Math. Ann. 263, 365–370 (1983)

Mathematische Annalen

Kac-Moody Lie Algebras and the Universal Element for the Category of Nilpotent Lie Algebras

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0. Introduction

The problem of classifying nilpotent Lie algebras is studied for the first time in 1891 by a student of Engel, Umlauf, who gives the complete list over C up to the dimension 6 and a certain complex family at the dimensions 7, 8 and 9 [15]. Later on, several authors worked out the subject: Chevalley [3], Dixmier [4], Morozov [11], Vergne [16].

The introduction in 1972 by Bratzlavsky [2] and Favre [5, 6] of the root systems for the nilpotent Lie algebras constitutes an important step in the classification (Gurevic applied already this notion to the study of metabelian Lie algebras [7]).

By using these root systems we establish a link between nilpotent Lie algebras and Kac-Moody Lie Algebras (which generalise semi-simple Lie algebras and are of infinite dimension).

In [12] (published in [13]) we started with the maximal rank case and in [14] we continued with certain nilpotent Lie algebras [see 2.11(b)]. Here we give the main theorem (2.10) in the general case.

All the structures are over an algebraically closed field K of characteristic 0.

1. Root System and Cartan Matrix Associated to a Nilpotent Lie Algebra

1.1. All through 1, $\mathfrak{g}$ is a nilpotent Lie algebra of finite dimension, $\operatorname{Der}\mathfrak{g}$ its derivation algebra, $\operatorname{Aut}\mathfrak{g}$ its automorphism group.

1.2. A torus on $\mathfrak{g}$ is a commutative subalgebra of $\operatorname{Der}\mathfrak{g}$ whose elements are semi-simple endomorphisms. We denote $\mathscr{T}_{\mathfrak{g}}$ the set of maximal tori on $\mathfrak{g}$ (i.e. tori not contained in any other tori).

1.3. **Mostow's Theorem** (4.1 of [10]). *If* $T, T' \in \mathscr{T}_{\mathfrak{g}}$ *there exists* $\theta \in \operatorname{Aut}\mathfrak{g}$ *such that* $\theta T \theta^{-1} = T'$.

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1.4. Let $T\in\mathscr{T}_{\mathfrak{g}}$ and consider the root space decomposition of $\mathfrak{g}$ relatively to T:

$$\mathfrak{g}=\sum_{\beta\in R(T)}\mathfrak{g}^{\beta},$$
$$\mathfrak{g}^{\beta}=\{x\in\mathfrak{g}\,;tx=\beta(t)x\ \ \forall t\in T\},$$
$$R(T)=\{\beta\in T^*\,;\mathfrak{g}^{\beta}\neq(0)\}\quad\text{(root system)}.$$

We denote:

$$R^1(T)=\{\beta\in R(T)\,;\mathfrak{g}^{\beta}\not\subset[\mathfrak{g},\mathfrak{g}]\},$$
$$l_{\beta}=\dim(\mathfrak{g}^{\beta}/[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{g}^{\beta})\quad\forall\beta\in R^1(T).$$
$$d_{\beta}=\dim\mathfrak{g}^{\beta}\quad\forall\beta\in R^1(T).$$

The map: $\beta\mapsto d_{\beta}\quad R^1(T)\to\mathbb{N}^*$ gives the partition:

$$R^1(T)=R^1(T)_{p_1}\cup\ldots\cup R^1(T)_{p_q},$$
$$p_1<\ldots<p_q\quad R^1(T)_{p_i}\neq\emptyset,$$
$$R^1(T)_p=\{\beta\in R^1(T)\,;d_{\beta}=p\}.$$

Let $s_i=\#R^1(T)_{p_i}$ and $s=s_1+\ldots+s_q$; we number the elements of $R^1(T):\beta_1\ldots\beta_s$ in such a way that:

$$R^1(T)_{p_1}=\{\beta_1\ldots\beta_{s_1}\},\qquad R^1(T)_{p_2}=\{\beta_{s_1+1}\ldots\beta_{s_2}\},\ldots.$$

Let $d_i=d_{\beta_i}$, $l_i=l_{\beta_i}$ and $l=l_1+\ldots+l_s$ (one checks that $l=\dim\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$). Let $\mathfrak{S}_s^{s_1\ldots s_q}$ be the group of permutations of $\{1\ldots s\}$ which leave $\{1\ldots s_1\}$, $\{s_1+1\ldots s_2\}\ldots$ invariant.

1.5. Let $T'\in\mathscr{T}_{\mathfrak{g}}$ be another maximal torus; then there exists $\theta\in\operatorname{Aut}\mathfrak{g}$ such that $\theta T\theta^{-1}=T'$ (1.3). For T', we use the notations of 1.4 with prime. The map:

$$\check{\theta}:T^*\to T'^*,\qquad\beta\mapsto\check{\theta}\beta,\qquad\check{\theta}\beta(\theta t\theta^{-1})=\beta(t)\quad\forall\beta\in T^*,\quad\forall t\in T,$$

is a vector space isomorphism and one has obviously:

$$\theta\mathfrak{g}^{\beta}=\mathfrak{g}^{\check{\theta}\beta}\quad\forall\beta\in R(T)$$

therefore $d'_{\check{\theta}\beta}=d_{\beta}\ \forall\beta\in R^1(T)$ which gives:

$$q'=q,\qquad p'_i=p_i,\qquad s'_i=s_i,\qquad 1\leqq i\leqq q,\qquad s'=s.$$

Since $\theta[\mathfrak{g},\mathfrak{g}]=[\mathfrak{g},\mathfrak{g}]$, one has $l'_{\check{\theta}\beta}=l_{\beta}\ \forall\beta\in R^1(T)$. The map $\check{\theta}$ induces a bijection between: $R(T)$ and $R(T')$, $R^1(T)$ and $R^1(T')$, $R^1(T)_{p_i}$ and $R^1(T')_{p_i}$ $1\leqq i\leqq q$; thus there exists $\tau\in\mathfrak{S}_s^{s_1\ldots s_q}$ such that

$$\check{\theta}\beta_a=\beta'_{\tau a}\qquad 1\leqq a\leqq s.$$

In conclusion, we have the

Lemma. *The integers* q, $p_1\ldots p_q$, $s_1\ldots s_q$, s, $d_1\ldots d_s$, $l_1\ldots l_s$, l *defined in* 1.4. *are invariants of* $\mathfrak{g}$.

If $T, T'\in\mathscr{T}_{\mathfrak{g}}$ *there exists* $\theta\in\operatorname{Aut}\mathfrak{g}$ *and* $\tau\in\mathfrak{S}_s^{s_1\ldots s_q}$ *such that* $\theta\mathfrak{g}^{\beta_a}=\mathfrak{g}^{\beta'_{\tau a}}$ $1\leqq a\leqq s$.

1.6. Let $f:\{1,\ldots,l\}\to\{1,\ldots,s\}$ be defined by

$$\begin{aligned} f(i)&=1 \quad \text{if} \quad 1\leqq i\leqq l_1\\ &=2 \quad \text{if} \quad l_1<i\leqq l_1+l_2\\ &\cdots\cdots\cdots\cdots\cdots\\ &=s \quad \text{if} \quad l_1+\ldots+l_{s-1}<i\leqq l. \end{aligned}$$

(I am thankful to the referee for his suggestion to introduce f instead of a double numbering in the original version.)

Since f is onto, there exists $g:\{1,\ldots,s\}\to\{1,\ldots,l\}$ such that $f\circ g=\mathrm{Id}$. Obviously

$$\mathfrak{S}_s^{s_1\ldots s_q}\to\mathfrak{S}_l,$$
$$\sigma\to\hat{\sigma}=g\circ\sigma\circ f,$$

is an one-to-one group homomorphism. Define an action of $\mathfrak{S}_s^{s_1\ldots s_q}$ on the set of $l\times l$ matrices by setting:

$$\sigma(A_{ij})_{1\leqq i,j\leqq l}=(A_{\hat{\sigma}i\hat{\sigma}j})_{1\leqq i,j\leqq l}$$

1.7. **Theorem.** *For $i,j\in\{1,\ldots,l\}$ $i\neq j$ let*

$$-A_{ij}(T)=\mathrm{Min}\{-n\in\mathbb{N};\ (\mathrm{ad}v)^{-n+1}w=0\ \forall v\in\mathfrak{g}^{\beta_{f(i)}}\ \forall w\in\mathfrak{g}^{\beta_{f(j)}}\}$$

with $(\mathrm{ad}0)^0=0$ *and let* $A_{ii}(T)=2$ *for* $i=1,\ldots,l$. *Then*

(1) $A(T)=(A_{ij}(T))_{1\leqq i,j\leqq l}$ *is a Cartan matrix* (2.1).

(2) *The* $\mathfrak{S}_s^{s_1\ldots s_q}$*-orbit of* $A(T)$ *is an invariant of* $\mathfrak{g}$.

Proof. (1) Since $\mathrm{ad}v$ is nilpotent, $A_{ij}(T)$ is a well-defined non-positive integer for $i\neq j$; if $[v,w]=0$ then $[w,v]=0$, therefore $A_{ij}(T)=0$ implies $A_{ji}(T)=0$. Since $A_{ii}(T)=2$ by definition, $A(T)$ is a Cartan matrix.

(2) Let $T'\in\mathscr{T}_{\mathfrak{g}}$, by 1.5 there exist $\theta\in\mathrm{Aut}\,\mathfrak{g}$ and $\tau\in\mathfrak{S}_s^{s_1\ldots s_q}$ such that $\theta\mathfrak{g}^{\beta_a}=\mathfrak{g}^{\beta'_{\tau a}}$ $1\leqq a\leqq s$; if $v\in\mathfrak{g}^{\beta_a}$ and $w\in\mathfrak{g}^{\beta_b}$ and if $i,j\in\{1,\ldots,l\}$ are such that $f(i)=a$, $f(i)=b$, then $(\mathrm{ad}v)^{-A_{ij}(T)+1}w=0$; thus $(\mathrm{ad}\,\theta v)^{-A_{ij}(T)+1}\theta w=0$ with $\theta v\in\mathfrak{g}^{\beta'_{\tau a}}=\mathfrak{g}^{\beta'_{f(\hat{\tau}i)}}$ and $\theta w\in\mathfrak{g}^{\beta'_{\tau b}}=\mathfrak{g}^{\beta'_{f(\hat{\tau}j)}}$; therefore $-A_{\hat{\tau}i\hat{\tau}j}(T')\leqq -A_{ij}(T)$, and by symmetry $A_{\hat{\tau}i\hat{\tau}j}(T')=A_{ij}(T)$ which proves that $\tau A(T')=A(T)$.

1.8. *Definition.* We choose arbitrarily A in the $\mathfrak{S}_s^{s_1\ldots s_q}$-orbit of $A(T)$ $\left(\text{which has at most } \dfrac{s!}{s_1!\ldots s_q!} \text{ elements}\right)$ and we say by an abuse of language: "$\mathfrak{g}$ is of type A" or "A is the Cartan matrix of $\mathfrak{g}$". We denote:

$$\mathscr{T}_{\mathfrak{g}}(A)=\{T\in\mathscr{T}_{\mathfrak{g}};\ A(T)=A\},$$
$$\mathfrak{S}_s^{s_1\ldots s_q}(A)=\{\sigma\in\mathfrak{S}_s^{s_1\ldots s_q};\ \sigma A=A\}.$$

1.9. **Lemma.** *If T, $T'\in\mathscr{T}_{\mathfrak{g}}(A)$ there exist $\theta\in\mathrm{Aut}\,\mathfrak{g}$ and $\tau\in\mathfrak{S}_s^{s_1\ldots s_q}(A)$ such that:*

$$\theta\mathfrak{g}^{\beta_a}=\mathfrak{g}^{\beta'_{\tau a}}\quad\forall a=1\ldots s.$$

Proof. By 1.5 there exists $\theta\in\mathrm{Aut}\,\mathfrak{g}$ and $\tau\in\mathfrak{S}_s^{s_1\ldots s_q}$ such that $\theta\mathfrak{g}^{\beta_a}=\mathfrak{g}^{\beta'_{\tau a}}$; by the proof of 1.7 (2), $\tau A(T')=A(T)$; therefore $\tau A=\tau A(T')=A(T)=A$.

1.10. We denote by $msg(\mathfrak{g})$ the set of minimal systems of generators of $\mathfrak{g}$; by [1, Sect. 4, ex. 4, p. 119]: $(x_1 \ldots) \in msg(\mathfrak{g})$ if and only if $(x_1 + [\mathfrak{g}, \mathfrak{g}], \ldots)$ is a basis of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ therefore each element of $msg(\mathfrak{g})$ is an l-tuple $(x_1 \ldots x_l)$ where $l = \dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

Let $T \in \mathcal{T}_{\mathfrak{g}}(A)$ and denote:

$$msg(T) = msg(\mathfrak{g}) \cap ((\mathfrak{g}^{\beta_1})^{l_1} \times \ldots \times (\mathfrak{g}^{\beta_s})^{l_s}).$$

For all $(x_1 \ldots x_l) \in msg(T)$ one has:

$$(\operatorname{ad} x_i)^{-A_{ij}+1} x_j = 0, \quad 1 \leqq i \neq j \leqq l.$$

2. Kac-Moody Lie Algebras and the Universal Element for the Category of Nilpotent Lie Algebras

2.1. An $l \times l$ matrix with entries in $\mathbb{Z}$, $A = (A_{ij})$ is a Cartan matrix if:

a) $A_{ii} = 2 \quad \forall i = 1, \ldots, l$,

b) $A_{ij} \leqq 0 \quad \forall i, j = 1, \ldots, l, \quad i \neq j$,

c) $(A_{ij} = 0 \text{ if } A_{ij} = 0) \quad \forall i, j = 1, \ldots, l, \quad i \neq j$.

2.2. Let $\|A\| = \operatorname{Sup}\{|A_{ij}|; i \neq j\}$; denote h the Coxeter number of A if A is of finite type and $+\infty$ if not.

For $n \in \mathbb{N}$ such that $\|A\| < n \leqq h + 1$ let $\operatorname{Nil} p_n(A)$ be the category of nilpotent Lie algebras of nilpotency index n and of type A.

2.3. To A one can associate a Lie algebra (called a Kac-Moody Lie algebra) defined by generators:

$$\{f_1 \ldots f_l \, h_1 \ldots h_l \, e_1 \ldots e_l\},$$

and relations:

$$[h_i e_j] = A_{ij} e_j, \quad [h_i f_j] = -A_{ij} f_j, \quad (\forall i, j = 1 \ldots l),$$
$$[h_i h_j] = 0, \quad [e_i f_i] = h_i, \quad (\forall i, j = 1 \ldots l),$$
$$[e_i f_j] = 0, \quad (\operatorname{ad} e_i)^{-A_{ij}+1} e_j = 0 = (\operatorname{ad} f_i)^{-A_{ij}+1} f_j, \quad (\forall i \neq j).$$

[8, 9]. Actually, this definition is not the usual one (ordinarily one has to take a quotient of the above algebra).

2.4. Let $H = Kh_1 + \ldots + Kh_l$; denote $\mathcal{L}_+(A)$ (resp. $\mathcal{L}_-(A)$) the subalgebra of $\mathcal{L}(A)$ generated by $\{e_1 \ldots e_l\}$ (resp. $\{f_1 \ldots f_l\}$). One shows that:

$$\mathcal{L}(A) = \mathcal{L}_-(A) \oplus H \oplus \mathcal{L}_+(A).$$

One can define $\mathcal{L}_+(A)$ by generators: $\{e_1 \ldots e_l\}$ and relations:

$$(\operatorname{ad} e_i)^{-A_{ij}+1} e_j = 0 \quad \forall i, j = 1 \ldots l, \quad i \neq j.$$

2.5. Let $L \equiv L_n(A) \equiv \mathcal{L}_+(A)/C^{n+1}\mathcal{L}_+(A)$ where $(C^p \mathcal{L}_+(A))_{p \geqq 1}$ is the descending central series of $\mathcal{L}_+(A)$ with $C^1 \mathcal{L}_+(A) = \mathcal{L}_+(A)$.

By abuse of notation, write e_i instead of $e_i + C^{n+1}\mathcal{L}_+(A)$. One can define $L_n(A)$ by generators: $\{e_1 \ldots e_l\}$ and relations:

$$(\operatorname{ad} e_i)^{-A_{ij}+1} e_j = 0 \quad \forall i, j = 1 \ldots l, \quad i \neq j;$$
$$[e_{i_1} \ldots e_{i_{n+1}}] = 0 \quad \forall i_k = 1 \ldots l \quad \forall k = 1 \ldots n+1.$$

2.6. If A is of finite type $\mathscr{L}_+(A)$ is a nilpotent Lie algebra of nilpotency index $h+1$; therefore $C^n\mathscr{L}_+(A)\supsetneqq C^{n+1}\mathscr{L}_+(A)$ (since $n\leqq h+1$) thus $C^nL\neq(0)$. If A is not of finite type then $C^p\mathscr{L}_+(A)\supsetneqq C^{p+1}\mathscr{L}_+(A)$ $\forall p\geqq 1$ (one says that $\mathscr{L}_+(A)$ is pro-nilpotent) thus $C^nL\neq(0)$.

In both cases, one has $C^{n+1}L=(0)$; therefore $L_n(A)$ is a nilpotent Lie algebra of nilpotency index n.

2.7. One defines derivations $\partial_1\ldots\partial_l$ of L by setting $\forall i,j=1\ldots l$

$$\partial_i e_j=\begin{cases} e_i & i=j \\ 0 & i\neq j \end{cases}.$$

Let $(\alpha_1\ldots\alpha_l)$ be the dual basis of $(\partial_1\ldots\partial_l)$ and denote $D=K\partial_1+\ldots+K\partial_l$. It is easy to show that:

$$D\in\mathscr{T}_L,\qquad R^1(D)=\{\alpha_1\ldots\alpha_l\},\qquad L^{\alpha_i}=Ke_i\quad\forall i=1\ldots l,$$

$$d_i=l_i=1\quad\forall i=1\ldots l,\qquad q=1,\qquad p_1=1,\qquad s_1=l,\qquad \mathfrak{S}_s^{s_1\ldots s_q}=\mathfrak{S}_l$$

(permutation group of $\{1\ldots l\}$). We are therefore in the case 2.11(b).

2.8. Let $i,j\in\{1\ldots l\}$ $i\neq j$ and assume that $(\operatorname{ad}e_i)^{-A_{ij}}e_j=0$ in L, then $(\operatorname{ad}e_i)^{-A_{ij}}e_j\in C^{n+1}\mathscr{L}_+(A)$ in $\mathscr{L}_+(A)$; but $C^{n+1}\mathscr{L}_+(A)$ is spanned as a vector space by the $[e_{i_1}\ldots e_{i_r}]$'s such that $r\geqq n+1$, thus $-A_{ij}+1\geqq n+1$ (since $(\operatorname{ad}e_i)^{-A_{ij}}e_j=[e_i\ldots e_ie_j]$ with $-A_{ij}+1$ terms); this contradicts $\|A\|<n$; therefore $(\operatorname{ad}e_i)^{-A_{ij}}e_j\neq 0$, it follows by 2.11(b) that $A(D)=A$; so L is of type A and $D\in\mathscr{T}_L(A)$.

2.9. **Lemma.** *The Lie algebra $L_n(A)$ is an object of* $\operatorname{Nil}p_n(A)$.

Proof. It follows from 2.6 and 2.8.

2.10. **Theorem.** *The Lie algebra $L_n(A)$ is an universal element for the category* $\operatorname{Nil}p_n(A)$ *i.e. any object of* $\operatorname{Nil}p_n(A)$ *is a quotient of $L_n(A)$.*

More precisely: for any object $\mathfrak{g}$ of $\operatorname{Nil}p_n(A)$, *for any $T\in\mathscr{T}_{\mathfrak{g}}(A)$ and any $x=(x_1\ldots x_l)\in msg(T)$, the map $\{e_1\ldots e_l\}\to\mathfrak{g}$ $e_i\mapsto x_i$ extends uniquely to an homomorphism $\pi\equiv\pi(T,x):L_n(A)\to\mathfrak{g}$ which is onto; and one has $\mathfrak{g}\cong L_n(A)/\operatorname{Ker}\pi$.*

Proof. In $\mathfrak{g}$ one has at least the relations: $(\operatorname{ad}x_i)^{-A_{ij}+1}x_j=0$ (1.10) and $[x_{i_1}\ldots x_{i_{n+1}}]=0$; whereas in L the defining relations are $(\operatorname{ad}e_i)^{-A_{ij}+1}e_j=0$ and $[e_{i_1}\ldots e_{i_{n+1}}]=0$ (2.5); whence the existence of an homomorphism $\pi:L\to\mathfrak{g}$ such that $\pi e_i=x_i$; π is onto because $(x_1\ldots x_l)\in msg(\mathfrak{g})$; π is unique because $(e_1\ldots e_l)\in msg(L)$.

2.11. *Particular cases.* (a) Assume

$$l_a=1\quad\forall a=1,\ldots,s,$$

then $s=l$ (since $l_1+\ldots+l_s=l$), $f=\operatorname{Id}$, $\hat\sigma=\sigma$ $\forall\sigma\in\mathfrak{S}_l^{s_1\ldots s_q}$ and for $i\neq j$:

$$-A_{ij}(T)=\operatorname{Min}\{-n\in\mathbb{N};(\operatorname{ad}v)^{-n+1}w=0\quad\forall v\in\mathfrak{g}^{\beta_i}\ \forall w\in\mathfrak{g}^{\beta_j}\}.$$

(b) Assume that

$$d_a=1\quad\forall a=1,\ldots,s,$$

(since $l_a \leqq d_a$, it is a subcase of the preceeding one). Then, with the notations of 1.4, we have $q=1$, $p_1=1$, $s_1=l$, $\mathfrak{S}_s^{s_1 \dots s_q}=\mathfrak{S}_l$. Furthermore for all $i=1 \dots l$ there exists $x_i \in \mathfrak{g}^{\beta_i} \backslash (0)$ such that $\mathfrak{g}^{\beta_i}=Kx_i$, therefore

$$\forall i \neq j, \qquad -A_{ij}(T)=\operatorname{Min}\{-n \in \mathbb{N}; (\operatorname{ad} x_i)^{-n+1} x_j = 0\},$$

thus $-A_{ij}(T)$ is the unique integer such that $(\operatorname{ad} x_i)^{-A_{ij}(T)} x_j \neq 0$ and $(\operatorname{ad} x_i)^{-A_{ij}(T)+1} x_j = 0$ which is the definition given in [12–14]. This subcase is studied in detail in [14].

Acknowledgements. All my thanks to Professors M. Vergne and V. G. Kac for their kind invitation to the Massachusetts Institute of Technology (Cambridge, USA). It is during this stay that this work began with the help of Professor V. G. Kac. I also thank D. H. Peterson (MIT, Dept. of Math.) for the interest he took in my work.

References

1. Bourbaki, N.: Groupes et algèbres de Lie. Chap. 1. Paris: Hermann 1968
2. Bratzlavsky, F.: Sur les algèbres admettant un tore d'automorphismes donné. J. Algebra **30**, 305–316 (1974)
3. Chevalley, C.: Unpublished work [quoted by J. Dixmier in (4)]
4. Dixmier, J.: Sur les représentations unitaires des groupes de Lie nilpotentes III. Canad. J. Math. **10**, 321–348 (1958)
5. Favre, G.: Système de poids sur une algèbre de Lie nilpotente. Thesis 1972, Ecole Polytechnique Fédérale de Lausanne
6. Favre, G.: Système de poids sur une algèbre de Lie nilpotente. Manus. Math. **9**, 53–90 (1973)
7. Gurevic, G.R.: On metabelian Lie algebras. Trudy Sem. Vektor. Tenzor. Anal. **12**, 9–61 (1963)
8. Kac, V.G.: Simple irreducible graded Lie algebras of finite growth. Math. USSR-Izv. **2**, 1271–1311 (1968)
9. Moody, R.V.: A new class of Lie algebras. J. Algebra **10**, 211–230 (1968)
10. Mostow, G.D.: Fully reducible subgroups of algebraic groups. Am. J. Math. **78**, 200–221 (1956)
11. Morozov, V.V.: Classification of nilpotent Lie algebras of sixth order (in Russian; MR 24A 190). Ivz. Ucebn. Zave denii Mat. n° 4 (5), 161 (1958), USSR
12. Santharoubane, L.J.: Structure et cohomologie des algèbres de Lie nilpotentes. Thesis (1979), University of Paris 6
13. Santharoubane, L.J.: Kac-Moody Lie algebras and the classification of Nilpotent Lie algebras of maximal rank. Canadian J. of Math. (to appear)
14. Santharoubane, L.J.: Kac-Moody Lie algebras and the classification of certain nilpotent Lie algebras. Manuscript
15. Umlauf, K.A.: Über die Zusammensetzung der endlichen kontinuierlichen Transformationsgruppen, insbesondere der Gruppen vom Range Null. Leipzig SS 80, 1891
16. Vergne, M.: Variétés des algèbres de Lie nilpotentes. Thesis, University of Paris (1966)

Received November 30, 1982; in revised form January 11, 1983