,\eta(s) \right] - \left[ \text{sign}\,Z_0' - \frac{1}{c}\,\text{sign}\,Z_0 \right].$$

Now the additivity of the signature shows that

(4.24)         $$\text{sign}\,Z_0' = \text{sign}\,Z', \quad \text{sign}\,Z_0 = \text{sign}\,Z$$

while (4.20) applied with $W = S$, where we do have a cyclic group action, gives

(4.25)         $$\eta(S') - \frac{1}{c}\,\eta(S) = -4s(a, c).$$

Substituting (4.24) and (4.25) in (4.23) then gives (4.22) as required. Note that, since $S$ has an orientation reversing isometry (for the standard metric), $\eta(S) = 0$ and (4.25) then shows that the Dedekind sum $s(a, c)$ is (up to a factor $-4$) given by the $\eta$-invariant of the appropriate lens space.

If we apply these ideas for dimension 2, i.e. when $W$ is the circle, then for a rotation with angle $\theta$,

$$\eta(\theta, 0) = -i \cot \theta/2.$$

As pointed out in [7, Sect. 2], taking the Fourier transform over a finite group of rotations leads naturally to the invariants $\eta_\alpha(0)$ associated to characters of $\alpha$ of $\pi_1$. Moreover it was verified in [7] that if $\alpha$ takes the generator of $\pi_1$ to $\exp(2\pi i a)$ then

$$-\tfrac{1}{2}\eta_\alpha(0) = a - [a] - \tfrac{1}{2} = ((a)),$$

the function which enters in the original version (1.2) of the Dedekind sum.

## 5. Families of Elliptic Curves

We shall now come to the main topic of the paper and consider fibrations $Z \to X$ where the fibre $M$ is a 2-torus. The local coefficient system over $X$ given by $H^1(M)$ then arises from a representation

$$\pi_1(X) \to SL(2, Z).$$

As noted in Sect. 4 we have

(5.1)                                $\mathrm{sign}\,(X, H^1(M)) = \mathrm{sign}\,(Z).$

Now since

(5.2)                    $H^q(SL(2, Z), Q) = 0, \quad \text{for} \quad q = 1, 2,$

the signature cohomology class of Sect. 2 for the group

$$SL(2, R) = SU(1,1)$$

vanishes (over $Q$) when restricted to $SL(2, Z)$ and there is then a unique function

(5.3)                                $\phi: \ SL(2, Z) \to Q$

whose coboundary is (minus) the signature cocycle. More specifically if $X_3$ is the sphere minus 3 discs and if the monodromies around the 3 boundary circles are given by $A, B, (AB)^{-1} \in SL(2, Z)$, then the signature of $X_3$ with the local coefficient system is

(5.4)                    $\mathrm{sign}\,(A, B) = \phi(AB) - \phi(A) - \phi(B).$

The additivity of the signature then gives the general formula for (5.1)

(5.5)                                $\mathrm{sign}\,Z = -\sum_j \phi(A_j),$

where the $A_j$ are the monodromy matrices around the bounding circles $S_j$ of $X$, given by the action on $H^1(M)$.

Geometrically the monodromy is more naturally thought of as the induced action $A^*$ on $H_1(M)$. However, the canonical duality between $H^1(M)$ and $H_1(M)$ means that $A$ and $A^*$ (in dual bases) are conjugate in $SL(2, Z)$. Since all our invariants will be class functions it essentially makes no difference which definition we adopt.

*Remark.* The definition of $\phi$ and the formula (5.5) are due to W. Meyer [25].

We shall now explain how (5.5) fits into the general situation of Theorem (2.13). When $G = SU(1,1)$ the central extension $G_2 \to G$, restricted to $\Gamma = SL(2, Z)$, has [by (5.2)] a canonical splitting $\sigma_1$, provided we extend the $Z$-kernel to $Q$. Meyer's invariant $\phi$ is then defined as the ratio of $\sigma_1$ to the signature section $\sigma$ of Sect. 2:

$$\phi = \sigma_1/\sigma.$$

If we interpret $\sigma$ and $\sigma_1$ as defining trivializations of (some power of) the line-bundle $L^2$ of (2.13) then $\sigma_1$ gives a trivialization over $X$ while $\sigma$ gives trivializations only

over the boundary components $S_j$. The relative Chern class $c_1(L^2, \sigma)$ is then clearly given by

$$c_1(L^2, \sigma) = -\sum_j \phi(A_j)$$

so that (5.5) coincides with (2.13).

The extension $G_2 \to G$ of Sect. 2 has class $(2, -2)$ when $G = U(1,1)$. Restricting to $SU(1,1)$ this gives 4 times the generator [since the class $(1,1)$ generates the kernel]. Since for $\Gamma = SL(2, Z)$

$$H^2(\Gamma, Z) \cong \operatorname{Hom}(\Gamma, U(1)) \cong Z_{12}$$

it follows that we only have to adjoin $\frac{1}{3}$ to construct $\sigma_1$. Equivalently $3\phi$ is integral as was proved by Meyer. Thus $\sigma_1$ gives naturally a trivialization of the line-bundle $L^6$.

From the definition of $\phi$, or from the properties of $\sigma$ in Sect. 2, we have

(5.6)         $\phi(1) = 0$

$$\phi(A^{-1}) = -\phi(A)$$

$$\phi(BAB^{-1}) = \phi(A) \quad \text{if} \quad B \in SL(2, Z),$$

$$= -\phi(A) \quad \text{if} \quad B \in GL(2, Z) \quad \text{with} \quad \det B = -1.$$

Next we shall define another function

(5.7)                                      $\chi \colon SL(2, Z) \to Q$

which will turn up later in connection with the Dedekind $\eta$-function. If $A$ is elliptic we put $\chi(A) = 0$. Otherwise let $\exp t\alpha$ with $A = \exp \alpha$ be a one-parameter subgroup of $U(1,1)$: it is uniquely determined (modulo the centre) by $A$. As in Sect. 2 let $\sigma_0$ be the canonical splitting of the extension $G_2 \to G$ restricted to this one-parameter group. Then $\sigma_0(A)$ is well-defined and we put

(5.8)                                      $\chi(A) = \sigma_1(A)/\sigma_0(A)$.

As before we can interpret this in terms of the line-bundle $L^6$. The splitting $\sigma_1$ gives an $SL(2, Z)$-invariant trivialization of $L^6$ (unique up to homotopy) while $\sigma_0$ gives an $A$-invariant trivialization. These two trivializations differ by the integer $3\chi(A)$.

Note that $\chi$ satisfies the same relations (5.6) as $\phi$ and in addition, since $\sigma_1$ and $\sigma_0$ are both homomorphisms on the one-parameter subgroup:

(5.9)                                      $\chi(A^k) = k\chi(A)$.

With these definitions we see that Proposition (2.15) implies

(5.10) **Proposition.** *If $A \in SL(2, Z)$ is hyperbolic then $\phi(A) = \chi(A)$.*

*Remark.* $\phi$ is defined via the signature while $\chi$ is a type of Chern class. Thus (5.10) is a version of the signature theorem, relating signature invariants to characteristic classes. In Sect. 2 we gave a cohomological proof (leading to (5.10)) while in Sect. 3 we gave an analytical version.

We shall return later in Sect. 6 to the question of evaluating $\phi$ explicitly on various classes of elements in SL $(2, Z)$, but we shall now relate $\phi$ to $\eta$-invariants. For this we need to introduce metrics and we shall always take flat metrics on the 2-torus, normalized to have unit total area. Such metrics (or conformal structures) are then parametrized by the upper half-plane $H$ modulo the action of SL $(2, Z)$. To fix our notation let $(y, x; \tau)$ be coordinates for $R^2 \times H$ and hence local coordinates for $T^2 \times H$, where $T^2 = R^2/Z^2$ is the quotient by the standard integer lattice. We take

$$\omega_\tau = dx + \tau \, dy$$

to be the holomorphic differential on $T^2 \times \{\tau\}$. The group SL $(2, R)$ acts on $R^2 \times H$ as follows. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A(y, x; \tau) = \left( dy - cx; \; -by + ax; \; \frac{a\tau + b}{c\tau + d} \right).$$

The induced action takes $\omega_\tau$ into $\omega_\tau (c\tau + d)^{-1}$, while $d\tau$ goes into $d\tau(c\tau + d)^{-2}$. Thus $\omega_\tau^2/d\tau$ is invariant and this enables us to identify canonically the square of the bundle of holomorphic differentials along the (torus) fibres with the cotangent bundle of $H$. If $\tau = u + iv$ then $A(v) = v|c\tau + d|^{-2}$, and, for the torus metric with area normalized to 1, we have $|\omega_\tau|^2 = 2\tau$. Thus $\dfrac{i}{2\tau} \cdot \omega \wedge \bar{\omega}$ is the Kähler form on the torus over the point $\tau \in H$.

*Remark.* Note that the complex orientation of the torus has here been chosen *opposite* to the standard orientation of $R^2$.

A fibration $Z \to X$ with 2-torus fibres then acquires a metric by picking a metric on $X$ and an equivariant map $f$ of the universal covering $\tilde{X}$ to $H$, i.e.

$$f(g(x)) = \alpha(g) f(x),$$

where $\alpha: \pi_1(X) \to$ SL $(2, Z)$ is the monodromy [action on $H_1(T^2)$] of the fibration. If the metric on $X$ is a product near the boundary and if $f$ is chosen appropriately then the metric on $Z$ will also be a product near the boundary. Moreover, for each bounding circle of $X$ with monodromy $A \in$ SL $(2, Z)$, the metric on the component $W(A)$ of $\partial Z$ is induced by an $A$-invariant path $R \to H$.

Given such a metric on $W(A)$ we can, as in Sect. 4, consider rescaling the base metric by $\varepsilon^{-2}$. In particular we shall here take $\varepsilon = m^{-1}$ with $m$ an *integer* and consider the integer adiabatic limit

(5.11) $$\eta^0(W(A)) = \lim_{m \to \infty} \eta(W_m(A)),$$

where $W_m(A)$ is $W(A)$ with the rescaled metric. Without invoking the general results of Bismut and Freed [9] we shall now give a direct proof of the following:

(5.12) **Proposition.** *For any* $A \in$ SL $(2, Z)$ *the integer adiabatic limit* $\eta^0(W(A))$, *defined by* (5.11), *exists and is independent of the metric. Moreover* $\eta^0(W(A)) = \phi(A)$.

*Proof.* Let $Z \to X$ be any 2-torus fibration endowed with a metric as described above, and let $Z(m) \to X$ be the new fibration obtained by dividing each torus

by its points of period $m$. Then $Z \to Z(m)$ is a covering of degree $m^2$ and the covering transformations are isometries, so that $Z(m)$ inerits a metric (induced by the same path $R \to H$). Thus $Z(m)$ is essentially the same manifold as $Z$ but with its fibre metric shrunk by $m^{-2}$. Since $\eta$ is a conformal invariant it follows that $\eta(\partial Z(m)) = \eta(\partial Z_m)$ where $Z_m$ is the manifold $Z$ with base metric stretched by the factor $m^2$. Now apply (4.1) to $Z(m)$ and we get

$$\operatorname{sign} Z(m) = \frac{1}{3} \int_{Z(m)} p_1 - \eta(\partial Z(m))$$

or

$$\operatorname{sign} Z = \frac{1}{3m^2} \int_Z p_1 - \eta(\partial Z_m).$$

Letting $m \to \infty$ we deduce

(5.13)                                      $$\operatorname{sign} Z = - \lim_{m \to \infty} \eta(\partial Z_m).$$

If we knew the limit $\eta^0$ existed for all $W(A)$ then (5.13) would yield

(5.13)'                                     $$\operatorname{sign} Z = - \sum_i \eta^0(W(A_k))$$

summed over the boundary components. Moreover, the metrics over these components can be varied independently and so this formula shows each boundary term is independent of the metric. Finally, comparison with (5.5) which defines $\phi$ uniquely would prove the equality $\phi(A) = \eta^0(W(A))$ for all $A$. To prove the limit exists we can argue as follows. First, if $A$ is elliptic, $W(A)$ has as finite covering the product $T^2 \times S^1$, and we can choose the product with an $A$-invariant metric. The general arguments concerning the behaviour of $\eta$ under finite coverings explained in Sect. 4 show that $\eta(W(A))$ is a topological invariant and in particular scale invariant [note that $T^2 \times S^1$ has an orientation reversing isometry so $\eta(T^2 \times S^1) = 0$]. Thus the limit (5.11) trivially exists for elliptic $A$ and invariant metrics. Since the elliptic elements generate SL$(2, Z)$ we can use (5.13) inductively (with $X = X_3$ the sphere minus three discs) to show that the limit (5.11) exists for all $A$. This completes the proof.

In view of the fact that $\eta^0(W(A))$ is independent of the metric we shall also denote it by $\eta^0(A)$.

We now turn to Quillen's determinant line-bundle $\mathscr{L}$ over $X$, determined by the metric fibration $Z \to X$. In Sect. 4 using the general results of Bismut and Freed, we derived formula (4.7). Comparing it with (5.13)' we see that the first Chern form $c_1(\mathscr{L})$ integrates to zero over $X$. Applying this when $X$ is a small disc mapped into $H$ we see easily that, for the universal family $T^2 \times H$, and hence for any fibration $Z \times X$, $c_1(\mathscr{L}) \equiv 0$. In other words we have the important result:

(5.14) **Proposition.** *For a 2-torus fibration $Z \to X$ the Quillen determinant line-bundle is flat.*

Since this is a key result we shall give several alternative and more direct proofs. Note first that it is sufficient to consider the universal case when $Z = T^2 \times H$ and that this is naturally holomorphic. Thus the general formula (4.4) reduces here to

$$c_1(\mathscr{L}) = -\frac{1}{2} \lim_{\iota \to 0} \int -\frac{c_1(Z_\iota)^2}{3},$$

where $Z_\iota$ is $Z$ with the $\mathscr{H}$-metric stretched by $\varepsilon^{-2}$. Now the holomorphic tangent bundle $T(Z)$ splits in an exact sequence

$$0 \to T(\text{fibre}) \to T(Z) \to T(\text{base}) \to 0.$$

If this was an orthogonal direct sum then the Chern form of $Z$ would be the sum of the Chern forms of fibre and base: in particular, since the Chern form of the (torus) fibre is zero, $c_1(Z)$ would come from the base. In fact the sequence is not a direct sum but it becomes one in the adiabatic limit when $\varepsilon \to 0$. This means that

$$c_1(Z_r) = \pi^* c_1(\text{base}) + 0(\varepsilon)$$

and so $c_1(Z_\iota)^2 = 0(\varepsilon)$, which gives $c_1(\mathscr{L}) = 0$ proving (5.14).

Actually we have here reversed an argument of [9]. Formula (4.4) is really deduced from the formula

(5.15) $$c_1(\mathscr{L}) = -\frac{1}{2} \int_{T^2} L_k(p(\text{fibre})),$$

where $p$ (fibre) stands for the Pontrjagin form of the tangent bundle along the fibres. In our case this is a holomorphic line-bundle which (including its metric) is pulled back from the base. Thus its Chern form comes from the base and hence $p_1 = c_1^2 = 0$ which, by (5.15), implies $c_1(\mathscr{L}) = 0$.

Yet a further proof of (5.14) comes from a direct computation of its curvature. This will also introduce explicitly the Dedekind $\eta$-function. Observe first that the metric on the universal space $T^2 \times H$ is Kähler so that the Bismut-Freed connection on $\mathscr{L}$ coincides with that introduced by Quillen using the holomorphic structure. Actually, it will be convenient first to consider the Quillen line-bundle $\mathscr{L}_\chi$ associated to the family of operators $\bar{\partial}_\chi$ for a fixed character $\chi$ of the fundamental group of the torus. If $\chi \neq 1$, then $\bar{\partial}_\chi$ is invertible and the determinant of the Laplacian $\bar{\partial}_\chi^* \bar{\partial}_\chi$ (defined by its $\zeta$-function) was computed in [33] based on Kronecker's second limit formula as expounded in [34]. In [33] the metric used was that giving total area $r = \text{Im}(\tau)$ for the torus with periods 1 and $\tau \in H$. However, a change of scale does not alter the determinant in view of the fact that the relevant $\zeta$-function vanishes at $s = 0$; in general

(5.16) $$\exp(-\zeta'_{k,\Lambda}(0)) = \exp(-\zeta'_{\Lambda}(0)) \exp(-\zeta(0)\log k).$$

The formula in [33, Theorem (4.1)] shows in particular that

$$\det \bar{\partial}_\chi^* \bar{\partial}_\chi = |f_\chi(\tau)|^2,$$

where $f_\chi(\tau)$ is a *holomorphic* function of $\tau \in H$. According to Quillen's definition of the connection on $\mathscr{L}_\chi$ this means that $\mathscr{L}_\chi$ is flat since it has a holomorphic section $f_\chi(\tau)^{-1} \det \bar{\partial}_\chi$ of norm 1.

Letting $\chi \to 1$ so that $\bar{\partial}_\chi \to \bar{\partial}$ it follows that the Quillen line-bundle of the $\bar{\partial}$-family over $H$ is also flat. Now the signature operator $D = \bar{\partial} + \bar{\partial}_1$, where $\bar{\partial}_1$ is the $\bar{\partial}$-operator on $\Omega^{1,0}$-forms. But on an elliptic curve multiplication by the basic holomorphic differential converts $\bar{\partial}$ into $\bar{\partial}_1$. More precisely $\bar{\partial}_1$ is the tensor product of $\bar{\partial}$ and the identity on the canonical line-bundle. Their determinant line-bundles are therefore isomorphic and so the determinant line-bundle $\mathscr{L}$ of $D$ is also flat, as asserted in (5.14).

For the $\bar{\partial}$-operator (unlike $\bar{\partial}_\chi$ with $\chi \neq 1$) we have to separate out the 0-eigenvalue. The calculation in [33] applied to the non-zero eigenvalues, and adjusted for our normalization of the metric, gives

$$(5.17) \qquad |\det' \bar{\partial}| = |\eta(\tau)(2v)^{\frac{1}{2}}|,$$

where $\eta(\tau)$ is the Dedekind $\eta$-function, defined in (1.1). Here $\det'$ indicates that, in defining the $\zeta$-function and so the determinant, we have omitted the 0-eigenvalue. Note that the modified $\zeta$-function no longer vanishes at $s = 0$: it gives the value $-1$. As a result scale changes alter $\det'$ as given by (5.16) and this has been incorporated in passing from the formula of [33] to (5.17).

Since our operator $D$ is essentially two copies of $\bar{\partial}$ (5.17) leads to

$$(5.18) \qquad |\det' D| = |\eta(\tau)^4 2v|.$$

Now recall the factorization (4.6) expressing $\mathscr{L}$ as the tensor product of $\mathscr{L}'$ and $\mathscr{H}$, corresponding to the non-zero and zero eigenvalues respectively. $\mathscr{L}'$ has a holomorphic section $\det' D$ whose norm is given by (5.18), while (4.12) identifies $\mathscr{H}^*$ as the square of the bundle of holomorphic differentials along the fibres. Thus $\mathscr{H}$ has the holomorphic section $\omega_\tau^{-2}$ whose norm is $(2v)^{-1}$. The product $\omega_\tau^{-2} \det' D$ is therefore a holomorphic section of $\mathscr{L}$ with norm $|\eta(\tau)^4|$, the $v$-factors cancelling out. Identifying $\mathscr{H}^*$ with the cotangent bundle $T^*(H)$, by $\omega_\tau^2 \to d\tau$ it follows that $\mathscr{L}^*$ can be holomorphically identified with $T^*(H)$ with a norm for which

$$(5.19) \qquad \eta(\tau)^4 \, d\tau$$

has norm 1. This characterizes (5.19) up to a constant factor of norm 1.

The line-bundle $\mathscr{L}^*$ over $H$ is acted on naturally by SL $(2, Z)$. It follows that the form (5.19) is invariant, up to roots of unity, by SL $(2, Z)$. This is the well-known modular property of the Dedekind $\eta$-function.

We are now in a position to take up the problem investigated by Dedekind concerning the transformation properties of $\log \eta(\tau)$ under SL $(2, Z)$. We shall concentrate on the interesting case of *hyperbolic* elements $A$ of SL $(2, Z)$. Such an element has two fixed points, $\alpha$, $\beta$ on the real axis (say $\alpha < \beta$) and the differential

$$(5.20) \qquad \omega_A = \frac{du}{u}, \qquad u = i \frac{\tau - \alpha}{\tau - \beta} = \frac{(\beta - \alpha) \, d\tau}{(\tau - \alpha)(\tau - \beta)}$$

is invariant under $A$ and under the whole one-parameter group $u \to \lambda u$ in PSL $(2, R)$ determined by $A$. Hence

$$f(\tau) = \frac{\eta(\tau)^4 \, d\tau}{\omega_A}$$

is a function invariant (up to 6th roots of units) by $A$. Thus

(5.21)                      $\log f(A\tau) - \log f(\tau) = 2\pi i N(A)$

for some rational number $N(A)$ with denominator dividing 6. Here $\log f$ is some definite branch of the logarithm in the upper half-plane: it is immaterial which. Since the functions $f(\tau)$ and $\eta(\tau)^4$ differ by the elementary factors in (5.20) it is clear that Dedekind's problem is essentially equivalent to determining $N(A)$.

In fact it is just a matter of reinterpretation to see that $N(A) = -\frac{1}{2}\chi(A)$, the topological invariant defined in (5.8). To see this recall that (as noted in Sect. 4) the line bundle $L$ of Sect. 2 is just the dual $\mathscr{L}^*$ of the Quillen line-bundle $\mathscr{L}$, and this can also be identified with $T^*(H)$. The differential form $\eta(\tau)^4 d\tau$ raised to the 6-th power is then a section of $(\mathscr{L}^*)^6 = L^6$ which is SL$(2, Z)$-invariant. Since it is nowhere zero it defines a trivialization of $L^6$ and so (by uniqueness) it must define the (homotopy) trivialization given by the SL $(2, Z)$-invariant splitting $\sigma_1$ described at the beginning of this section. Similarly $\omega_A$ defines the trivialization given by $\sigma_0$. Thus $6N(A)$ measures the difference of these two trivializations of $L^6$ which [by the observation following (5.8)] is equal to $3\chi(A)$. There is also a sign change because (as noted earlier) the orientation of the complex torus we have taken corresponds to the opposite of the standard orientation of $R^2$. Thus we have

(5.22)                      $\log f(A\tau) - \log f(\tau) = -\pi i \chi(A)$

so that $\chi(A)$ essentially describes the behaviour of $\log \eta(\tau)$ under the transformation $A$.

The number $\chi(A)$ can also be related to the monodromy of the line-bundle $\mathscr{L}^*$. Because of SL$(2, Z)$-invariance we can consider $\mathscr{L}^*$ as a line-bundle on the quotient $H_A$ of the upper half-plane by the infinite cyclic group generated by $A$. Moreover $\mathscr{L}^*$ has its standard trivialization, defined by the one-parameter group through $\pm A$. Since $\mathscr{L}^*$ is flat the fundamental loop of $H_A$ gives rise to a well-defined logarithmic monodromy $-\pi i \mu(A)$, for some real-valued invariant $\mu(A)$. Now identifying $\mathscr{L}^*$ with the bundle of differentials on $H_A$ with its natural basis $\omega_A$ and using the fact that $\eta(\tau)^4 d\tau$ is a covariant constant section of $\mathscr{L}^*$ it follows from (5.22) that

(5.23)                              $\mu(A) = \chi(A)$.

*Remark.* The preceding discussion applies just as well to parabolic $A$. By conjugation we may take

$$A = \pm \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

giving the translation $\tau \to \tau + k$ on the upper half-plane. The $A$-invariant differential defined by the one-parameter group is just $d\tau$ so that $f(\tau) = \eta(\tau)^4$. Formula (5.22) then enables one trivially to evaluate $\chi(A)$ as

(5.24)                              $\chi(A) = -k/3$.

The monodromy invariant $\mu(A)$ can be computed directly from its definition as follows. Take the semi-circle $S$ in $H$ with diameter $(\alpha, \beta)$, the fixed points of $A$. This gives the unique $A$-invariant geodesic and descends to a closed geodesic $S_A$ in the

quotient $H_A$. Since $\mathcal{H} \cong T(H)$ and $S$ is a geodesic, parallel transport for $\mathcal{H}$ along $S$ coincides with the one-parameter group action, so that $\mathcal{H}$ acquires a natural trivialization with basis $\omega_A^{-1}$. Thus $\pi i \mu(A)$ is the logarithmic monodromy round $S_A$ of the (trivial) line-bundle $\mathcal{L}'$, so that formally (since $D$ is essentially two copies of $\bar{\partial}$)

$$(5.25) \qquad \pi \mu(A) = 2 \Delta_S \arg(\det' \bar{\partial}),$$

where $\Delta_{S_A}$ is the variation in the argument of the determinant on going round $S_A$. The rigorous definition of the connection [9] means that we have to use a $\zeta$-function regularization. Moreover, since $\bar{\partial}$ takes functions to $(0,1)$-forms we have to use the natural connection on the space of $(0,1)$-forms in computing the variation. This means we should use the $A$-invariant basis

$$\sqrt{\frac{\overline{\beta - \alpha}}{2}} \; \frac{dx + \bar{\tau} dy}{\sqrt{(\bar{\tau} - \alpha)(\bar{\tau} - \beta)}}$$

essentially the square-root of (5.20). Relative to the standard exponential basis $\exp(2\pi i(my + nx))$, the family of $\bar{\partial}$-operators along $S$ can be simultaneously diagonalized with eigenvalues

$$(5.26) \qquad \lambda_{(m,n)}(\tau) = -\frac{C}{v}(m - n\tau)\sqrt{(\bar{\tau} - \alpha)(\bar{\tau} - \beta)}, \quad C = \sqrt{\frac{\overline{\beta - \alpha}}{2}}.$$

As we move once round $S_A$ we move by $A$ along the semi-circle $S$ and the eigenvalues given by (5.26) get transformed into one another. A direct verification shows that

$$(5.27) \qquad \lambda_{A(m,n)}(A(\tau)) = \lambda_{(m,n)}(\tau).$$

Here $A$ acts on the lattice $Z^2$ of characters dually to its action on the $(y, x)$ variables, so that

$$A(m, n) = (am + bn, cm + dn), \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now the rigorous version of (5.25) becomes

$$(5.28) \qquad \mu(A) = -\frac{2i}{\pi}\left\{ \sum_{(m,n)}' \int |\lambda_{(m,n)}(\tau)|^{-s} d(\log \lambda_{(m,n)}(\tau)) \right\}_{s=0}.$$

Here the integral is taken along any fundamental arc $(\tau_0, A(\tau_0))$ for the action of $A$ on the semi-circle, the sum $\Sigma'$ is over non-zero lattice points and $s$ is put equal to zero after analytic continuation. Note that $\mu(A)$, as given by (5.28), is real because $|\det \bar{\partial}'|$ is unambiguously defined.

Before proceeding to analyze formula (5.28) we shall make a brief notational digression. From the definition of $\mu$ we have

$$\mu(-A) = \mu(A) = -\mu(A^{-1}).$$

Hence, replacing $A$ by $-A$ if necessary, we can assume

$$(5.29) \qquad \operatorname{Tr} A = a + d > 0.$$

Moreover, replacing $A$ by $A^{-1}$ if necessary, we can then ensure

(5.30)                                            $c < 0$.

Assumption (5.29) means that $A$ lies on a one-parameter group $A_t$. In terms of the variable $u$ defined in (5.20) $A_t$ is given by

(5.31)                        $A_t(u) = \varrho^t u, \qquad \varrho = \lambda_- / \lambda_+ = \lambda^2 < 1,$

where $\lambda_\pm$ are the eigenvalues of $A$ given by

(5.32)                        $\lambda_\pm = \dfrac{(a+d) \pm \sqrt{\Delta}}{2}, \qquad \Delta = (a+d)^2 - 4.$

The condition (5.30) means that

$$\alpha = \frac{(a-d) + \sqrt{\Delta}}{2c}, \qquad \beta = \frac{(a-d) - \sqrt{\Delta}}{2c},$$

from which it follows that $A_1 = A$. Since $\varrho < 1$ in (5.31) it follows that, for any point $\tau$ on the semi-circle $S$ with diameter $(\alpha, \beta)$, we have

(5.33)                        $A_t(\tau) \to \alpha \quad$ as $\quad t \to +\infty$

                                    $A_t(\tau) \to \beta \quad$ as $\quad t \to -\infty$.

The action of $A$ on the lattice $Z^2$ extends to an action of $A_t$ on the plane $R^2 \supset Z^2$, with coordinates $Y, X$ naturally dual to $y, x$. This action of $A_t$ preserves the quadratic form

(5.34)                        $N(Y, X) \equiv cY^2 + (d-a)XY - bX^2$

                                    $= c(Y - \alpha X)(Y - \beta X),$

and the orbit of any point $(Y_0, X_0) \neq (0,0)$ is one branch of the hyperbola

(5.35)                        $N(Y, X) = N(Y_0, X_0).$

Moreover, as $t$ increases, the branches are traversed in the positive sense if $N < 0$ and in the negative sense if $N > 0$ (see figure).
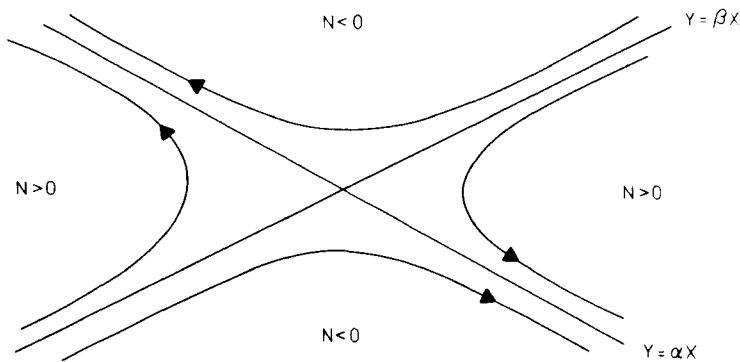
We now extend the definition of $\lambda_{(m,n)}(\tau)$ in (5.26) to non-integer values $(Y, X)$. Then (5.27) continues to hold with $A$ replaced by $A_t$ and $(m, n)$ by $(Y, X)$.

We now return to formula (5.28) and consider all the terms arising from a single $A$-orbit, i.e. from all lattice points

$$(m_k, n_k) = A_k(m_0, n_0) \qquad k \in Z,$$

obtained by applying $A_k = A^k$ to a given point $(m_0, n_0)$. For brevity denote by $f(m, n; \tau)$ the expression being integrated in (5.28). Then (5.27) implies

$$\int_{\tau_0}^{A\tau_0} f(m_k, n_k; \tau) = \int_{\tau_0}^{A\tau_0} f(m_0, n_0; A^{-k}\tau) = \int_{A^k\tau_0}^{A^{k+1}\tau_0} f(m_0, n_0; \tau).$$

Summing over the whole $A$-orbit, and noting (5.33), then leads to the single integral

(5.36)                                  $\int_{\beta}^{\alpha} f(m_0, n_0; \tau).$

Note also that this integral is unchanged if we now replace $(m_0, n_0)$ by any point $(Y, X)$ lying on the same branch of the hyperbola (5.35). This follows from (5.27) by the substitution $\tau \to A_t(\tau)$ with $t$ chosen so that

$$A_t(Y, X) = (m_0, n_0).$$

On each branch of the hyperbola choose the unique point $(Y, X)$ with

(5.37)                          $Y = \gamma_\varepsilon X, \qquad \varepsilon = \pm 1,$

where $\gamma_1, \gamma_{-1}$ are fixed constants with $\gamma_{-1} < \alpha < \gamma_1 < \beta$. Then

$$N(\gamma_1, 1) > 0, \qquad N(\gamma_{-1}, 1) < 0$$

so that

(5.38)                          $\varepsilon = \operatorname{sign} N(Y, X).$

Moreover, the value of $X$ is determined by

(5.39)                          $X^2 = \dfrac{N(Y, X)}{N(\gamma_\varepsilon, 1)},$

the choice of square-root depending on the branch.

Computing the integral (5.36) by using this value of $(Y, X)$ and recalling the formula (5.26) [with $(m, n)$ replaced by $(Y, X)$] we get

(5.40)      $|CX|^{-s} \displaystyle\int_{\beta}^{\alpha} \left| \dfrac{\gamma_\varepsilon - \tau}{v} \right|^{-s} d\log\left\{ \left( \dfrac{\gamma_\varepsilon - \tau}{v} \right) \sqrt{(\bar{\tau} - \alpha)(\bar{\tau} - \beta)} \right\}.$

Since $v = \operatorname{Im}\tau \to 0$ as $\tau \to \alpha$ or $\tau \to \beta$ this integral converges for $\operatorname{Re} s > 0$ and is holomorphic in $s$. Moreover for $\operatorname{Re} s = 0$ the divergent term arises from $d\log v$. This gives rise to a simple pole at $s = 0$ with *real* residue. Thus we can rewrite (5.40) briefly as

(5.41)                          $|CX|^{-s} I_\varepsilon(s),$

where $I_r(s)$ is meromorphic in $s$, depends only on $\varepsilon$ (and $\alpha, \beta$) and near $s = 0$.

(5.42)
$$I_r(s) = \frac{R_r}{s} + (P_r + iQ_r) + \ldots,$$

where $P$, $Q$, $R$ are real and

(5.43)
$$iQ_r = \int_\beta^\alpha d \log [(\gamma_r - \tau) \, \sqrt{(\bar{\tau} - \alpha)(\bar{\tau} - \beta)}].$$

Now, on traversing the semi-circle $S$ from $\beta$ to $\alpha$, one finds the following variation in the arguments

$$\Delta \arg(\gamma_1 - \tau) = \pi$$
$$\Delta \arg(\gamma_{-1} - \tau) = 0$$
$$\Delta \arg \sqrt{(\bar{\tau} - \alpha)(\bar{\tau} - \beta)} = -\pi/2.$$

Hence, from (5.43), we get

(5.44)
$$Q_\varepsilon = \varepsilon \pi/2.$$

Substituting (5.40) [i.e. (5.41)], for the contribution of each $A$-orbit in (5.28) we get

$$\mu(A) = -\frac{2i}{\pi} \{ \Sigma \, |CX|^{-s} I_r(s) \}_{s=0},$$

where the sum is taken over all $A$-orbits. Writing $\Sigma = \Sigma^+ + \Sigma^-$ where $\Sigma^\pm$ is the sum over all $A$-orbits with sign $N = \pm 1$, and using (5.39) we have

(5.45)
$$\mu(A) = -\frac{2i}{\pi} \{ C_+^s I_1(s) \Sigma^+ \, |N|^{-s/2} + C_-^s I_{-1}(s) \Sigma^- \, |N|^{-s/2} \}_{s=0},$$

where $C_\pm$ are the constants

$$C_r = C^{-1} \, |N(\gamma_r, 1)|^{\frac{1}{2}}.$$

We now consider the two "$\zeta$-functions", associated to $A$.

$$\zeta_A^\pm(s) = \Sigma^\pm \, |N|^{-s}$$

and the $L$-function, given by this difference,

(5.46)
$$L_A(s) = \zeta_A^+(s) - \zeta_A^-(s) = \Sigma (\operatorname{sign} N) \, |N|^{-s}$$

the sums being over the appropriate $A$-orbits of the lattice and $N$ being as before the quadratic form (5.34) defined by $A$. In a slightly different notation these are familiar quantities in the theory of real quadratic fields (here the field is generated by the eigenvalues of $A$). In particular all these functions have meromorphic continuations in $s$ [from the region $\operatorname{Re}(s) > 1$ of convergence] and are finite at $s = 0$. The values of the functions and their derivatives at $s = 0$ are all real. Using these facts and the value (5.44), for the relevant coefficient $Q$ in the expansion (5.42), formula (5.45) leads to

(5.47)
$$\mu(A) = L_A(0).$$

We recall that, in establishing (5.47), we made the inessential assumptions (5.29) and (5.30). We can now remove this restriction, and still have formula (5.47), provided we redefine the quadratic form $N$ by

(5.48)     $$N(Y, X) = \operatorname{sign}(a+d)(cY^2 + (d-a)XY - bX^2).$$

Finally therefore we have established the following:

(5.49) **Proposition.** *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a hyperbolic element of* $\operatorname{SL}(2, \mathbb{Z})$ *with associated quadratic form* $N(Y, X)$ *given by* (5.48). *Define the L-series* $L_A(s)$ *by*

$$L_A(s) = \sum \frac{\operatorname{sign} N}{|N|^s},$$

*where the sum is over (non-zero) A-orbits of the integer lattice. Let* $\mathscr{L}$ *be the determinant line-bundle over H for the family of signature operators of the torus bundle over H, and let* $\pi i \mu(A)$ *be the logarithmic monodromy of* $\mathscr{L}$ *round the basic loop* $S_A$ *in* $H_A$ *relative to the trivialization determined by the one-parameter group through* $\pm A$. *Then*

$$\mu(A) = L_A(0).$$

*Remarks.* 1) The calculations involved in deriving this formula for $\mu(A)$ are similar to calculations in [34], based on ideas of Hecke, for computing $L_A(1)$. The functional equation for $L_A(s)$ enables one to determine the values for $s = 0, 1$ from one another, but $s = 0$ is the value which is most natural from the geometric point of view.

2) The $L$-function $L_A(s)$ in (5.49) is essentially the same as the Shimizu $L$-function introduced in [16]. We embed the lattice $\mathbb{Z}^2$ in the quadratic field $K = Q(\sqrt{\Delta})$ by sending

$$(1, 0) \to -1, \qquad (0, 1) \to \alpha.$$

For the dual lattice we replace $\alpha$ by $-\alpha^{-1}$. Hopefully our sign conventions are consistent with those of [16].

Hirzebruch [16] showed that

(5.50)                              $$L_A(0) = \delta(A)$$

where $\delta(A)$, the "signature defect", is closely related to Meyer's invariant $\phi(A)$. It is defined by the formula

(5.51)                              $$\operatorname{sign} Z = \tfrac{1}{3} p_1(Z, \partial Z) - \delta(A),$$

where $Z$ is any oriented 4-manifold with boundary $W(A)$ (the 2-torus bundle fibred over $S^1$ defined by $A$), and the relative Pontrjagin class $p_1$ is defined by the natural parallelism on $W(A)$ given by the eigenvectors of $A$. We shall shortly give a direct proof that

(5.52)                              $$\delta(A) = \phi(A).$$

Thus Hirzebruch's result essentially identifies the analytic invariant $L_A(0)$ with the topological invariant $\phi(A)$. Hirzebruch used the resolution of "cusp singularities"

to construct explicit choices of $Z$ and compared this with the explicit evaluation of $L_A(0)$ (see [16]).

In [5] a direct analytical proof based on [6] was given that

(5.53)                               $$L_A(0) = \eta(A) = \delta(A),$$

where $\eta(A) = \eta(W(A))$ with the standard metric given by the $A$-invariant geodesic $S$ in $H$. Moreover, the result in [5] applied to totally real fields of any degree whereas here we are discussing only the quadratic case.

In our approach, combining (5.10), (5.12), (5.23), and (5.49) we get an independent direct proof of the equalities

(5.54)                               $$L_A(0) = \eta^0(A) = \phi(A).$$

We will shortly give a direct proof of:

(5.55)                               $$\eta(A) = \eta^0(A)$$

so that [in view of (5.52)], (5.53) is essentially the same as (5.54). Thus the Bismut-Freed results which we have used can be viewed as a generalization of the results in [5] (at least for quadratic fields). In fact the techniques used in both papers have common features, notably the use of the "adiabatic" limit.

As promised we shall now give direct proofs of (5.52) and (5.55) in the form of two lemmas. We begin with the latter.

(5.56) **Lemma.** *For any hyperbolic element $A$ of* SL$(2, Z)$, $\eta(W(A)) = \eta^0(W(A))$, *i.e. the adiabatic limit is unnecessary.*

*Proof.* We have to show that $\eta(W(A))$ is unchanged if we rescale the torus metric, or equivalently if we rescale the circle metric. Now the standard metric on the universal covering $R^2 \times R$ of $W(A)$ is

$$ds^2 = e^{\lambda t} dx^2 + e^{-\lambda t} dy^2 + dt^2$$
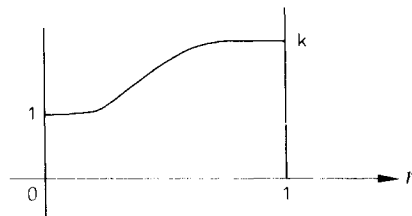


relative to $(x, y)$ coordinates of $R^2$ given by the eigenvectors of $A$, with $e^{\pm \lambda}$ being the corresponding eigenvalue of $A^2$. Now let $\varrho(r)$ be a function of the real variable $r$ with graph as indicated. Consider the metric on $R^2 \times R \times R$ given by

(5.57)                     $$e^{\lambda t} dx^2 + e^{-\lambda t} dy^2 + \varrho(r)^2 dt^2 + dr^2.$$

This induces a metric on the 4-manifold $Z = W(A) \times I$ which is a product near the boundary, giving the standard metric on $W(A) \times 0$ and this metric rescaled by a

factor $k^2$, in the circle direction, on $W(A) \times 1$. By the main result of [6] the difference of the $\eta$-invariants for the two boundary components is therefore given by $\frac{1}{3} \int_Z p_1$, where $p_1$ is the Pontrjagin form for the metric (5.57). It is therefore sufficient to show that $p_1 = 0$. But (5.57) is conformally equivalent (by the factor $e^{-\lambda t}$) to

$$dx^2 + \{e^{-2\lambda t} dy^2 + e^{-\lambda t} \varrho(r)^2 dt^2 + e^{-\lambda t} dr^2\}$$

which is a product metric on $R^1 \times R^3$. Now for such product metrics $p_1 = 0$ trivially, and quite generally $p_1$ is conformally invariant. Hence $p_1 = 0$ and the lemma is established.

Next we prove:

(5.58) **Lemma.** *For any hyperbolic element $A$ of* SL$(2, Z)$, *the signature defects of Hirzebruch and Meyer coincide, i.e.* $\delta(A) = \phi(A)$.

*Proof.* Assume first that $A$ lies in the commutator subgroup $\Gamma'$ of $\Gamma = $ SL$(2, Z)$. Then we can choose the 4-manifold $Z$ with boundary $W(A)$ to be a 2-torus bundle over a surface $X$ with just one bounding circle: we use the appropriate representation $\pi_1(X) \to$ SL$(2, Z)$. Then we apply the argument used earlier, mapping $Z \to Z(m)$, where each torus is factored by the points of period $m$. Since $Z(m)$ is essentially the same manifold as $Z$, while the relative $p_1$ gets divided by $m^2$, (5.51) applied to $Z(m)$ shows that the $p_1$ term must vanish. Comparison with (5.5) then shows $\delta(A) = \phi(A)$. In general since $\Gamma'$ has finite index in $\Gamma$ (in fact index 12), we get $\delta(A^k) = \phi(A^k)$ for some integer $k$. It is therefore sufficient to establish

(5.59)                          $\phi(A^k) = k\phi(A), \qquad \delta(A^k) = k\delta(A)$.

The first part of (5.59) follows from (5.9) and (5.10). For $\delta$ we shall use the fact that we can always choose the 4-manifold $Z$ with boundary $W(A)$ to have a map to $S^1$ extending the projection $W(A) \to S^1$. This follows easily from the vanishing of the oriented cobordism groups in dimensions 2 and 3. Note also that the explicit model for $Z$ given by the cusp resolution of Hirzebruch [16], which we shall meet later, has this property. We can then take the $k$-fold cover $\tilde{Z}$ of $Z$ induced by the $k$-fold cover of $S^1$. Then $\partial\tilde{Z} = W(A^k)$. Comparing (5.51) for $Z$ and $\tilde{Z}$, and using the multiplicativity of the $p_1$-term we see that

$$\delta(A^k) - k\delta(A) = -[\text{sign } \tilde{Z} - k \text{ sign } Z].$$

But for such a cyclic covering without fixed points the general formula (4.16) gives

$$\eta(A^k) - k\eta(A) = -[\text{sign } \tilde{Z} - k \text{ sign } Z].$$

This is true for all metrics hence $\eta$ can be replaced by $\eta^0$ [or appeal to (5.56)]. Then (5.12) replaces $\eta^0$ by $\phi$ and we already know (5.59) for $\phi$. Thus $\delta(A^k) = k\delta(A)$ as required.

*Remark.* The last part of this proof is somewhat circuitous. It uses [6] but not [9], nor does it depend on explicit computations. An alternative and more natural method is to use the manifold $Z$ constructed by Hirzebruch [16], and to verify sign $\tilde{Z} = k$ sign $Z$ directly.

It may now be convenient to summarize all our results in an omnibus theorem:

(5.60) **Theorem.** *Let $A \in \mathrm{SL}\,(2, Z)$ be hyperbolic. Then the following invariants of $A$ coincide.*

1) *Meyer's signature invariant $\phi\,(A)$ (see (5.3)).*

2) *Hirzebruch's signature defect $\delta\,(A)$.*

3) *The invariant $\chi\,(A)$ describing the transformation properties of $\log \eta\,(\tau)$ under $A$ (see (5.22)).*

4) *$\mu\,(A)$ the logarithmic monodromy (divided by $\pi i$) of Quillen's determinant line-bundle $\mathscr{L}$.*

5) *The value $L_A(0)$ of the Shimizu L-function (see (5.49)).*

6) *The Atiyah-Patodi-Singer invariant $\eta\,(A)$.*

7) *The "adiabatic limit" $\eta^0\,(A)$.*

*Remark.* For brevity we have not spelled out in the theorem all the relevant data on which these invariants depend, for example the parallelism of $W(A)$ [to define $\delta(A)$], the trivialization of $\mathscr{L}$ needed to define $\mu\,(A)$, or the choice of metric on $W(A)$ to define $\eta\,(A)$ or $\eta^0\,(A)$. These were explained earlier.

Because of the large number of quantities (all equal) involved in the theorem, it may be helpful to recall briefly the way in which they are related and the order in which the equalities are established.

The two invariants $\phi$ and $\delta$ are of very similar cohomological character, both being "signature defects". The main difference is that $\phi$ is only defined for torus bundles over $S^1$ whereas $\delta$ is defined for all parallelized 3-manifolds. Despite its apparent analytical nature $\chi$ is also a cohomological invariant as is clear from our original definition (5.8). (See also the remarks in the introduction.) The equality $\phi = \chi$ is, as noted before, a "signature theorem" and is essentially proved in Sect. 2 (with an alternative in Sect. 3). The monodromy $\mu$ uses the analyticity of $\eta\,(\tau)$ and its appearance in the formula for $|\det \bar{\partial}'|$, but the equality $\chi = \mu$ is then an immediate consequence. The formula $\mu\,(A) = L_A(0)$ is a classical but straightforward computation. The equality $\mu = \eta^0$ is a refinement of the general Bismut-Freed result and is proved by identifying both terms with $\phi$. The equality $\phi\,(A) = \eta^0\,(A)$ is in fact true for all $A \in \mathrm{SL}\,(2, Z)$, not just hyperbolic $A$. Finally $\eta = \eta^0$ was established by a direct elementary computation based on conformal invariance.

## 6. Computations and Dedekind Sums

We shall now show how to give explicit formulae for the invariant $\phi$ of Sect. 5. We recall that

$$\phi:\ \mathrm{SL}\,(2, Z) \to Q$$

is a class-function, and that the conjugacy classes in $\mathrm{SL}\,(2, Z)$ are of 3 types

  (i) elliptic,

  (ii) parabolic,

  (iii) hyperbolic.

Moreover, there are few classes in (i) and (ii), most classes being hyperbolic. We shall begin by considering the elliptic classes. In principle, we could use the explicit

determination of $\sigma(A)$ in Sect. 2 to compute $\phi(A)$, but we shall instead use the fixed-point methods described in Sect. 4.

If $A \in \mathrm{SL}(2, Z)$ is of finite order $N$ then the associated 3-manifold $W(A)$ is the quotient of

$$\widetilde{W} = T^2 \times S^1$$

by a cyclic group of order $N$ acting simultaneously on both factors. Moreover, the (product) metric can be chosen so that this action is isometric. Since $S^1$ and hence $\widetilde{W}$ admits an orientation-reversing isometry we have $\eta(\widetilde{W}) = 0$. Hence applying (4.16) and (4.18) for the finite covering $\widetilde{W} \to W$ and, noting that the quadratic form on $H^2(T^2 \times D^2)$ is zero, we get

$$(6.1) \qquad\qquad \eta(W) = -\frac{1}{N} \sum_{k=1}^{N-1} \sum_P \cot\frac{\alpha}{2} \cot\frac{\beta}{2},$$

where the second summation is over the fixed points of $A^k$ acting on $T^2 \times D^2$ (where $D^2$ is the unit disc), and $\alpha$, $\beta$ are the corresponding rotation angles.

As noted in Sect. 5 the adiabatic limit is irrelevant for elliptic elements so that by (5.12)

$$\eta(W(A)) = \eta^0(W(A)) = \phi(A)$$

are all given by (6.1), and it is then a simple matter to carry out the computation explicitly in each case. Since $\phi(1) = 0$, $\phi(A^{-1}) = -\phi(A)$ we also have $\phi(-1) = 0$ and it is enough to consider the following 3 cases.

(i) $A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ of order 6; $A$ and $A^5$ have one fixed point with both angles $\pi/3$, $A^2$ and $A^4$ have 3 fixed points with all angles $2\pi/3$, $A^3$ has 4 fixed points with angles $\pi$;

$$\phi(A) = -\tfrac{1}{6}\{3 + 3(\tfrac{1}{3}) + 0 + 3(\tfrac{1}{3}) + 3\} = -\tfrac{4}{3};$$

(ii) $A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ of order 3, fixed points as above (for $A^2$)

$$\phi(A) = -\tfrac{1}{3}\{1 + 1\} = -\tfrac{2}{3};$$

(iii) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of order 4: $A$ and $A^3$ have 2 fixed points with both angles $\pi/2$ while $A^2$ has 4 fixed points with angles $\pi$;

$$\phi(A) = -\tfrac{1}{4}\{2 + 2\} = -1.$$

We turn next to the parabolic elements. Perhaps the most elegant and interesting way to deal with these is to consider *elliptic surfaces*, i.e. complex analytic families of elliptic curves arising from a holomorphic map

$$f: Z \to X$$

of a compact analytic surface $Z$ onto an algebraic curve $X$. The generic fibre $f^{-1}(x)$ is assumed to be an elliptic curve but there are special exceptional fibres which degenerate. We will assume there are no multiple fibres. The monodromy round each such fibre is then [20] parabolic or elliptic.

The signature of such an elliptic surface is given in terms of its Chern classes $c_1$, $c_2$ by

$$\text{sign}(A) = \frac{c_1^2 - 2c}{3}.$$

Now the canonical divisor lies purely in the fibres so that $c_1^2 = 0$ while the Euler number $c_2$ is easily seen to be given by the Euler numbers of the exceptional fibres.

$$c_2 = \sum_i e(F_i)$$

so that

$$\text{sign}(Z) = -\frac{2}{3} \sum_i e(F_i).$$

On the other hand letting $Z' = Z - \bigcup_i F_i$ we have

$$\text{sign}(Z) = \text{sign}(Z') + \sum_i \text{sign}\, F_i,$$

where for simplicity we have put $F_i$ instead of $f^{-1}(D_i)$ with $D_i$ a small disc around $x_i = f(F_i)$.

Hence, for $Z'$, we have

(6.2) $$\text{sign}(Z') = -\sum_i \left( \frac{2}{3} e(F_i) + \text{sign}\, F_i \right).$$

Comparing (6.2) with (5.5) strongly suggests that

(6.3) $$\phi(A) = \tfrac{2}{3} e(F) + \text{sign}(F),$$

where $A$ is the monodromy around the exceptional fibre $F$. Here the monodromy is defined by the external orientation (i.e. as the boundary of $Z'$). However, this coincides with the standard algebraic-geometric convention for the monodromy (relative to the internal orientation near $F$) because of the different orientation used for the torus (see Sect. 5).

There are various possible ways to prove (6.3). The most direct would be to replace $\phi(A)$ by $\eta^0(A)$ and to deduce (6.3) by differential-geometric methods, from (4.1) applied to a neighbourhood of $F$ (taking the adiabatic limit). A second approach would be to exhibit sufficiently many global examples of elliptic surfaces so that (6.3) would follow from (6.2) by linear independence. We shall adopt a variant of this method.

The degenerate fibres $F$ have been classified by Kodaira [20] and the corresponding monodromy matrices $A$ are either elliptic or conjugate to $\pm U^k$ with $k > 0$, where

(6.4) $$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For the elliptic cases (6.3) can be directly verified using our formula above for $\phi(A)$ and Kodaira's description of the degenerate fibres. Note that, if $F$ has $r$ components, then $\text{sign}(F) = -(r-1)$.

To compute $\phi(U)$ we consider the standard elliptic surface $Z$ obtained by blowing up 9 general points in the plane. Each $F$ is then a nodal cubic and the corresponding monodromy is just $U$. Since all terms in the summation in (6.2) [and (5.5)] are equal we can deduce (6.3) for $A = U$. Moreover sign $F = 0$, $e(F) = 1$ so that

(6.5)                                             $$\phi(U) = \tfrac{2}{3}.$$

To compute $\phi(U^k)$ take the standard fibration $Z \to P_1$ above, consider the induced fibration $Z' \to P_1$ given by the $k$-fold cover $P_1 \to P_1$ ($y = x^k$), and then resolve its singularities to give $Z'' \to P_1$. If we choose the branch point $x = 0$ to give an exceptional fibre of $Z$ then $y = 0$ gives an exceptional fibre of $Z''$ with monodromy $U^k$. Moreover, the exceptional fibres of $Z''$ are either of type $U$ or $U^k$. Since (6.3) has been proved for $U$ the global formula (6.2) proves (6.3) also for $U^k$. The corresponding fibre $F_k$ consists of a cycle of $k$ rational curves so that

$$\text{sign}(F_k) = -(k-1), \qquad e(F_k) = k$$

(6.6)                                   $$\phi(U^k) = -k/3 + 1.$$

An alternative way of deriving (6.6) is to consider the cyclic cover $W(U^k) \to W(U)$ and apply the methods described at the end of Sect. 4. The 3-manifold $W(U^k)$ is the boundary of a neighbourhood $Z_k$ of the exceptional fibre $F_k$ and the cyclic group of order $k$ acts naturally without fixed points. Thus (taking care with orientation conventions) we get

$$\phi(U^k) = k\phi(U) - (k-1)$$

which, together with (6.5), yields (6.6).

Similarly, to compute $\phi(-U^k)$ we can consider the double covering $W(U^{2k}) \to W(-U^k)$. The induced involution $\sigma$ on $Z_{2k}$ has 4 fixed points and $\text{sign}(Z_{2k}, \sigma) = -1$, so that by (4.16), (and recalling again that we have the "wrong" orientation)

$$\phi(-U^k) - \tfrac{1}{2}\phi(U^{2k}) = -\tfrac{1}{2}$$

(6.7)                                   $$\phi(-U^k) = \frac{-k}{3}.$$

This completes the computation of $\phi$ for all elliptic and parabolic elements, and our results agree with those of Meyer which are based directly on the defining property (5.3) of $\phi$ and the explicit computation of $\text{sign}(A, B)$.

We come now to the more interesting case of hyperbolic elements. Here we shall work with actual *matrices*, not just conjugacy classes, and we begin with the simple case

(6.8)                     $$A = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \qquad a > 2.$$

As usual we can form the 3-manifold $W(A)$, fibred over $S^1$ with $T^2$ as fibre and $A$ as monodromy. We shall construct explicitly a 4-manifold $Z$ with boundary $W(A)$ and use (5.51) to compute the signature defect $\delta(A)$, using the standard parallelism of $W(A)$ given by the eigenvectors of $A$. In view of (5.52) we have $\phi(A) = \delta(A)$ and so $\phi(A)$ will be computed in this case. Moreover, our manifold $Z$ will be a complex manifold and we can replace $p_1$ by $c_1^2 - 2c_2$.

The construction of $Z$ and the computation of $\delta(A)$ is a special case of the construction given in [16] by Hirzebruch. In fact $Z$ is just the neighbourhood of a nodal rational curve $\Gamma$ with normal degree $-a$. The canonical divisor of $Z$ is $-\Gamma$ so that, allowing for the node,

$$c_1^2 = \Gamma^2 = -a + 2$$
$$c_2 = e(\Gamma) = 1$$
$$\text{sign } Z = -1.$$

Now our orientation of $W(A)$ turns out to be opposite to the orientation of $\partial Z$, induced by the complex orientation of $A$. Hence

(6.9)                 $$\delta(A) = -\left\{\frac{c_1^2 - 2c_2}{3} - \text{sign } Z\right\} = \frac{a}{3} - 1.$$

These calculations are similar to those made above for the exceptional fibres of elliptic surfaces except that there the signature term was zero. One must of course check that the trivialization of the canonical bundle on $W(A) = \partial Z$ given holomorphically is consistent with our parallelism.

Since we shall be using $Z$ again shortly it is convenient to give its explicit construction at this stage. We start with the line-bundle of degree $-a$ over the projective line $P_1$. If we parametrize $P_1$ by two local coordinates $u_0$ and $v_1$ with $v_1 = u_0^{-1}$, the corresponding fibre coordinates $u_1$ and $v_0$ will be related by $u_1 = u_0^a v_0$. If we now make the further identification $(u_0, u_1) \to (v_0, v_1)$ we clearly find the nodal rational curve $\Gamma$ with normal degree $-a$ (see figure)
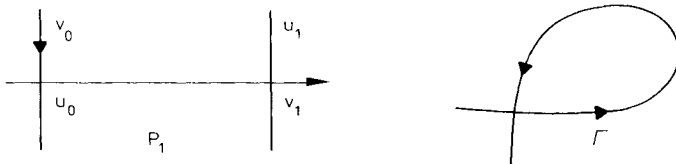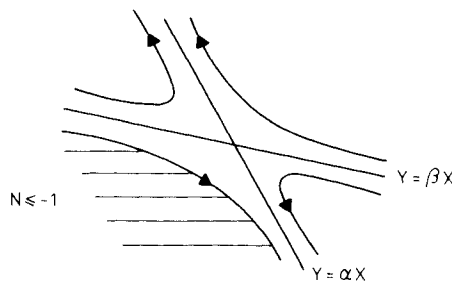


To be precise we must describe appropriate neighbourhoods for the identification. This will then give us an explicit neighbourhood $Z$ of $\Gamma$ with boundary $W(A)$. Taking logarithms we get the equations

$$\log u_1 = a \log u_0 + \log v_0$$
$$\log v_1 = -\log u_0$$

exhibing the matrix $A$. Taking absolute values and putting $Y = \log|u|$, $X = \log|v|$ we have as in Sect. 5 the figure associated to the quadratic form

$$N(Y, X) - (Y^2 + aXY + Y^2)$$

Note that the roots $Y = \alpha X$, $Y = \beta X$ of $N(Y, X) = 0$ are both negative as indicated in the figure. The matrix $A$ acts on the $Y$, $X$ plane preserving $N(Y, X)$ and hence the hyperpolas $N(Y, X) = $ constant. Consider the region $N(Y, X) \leqq -1$ and take the component containing the negative quadrant (shaded region). The manifold $Z$ is defined by this constraint on the absolute values and it is clear that the boundary is precisely $W(A)$. Our orientation of $W(A)$ is opposite to that coming from the complex structure of $Z$ because we take the imaginary parts (of the logarithms) first (giving the torus), followed by the real parts (illustrated in the figure).

Now let us pass to a general hyperbolic element

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

As before, without loss of generality, we may assume $c < 0$, $a + d > 0$. We define

$$B = \begin{pmatrix} a & -bc \\ -1 & d \end{pmatrix}$$

and note that $B = DB'D^{-1}$ where

$$B' = \begin{pmatrix} a+d & 1 \\ -1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}.$$

Hence, by (6.9) we know $\phi(B)$, namely

(6.10)                               $$\phi(B) = \phi(B') = \frac{a+d}{3} - 1.$$

We shall compute $\phi(A)$ by relating $W(A)$ to $W(B)$ and then using (6.10). So let

$$C: \ Z^2 \to Z^2$$

be the embedding of lattices given by

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -c \end{pmatrix}.$$

Since $AC = CB$ the matric $C$ induces a $|c|$-fold covering $T^2 \to T^2$ which is compatible with the actions of $B$ and $A$ on the two tori. Hence we get an induced $|c|$-fold covering

(6.11)                                    $$W(B) \to W(A).$$

Now let us introduce the manifold $Z$ constructed above (with $a + d$ replacing $a$) whose boundary is $W(B)$. We recall that $Z$ arises from the line-bundle of degree $-(a+d)$ over $P_1$ by suitable identifications. Now the cyclic group of order $|c|$ acts naturally on the line-bundle. In terms of the local coordinates this is given by

$$(u_0, v_0) \to (\zeta u_0, \zeta^{-a} v_0)$$
$$(u_1, v_1) \to (\zeta^d u_1, \zeta^{-1} v_1),$$

where $\zeta$ is a primitive root of unity. Note that these formulae are consistent with the equations of the line-bundle

$$u_1 = u_0^{a+d} v_0$$
$$v_1 = u_0^{-1}.$$

The identification $(u_0, v_0) \to (u_1, v_1)$ which produces $Z$ is *not* compatible with this action. However, the identification just corresponds to replacing $\zeta$ by another primitive root, namely $\zeta^d$ (note that $ad - bc = 1$, so that $a \equiv d^{-1} \bmod c$). Hence we get a well-defined *equivalence relation* on $Z$. This induces the covering $W(B) \to W(A)$ on the boundary and has a unique fixed point of type $(1, -a)$.

We are now just in the position discussed at the end of Sect. 4 for computing the deviation from multiplicativity of $\eta$-invariants. Moreover, this deviation is independent of the choice of metric and so we can pass to the adiabatic limit $\eta^0$. Since we have shown (5.12) that $\eta^0(W(A)) = \phi(A)$, and since $W(A)$ has the negative orientation of $\partial Z$, formula (4.22) with $(a, c) \to (-a, -c)$ gives

$$(6.12) \qquad -\phi(A) + \frac{1}{|c|} \phi(B) = -4s(-a, |c|) - \left( \operatorname{sign} Z' - \frac{1}{|c|} \operatorname{sign} Z \right)$$

$$= 4s(a, c) - \left( -1 + \frac{1}{|c|} \right).$$

Substituting the formula (6.10) for $\phi(B)$ and recalling that $c < 0$ we get the formula for $\phi(A)$ in the hyperbolic case:

$$(6.13) \qquad \phi(A) = -\frac{a+d}{3c} - 4s(a, c) - 1 \quad \text{for} \quad c < 0, \; a + d > 0.$$

Since $\phi(-A) = \phi(A) = -\phi(A^{-1})$ and $s(a, c) = s(d, c)$ we can drop the conditions on $c$ and $a + d$, giving finally

$$(6.14) \qquad \phi(A) = -\frac{a+d}{3c} + 4 \operatorname{sign} c \cdot s(a, c) + \operatorname{sign} c(a + d) \quad \text{for} \quad c \neq 0.$$

This formula was established by Meyer [25] by quite different methods.

As explained in Sect. 4 this approach to computing $\phi(A)$ arises from the natural cobordism implicitly constructed above between the 3-manifold $W(A)$ and the lens space $L(|c|, -a)$. Formula (6.13) expresses the difference of their signature defects in terms of the relative (rational) Pontrjagin class of the cobordism. To get the cobordism we remove a neighbourhood of the fixed point in $Z$ and pass to the quotient, i.e. we cut out the singular point of $Z'$. Note that, if $A$ is a diffeomorphism of any manifold $T$ and $T = \partial X$, there is a natural cobordism between $W(A)$ (the fibre bundle over $S^1$ with $T$ as fibre and $A$ as monodromy) and $M(A) = X \cup X$ (the double of $X$ using $A$ to glue the common boundary). When $T = T^2$, $X = S^1 \times D^2$ and $M(A)$ is a lens space.

If we resolve the cyclic singularity of $Z'$ we get a manifold $\tilde{Z}(A)$, still having $W(A)$ as boundary. This contains a cycle of rational curves consisting of the resolution of $P$ together with the transform of $\Gamma'$ (image of $\Gamma$ in $Z'$). This cycle is just the Hirzebruch resolution [16] of the "cusp" associated to $A$. Thus a cusp singularity may be resolved in two steps, the first leading to a cyclic singularity lying on a unique exceptional curve and the second being the standard resolution of a cyclic singularity. This relationship between cusp and cyclic singularities can again be viewed as the general explanation of the formula (6.13). As in Hirzebruch [16, p. 44] the manifold $\tilde{Z}(A)$, containing the resolution of the cusp, can be used to compute the signature defect $W(A)$. The resulting formula is different from (6.13).

It involves the integers occurring in the Euclidean algorithm for the pair $(a, c)$. By comparing this with (6.13) we get another formula for the Dedekind sum which is proved differently by Rademacher [31]. This formula can also be obtained by just using the resolution of a cyclic singularity to compute its signature defect.

The relation between $W(B)$ and $W(A)$ which we have just used to compute $\phi(A)$ from $\phi(B)$ in the hyperbolic case can also be used in the elliptic and parabolic cases. There are minor modifications in the answers due to sign changes. The manifold $Z(B)$ exists in all cases as the boundary of a nodal rational curve $\Gamma$ of degree $-(a+d)$. The difference is that, for $a+d>0$,

$$\Gamma^2 = -(a+d)+2$$

is negative in the hyperbolic case, zero in the parabolic case and positive in the elliptic case. Formula (6.10) for $\phi(A)$ in the hyperbolic case generalizes to

(6.15)                                  $$\phi(B) = \frac{a+d}{3} + \varepsilon,$$

where $\varepsilon = -1, 0, +1$ according as $B$ is hyperbolic, parabolic or elliptic. This can be checked directly from the explicit formulae for the elliptic and parabolic elements given earlier. In fact for the parabolic case we gave a direct geometric proof on the lines of (6.9) in the hyperbolic case. Now apply (6.12), substituting $\phi(B)$ from (6.15) and noting that

$$\text{sign}(Z) = \text{sign}(Z') = \varepsilon.$$

This gives

(6.16)            $$\phi(A) = -\frac{a+d}{3c} - 4s(a, c) + \varepsilon \quad \text{for} \quad c < 0, \, a+d > 0$$

and hence the general formula (see [26]):

(6.17)     $$\phi(A) = -\frac{a+d}{3c} + 4 \,\text{sign}\, c \cdot s(a, c) - \varepsilon \,\text{sign}\, c(a+d) \quad \text{for} \quad c \neq 0.$$

Of course, if $c = 0$, $A$ is parabolic and we have the elementary formulae for $\phi(A)$, given in (6.6) and (6.7).

The Dedekind formula (1.3) for the transformation of $\log \eta(\tau)$ under $A$, in the elliptic or parabolic case, is then easily deduced from the formula (6.17) for $\phi(A)$. The argument is formally similar to the hyperbolic case but much more elementary. For parabolic $A$ our computations of $\phi(A)$ in (6.6) and (6.7) and of $\chi(A)$ in (5.24) show that

(6.18)                          $$\phi(A) = \chi(A) + 1 \quad \text{if} \quad A = U^k$$

                               $$= \chi(A) \quad \text{if} \quad A = -U^k.$$

Since $\chi(A)$ essentially describes the effect of $A$ on $\log \eta(\tau)$ formulae (6.18) and (6.17) lead to (1.3). In the elliptic case we took $\chi(A) = 0$ because, for a finite group, the rational cohomology is trivial and $\eta(\tau)^{24} \, d\tau^6$ can be homotopically identified as giving the unique equivariant trivialization of $T^{-6}$ ($T$ the tangent bundle of $H$). The formulae at the beginning of this section for $\phi(A)$ with $A$ elliptic therefore replace (6.18), and (6.17) then leads to (1.3).

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