Math. Ann. 274, 157–176 (1986)

Mathematische Annalen

Parallelizability of Homogeneous Spaces, II

Wilhelm Singhof and Dieter Wemmer

Mathematisches Institut, Universität Köln, Weyertal 86–90, D-5000 Köln 41, Federal Republic of Germany

1. Introduction

In this paper, we study the parallelizability of manifolds of the form G/H where G is a simple simply connected compact Lie group and H a closed connected subgroup. That G is simple means that its Lie algebra $\mathfrak{g}$ is simple; some authors call such a group almost simple.

Theorem 1. *Let G be a simple 1-connected compact Lie group and H a closed connected subgroup. Then G/H is stably parallelizable if and only if the adjoint representation Ad_H of H is contained in the image of the restriction map of real representation rings $RO(G) \to RO(H)$.*

Observe that Theorem 1 allows to construct explicit framings of such manifolds G/H as soon as they are stably parallelizable: Extend e.g. the method of Proposition 4.1 of [20] to virtual representations.

One direction of Theorem 1 is well known: Atiyah and Hirzebruch [3] introduced a homomorphism of rings $\alpha: RO(H) \to KO(G/H)$ as follows: If $\varrho: H \to \mathrm{GL}(V)$ is a representation of H in a vector space V, let $\alpha(\varrho)$ be the vector bundle over G/H with total space $G \times_H V$. In [9] it was observed that the tangent bundle $\tau(G/H)$ equals $\alpha(\iota)$ where $\iota = \mathrm{Ad}_G|H - \mathrm{Ad}_H$ is the isotropy representation. Naturality of α then implies that G/H is stably parallelizable if Ad_H is contained in the image of the restriction map $RO(G) \to RO(H)$ (cf. I (2.4) – by I we denote [25]).

The opposite direction is quite cumbersome in that it depends on the explicit classification of all stably parallelizable homogeneous spaces of the above type in Theorem 2 below; it would be interesting to have an a priori proof.

We do not know whether Theorem 1 remains true when G is a product of simple groups. On the other hand, the result is wrong if G is not assumed to be simply connected. Here is a counter-example: Take $G = \mathrm{SO}(8)$ and $H = \mathrm{SU}(4)$ and embed H into G by realification. Then Ad_H is not contained in $\mathrm{Im}(RO(G) \to RO(H))$. However, the inclusion $\mathrm{SO}(7) \to \mathrm{SO}(8)$ induces a diffeomorphism $\mathrm{SO}(7)/\mathrm{SU}(3) \to \mathrm{SO}(8)/\mathrm{SU}(4)$, and $\mathrm{SO}(7)/\mathrm{SU}(3)$ is stably parallelizable by I (2.4).

Theorem 2. *Let G and H be as in Theorem 1. Then G/H is stably parallelizable if and only if either H is abelian, or if G/H is diffeomorphic to one of the following manifolds:*

(i) *the real, complex, or quaternionic Stiefel manifolds;*

(ii) $SU(n)/(SU(2)\times \dots \times SU(2)) = X_{n,k}$ *where k denotes the number of factors* SU(2), *with* $2k \leqq n$;

(iii) SU(4)/SO(4);

(iv) $SU(4)/H_0$, *where* H_0 *is the subgroup isomorphic to* SU(2) *consisting of all matrices* $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ *with* $A \in SU(2)$;

(v) $Sp(n)/(Sp(1)\times \dots \times Sp(1)) = Y_{n,k}$ *where k denotes the number of factors* Sp(1), *with* $k \leqq n$;

(vi) $Sp(n)/SU(2)$, $n \geqq 2$;

(vii) $Spin(n)/Ad_H(H)$ *with H 1-connected and* $n \geqq \max\{5, \dim H\}$;

(viii) $Spin(n)/SU(2)$, $n \geqq 5$;

(ix) $Spin(n)/SU(3)$, $n \geqq 6$;

(x) $Spin(n)/SU(4)$, $8 \leqq n \leqq 10$;

(xi) Spin(10)/SU(5);

(xii) $Spin(n)/(SU(2)\times SU(2))$ *for* $8 \leqq n \leqq 12$;

(xiii) $Spin(12)/(SU(2)\times SU(2)\times SU(2))$;

(xiv) $Spin(n)/H_0$ *for* $n = 8, 9$, H_0 *as in* (iv);

(xv) $Spin(n)/Sp(2)$ *for* $n = 8, 9$;

(xvi) $Spin(n)/K_0$, *where* K_0 *is the universal cover of the group* $\left\{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \middle| A \in SO(4)\right\}$ *for* $n = 8, 9$;

(xvii) $Spin(n)/G_2$ *for* $n \geqq 7$;

(xviii) $F_4/SU(3)$, $F_4/Spin(8)$, $E_6/Spin(8)$, $E_7/Spin(8)$.

Remarks. 1. The embeddings in Theorem 2 are supposed to be the obvious ones. In particular, we consider the inclusions $Sp(n) \subsetneqq SU(2n) \subsetneqq SO(4n)$ and their lifts into Spin(4*n*).

2. Theorem 2 does not list all the pairs (G, H) such that G/H is stably parallelizable. On the other hand it is not true that the listed manifolds are pairwise non-diffeomorphic. We had to be somewhat inconsequent in order to have a reasonably short and clear statement.

3. In the course of the proof, we will also find a number of stably parallelizable quotients of SO(*n*). We didn't pursue their classification further.

4. Recall from *I* (3.4) that $p_1(G/H) \neq 0$ if *H* is neither abelian nor semisimple. Therefore it suffices for the proof of Theorem 2 to assume *H* semisimple.

Theorem 3. *Let G and H be as in Theorem 1 and suppose that G/H is stably parallelizable but* not *parallelizable. Then G/H is a sphere or* $\operatorname{rank} G = \operatorname{rank} H$.

Observe that if $\operatorname{rank} G = \operatorname{rank} H$ then $\chi(G/H) = \# W(G)/\# W(H) \neq 0$ so that G/H is certainly not parallelizable. Here is a list of all stably parallelizable homogeneous spaces G/H which are not parallelizable (G and H as in Theorem 1):

(i) G/T, T a maximal torus of G;
(ii) the spheres S^n, $n \neq 1, 3, 7$;
(iii) $Y_{n,n} = \mathrm{Sp}(n)/\mathrm{Sp}(1)^n$;
(iv) $F_4/\mathrm{Spin}(8)$.

The proof of the Theorem 2 is based on the methods developed in [*I*] and, moreover, on the consideration of Stiefel-Whitney classes. These are of course also discussed by Borel-Hirzebruch [9] and in [29], but only in the restricted situation in which there is a satisfactory theory of 2-roots. Therefore we had to establish a fairly general criterion for the non-vanishing of Stiefel-Whitney classes (Sects. 4 and 5).

In Sect. 2 we use the index of a Lie group homomorphism as introduced by Dynkin to obtain a necessary condition for the parallelizability of homogeneous spaces.

In Sects. 3, 6, 7, and 8 we study quotients of the exceptional groups, of $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, and $\mathrm{Spin}(n)$, respectively. We make extensive use of Hsiang's computations [17].

Finally, in Sect. 9 we show the parallelizability of the quotients $\mathrm{SO}(n)/\mathrm{Ad}_H(H)$ for $n \geqq \dim H$. This is independent from the earlier sections.

This paper is a combination of a 1983 IHES preprint of the first author and the thesis [32] of the second author; we refer to [32] for details of several hard computations

2. The Index of a Lie Group Homomorphism

Let G be a simple compact Lie group with Lie algebra $\mathfrak{g}$. There exists a unique $\mathrm{Ad}\,G$-invariant scalar product $(\cdot|\cdot)_G$ on $\mathfrak{g}$ with $(\mu_G|\mu_G)_G = 2$ where μ_G is the highest root of G. If H is another such group and $\varrho: H \to G$ a homomorphism with induced homomorphism $\varrho_*: \mathfrak{h} \to \mathfrak{g}$ then, following Dynkin [13], we define its *index* $I(\varrho) \in \mathbb{R}$ by

$$(\varrho_*(x)|\varrho_*(y))_G = I(\varrho) \cdot (x|y)_H \quad \text{for} \quad x, y \in \mathfrak{h}.$$

This number is topologically relevant because of

(2.1) **Proposition.** *If G and H are moreover simply connected, the homomorphism $(B\varrho)^*: H^4(BG;\mathbb{Z}) \to H^4(BH;\mathbb{Z})$ is multiplication by $I(\varrho)$. (Observe that these cohomology groups can be unambiguously identified with $\mathbb{Z}$ by stipulating that the sum of the squares of the roots be negative.)*

This proposition is stated in [23]; when G and H are classical groups, a proof is given in [14, Sect. 15]. For the sake of completeness, we reduce the proof of the general statement to the special case $H = \mathrm{SU}(2)$, $G = \mathrm{SU}(n)$ by an argument similar to [4, Sect. 8]:

Let K_μ be the rank one subgroup of an arbitrary simple group H associated to its highest root μ. Since K_μ is isomorphic to $\mathrm{SU}(2)$, we obtain an embedding $j: \mathrm{SU}(2) \to H$, and $I(j) = 1$ by definition of the index. On the other hand, j induces an isomorphism $j_*: \pi_3(\mathrm{SU}(2)) \to \pi_3(H)$ by [10, Proposition 10.2]. Applying the theorem of Hurewicz and transgression, we find that $(Bj)^*$ is the identity on H^4.

The index is multiplicative with respect to the composition of homomorphisms. Hence our assertion follows if G happens to be $\mathrm{SU}(n)$. The general case is then obtained by comparing with a non-trivial homomorphism from G to $\mathrm{SU}(n)$.

As a first application, we give a criterion for the vanishing of the first Pontrjagin class of a homogeneous space. If φ is a complex representation of H we define $I(\varphi)$ unambiguously by considering φ as a homomorphism $H \to \mathrm{SU}(n)$. If φ is a real representation, put $I(\varphi) := I(\varphi \otimes \mathbb{C})$.

(2.2) **Proposition.** *Let H, G be simply connected compact Lie groups and assume that G is simple and $H = H_1 \times \ldots \times H_m$ with simple factors H_j. Let $\varrho : H \to G$ be a non-trivial homomorphism, $\varrho_j := \varrho | H_j$ and $H_\varrho := \operatorname{Im} \varrho$. Then the following assertions are equivalent:*

(i) $p_1(G/H_\varrho) = 0$.

(ii) *The vector $(I(\mathrm{Ad}_{H_1}), \ldots, I(\mathrm{Ad}_{H_m}))$ is an integral multiple of $(I(\varrho_1), \ldots, I(\varrho_m))$.*

Proof. Let $\varphi : G/H_\varrho \to B\,\mathrm{SU}(N)$ be the classifying map for the complexification of the stable tangent bundle. The following diagram is commutative up to sign (cf. I.3.):

$$(*)\qquad \begin{array}{ccccc} & & H^4(BH) & \xleftarrow{(B\mathrm{Ad}_H)^*} & H^4(B\,\mathrm{SU}(N)) \\ & {\scriptstyle (B\varrho)^*}\nearrow & \uparrow{\scriptstyle (B\pi)^*} & \swarrow{\scriptstyle (B\mathrm{Ad}_{H_\varrho})^*} & \downarrow{\scriptstyle \varphi^*} \\ H^4(BG) & \xrightarrow{q^*} & H^4(BH_\varrho) & \xrightarrow{j^*} & H^4(G/H_\varrho) \end{array}$$

Let $c_2 \in H^4(B\,\mathrm{SU}(N))$ be the universal Chern class. Since the lower row of $(*)$ is exact, $p_1(G/H_\varrho) = 0$ iff $(B\,\mathrm{Ad}_{H_\varrho})^* c_2 \in \operatorname{Im} q^*$. This is equivalent with $(B\,\mathrm{Ad}_H)^* c_2 \in \operatorname{Im}(B\varrho)^*$ since $(B\pi)^*$ is injective: In fact, $N := \operatorname{Ker}(H \to H_\varrho)$ is finite, and therefore $H^i(BN) = 0$ for odd integers i; apply then the Serre exact sequence of the fibration $BN \to BH \to BH_\varrho$.

Assertion (2.2) now follows from (2.1).

Thus the vanishing of p_1 can be read off from Dynkin's table of indices [13, Table 5] (see also [21] for corrections).

For our next application, we extend the definition of the index to homomorphisms $\varrho : H \to G$ where G is simple but H is only semisimple: Let $\pi : H_1 \times \ldots \times H_m \to H$ be the universal covering of H with simple groups H_j. Then we put

$$I(\varrho) := \sum_j I(\varrho \circ \pi | H_j).$$

(2.3) **Proposition.** *Let G be a simple, simply connected Lie group and H a semisimple subgroup with inclusion $\varrho : H \to G$. Denote by φ_G a fundamental representation of G of lowest dimension and assume that G/H is stably parallelizable. Then $I(\mathrm{Ad}_H)$ is divisible by $I(\varphi_G) \cdot I(\varrho)$.*

The proof of (2.3) requires some preparations: Since $I(\varphi_1 + \varphi_2) = I(\varphi_1) + I(\varphi_2)$ for representations φ_i of H we can consider I as an additive homomorphism $R(H) \to \mathbb{Z}$.

I is not multiplicative; instead we have [13, p. 133]

(2.4) $$I(\varphi_1 \cdot \varphi_2) = I(\varphi_1) \cdot \dim \varphi_2 + \dim \varphi_1 \cdot I(\varphi_2).$$

We will show the following assertion which contains (2.3) as a special case:

(2.5) *Let* $\alpha: R(H)\to K(G/H)$ *be the usual map. If* $\psi\in\operatorname{Ker}\alpha$ *then* $I(\varphi_G)\cdot I(\varrho)$ *divides* $I(\psi)$.

Denote by $\iota: S^1\hookrightarrow \mathrm{SU}(2)$ the usual inclusion and write $R(S^1)=\mathbb{Z}[z,z^{-1}]$. The additive homomorphism $\kappa: R(S^1)\to\mathbb{Q}$ defined by $\kappa(z^k)=k^2/2$ has the following property: Let J be the ideal of $R(S^1)$ generated by the elements $\varphi\circ\iota-\dim\varphi$ with $\varphi\in R(\mathrm{SU}(2))$. Then $\kappa(J)\subsetneqq\mathbb{Z}$ and

$$\kappa((\varphi\circ\iota-\dim\varphi)\cdot\lambda)=I(\varphi)\cdot\dim\lambda \tag{*}$$

for $\varphi\in R(\mathrm{SU}(2))$ and $\lambda\in R(S^1)$. This can be verified by direct computation.

From Table 5 of [13] we deduce that $I(\varphi_G)$ divides $I(\sigma)$ for $\sigma\in R(G)$; therefore

$$I(\varphi_G)\cdot I(\varrho)\mid I(\sigma|H). \tag{**}$$

With the notation as above, choose homomorphisms $g_j: \mathrm{SU}(2)\to H_j$ with index 1 and let g be the composition

$$\mathrm{SU}(2)\xrightarrow{d}\mathrm{SU}(2)^m\xrightarrow{g_1\times\ldots\times g_m}H_1\times\ldots\times H_m\xrightarrow{\pi}H.$$

Then

$$I(\psi)=I(\psi\circ g)\quad\text{for}\quad \psi\in R(H). \tag{***}$$

Denote the subgroup $g\circ\iota(S^1)$ of H again by S^1 and consider $\alpha: R(S^1)\to K(G/S^1)$. By naturality of α, we have $\alpha(\psi|S^1)=0$. Now we can apply Snaith's theorem [26, Theorem 5.5] to the pair (G,S^1) and obtain

$$\psi|S^1-\dim\psi=\sum_i(\sigma_i|S^1-\dim\sigma_i)\lambda_i\in J$$

with $\sigma_i\in R(G)$, $\lambda_i\in R(S^1)$. Hence, by (*) and (***),

$$I(\psi)=I(\psi\circ g)=\kappa(\psi|S^1-\dim\psi)=\sum_i I(\sigma_i\circ g)\cdot\dim\lambda_i$$
$$=\sum_i I(\sigma_i|H)\cdot\dim\lambda_i,$$

and we are done by (**).

3. Quotients of the Exceptional Groups

Proposition (2.3) is a quite powerful tool for quotients of the exceptional groups because $I(\varphi_G)$ is relatively large in these cases. In fact, for $G=G_2,\ldots,E_8$, the numbers $I(\varphi_G)$ are 2, 6, 6, 12, 60.

In this section, G will then be a simply connected exceptional group and H a simple subgroup such that G/H is stably parallelizable.

(3.1) *H is locally isomorphic to* $\mathrm{SU}(2)$, $\mathrm{SU}(3)$, G_2, $\mathrm{Spin}(8)$, F_4, E_6, *or* E_7.

For otherwise $0\neq p_2(G/H)\in H^8(G/H;\mathbb{R})$ by [I], (3.2), since for the exceptional groups $H^8(BG;\mathbb{R})=H^4(BG;\mathbb{R})\cdot H^4(BG;\mathbb{R})$.

There is a natural sequence of inclusions

$$\mathrm{SU}(2)\to\mathrm{SU}(3)\to G_2\to\mathrm{Spin}(8)\to F_4\to E_6\to E_7\to E_8\,;$$

these embeddings are up to conjugation determined by having index 1.

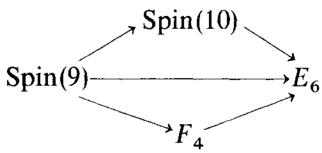
From (2.3), (3.1) and Dynkin's tables 5, 16, and 25 [13] we conclude:

(3.2) *The pair* (G,H) *is isomorphic to* $(G_2,\mathrm{SU}(2))$, $(G_2,\mathrm{SU}(3))$, $(F_4,\mathrm{SU}(3))$, $(F_4,\mathrm{Spin}(8))$, $(E_6,\mathrm{Spin}(8))$, $(E_7,\mathrm{Spin}(8))$, $(E_6,\mathrm{SU}(3))$, (E_6,F_4), (E_7,E_6) *with the natural embeddings.*

Remark. The first six pairs lead to stably parallelizable quotients which appear in items (i) and (xviii) of Theorem 2; we have to show that the last three pairs don't give stably parallelizable quotients.

(3.3) E_6/F_4 *is not stably parallelizable.*

Proof. By [13] there exists an index 1 subgroup Spin(10) of E_6. Therefore we have a commutative diagram of inclusions



from which we get a map

$$q: S^9=\mathrm{Spin}(10)/\mathrm{Spin}(9)\to E_6/F_4\,.$$

By an argument similar to *I* (6.4) it can be shown that $q^*\tau(E_6/F_4)$ represents the non-trivial element in $\widetilde{KO}(S^9)$.

(3.4) E_7/E_6 *is not stably parallelizable.*

Proof. Denote E_7/E_6 by X. The integral cohomology of X was calculated in [31, p. 363]; additively, we have:

$$H^*(X;\mathbb{Z})=\mathbb{Z}\cdot\{1,z_{10},z_{18},z_{37},z_{45},z_{55}\}\oplus\mathbb{Z}_2\cdot z_{28} \text{ with } \deg z_i=i\,. \tag{1}$$

For dimensional reasons, the following is then clear:

The Atiyah-Hirzebruch spectral sequence converging to $K(X)$ collapses, and $\tilde{K}(X)=\mathbb{Z}a_{10}\oplus\mathbb{Z}a_{18}\oplus\mathbb{Z}_2a_{28}$ with indices denoting filtration. (2)

Comparing with real K-theory, we find:

$\widetilde{KO}(X)$ is a group of 8 elements, $r:\tilde{K}(X)\to\widetilde{KO}(X)$ is onto and hence $r(a_{10}+b)\neq 0$ for $b\in\mathbb{Z}a_{18}\oplus\mathbb{Z}_2a_{28}$. (3)

Let $\tau\in\widetilde{KO}(X)$ be the tangent bundle, φ a fundamental representation of E_6 of lowest dimension and $\eta:=\alpha(\varphi-27)\in\tilde{K}(X)$.

$$r(\eta)=\tau\,. \tag{4}$$

Information on the integral cohomology of BE_6 and BE_7 can be found in [8, 1, 18, 19]; we conclude that the natural map $j: X \to BE_6$ induces

$$j^*: H^{10}(BE_6) = \mathbb{Z} \xrightarrow{\cdot 2} H^{10}(X) = \mathbb{Z}.$$

Let $c_5 \in H^{10}(B\mathrm{SU}; \mathbb{Z})$ be the universal Chern class. Then $c_5(\eta) = j^*(B\varphi)^*(c_5)$. There exists an embedding $f: \mathrm{SU}(6) \to E_6$ with $\varphi \circ f = \varrho_2 + 2\bar{\varrho}_1$ (cf. [13, p. 206]; $\varrho_1, \varrho_2, \ldots$ are the fundamental representations of SU(6)). Thus $(Bf)^*(B\varphi)^*(c_5) = -12c_5$. We conclude:

$$c_5(\eta) = k \cdot z_{10} \in H^{10}(X; \mathbb{Z}) \quad \text{with } k|24. \tag{5}$$

Write $\eta = ma_{10} + b$ as above. Denote by ch^5 the homogeneous part of degree 10 of the Chern character. Then $ch^5(\eta) = m \cdot ch^5(a_{10})$. On the other hand, by (1), (2) and the integrality theorem, $ch^5(\eta) = \frac{k}{24} \cdot z_{10}$. Comparison yields by (5) that $k = 24$ and $m = 1$. Thus we are done by (3) and (4).

(3.5) $E_6/\mathrm{SU}(3)$ *is not stably parallelizable.*

Proof. Write $Y := E_6/\mathrm{SU}(3)$ and $Z := E_6/F_4$. There are inclusions $\mathrm{SU}(3) \xrightarrow{i} F_4 \xrightarrow{j} E_6$; therefore we have a fibration $F_4/\mathrm{SU}(3) \xrightarrow{\iota} Y \xrightarrow{\pi} Z$. The tangent bundle of Y is given by $\tau(Y) = \pi^*\tau(Z) \oplus \eta$, where η is the bundle along the fibres, and $\eta = \alpha(\mathrm{Ad}_{F_4}|\mathrm{SU}(3) - \mathrm{Ad}_{\mathrm{SU}(3)})$. A standard calculation shows that η is stably trivial. Therefore $\tau(Y)$ is stably equivalent to $\pi^*\tau(Z)$.

By [2, 2.6], $j^*: H^*(E_6; \mathbb{Z}/2) \to H^*(F_4; \mathbb{Z}/2)$ is surjective. The Serre spectral sequence implies that $H^*(Z; \mathbb{Z}/2) \to H^*(E_6; \mathbb{Z}/2)$ is injective; consequently:

(3.6) $\pi^*: H^*(Z; \mathbb{Z}/Z) \to H^*(Y; \mathbb{Z}/2)$ *is injective.*

Furthermore,

$$H^*(E_6; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_9, x_{15}, x_{17}, x_{23})$$

by [2, 2.7]. Since it is well known ([7], Chap. V) that

$$i^*: H^*(F_4; \mathbb{Z}/2) \to H^*(\mathrm{SU}(3); \mathbb{Z}/2)$$

is surjective, we have that $i^* \circ j^*$ is surjective. Therefore the mod 2 Serre spectral sequence of the fibration $\mathrm{SU}(3) \to E_6 \to Z$ collapses, and we conclude that up to dimension 10 the cohomology of Z is given by

$$H^i(Z; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{for } i = 0, 6, 9; \\ 0 & \text{for } i = 1, 2, 3, 4, 5, 7, 8, 10. \end{cases} \tag{3.7}$$

By [2, 2.5] we know that $H^*(E_6/F_4; \mathbb{Z}) \cong \Lambda(z_9, z_{17})$. Hence (3.6) and (3.7) yield

(3.8) $\pi^*: H^9(Y; \mathbb{Z}/2) \to H^9(Z; \mathbb{Z}/2)$ *is an isomorphism.*

Now apply the Atiyah-Hirzebruch spectral sequence to the spaces and maps

$$Y \xrightarrow{\pi} Z \xleftarrow{q} S^9$$

where q is as in (3.3). Since $q^*\tau(Z) - \dim Z$ is the non-trivial element in $\widetilde{KO}(S^9)$ and since $\tau(Y)$ is stably equivalent to $\pi^*\tau(Z)$, we conclude from (3.7) and (3.8) that $\tau(Y)$ is not stably trivial.

(3.9) *Quotients of the exceptional group G by non-simple subgroups are not stably parallelizable.*

Proof. Let $f: H_1 \times \ldots \times H_k \to G$ be a Lie group homomorphism with simple groups H_i such that $G/\mathrm{Im} f$ is stably parallelizable. Then we have principal fibrations $G/K_i \to G/\mathrm{Im} f$, with

$$K_i = f(1 \times \ldots \times 1 \times H_i \times 1 \times \ldots \times 1),$$

and by *I* (2.6.c) their total spaces are stably parallelizable. By (3.2) we see that $2\,\mathrm{Rank}\, K_i \geqq \mathrm{Rank}\, G$, hence $k=2$, and (G, K_i) is isomorphic to $(G_2, \mathrm{SU}(2))$ or $(F_4, \mathrm{SU}(3))$. The classification of subgroups of maximal rank tells us that there are no homomorphisms f of this type.

4. Stiefel-Whitney Classes of Homogeneous Spaces

In this section, let G be a compact connected Lie group and H a closed connected subgroup. Then G/H is orientable, that is, $w_1(G/H)=0$. It is also easy to deal with $w_2(G/H)$:

(4.1) Let T be a maximal torus of H and consider the natural maps

$$G/H \xrightarrow{j} BH \xleftarrow{i} BT.$$

In cohomology, they induce

$$H^2(G/H;\mathbb{Z}_2) \xleftarrow{j^*} H^2(BH;\mathbb{Z}_2) \rightarrowtail^{i^*} H^2(BT;\mathbb{Z}_2) \xleftarrow{\text{red}} H^2(BT;\mathbb{Z})$$

where red denotes reduction of coefficients. Let $\mathbb{S}^+ \subset H^2(BT;\mathbb{Z})$ denote the set of positive roots of H. Then

$$w_2(G/H) = j^*(x) \quad \text{with} \quad i^*(x) = \mathrm{red}\Big(\sum_{\alpha\in\mathbb{S}^+} \alpha\Big).$$

When is $w_2(G/H)=0$? Recall the following definition [30, p. 59]: H is called *acceptable* if $\frac{1}{2}\sum_{\alpha\in\mathbb{S}^+} \alpha$ is a weight of H. Introduce an Ad-invariant scalar product on $\mathfrak{h} = \mathrm{Lie}(H)$. Then we have as an immediate consequence of (4.1):

(4.2) **Corollary.** *Let G be simply connected. The following assertions are equivalent:*
a) *G/H admits a spin structure, that is, $w_2(G/H)=0$.*
b) $\mathrm{Ad}: H \to \mathrm{SO}(\mathfrak{h})$ *lifts to* $\mathrm{Spin}(\mathfrak{h})$.
c) *H is acceptable.*

Here is a complete list of those simple groups which are *not* acceptable:

(4.3) (i) $\mathrm{SU}(n)/N$ where N is a subgroup of the center of $\mathrm{SU}(n)$ with $v_2(n) = v_2(\mathrm{ord}(N)) \geqq 1$; here, as usually, $v_2(n)$ is the exponent of 2 in the decomposition of n into prime powers.

(ii) $SO(2n+1)$, $n \geqq 1$
(iii) $P\,Sp(n)$ with $n \equiv 1(4)$ or $n \equiv 2(4)$.
(iv) $P\,SO(2n)$ with $n \equiv 2(4)$ or $n \equiv 3(4)$.
(v) $Ss(4n)$, the semi-spinor group, with $n \equiv 1(2)$.
(vi) $Ad(E_7)$.

For the higher Stiefel-Whitney classes, there is a useful criterion for which we need some notation: Write $R(S^1) = \mathbb{Z}[z, z^{-1}]$ in the usual way; if $\varrho \in R(S^1)$ we have $\varrho = \sum_i N_i(\varrho) z^i$ with $N_i(\varrho) \in \mathbb{Z}$. Define $N^{\text{odd}}(\varrho)$ by

$$N^{\text{odd}}(\varrho) := \sum_i N_{2i+1}(\varrho).$$

(4.4) **Proposition.** *Let H be a closed connected subgroup of $SU(n)$, $\varrho: H \hookrightarrow SU(n)$, and let $\iota: S^1 \hookrightarrow H$ be an embedding. Assume that $N^{\text{odd}}(\mathrm{Ad}_H \circ \iota) \neq 0$ and that $\nu := \nu_2(N^{\text{odd}}(\mathrm{Ad}_H \circ \iota)) \leqq \nu_2(N^{\text{odd}}(\varrho \circ \iota))$. Then*

$$w_{2^\nu}(SU(n)/H) \neq 0.$$

Proof. Let $q: SU(n)/S^1 \to SU(n)/H$ be the natural projection. We show that $w_{2^\nu}(q^* \tau(SU(n)/H)) \neq 0$:

Write $M_i := N_i(\mathrm{Ad} \circ \iota)$; then $M_i = M_{-i}$; and in $RO(S^1)$, we have $\mathrm{Ad} \circ \iota$ stably equivalent to $r\left(\sum_{i>0} M_i z^i\right)$, where

$r: R(S^1) \to RO(S^1)$ is realification.

There is the standard fibration $SU(n)/S^1 \xrightarrow{j} BS^1 \xrightarrow{\varphi} B\,SU(n)$. Let η be the canonical complex line bundle over BS^1, that is, $\eta = \alpha(z)$ with $\alpha: R(S^1) \to K(BS^1)$. Since $\tau(SU(n)/H)$ is stably equivalent to $-\alpha(\mathrm{Ad})$ in $KO(SU(n)/H)$, naturality of α gives that $q^* \tau(SU(n)/H)$ is stably equivalent to $-\alpha(\mathrm{Ad} \circ \iota)$, and this in turn is stably equivalent to $-j^* r\left(\sum_{i>0} M_i \eta^i\right)$. So it suffices to show that the first non-zero Stiefel-Whitney class of $j^* r\left(\sum_{i>0} M_i \eta^i\right)$ is w_{2^ν}.

Now the total Stiefel-Whitney class of this bundle is $j^* \operatorname{red}\left(c_*\left(\sum_{i>0} M_i \eta^i\right)\right)$, where c_* denotes the total Chern class. As an easy application of the Cartan formula, one has:

The smallest integer $s \geqq 1$ with $\operatorname{red} c_s\left(\sum_{i>0} M_i \eta^i\right) \neq 0$ is $s = 2^{\nu-1}$.

Writing $H^*(BS^1; \mathbb{Z}) = \mathbb{Z}[y]$ with $y = c_1(\eta)$, this means that

$$w_i(q^* \tau(SU(n)/H) = 0 \quad \text{for} \quad 0 < i < 2^\nu$$

and $w_{2^\nu}(q^* \tau(SU(n)/H)) = j^* \operatorname{red}(y^{2^{\nu-1}})$. By the Gugenheim-May theorem [16], this will be non-zero if we can show that

$$\varphi^* = 0: H^i(B\,SU(n); \mathbb{Z}_2) \to H^i(BS^1; \mathbb{Z}_2) \quad \text{for} \quad 0 < i \leqq 2^\nu.$$

But this is straightforward: Let T be a maximal torus of $SU(n)$ and consider φ as a map from BS^1 to BT. We can choose $x_1, \ldots, x_n$ in $H^2(BT; \mathbb{Z}_2)$ such that

$$H^*(BT; \mathbb{Z}_2) = \mathbb{Z}_2[x_1, \ldots, x_n]/(\sum x_i),$$

$$\varphi^*(x_i) = \begin{cases} \operatorname{red} y & \text{for} \quad 1 \leqq i \leqq N^{\text{odd}}(\varrho \circ \iota) \\ 0 & \text{for} \quad N^{\text{odd}}(\varrho \circ \iota) < i \leqq n. \end{cases}$$

Let σ_i be the i^{th} elementary symmetric function in $x_1, \ldots, x_n$. Then $H^*(B\,\mathrm{SU}(n); \mathbb{Z}_2)$ is the subring of $H^*(BT; \mathbb{Z}_2)$ generated by the σ_i, and an easy calculation with binomial coefficients shows that $\varphi^*(\sigma_i)=0$ for $1 \leqq i < 2^\mu$ with $\mu = v_2(N^{\text{odd}}(\varrho \circ \iota))$. Since $\dim \sigma_i = 2i$ and, by hypothesis, $v \leqq \mu$, the result follows.

Remark. This proposition can obviously be generalized to a criterion for the non-vanishing of Stiefel-Whitney classes for any homogeneous vector bundle, that is, a vector bundle in $\mathrm{Im}(\alpha: RO(H) \to KO(\mathrm{SU}(n)/H))$. The following corollary is, strictly speaking, a consequence of this generalization.

(4.5) **Corollary.** *Let H be a closed connected subgroup of* $\mathrm{Sp}(n)$, $\varrho: H \hookrightarrow \mathrm{Sp}(n)$, *and let $\iota: S^1 \hookrightarrow H$ be an embedding. Let $\psi: \mathrm{Sp}(n) \to \mathrm{SU}(2n)$ be the natural homomorphism. Assume that $N^{\text{odd}}(\mathrm{Ad}_H \circ \iota) \neq 0$ and that*

$$v := v_2(N^{\text{odd}}(\mathrm{Ad}_H \circ \iota)) \leqq v_2(N^{\text{odd}}(\psi \circ \varrho \circ \iota)).$$

Then

$$w_{2^v}(\mathrm{Sp}(n)/H) \neq 0.$$

Proof. Apply (4.4), as generalized in the preceding remark, to the vector bundle $\xi = \alpha(\mathrm{Ad}_H \circ \iota)$ over $\mathrm{SU}(2n)/S^1$. We find that $w_{2^v}(\xi) \neq 0$. Consider the natural maps

$$\mathrm{SU}(2n)/S^1 \xleftarrow{\bar{\psi}} \mathrm{Sp}(n)/S^1 \xrightarrow{q} \mathrm{Sp}(n)/H.$$

Since $\bar{\psi}^*(\xi)$ is stably equivalent to $-q^*\tau(\mathrm{Sp}(n)/H)$, it suffices to show that $\bar{\psi}^*(w_{2^v}(\xi)) \neq 0$. This, however, follows from the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Sp}(n)/S^1 & \longrightarrow & BS^1 & \longrightarrow & B\,\mathrm{Sp}(n) \\
\downarrow{\scriptstyle \bar{\psi}} & & \| & & \downarrow{\scriptstyle B\psi} \\
\mathrm{SU}(2n)/S^1 & \longrightarrow & BS^1 & \longrightarrow & B\,\mathrm{SU}(2n),
\end{array}$$

from the Gugenheim-May theorem and the fact that

$$(B\psi)^*: H^*(B\,\mathrm{SU}(2n)) \to H^*(B\,\mathrm{Sp}(n))$$

is surjective [6].

5. Some Applications of Stiefel-Whitney Classes

In *I* (2.5), we observed that $\mathrm{SO}(\mathfrak{h})/\mathrm{Ad}(H)$ is always stably parallelizable, $\mathfrak{h} = \mathrm{Lie}(H)$ being endowed with an Ad-invariant scalar product. If $\mathfrak{h}_{\mathbb{C}}$ denotes the complexification of $\mathfrak{h}$, it seems therefore natural to ask whether $\mathrm{SU}(\mathfrak{h}_{\mathbb{C}})/\mathrm{Ad}(H)$ is also stably parallelizable; it is obvious that the complexification of its tangent bundle is stably trivial.

The following proposition was stated without proof in [17]; a long case by case proof was given in [29].

(5.1) **Proposition.** *If H is not abelian then there exists an $i > 0$ such that*

$$w_i(\mathrm{SU}(\mathfrak{h}_{\mathbb{C}})/\mathrm{Ad}(H)) \neq 0.$$

Proof. It is easy to check that for each non-abelian H the group $\mathrm{Ad}(H)$ contains a subgroup which is isomorphic to $\mathrm{SO}(3)$. For any closed subgroup K of a Lie group G, the restriction $\mathrm{Ad}_G|K$ contains Ad_K as a subrepresentation. Apply this observation to the pair $(G, K)=(\mathrm{Ad}(H), \mathrm{SO}(3))$: Since $\mathrm{Ad}_{\mathrm{SO}(3)}$ is just the inclusion $\mathrm{SO}(3) \hookrightarrow \mathrm{SU}(3)$, we find that $\mathrm{Ad}_{\mathrm{Ad}(H)}|\mathrm{SO}(3)$ contains this inclusion. Now apply (4.4), where $\imath$ is the following composite:

$$\imath : S^1 = \mathrm{SO}(2) \hookrightarrow \mathrm{SO}(3) \hookrightarrow \mathrm{Ad}(H).$$

We have seen that $\mathrm{Ad}_{\mathrm{Ad}(H)} \circ \imath$ contains the inclusion $S^1 = \mathrm{SO}(2) \hookrightarrow \mathrm{SU}(2)$ as a subrepresentation, that is, $\mathrm{Ad} \circ \imath$ contains $z + z^{-1}$. This verifies the hypothesis $N^{\mathrm{odd}}(\mathrm{Ad} \circ \imath) \neq 0$ in (4.4). The second hypothesis of (4.4) is trivially satisfied since $\varrho = \mathrm{Ad}$. The result follows.

Next we consider the fourth Stiefel-Whitney class. We need the following auxiliary observation:

(5.2) **Lemma.** *Let H be a simple Lie group and $g : \mathrm{SU}(2) \to H$ the standard homomorphism with image K_μ as in the proof of* (2.1). *Let φ be the identical representation of* SU(2). *Then $\mathrm{Ad}_H \circ g$ is stably the same as $\varphi^2 + (I(\mathrm{Ad}_H) - 4) \cdot \varphi$ in* $R(\mathrm{SU}(2)) = \mathbb{Z}[\varphi]$.

Proof. From [11, Ch. VI.1, Proposition 8(ii)] we deduce that

$$\mathrm{Ad}_H \circ g = \varphi^2 + n \cdot \varphi + l, \qquad n, l \in \mathbb{Z}.$$

Applying the index to both sides gives the desired result since $I_g = 1$.

(5.3) **Proposition.** *Let H and G be compact connected simple Lie groups. Furthermore, let G be 1-connected and $f : H \to G$ an embedding with even index I_f. With the notation of Sect. 4, assume that $v_2(I(\mathrm{Ad}_H)) = 1$. Then $w_4(G/H) \neq 0$.*

The proof is a modification of the proof of (4.4) in that the Gugenheim-May theorem is simply replaced by the Serre exact sequence:

We choose for $\imath$ the composition $S^1 \hookrightarrow \mathrm{SU}(2) \overset{g}{\hookrightarrow} H$ where g is as in (5.2). Comparing the exact mod 2 Serre cohomology sequences of the fibrations $G/S^1 \to BS^1 \to BG$ and $G/\mathrm{SU}(2) \to B\,\mathrm{SU}(2) \to BG$ we find that

$$H^i(BS^1; \mathbb{Z}/2) \to H^i(G/S^1; \mathbb{Z}/2) \text{ is injective for } i \leqq 4.$$

By the proof of (4.4) it remains to compute $N^{\mathrm{odd}}(\mathrm{Ad}_H \circ g \circ \imath)$. From (5.2) we see that

$$v_2(N^{\mathrm{odd}}(\mathrm{Ad}_H \circ g \circ \imath)) = v_2(2 \cdot (I(\mathrm{Ad}_H) - 4)) = 2.$$

The simple, simply connected groups H with $v_2(I(\mathrm{Ad}_H)) = 1$ are $\mathrm{SU}(2n+1)$, $\mathrm{Spin}(2n+1)$ with $n \geqq 3$, $\mathrm{Sp}(2n)$, and F_4.

(5.4) **Corollary.** *Let $G = \mathrm{SU}(N)$ or $\mathrm{Sp}(N)$ and let H be a subgroup locally isomorphic to $\mathrm{Spin}(2n+1)$ with $n \geqq 3$ or F_4. Then*

$$w_4(G/H) \neq 0.$$

This is clear since any representation of H and hence any homomorphism $f : H \to G$ has even index.

6. Quotients of SU(*n*)

(6.1) **Lemma.** *Let H be a subgroup of* $\mathrm{SU}(n)$ *such that* $\mathrm{SU}(n)/H$ *is stably parallelizable, and suppose that H is locally isomorphic, but* not *isomorphic, to* $\mathrm{SU}(2)^m=\mathrm{SU}(2)\times\ldots\times\mathrm{SU}(2)$. *Then* $\mathrm{SU}(n)/H$ is $\mathrm{SU}(4)/\mathrm{SO}(4)$.

Proof. For $m=1$ the assertion follows from (2.2) and (5.1). So assume that

$$H=\mathrm{Im}(\varrho:\mathrm{SU}(2)^m\to\mathrm{SU}(n))$$

with $m\geqq 2$. Then $\varrho=\varrho^{(1)}+\ldots+\varrho^{(v)}$ with $\varrho^{(i)}=\varrho^{(1,i)}\times\ldots\times\varrho^{(m,i)}$, the $\varrho^{(j,i)}$ being irreducible representations of SU(2). Let $\iota_r:\mathrm{SU}(2)\to\mathrm{SU}(2)^m$ be the inclusion as the r^{th} factor. Then [e.g. by *I* (2.6c)] $\mathrm{SU}(n)/\mathrm{Im}(\varrho\circ\iota_r)$ is stably parallelizable. As we saw in the case $m=1$ above, $\varrho\circ\iota_r$ has to be injective; therefore, by *I* (5.1), either $\varrho\circ\iota_r=\varrho_1+N_0$ or $=2\varrho_1$. On the other hand, $\varrho\circ\iota_r=\sum_{i=1}^{n}\left(\prod_{j\neq r}\dim\varrho^{(j,i)}\right)\varrho^{(r,i)}$. So either each $\varrho^{(i)}$ is of the form $1\times\ldots\times 1\times\varrho_1\times 1\times\ldots\times 1$, or $m=2$ and $\varrho=\varrho_1\times\varrho_1$. In the first case, $H\cong\mathrm{SU}(2)^m$, and in the second case, $\mathrm{SU}(n)/H=\mathrm{SU}(4)/\mathrm{SO}(4)$.

(6.2) **Lemma.** *Let $n\geqq k\geqq 2$, and assume that* $\mathrm{SU}(n)/\mathrm{SO}(k)$ *is stably parallelizable. Then $k=2$, or $n=k=4$.*

Proof. By (4.2) and (4.3), k has to be even. In *I*, Sect. 4, we studied the spaces $\mathrm{Sp}(n)/\mathrm{SU}(k)$. Imitating the proof of *I*, (4.2), we find: If $\mathrm{SU}(n)/\mathrm{SO}(n)$ is stably parallelizable then so is $\mathrm{SU}(n-1)/\mathrm{SO}(n-2)$. Since $\mathrm{SU}(5)/\mathrm{SO}(4)$ is not stably parallelizable by (6.1), the result follows.

(6.3) **Lemma.** *Let K be a subgroup of* $\mathrm{SU}(n)$ *such that* $\mathrm{SU}(n)/K$ *is stably parallelizable, and assume that K is locally isomorphic to* $\mathrm{SU}(k)$. *Then K is isomorphic to* $\mathrm{SU}(k)$.

Proof. This follows from the non-vanishing of Pontrjagin classes as computed in [17, Theorem 1].

(6.4) **Lemma.** *For $n\geqq 2k$, consider the natural embedding of* $\mathrm{Sp}(k)$ *into* $\mathrm{SU}(n)$. *Assume that $k>1$ and that* $\mathrm{SU}(n)/\mathrm{Sp}(k)$ *is stably parallelizable. Then $n=4$ and $k=2$.*

Proof. The inclusion $\mathrm{SU}(2n-1)\to\mathrm{SU}(2n)$ induces a diffeomorphism

$$\mathrm{SU}(2n-1)/\mathrm{Sp}(n-1)\to\mathrm{SU}(2n)/\mathrm{Sp}(n).$$

Therefore, it is sufficient to show that $\mathrm{SU}(5)/\mathrm{Sp}(2)$ is not stably parallelizable. Now $\mathrm{SU}(4)/\mathrm{Sp}(2)=S^5$ and $\mathrm{SU}(5)/\mathrm{SU}(4)=S^9$ so that there is a fibre bundle

$$S^5\xrightarrow{i}\mathrm{SU}(5)/\mathrm{Sp}(2)\xrightarrow{p}S^9.$$

Arguing as in *I*, Sect. 6, we find that $\tau(\mathrm{SU}(5)/\mathrm{Sp}(2))-14=p^*(z)$, where z is the non-zero element of $\widetilde{KO}(S^9)$, and that $p^*:KO(S^9)\to KO(\mathrm{SU}(5)/\mathrm{Sp}(2))$ is injective.

The study of the other quotients of SU(*n*) now goes along familiar lines:

(6.5) **Lemma.** *Let H be a closed connected non-abelian subgroup of* $\mathrm{SU}(n)$ *such that* $\mathrm{SU}(n)/H$ *is stably parallelizable. Then* $\mathrm{SU}(n)/H$ *is diffeomorphic to a Stiefel manifold, to* $\mathrm{SU}(n)/(\mathrm{SU}(2)\times\ldots\times\mathrm{SU}(2))$, *to* $\mathrm{SU}(4)/H_0$, *or to* $\mathrm{SU}(4)/\mathrm{SO}(4)$.

Proof. By *I* (3.4), H is semisimple. If H is simple and not covered by the preceding lemmas and *I* (5.1), proceed as follows: Look at Pontrjagin classes [17]; if p_1, p_2, and p_3 vanish, look at Stiefel-Whitney classes (Sect. 5 or [29]); in the few cases where they also vanish, do calculations in the complex representation ring using [26].

If H is semisimple but not simple, reread the proof of (6.1).

7. Quotients of Sp(n)

The non-parallelizability of these manifolds will be an easy consequence of what we have done so far and of the following observation:

(7.1) *Suppose that* $G = \mathrm{Sp}(n_1) \times \ldots \times \mathrm{Sp}(n_s)$ *and that* H *is a subgroup of* G *isomorphic to* $\mathrm{Sp}(k_1) \times \ldots \times \mathrm{Sp}(k_r)$. *Then*

$$\bar{\alpha}: RO(H) \otimes_{RO(G)} \mathbb{Z} \to KO(G/H)$$

is injective.

The proof is essentially the same as that of Proposition 5.1 of [26], replacing complex K-theory by real K-theory and taking into account that $H^i(BG;\mathbb{Z})$ and $H^i(BH;\mathbb{Z})$ are 0 for $i \not\equiv 0 \pmod 4$.

In order to apply (7.1), one has to know explicitly the structure of $RO(\mathrm{Sp}(n))$:

(7.2) *Write* $R(\mathrm{Sp}(n)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n]$, *where* λ_i *are the fundamental representations, with* λ_{2i} *orthogonal and* λ_{2i+1} *symplectic. Then* $RO(\mathrm{Sp}(n))$ *is the subring of* $R(\mathrm{Sp}(n))$ *generated by*

$$\lambda_2, \lambda_4, \lambda_6, \ldots;$$
$$2\lambda_1, 2\lambda_3, 2\lambda_5, \ldots;$$
$$\lambda_i \lambda_j \text{ for all odd } i, j.$$

See [12, VI, 4.8].

(7.3) **Corollary.** *The 30-dimensional manifold* $\mathrm{Sp}(4)/(\mathrm{SU}(2) \times \mathrm{SU}(2))$ *is not stably parallelizable.*

(7.4) **Lemma.** *Let* H *be a subgroup of* $\mathrm{Sp}(n)$ *such that* $\mathrm{SU}(2n)/H$ *is stably parallelizable. Then* $\mathrm{Sp}(n)/H$ *is diffeomorphic to* $\mathrm{Sp}(n)/(\mathrm{Sp}(1) \times \ldots \times \mathrm{Sp}(1))$ *or to* $\mathrm{Sp}(2)/\mathrm{SU}(2)$.

Proof. This follows from (6.5).

(7.5) **Lemma.** *Let* H *be a subgroup of* $\mathrm{Sp}(n)$ *such that* $\mathrm{Sp}(n)/H$ *is stably parallelizable, but* $\mathrm{SU}(2n)/H$ *is not. Then* $\mathrm{Sp}(n)/H$ *is diffeomorphic to* $\mathrm{Sp}(n)/\mathrm{SU}(2)$, $n > 2$, *or to* $\mathrm{Sp}(n)/\mathrm{Sp}(k)$.

This follows by considering Pontrjagin and Stiefel-Whitney classes.

In order to settle the question of parallelizability, it remains to show:

(7.6) $\mathrm{Sp}(n)/\mathrm{Sp}(1)^k = Y_{n,k}$ *is parallelizable for* $n > k$.

For $n \geqq k+2$, $Y_{n,k}$ admits a free $\mathbb{Z}_2 \times \mathbb{Z}_2$-action and is hence parallelizable by [27], Lemma 3.4. So it suffices that the semicharacteristic $k(Y_{n+1,n}; \mathbb{Z}_2)$ is zero. By an easy spectral sequence argument, $H^{\text{even}}(Y_{n+1,n}) \cong H^*(Y_{n,n})$; and $\dim H^*(Y_{n,n}) = n!$

8. Quotients of Spin(n)

As the parallelizable quotients of Spin(n) are more numerous than those of the other simple groups it is not surprising that the amount of technical work is quite formidable in this situation. We refer to [32] for fuller details. We begin with a modification of the techniques of Hsiang [17] to obtain at least sufficient criteria for the non-vanishing of the low-dimensional Pontrjagin classes of Spin quotients:

Let $\psi: H \to SO(n)$ be a real representation of a compact connected Lie group such that the lift $\bar{\psi}: H \to \text{Spin}(n)$ exists. Let T be a maximal torus of H and $\mathbb{S}^+(H)$ the set of positive roots of H with respect to T. We define $P^k H \in H^{2k}(BT; \mathbb{Z})$ as the homogeneous components of $\prod_{\alpha \in \mathbb{S}^+(H)} (1-\alpha^2)$. More generally, let $\Omega^+(\psi)$ be the positive weights of ψ and let $P^k \psi \in H^{2k}(BT; \mathbb{Z})$ be the homogeneous components of $\prod_{\omega \in \Omega^+(\psi)} (1-\omega^2)$. Finally, let

$$P^2\bar{\psi} = \tfrac{1}{2} P^2 \psi \in H^4(BT; \mathbb{Z}),$$

$$P^4\bar{\psi} = \tfrac{1}{2}(P^4\psi - (\tfrac{1}{2}P^2\psi)^2) \in H^8(BT; \mathbb{Z}),$$

$$P^6\bar{\psi} = \begin{cases} \tfrac{1}{2}P^6\psi & \text{if } \tfrac{1}{2}P^6\psi \in H^{12}(BT;\mathbb{Z}) \\ P^6\psi & \text{otherwise}. \end{cases}$$

(8.1) **Proposition.** *Let $n \geq 13$. If $p_i(\text{Spin}(n)/\bar{\psi}(H)) = 0$ for $i = 1, 2, 3$ then $P^{2k}H$ is contained in the ideal of $H^*(BT; \mathbb{Z})$ generated by $P^2\bar{\psi}$, $P^4\bar{\psi}$, $P^6\bar{\psi}$ for $k = 1, 2, 3$.*

The corresponding result for quotients of SO(n) can be found in [17]. The cohomology groups $H^{4k-1}(\text{Spin}(n); \mathbb{Z})$ are infinite cyclic by [7, 12.1]. Let y_{4k-1} be generators. Following [17] it suffices to show:

a) y_3, y_7, and $2y_{11}$ are universally transgressive.

b) $\bar{\varrho}^*(\tau(y_3)) = P^2\bar{\psi}$, $\bar{\varrho}^*(\tau(y_7)) = P^4\bar{\psi}$, $\bar{\varrho}^*(\tau(2y_{11})) = P^6\psi$, where τ is the transgression of the universal bundle

$$\text{Spin}(n) \to E\,\text{Spin}(n) \to B\,\text{Spin}(n)$$

and $\bar{\varrho}$ is the composition $BT \to BH \xrightarrow{B\bar{\psi}} B\,\text{Spin}(n)$.

The relevant cohomological information for a) is contained in [7] and [24, 6.5]. For b), consider the following commutative diagram:

$$\begin{array}{ccccc} & & B\,\text{Spin}(n) & \longleftarrow & BS \\ & \overset{\bar{\varrho}}{\nearrow} & \downarrow{\scriptstyle B\pi} & & \downarrow \\ BT & \xrightarrow{\varrho} & B\,SO(n) & \longleftarrow & BS', \end{array}$$

where S and S' are the standard maximal tori. Observe that $P^{2k}\psi$ is the pull-back under ϱ of the k^{th} universal Pontrjagin class [9]. In order to identify the images of the low-dimensional Pontrjagin classes under $(B\pi)^*$ we can work in the cohomology of BS, cf. [15].

The following two criteria are immediate from (2.3) resp. from I (2.6.b):

Let $\bar{\psi}$ be a homomorphism from the semisimple group H into $\mathrm{Spin}(n)$ such that $\mathrm{Spin}(n)/\bar{\psi}(H)$ is stably parallelizable.

(8.2) $2 \cdot I(\bar{\psi})$ *divides* $I(\mathrm{Ad}_H)$.

(8.3) *If* $\bar{\psi}$ *factors as* $H \xrightarrow{\varphi} \mathrm{SU}(m) \to \mathrm{Spin}(n)$ *then* $\mathrm{SU}(m)/\varphi(H)$ *is stably parallelizable.*

Application of (8.3), (8.2), (8.1) resp. (5.3) yields the desired non-parallelizability results for quotients of $\mathrm{Spin}(n)$ by simple groups H except in the following four cases:

(1) $\mathrm{Spin}(14)/\bar{\psi}(\mathrm{Spin}(6))$ with $\psi = \sigma_1 + \Delta^+ + \Delta^-$;

(2) $\mathrm{Spin}(26)/\bar{\psi}(F_4)$ where $\psi = \varphi_{F_4}$ is the first fundamental representation;

(3) $\mathrm{Spin}(10)/\Delta(\mathrm{Spin}(7))$;

(4) $\mathrm{Spin}(n)/\mathrm{SU}(k)$ with $n \geqq 2k$.

In case (1), consider the fibration

$$\mathrm{Spin}(6)/\mathrm{SU}(2) \to \mathrm{Spin}(14)/g(\mathrm{SU}(2)) \to \mathrm{Spin}(14)/\psi(\mathrm{Spin}(6))$$

where $g = 4\varphi + 6$. Observe that the bundle along the fibres is trivial. We know already that the total space is not stably parallelizable. Therefore, the base isn't.

Similarly, in case (2) we use the fibration

$$F_4/\mathrm{SU}(3) \to \mathrm{Spin}(27)/g(\mathrm{SU}(3)) \to \mathrm{Spin}(27)/F_4$$

with $g = 3(\varrho_1 + \bar{\varrho}_1) + 9$.

In case (3), we have the fibration

$$\mathrm{Spin}(9)/\Delta(\mathrm{Spin}(7)) \to \mathrm{Spin}(10)/\Delta(\mathrm{Spin}(7)) \to S^9.$$

The fiber is diffeomorphic to S^{15} [33, Appendix A]; the tangent bundle of the total space comes from the non-vanishing element of $\widetilde{KO}(S^9)$, cf. I (6.2) and I (6.3).

Finally, in case (4), we have the following result:

(8.4) **Lemma.** *$\mathrm{Spin}(n)/\mathrm{SU}(k)$ is stably parallelizable if and only if either $k = 2, 3$, and $n \geqq 2k$ or $k = 4, 5$, and $2k \leqq n \leqq 10$.*

Proof. Using the fact that the natural map

$$\mathrm{Spin}(2n-1)/\mathrm{SU}(n-1) \to \mathrm{Spin}(2n)/\mathrm{SU}(n)$$

is a diffeomorphism the proof is easily reduced to the case $\mathrm{Spin}(11)/\mathrm{SU}(4)$. We denote $\mathrm{Spin}(n)/\mathrm{SU}(4)$ by X_n.

(8.5) $$H^*(X_8; \mathbb{Z}) \cong \Lambda(z_6, z_7)$$

This follows from the fact that $\mathrm{SU}(4) \to \mathrm{Spin}(8)$ has index 1.

(8.6) $$H^*(X_9; \mathbb{Z}) \cong \Lambda(z_6, z_{15})$$

Apply the Serre spectral sequence of the fibration $X_8 \to X_9 \to S^8$ and the exact cohomology sequence of

$$X_9 \to B\,\mathrm{SU}(4) \to B\,\mathrm{Spin}(9).$$

The collapsing of the spectral sequence of $X_9 \to X_{10} \to S^9$ yields:

(8.7) $$H^*(X_{10};\mathbb{Z}) \cong \Lambda(z_6, z_9, z_{15}).$$

(8.8) *Up to dimension* 10, *the only non-vanishing cohomology groups of* X_{11} *are* $H^6(X_{11};\mathbb{Z}) \cong \mathbb{Z}$ *and* $H^{10}(X_{11};\mathbb{Z}) \cong \mathbb{Z}/2$.

Proof. Let $Y_n := \mathrm{Spin}(n)/\mathrm{SU}(5)$. Consider the following diagram of fibrations:

$$\begin{array}{ccccc} & & S^9 & = & S^9 \\ & & \downarrow & & \downarrow \\ X_9 & \longrightarrow & X_{11} & \longrightarrow & V_{11,2} \\ \| & & \downarrow & & \downarrow \\ Y_{10} & \longrightarrow & Y_{11} & \longrightarrow & S^{10} \end{array}$$

By [16], $H^{16}(Y_{11};\mathbb{R}) \cong \mathbb{R}$. Compare the spectral sequences of the two horizontal lines: Both collapse.

We can now conclude the proof of (8.4) as follows:

Let $p: X_{11} \to S^{10}$ be the fibration with fibre X_{10}. Its exact cohomology sequence together with (8.7) and (8.8) implies that $p^*: H^{10}(S^{10};\mathbb{Z}/2) \to H^{10}(X_{11};\mathbb{Z}/2)$ is bijective. Comparison of the Atiyah-Hirzebruch spectral sequences of X_{11} and S^{10} shows that $p^*: \widetilde{KO}(S^{10}) \to \widetilde{KO}(X_{11})$ is injective. Finally $\tau(X_{11}) - \dim X_{11} = p^*(\alpha)$ where α is the non-zero element of $\widetilde{KO}(S^{10})$.

Let now H be a semisimple but not simple subgroup of $\mathrm{Spin}(n)$. Most cases can be handled by the techniques introduced so far and offer no particular difficulties. There remains one hard case, namely $\mathrm{Spin}(13)/\mathrm{SU}(2)^2$.

(8.9) $\mathrm{Spin}(13)/\mathrm{SU}(2)^2$ *is not stably parallelizable.*

Proof. Let $Z_n := \mathrm{Spin}(n)/\mathrm{SU}(2)^2$. It is easy to see that Z_9 has the same cohomology groups as $V_{7,3} \times S^{15}$. The cohomology of Z_{13} up to dimension 12 can be computed from the spectral sequence of $Z_9 \to Z_{13} \to V_{13,4}$; there are no differentials in the interesting range. Let $p: Z_{13} \to S^{12}$ be the fibration with fibre Z_{12}.

From the cohomological facts we conclude that

$$p^*: \widetilde{KO}(S^{12}) \to \widetilde{KO}(Z_{13})$$

is non-trivial. The tangent bundle of Z_{13} is the image of a generator of $\widetilde{KO}(S^{12})$.

This completes the proof of Theorem 2 and Theorem 1. As to Theorem 3, parallelizability of $\mathrm{Spin}(n)/\mathrm{Ad}_H(H)$ will be shown in Sect. 9. In the remaining cases, it is not difficult to show the vanishing of the semicharacteristics, either by determination of the cohomology or by application of [27].

9. Parallelizability of SO(n)/Ad(H)

In order to complete the proof of Theorem 3, we have to consider the following situation: Let H be a compact connected Lie group of dimension n. Endow $\mathfrak{h}$ with an Ad-invariant scalar product and identify $\mathfrak{h}$ with Euclidean space $\mathbb{R}^n$. Then $\mathrm{Ad}(H)$ is identified with a subgroup of $\mathrm{SO}(n)$, and, for $m \geqq n$, with a subgroup of $\mathrm{SO}(m)$. The homogeneous space $\mathrm{SO}(m)/\mathrm{Ad}(H)$ is stably parallelizable.

(9.1) **Proposition.** *For any* $m \geqq n = \dim H$, $\mathrm{SO}(m)/\mathrm{Ad}(H)$ *is parallelizable.*

Remark. In the special case $H = \mathrm{SO}(3) \times \ldots \times \mathrm{SO}(3)$, this result was recently shown in [22] and [28].

Proof. We only have to show that $k(\mathrm{SO}(m)/\mathrm{Ad}(H); \mathbb{Z}_2) = 0$ if $\dim \mathrm{SO}(m)/\mathrm{Ad}(H)$ is odd. It suffices to assume that H is semisimple and with center $\{e\}$; then H is a product of centerfree simple groups. Our assertion is quite easy if H contains either no or at least three factors of type A_1; in the other cases it seems to be more subtle. We make again use of the fact [27, 3.4] that the semicharacteristic vanishes if there exists a free $\mathbb{Z}_2 \times \mathbb{Z}_2$-operation.

Let I_k be the $k \times k$ unit matrix; for $m \geqq 3k$, define $Y_{k,i} \in \mathrm{SO}(m)$, $i = 1, 2, 3$, by

$$Y_{k,1} = \mathrm{diag}(-I_k, -I_k, I_k, I_{m-3k}),$$
$$Y_{k,2} = \mathrm{diag}(-I_k, I_k, -I_k, I_{m-3k}),$$
$$Y_{k,3} = \mathrm{diag}(I_k, -I_k, -I_k, I_{m-3k}).$$

These, together with the identity, form a subgroup $\mathfrak{G}_k$ of $\mathrm{SO}(m)$ of type $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(9.2) *If H contains at least 3 factors of type A_1, then there is a free $\mathbb{Z}_2 \times \mathbb{Z}_2$-operation on* $\mathrm{SO}(m)/\mathrm{Ad}(H)$.

Indeed, we may write

$$\mathrm{Ad}(H) = \mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathrm{SO}(3) \times H'$$

with $H' \subseteqq \mathrm{SO}(m-9)$. The group $\mathfrak{G}_3$ operates freely by

$$A \cdot \mathrm{Ad}(H) \to A Y_{3,i} \cdot \mathrm{Ad}(H)$$

for $A \in \mathrm{SO}(m)$.

Now we have to introduce some more notation: Let $\{\alpha_1, \ldots, \alpha_r\}$ be a basis for the root system $\mathbb{S}$ of H, and let $\Gamma := \bigoplus_{i=1}^{r} \mathbb{Z}\alpha_i$. If $\varphi: \Gamma \to \{+1, -1\}$ is a group homomorphism, put

$$N(\varphi) := |\{\beta \in \mathbb{S} | \varphi(\beta) = -1\}|.$$

Finally, let $\mathfrak{M}(H) := \{N(\varphi) | \varphi \in \mathrm{Hom}(\Gamma, \{1, -1\})\}$. Then

$$\mathfrak{M}(H_1 \times H_2) = \{k_1 + k_2 | k_i \in \mathfrak{M}(H_i)\}.$$

(9.3) *If H is simple and* $\mathrm{Rank}(H) \geqq r$, *then, for any non-constant φ, we have* $N(\varphi) \geqq 2r$.

We begin with an easy graph-theoretical observation: Let T be a connected tree with r vertices, M a non-empty subset of the set of vertices of T, and $\mathscr{F}$ the set of all connected subgraphs of T containing exactly one vertex which lies in M. Then $|\mathscr{F}|\geqq r$.

Now choose T to be the Dynkin graph of H; let M be the set of those α_i for which $\varphi(\alpha_i)=-1$. If $\{\alpha_{i_1},\dots,\alpha_{i_k}\}\in\mathscr{F}$, then $\alpha_{i_1}+\dots+\alpha_{i_k}$ is a positive root of H and $\varphi(\alpha_{i_1}+\dots+\alpha_{i_k})=-1$. Hence there are at least $2r$ roots β with $\varphi(\beta)=-1$.

(9.4) *Suppose there is a natural number k such that* $2k\notin\mathfrak{M}(H)$ *and* $3k\leqq m$. *Then* $\mathbb{Z}_2\times\mathbb{Z}_2$ *acts freely on* $SO(m)/Ad(H)$.

Let $\mathfrak{G}_k$ act in the ordinary way, i.e. *from the left*, on $SO(m)/Ad(H)$. We have to show that this operation is free. If not, there would exist $A\in SO(m)$ with $Y_{k,i}A\cdot Ad(H)=A\cdot Ad(H)$, i.e. $A^{-1}Y_{k,i}A\in Ad(H)$. Let T be a maximal torus of H. Then there exists $X\in SO(m)$ with $X^{-1}Y_{k,i}X\in Ad(T)$, and therefore $u\in\mathfrak{t}$ with $X^{-1}Y_{k,i}X=Ad(\exp u)=e^{\mathrm{ad}\,u}$. Hence $e^{\mathrm{ad}\,u}$ has the eigenvalue 1 with multiplicity $m-2k$ and the eigenvalue -1 with multiplicity $2k$.

On the other hand, $e^{\mathrm{ad}\,u}$ has the eigenvalues $e^{\alpha(u)}$ for $\alpha\in\mathbb{S}$ and, in addition, the eigenvalue 1 with multiplicity $\mathrm{Rank}(H)$.

Hence the map $\varphi:\alpha\to e^{\alpha(u)}$ is a group homomorphism $\Gamma\to\{1,-1\}$ with $N(\varphi)=2k$, contradicting the assumption that $2k\notin\mathfrak{M}(H)$.

(9.5) *If H does not contain a factor of type* A_1, *there exists a free* $\mathbb{Z}_2\times\mathbb{Z}_2$*-action on* $SO(m)/Ad(H)$.

Indeed, by (9.3), we know that $2\notin\mathfrak{M}(H)$, and we can apply (9.4) with $k=1$.

(9.6) $\max\mathfrak{M}(H)\leqq 2/3\cdot\dim H$ *for any semisimple H. Moreover, if H contains a simple factor of* $\mathrm{rank}\geqq 3$, *then* $\max\mathfrak{M}(H)+2\leqq 2/3\dim H$.

For the proof, it suffices to consider the different simple groups separately. The verification is elementary but tedious and will not be given here. It turns out that more is true: For groups of high rank, the ratio between $\max\mathfrak{M}(H)$ and $\dim(H)$ approaches 1/2.

Anyhow, if H contains a simple factor of $\mathrm{rank}\geqq 3$, (9.6) allows to apply (9.4) with $k=1/2\max\mathfrak{M}(H)+1$. Therefore it remains to prove (9.1) for groups H which only contain factors of types A_2, B_2, G_2 and one or two factors of type A_1. Observe that $\mathfrak{M}(A_2)=\{0,4\}$, $\mathfrak{M}(B_2)=\{0,4,6\}$, $\mathfrak{M}(G_2)=\{0,8\}$. Consequently, (9.4) also applies with $k=1/2\max\mathfrak{M}(H)+1$ except in very few cases. Of these, only those with $\dim SO(m)/Ad(H)\equiv 3\,(\mathrm{mod}\,4)$ need to be considered [5]. So we are actually left with $H=A_1\times G_2$ and $H=A_1\times A_1\times B_2\times B_2$. Since $\mathfrak{M}(A_1\times G_2)=\{0,2,3,10\}$, we may apply (9.4) with $k=2$ in the first case. In the second case, we may finally apply the following observation whose proof is similar to that of (9.2):

(9.7) *If H contains two (not necessarily simple) factors of positive even dimension, then* $\mathbb{Z}_2\times\mathbb{Z}_2$ *acts freely on* $SO(m)/Ad(H)$.

This concludes the proof of (9.1).

Acknowledgements. We are grateful to V. Snaith and R. Stong for helpful correspondence. One of us (W.S.) would like to thank the IHES for its hospitality during the preparation of this paper.

References

1. Araki, S.: Differential Hopf algebras and the cohomology mod 3 of the compact exceptional groups E_7 and E_8. Ann. Math. **73**, 404–436 (1961)
2. Araki, S.: Cohomology modulo 2 of the compact exceptional groups E_6 and E_7. J. Math. Osaka Univ., Ser. A, **12**, 43–65 (1961)
3. Atiyah, M.F., Hirzebruch, F.: Vector bundles and homogeneous spaces. Proc. Symp. Pure Math. **3**, 7–38 (1961)
4. Atiyah, M.F., Hitchin, N.J., Singer, I.M.: Self-duality in four-dimensional Riemannian geometry. Proc. R. Soc. Lond. Ser. A, **362**, 425–461 (1978)
5. Becker, J.C.: The real semi-characteristic of a homogeneous space. Ill. J. Math. **25**, 577–588 (1981)
6. Borel, A.: Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. Ann. Math. **57**, 115–207 (1953)
7. Borel, A.: Sur l'homologie et la cohomologie des groupes de Lie compacts connexes. Am. J. Math. **76**, 272–342 (1954)
8. Borel, A.: Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes. Tohoku Math. J., Ser. II, **13**, 216–240 (1961)
9. Borel, A., Hirzebruch, F.: Characteristic classes and homogeneous spaces, I. Am. J. Math. **80**, 458–538 (1958)
10. Bott, R., Samelson, H.: Application of the theory of Morse to symmetric spaces. Am. J. Math. **80**, 964–1029 (1958)
11. Bourbaki, N.: Groupes et algèbres de Lie, Chap. IV–VIII. Paris: Hermann 1968, 1975
12. Bröcker, Th., Dieck, T. tom: Representations of compact Lie groups. Berlin, Heidelberg, New York, Tokyo: Springer 1985
13. Dynkin, E.B.: Semisimple subalgebras of semisimple Lie algebras. Am. Math. Soc. Transl., Ser. 2, **6**, 111–244 (1957)
14. Dynkin, E.B.: Topological characteristics of homomorphisms of compact Lie groups. Am. Math. Soc. Transl., Ser. 2, **12**, 301–342 (1959)
15. Feshbach, M.: The image of $H^*(BG;\mathbb{Z})$ in $H^*(BT;\mathbb{Z})$ for G a compact Lie group with maximal torus T. Topology **20**, 93–95 (1981)
16. Gugenheim, V.K.A.M., May, J.P.: On the theory and applications of differential torsion products. Mem. Am. Math. Soc. **142** (1974)
17. Hsiang, W.-Y.: On characteristic classes of compact homogeneous spaces and their applications in compact transformation groups. Bol. Soc. Bras. Mat. **10**, 87–162 (1979)
18. Kono, A., Mimura, M.: Cohomology mod 2 of the classifying space of the compact Lie group of type E_6. J. Pure Appl. Alg. **6**, 61–81 (1975)
19. Kono, A., Mimura, M., Shimada, N.: On the cohomology mod 2 of the classifying space of the 1-connected exceptional Lie group E_7. J. Pure Appl. Alg. **8**, 267–283 (1976)
20. Löffler, P., Smith, L.: Line bundles over framed manifolds. Math. Z. **138**, 35–52 (1974)
21. McKay, W.G., Patera, J.: Tables of dimensions, indices, and branching rules for representations of simple Lie algebras. New York, Basel: Dekker 1981
22. Miatello, I.D., Miatello, R.J.: On stable parallelizability of $\tilde{G}_{k,n}$ and related manifolds. Math. Ann. **259**, 343–350 (1982)
23. Oniščik, A.L.: Torsion of special Lie groups. Am. Math. Soc. Transl., Ser. 2, **50**, 1–4 (1966)
24. Quillen, D.: The mod 2 cohomology rings of extra-special 2-groups and spinor groups. Math. Ann. **194**, 197–212 (1971)
25. Singhof, W.: Parallelizability of homogeneous spaces, I. Math. Ann. **260**, 101–116 (1982)
26. Snaith, V.P.: Massey products in K-theory II. Proc. Camb. Phil. Soc. **69**, 259–289 (1971)
27. Stong, R.E.: Semicharacteristics and free group actions. Compos. Math. **29**, 223–248 (1974)
28. Stong, R.E.: Semicharacteristics of oriented Grassmannians (to appear)
29. Wang, M.Y.K.: Some remarks on the calculation of Stiefel-Whitney classes and a paper of Wu-Yi Hsiang's. Pac. J. Math. **112**, 431–444 (1984)

30. Warner, G.: Harmonic analysis on semi-simple Lie groups II. Berlin, Heidelberg, New York: Springer 1972
31. Watanabe, T.: The integral cohomology ring of the symmetric space E VII. J. Math. Kyoto Univ. **15**, 363–385 (1975)
32. Wemmer, D.: Stabil parallelisierbare homogene Räume. Thesis, Köln 1985
33. Whitehead, G.W.: Elements of Homotopy Theory. Berlin, Heidelberg, New York: Springer 1978

Received July 2, 1985; in revised form January 21, 1986