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Global Several-Parameter Bifurcation and Continuation Theorems: a Unified Approach Via Complementing Maps

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1. Introduction

In this paper we will prove a theorem whose sole assumption is that the Leray-Schauder degree of a certain compact vector field is nonzero, and whose conclusions imply several-parameter, multidimensional global refinements of the implicit function theorem, the Leray-Schauder continuation principle [3, 11], and the global bifurcation theorem of Rabinowitz [13].

Let X be a Banach space, m be a positive integer and $\mathcal{O}$ be an open subset of $\mathbb{R}^m \times X$. We will consider local and global properties of the set of solution of equations of the form

$$f(\lambda, x) \equiv x - F(\lambda, x) = 0, \quad (\lambda, x) \in \mathcal{O}, \tag{1.1}$$

where $F : \mathcal{O} \to X$ is a compact mapping. If one identifies X with $\{0\} \times X \subseteq \mathbb{R}^m \times X$, one can view f as a map of $\mathcal{O} \subseteq \mathbb{R}^m \times X$ into $\mathbb{R}^m \times X$, and it is a compact perturbation of the identity on $\mathbb{R}^m \times X$. However, the range of f is contained in a subspace of $\mathbb{R}^m \times X$ of codimension m, so that its Leray-Schauder degree, when it is defined, is zero. In a certain sense, (1.1) is underdetermined by m dimensions. As such, it is not unreasonable to expect that under stable conditions under which one has existence results for (1.1) one will also have "m-multiplicity" results for the set of solutions. Our principle aim is to give a precise realization to this expectation.

A map $g : \mathcal{O} \to \mathbb{R}^m$, which maps bounded subsets of $\mathcal{O}$ into bounded sets, will be called a *complement* for $f : \mathcal{O} \to X$ provided that the Leray-Schauder degree, $\deg((g,f), \mathcal{O}, 0)$, is defined and non-zero: the map $(g,f) : \mathcal{O} \to \mathbb{R}^m \times X$ is defined by $(g, f)((\lambda, x)) = (g(\lambda, x), f(\lambda, x))$. Since (g, f) is a compact perturbation of the identity on $\mathbb{R}^m \times X$, the above degree is defined precisely when $\{(\lambda,x) \in \mathcal{O} | (g,f)((\lambda, x)) = 0\}$ is compact. We emphasize that, at this point, we are making no assertion about the compact extendability of f to $\bar{\mathcal{O}}$, and in the case such an extension exists, no assertion about the zeros of (g,f) on $\partial\mathcal{O}$. In [6] we introduced the notion of a complementing mapping under the additional assumptions that g and f were both defined on $\bar{\mathcal{O}}$ and $(g,f)^{-1}(0) \cap \partial\mathcal{O} = \emptyset$. While, as we demonstrated in [6], these additional assumptions are crucial in obtaining sharp continuation results for

(1.1), in order to obtain a general result which is applicable to both continuation and bifurcation problems it is necessary to consider the more general notion of complementing mapping which we have defined above.

When the mapping $f:\mathcal{O}\to X$ has a complement, $g:\mathcal{O}\to\mathbb{R}^m$, one can make rather precise global m-dimensional assertions concerning the manner in which $f^{-1}(0)$ emanates from $(g,f)^{-1}(0)$: this is the point of Theorem 2.1. Moreover, and, what is, of course, of equal importance, there are a number of interesting situations when one can show that a mapping has a complement.

If one chooses $\lambda_0\in\mathbb{R}^m$, and, letting $\mathcal{O}_{\lambda_0}=\{x\in X|(\lambda_0,x)\in\mathcal{O}\}$, defines $f_{\lambda_0}:\mathcal{O}_{\lambda_0}\to X$ by $f_{\lambda_0}(x)=f(\lambda_0,x)$, then $f:\mathcal{O}\to X$ is complemented by $g:\mathcal{O}\to\mathbb{R}^m$ defined by $g(\lambda,x)=\lambda-\lambda_0$ if and only if $\deg(f_{\lambda_0},\mathcal{O}_{\lambda_0},0)\neq 0$ [6]. From this observation we are able to use Theorem 2.1 to deduce multidimensional global versions of the implicit function theorem and the Leray-Schauder continuation principle: these are Theorems 2.2 and 2.3, respectively.

In order to obtain our global bifurcation result, Theorem 2.4, from Theorem 2.1, we have to use a somewhat different type of complementing map (see Proposition 4.1).

In [6] we have given, under the assumptions that $X=\mathbb{R}^k$, that $\mathcal{O}$ is open and bounded with $\partial\mathcal{O}$ smooth, and that $f:\bar{\mathcal{O}}\to\mathbb{R}^k$ is smooth with 0 a regular value both of $f:\mathcal{O}\to\mathbb{R}^k$ and of $f:\partial\mathcal{O}\to\mathbb{R}^k$, a necessary and sufficient condition for f to have a complement: the condition is that $f^{-1}(0)\cap\mathcal{O}$ and $f^{-1}(0)\cap\partial\mathcal{O}$ be nonempty.

In [2], Alexander and Yorke proved a global version of the implicit function theorem, while in [1] Alexander and Antman proved a multidimensional version of the global bifurcation result of Rabinowitz [13]. The conclusions of [1] and [2] are stated in the form of existence results for essential mappings, and are not directly comparable to the two consequences of Theorem 2.1 to which we have just referred. On the other hand, there is a substantial difference between the topological methods uses in [1] and [2], and, in our opinion, our present techniques are simpler and more elementary. In [1] and [2], $\mathcal{O}$ is required to be homeomorphic of $\mathbb{R}^m\times X$, while no such requirement is needed here. Also, the global bifurcation conclusions which we give are more precise (see Remark 2.1).

The global continuation result we obtain, Theorem 2.3, is a refinement of the result obtained by the authors in [6], the results of [6] being an improvement of the continuation results of Massabò and Pejsachowicz [10].

After this introduction, the paper has three further sections. In Sect. 2 we state, and discuss, our results. Section 3 is devoted to a proof of our main result, Theorem 2.1. In the last section we prove some of the consequences of this theorem.

An announcement of our present results appeared in [5].

2.

Let A be a topological space and k be a non-negative integer. Then A is said to have (Lebesgue covering) dimension equal to k provided that k is the smallest integer with the property that whenever $\mathcal{F}$ is an open cover of A there exists a refinement, $\mathcal{F}'$, of $\mathcal{F}$, which also covers A, and no more than any $k+1$ members of $\mathcal{F}'$ have non-empty intersection. If A fails to have the above refinement property for each integer k, then A is said to have infinite dimension. If $a \in A$, A will be said to have dimension at least k at a if each neighborhood, in A, of a has dimension at least k.

We will only use the notion of dimension when A is a separable metric space, and consequently, here, the concept of dimension defined above coincides with the notion of inductive dimension (see Hurewicz and Wallman [8]).

In the absence of any manifold structure on $f^{-1}(0)$, the notion of topological dimension is the natural way in which to describe its size.

When (C, D) is a pair of normal topological spaces with $D \subset C$, by $\check{H}^m(C, D)$ we will denote the m-th Čech cohomology group with integral coefficients of the pair. Moreover, a map $h:(C, D) \to (\mathbb{R}^m, \mathbb{R}^m - 0)$ will be said to be cohomologically nontrivial provided that the induced homomorphism

$$g*: \check{H}^m(\mathbb{R}^m, \mathbb{R}^m - 0) \to \check{H}^m(C, D)$$

is non-trivial.

Recall that D is said to be cobounded in C if $\overline{C/D}$ is compact.

We refer the reader to Dold [4] and Spanier [14] for some of the results in homology and cohomology theory which we will use, and to Lloyd [9] for results concerning the Leray-Schauder degree which we will need.

Our basic result is the following.

Theorem 2.1. *Let m, X, $\mathcal{O}$, and f be as in the introduction, and assume that there exists a complement $g: \mathcal{O} \to \mathbb{R}^m$ for $f: \mathcal{O} \to X$. Then there exists a closed connected subset, C, of $f^{-1}(0)$ whose dimension at each point is at least m, and*

for each cobounded subset, D, of C with
$D \cap g^{-1}(0) = \emptyset$, $g:(C, D) \to (\mathbb{R}^m, \mathbb{R}^m - 0)$ (∗)
is cohomologically nontrivial.

In particular,

$$C \cap g^{-1}(0) \neq \emptyset, \tag{2.1}$$

and

$$\text{either } C \text{ is unbounded or } \bar{C} \cap \partial\mathcal{O} \neq \emptyset. \tag{2.2}$$

Remark 2.1. Observe that (2.1) and (2.2) follow immediately from (∗) by taking $D = C$ and $D = \emptyset$, respectively.

Remark 2.2. Another easy consequence of (∗) is the following: if $K \subset C$ is compact, with $g^{-1}(0) \cap C \subsetneq K$, then, for $1 \leq i \leq m$, g_i must change sign on $(C \backslash K) \cap \{(\lambda, x) | g_j(\lambda, x) = 0, j \neq i, 1 \leq j \leq m\}$. This will be crucial in proving our bifurcation results.

When f is continuously Fréchet differentiable, $(\lambda_0, x_0) \in \mathcal{O} \cap f^{-1}(0)$ and $\frac{\partial f}{\partial x}(\lambda_0, x_0)$ is a bijection, the classical implicit function theorem implies that in a neighborhood of (λ_0, x_0), $f^{-1}(0)$ is an m-manifold. On the other hand, in this situation, if one lets V be a small neighborhood of (λ_0, x_0) then $f: V \to X$ is complemented by $g: V \to \mathbb{R}^m$ defined by $g(\lambda, x) = \lambda - \lambda_0$: this follows from the well-known linearization property of the Leray-Schauder degree. Hence the following theorem, which is a straight forward consequence of Theorem 2.1, contains, as a special case, a global version of the implicit function theorem.

Theorem 2.2. *Let $m, X, \mathcal{O}$, and f be as in the introduction. Assume that $g:\mathcal{O}\to\mathbb{R}^m$ is continuous, and maps bounded sets into bounded sets. Suppose $V\subseteqq\mathcal{O}$ is open and $g:V\to\mathbb{R}^m$ complements $f:V\to X$. Let $K=((g,f)|_V)^{-1}(0)$. Then there exists a closed connected subset, C, of $f^{-1}(0)$ whose dimension at each point is at least m, $C\cap K\neq\emptyset$, and at least one of the following properties holds:*

$$\begin{aligned}&(a)\ C \text{ is unbounded.}\\ &(b)\ \bar{C}\cap\partial\mathcal{O}\neq\emptyset.\\ &(c)\ \bar{C}\cap\{(g,f)^{-1}(0)\backslash K\}\neq\emptyset.\end{aligned} \tag{2.3}$$

The above theorem is an immediate consequence of Theorem 2.1. Indeed, let $\mathcal{U}=\mathcal{O}\backslash\{(g,f)^{-1}(0)\backslash K\}$ and observe that $g:\mathcal{U}\to\mathbb{R}^m$ is a complement for $f:\mathcal{U}\to X$. So we apply Theorem 2.1, with $\mathcal{U}$ playing the role of $\mathcal{O}$.

When one applies Theorem 2.2 in the situation where $\frac{\partial f}{\partial x}(\lambda_0, x_0)$ is invertible and V is a small neighborhood of (λ_0, x_0) the conclusion is that the m-manifold, $f^{-1}(0)\cap V$, may be continued globally into a connected set, C, with $C\subseteqq f^{-1}(0)$ and C has dimension at least m at each point. Moreover, either C is unbounded, $\bar{C}\cap\partial\mathcal{O}\neq\emptyset$ or C curls back to $\mathcal{O}_{\lambda_0}$ at some point distinct from x_0.

In the situation where f and g are defined on $\bar{\mathcal{O}}$ with $(g,f)^{-1}(0)\cap\partial\mathcal{O}=\emptyset$ one can considerably sharpen the conclusion of Theorem 2.1.

Recall that if A is a locally compact space and $h:A\to\mathbb{R}^m$ is a proper continuous map, then h is called *essential among proper maps* provided that $\hat{h}:\hat{A}\to S^m$ is homotopically nontrivial, where $\hat{h}$ is the extension of h to $\hat{A}$, $\hat{A}$ being the one-point compactification of A. Clearly, if h is essential among proper maps then $h(A)=\mathbb{R}^m$.

Our second consequence of Theorem 2.1 is

Theorem 2.3. *Let f and g satisfy the assumptions of Theorem 2.1, in addition to which we assume f and g are defined on $\bar{\mathcal{O}}$ with $(g,f)^{-1}(0)\cap\partial\mathcal{O}=\emptyset$. Then there exists a closed connected subset, C, of $f^{-1}(0)\cap\mathcal{O}$ such that $C\cap g^{-1}(0)\neq\emptyset$ and C has dimension at least m at each point. Moreover, C satisfies at least one of the following properties:*

$$\begin{aligned}&(a)\ C \text{ is unbounded}\\ &(b)\ \dim(\bar{C}\cap\partial\mathcal{O})\geqq m-1, \text{ when } m>1, \text{ while } \bar{C}\cap\partial\mathcal{O}\\ &\quad \text{contains at least two points when } m=1.\end{aligned} \tag{2.4}$$

If, in addition, $g:\bar{C}\to\mathbb{R}^m$ is proper, then C satisfies at least one of the following two properties:

$$\begin{aligned}&(a)\ \dim(\bar{C}\cap\partial\mathcal{O})\geqq m-1\\ &(b)\ g:C\to\mathbb{R}^m \text{ is essential among proper maps, and in particu-}\\ &\quad \text{lar, } g(C)=\mathbb{R}^m.\end{aligned} \tag{2.5}$$

As a particular case of the above we obtain

Corollary 2.1. *Let $m, X, \mathcal{O}$, and f be as in the introduction, in addition to which we assume F is defined on $\bar{\mathcal{O}}$. Suppose $\lambda_0\in\mathcal{O}$ is such that $\deg(f_{\lambda_0}, \mathcal{O}_{\lambda_0}, 0)\neq 0$ and $0\notin f_{\lambda_0}(\partial\mathcal{O}_{\lambda_0})$. Let the following a-priori bound hold: if $\{(\lambda_n, x_n)\}\subseteqq\bar{\mathcal{O}}\cap f^{-1}(0)$ and $\{\lambda_n\}$*

is bounded, then $\{x_n\}$ *is bounded. Then there exists a closed connected subset, C, of* $f^{-1}(0)$, *whose dimension at each point is at least m,* $C \cap \mathcal{O}_{\lambda_0} \neq \emptyset$, *and at least one of the following two properties holds:*

$$\begin{aligned}&\text{(a) } \dim(\bar{C}\cap\partial\mathcal{O}) \geqq m-1, \text{ when } m>1, \text{ while } \bar{C}\cap\partial\mathcal{O} \text{ contains at least two points when } m=1. \\ &\text{(b) for each } \lambda\in\mathbb{R}^m \text{ there exists some } x\in X \text{ with } (\lambda, x)\in C.\end{aligned} \tag{2.6}$$

In the case when $m=1$ the above corollary is a slight refinement of the classical Leray-Schauder continuation principle (see [3, 11]). When $m>1$, if one slices the parameter space into one-dimensional subspaces passing through λ_0 one could apply the one dimensional result to f, with its parameter restricted to this subspace. The above corollary gives a description of how these slices of solution mesh together.

An m-dimensional continuation principle similar to the above corollary was obtained by Massabò and Pejsachowicz in [10], and by the authors in [6].

The final consequence of Theorem 2.1 which we will discuss is a bifurcation result. Here we assume, in addition to our standing assumptions on f, that

$$f(\lambda,0)=0 \quad \text{whenever} \quad (\lambda,0)\in\mathcal{O},$$

and we refer to $\{\mathbb{R}^m\times\{0\}\}\cap\mathcal{O}$ as the trivial solutions. We are interested in discussing the manner in which the nontrivial solutions of (1.1) branch away from the trivial solutions.

Recall that if $\lambda\in\mathbb{R}^m$ is such that 0 is an isolated point of $f_\lambda^{-1}(0)$ then the Leray-Schauder index, $\mathrm{ind}(f_\lambda, 0)$, is well-defined. For a subset, Γ, of $\mathbb{R}^m$, we will identify Γ with $\Gamma\times\{0\}\subseteqq\mathbb{R}^m\times X$, and for a subset, D, of $\mathbb{R}^m\times X$, denote by D_Γ the set $\{(\lambda,x)\mid(\lambda,x)\in D, \lambda\in\Gamma\}$. Moreover, for our mapping $f:\mathcal{O}\to X$, the map f_Γ will denote the restriction of f to $\mathcal{O}_\Gamma$. A point $\lambda^*\in\Gamma$ will be called a *Γ bifurcation point of f if* there exists a sequence $\{(\lambda_k,x_k)\}$ converging to $(\lambda^*,0)$, with $\{\lambda_k\}\subseteqq\Gamma$ and $\{x_k\}\subseteqq X\setminus\{0\}$, and $\{(\lambda_k,x_k)\}\subseteqq f^{-1}(0)$.

Theorem 2.4. *Suppose m, X,* $\mathcal{O}$, *and f are as in the introduction. Let* $\Gamma\subseteqq\mathcal{O}\cap\{\mathbb{R}^m\times\{0\}\}$ *be described by* $\Gamma=h^{-1}(0)$, *where* $h:\mathbb{R}^m\to\mathbb{R}^{m-1}$ *is continuously differentiable and has* 0 *as a regular value. Suppose* $\underline{\lambda}$ *and* $\bar{\lambda}$ *lie in the same component of* Γ, *neither* $\underline{\lambda}$ *nor* $\bar{\lambda}$ *are* Γ *bifurcation points of f and* $\mathrm{ind}(f_{\underline{\lambda}},0)\neq\mathrm{ind}(f_{\bar{\lambda}},0)$. *Then there exists a closed connected subset, C, of* $\{(\lambda,x)\mid(\lambda,x)\in\mathcal{O}, x\neq0, f(\lambda,x)=0\}$, *whose dimension at each point is at least m, and* $\bar{C}\cap[\underline{\lambda},\bar{\lambda}]\neq\emptyset$, *where* $[\underline{\lambda},\bar{\lambda}]$ *denotes any segment of* Γ *between* $\underline{\lambda}$ *and* $\bar{\lambda}$. *One of the following three properties is also satisfied by C:*

$$\begin{aligned}&\text{(a) } C \text{ is unbounded,} \\ &\text{(b) } \bar{C}\cap\partial\mathcal{O}\neq\emptyset, \\ &\text{(c) } \bar{C}\cap\{\Gamma\setminus[\underline{\lambda},\bar{\lambda}]\}\neq\emptyset.\end{aligned} \tag{2.7}$$

Remark 2.1. The conclusion of Theorem 2.4 does *not* hold when $\Gamma=\{\gamma(t)\mid(t\in\mathbb{R}\}$, where $\gamma:\mathbb{R}\to\mathbb{R}^m$ is an arbitrary continuously differentiable homeomorphism. For example, it does not hold if $m=1$, $\mathcal{O}=\mathbb{R}\times X$ and Γ is a bounded interval. Rather then state the most general, necessarily technical, hypotheses on Γ under which one can prove the bifurcation result, we have asserted that Γ has a trivial normal bundle. There are Γ, not necessarily included under this hypothesis, for which the conclusion also holds. One such situation is when $\Gamma=\{\gamma(t)|(t\in\mathbb{R}\}$, where $\gamma:\mathbb{R}\to\mathbb{R}^m$ is a continuously differentiable homeomorphism such that $|\gamma(t)|\to+\infty$ as $|t|\to+\infty$. That the theorem holds in this situation follows from the proof of Theorem 2.4 which we will give, together with the observation that for each $r>0$, the components of $\Gamma\cap B(0,r)$ have trivial normal bundles.

Remark 2.2. When one assumes simply that $\Gamma=\{\gamma(t)|t\in\mathbb{R}\}$, γ being a C^1 homeomorphism, it is possible to obtain a weaker conclusion: namely, if one replaces (2.7) (c) by

$$\text{(c')}\quad \bar{C}\cap\{\mathbb{R}^m\backslash[\underline{\lambda},\bar{\lambda}]\}\neq\emptyset, \tag{2.7}$$

the theorem is true. However, this latter conclusion is somewhat unsatisfactory, since, for instance, C could be a small m-disc transversal to $(\underline{\lambda},\bar{\lambda})$, and so the conclusion, in some sense, is not global. Again, an examination of our proof of Theorem 2.4 shows how to prove this latter result.

Remark 2.3. We observe that the section of C over Γ, C_Γ, need not be connected. However, it is not difficult to see from our proof of Theorem 2.1 that there is a connected component of C_Γ which satisfies one of the alternatives listed in (2.7).

Remark 2.4. In the case when $m=1$, Theorem 2.4 is a slight refinement of the global bifurcation theorem of Rabinowitz [13]. An m-dimensional generalization of the Rabinowitz theorem has been recently obtained by Alexander and Antman [1]. Theorem 2.4 improves the corresponding result of Alexander and Antman [1]: our assumptions are somewhat weaker, and we discuss the intersection of the component, C, with $\Gamma\backslash[\underline{\lambda},\bar{\lambda}]$.

Remark 2.5. The conclusions of our results are not topological, in the sense that C being unbounded is not a property preserved by homeomorphic change of variables. For this reason a certain amount of care has to be taken when making simplifying assumptions.

The notion of a complementing map, obviously, can be defined whenever (g,f) falls within a class of mappings for which there is defined a topological degree. Moreover, if this degree is sufficiently well developed, the results of this paper will carry over. Our results hold for substantially larger classes of mappings then we have considered here. We will return to this in a future paper in which we will also present applications of our present results in the study of nonlinear partial differential equations. Examples of nonlinear ordinary and partial differential equations which may be formulated naturally as an equation of type (1.1) may be found in [6, 7].

3.

We will prove Theorem 2.1. To do so, and also because it will be useful in other contexts, we will first prove a preliminary proposition.

We let $S=\{(\lambda,x)\in\mathcal{O}|f(\lambda,x)=0\}$, and observe that S is locally compact since F is a compact mapping.

Proposition 3.1. *Under the assumptions of Theorem 2.1, S has the following property:*

if D is a cobounded subset of S such that $D \cap g^{-1}(0) = \emptyset$, *then* $g:(S,D) \to (\mathbb{R}^m, \mathbb{R}^m - 0)$ *is cohomologically nontrivial.*

Proof. We first observe that $\check{H}^m(S,D) = \varinjlim \{\check{H}^m(V,W) \mid (V,W) \in T\}$, where T consists of pairs (V,W) with V a neighborhood, in $\mathcal{O}$, of S, and W a neighborhood, in $\mathcal{O}$, of D, such that $W \subset V$. But those pairs with the additional property that $W \cap g^{-1}(0) = \emptyset$ are a cofinal subfamily of T.

Consequently, to prove the result it will suffice to show that if (V,W) is a pair of open subsets of $\mathcal{O}$, $S \subset V$, $D \subset W$, $W \subset V$ and $g^{-1}(0) \cap W = \emptyset$, then $g:(V,W) \to (\mathbb{R}^m, \mathbb{R}^m - 0)$ is cohomologically nontrivial. Let (V,W) be such a pair.

Since $S \backslash W$ is bounded we may choose an $r > 0$ such that $S \backslash W \subseteqq B \equiv \{(\lambda,x) \in \mathbb{R}^m \times X \mid \|(\lambda,x)\| < r\}$. Then we see that

$$(g,f)^{-1}(0) \subseteqq S \backslash W \subseteqq V \cap B,$$

and so, since $\mathbb{R}^m \times X$ is normal, we may choose an open subset, $\mathcal{U}$, of $V \cap B$, such that

$$(g,f)^{-1}(0) \subseteqq \mathcal{U} \subseteqq \bar{\mathcal{U}} \subseteqq V \cap B.$$

But then $(g,f)^{-1}(0) \cap \partial\mathcal{U} = \emptyset$. Since $\partial\mathcal{U}$ is bounded and closed, and since F is compact, we may select $\varepsilon > 0$ such that $\|(g,f)(\lambda,x)\| \geqq \varepsilon > 0$, whenever $(\lambda,x) \in \partial\mathcal{U}$.

Using the basic approximation property for compact mappings we may choose a finite-dimensional subspace, H, of X and a continuous mapping $F_\varepsilon : \bar{\mathcal{U}} \to H$ such that

$$\|F(\lambda,x) - F_\varepsilon(\lambda,x)\| \leqq \varepsilon/2, \quad \text{whenever} \quad (\lambda,x) \in \bar{\mathcal{U}}.$$

It follows from the excision and homotopy invariance properties of degree that

$$\deg((g,f),\mathcal{O},0) = \deg((g,f),\mathcal{U},0) = \deg((g,f_\varepsilon),\mathcal{U},0),$$

where $f_\varepsilon(\lambda,x) = x - F_\varepsilon(\lambda,x)$, for $(\lambda,x) \in \bar{\mathcal{U}}$.

Consequently, if we let $\mathcal{U}' = \mathcal{U} \cap \{\mathbb{R}^m \times H\}$, and use the assumption that g complements f on $\mathcal{O}$, together with the reduction property for Leray-Schauder degree, it follows that

$$\deg((g,f_\varepsilon)|_{\mathcal{U}'},\mathcal{U}',0) \neq 0,$$

where now "deg" means Brouwer degree.

We now identify H with $\mathbb{R}^n$, for some positive integer n. Let $W' = W \cap \mathcal{U}'$ and let $Z = \{(\lambda,x) \in \mathcal{U}' \mid f_\varepsilon(\lambda,x) \neq 0\}$. Then we have the following maps of pairs;

$$f_\varepsilon : (\mathcal{U}', Z) \to (\mathbb{R}^n, \mathbb{R}^n - 0),$$

and

$$g : (\mathcal{U}', W') \to (\mathbb{R}^m, \mathbb{R}^m - 0).$$

Since $\mathcal{U}'$, W', and Z are open subsets of euclidean spaces, we can identify the Čech cohomology groups $\check{H}^i(\mathcal{U}', Z)$ and $\check{H}^i(\mathcal{U}', W')$, with the singular cohomology groups $H^i(\mathcal{U}', Z)$ and $H^i(\mathcal{U}', W')$ respectively, by means of a natural isomorphism. Let e_k be the canonical generator of the local cohomology group, $H^k(\mathbb{R}^k, \mathbb{R}^k - 0)$.

We shall compute the cup-product

$$\eta \equiv g^*(e_m) \cup f_\varepsilon^*(e_n) \in H^{m+n}(\mathscr{U}', W' \cup Z)$$

in terms of the homomorphism induced in cohomology by the following map of pairs:

$$h \equiv (g, f_\varepsilon) \equiv (g \times f_\varepsilon) \circ \Delta : (\mathscr{U}', W' \cup Z) \to (\mathbb{R}^{m+n}, \mathbb{R}^{m+n} - 0),$$

where Δ is the diagonal map.

Indeed, from the properties of the cup-product and cross-product, we obtain

$$\eta = \Delta^*(g^*(e_m) \times f^*(e_n)) = h^*(e_m \times e_n) = h^*(e_{m+n}).$$

On the other hand, from our assumptions, the set $K \equiv \mathscr{U}' \setminus \{W' \cup Z\}$ is a compact set containing $h^{-1}(0)$, so that if we let $O_K \in H_{m+n}(\mathscr{U}', \mathscr{U}' - K)$ be the fundamental class around K and O_{m+n} denotes the fundamental class around 0 in $\mathbb{R}^{m+n}$, it follows that

$$h_*(O_K) = \deg(h, \mathscr{U}', 0) \cdot O_{m+n} \quad \text{in} \quad H_{m+n}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} - 0),$$

(see Proposition 5.5 of [4]). H_{m+n} denoting the singular homology group, and h_* being the homeomorphism induced in homology.

Now we take the Kronecker product of $\eta \in H^{m+n}(\mathscr{U}', W' \cup Z) = H^{m+n}(\mathscr{U}', \mathscr{U}' \setminus K)$ with O_K. We obtain

$$\begin{aligned} \langle \eta, O_K \rangle &= \langle h^*(e_{m+n}), O_K \rangle \\ &= \langle e_{m+n}, h_*(O_K) \rangle = \deg(h, \mathscr{U}', 0), \end{aligned}$$

since $\langle e_{m+n}, O_{m+n} \rangle = 1$.

Thus $\eta \neq 0$, and so $g^*(e_m) \in H^m(\mathscr{U}', W') \cong \check{H}^m(\mathscr{U}', W')$ is nonzero.

Letting $i : (\mathscr{U}', W') \to (V, W)$ be the inclusion map, we have the following commutative diagram

$$\begin{array}{ccc} \check{H}^m(V, W) & \xleftarrow{\;g^*\;} & \\ \downarrow{\scriptstyle i^*} & & \check{H}^m(\mathbb{R}^m, \mathbb{R}^m - 0) \\ \check{H}^m(\mathscr{U}', W') & \xleftarrow{\;g^*\;} & \end{array}$$

It follows that $g : (V, W) \to (\mathbb{R}^m, \mathbb{R}^m - 0)$ is cohomologically nontrivial, and so the proposition is proven. □

Proof of Theorem 2.1. Recall that the m-th Čech cohomology group of S, with compact supports, $\check{H}_c^m(S)$, is defined by

$$\check{H}_c^m(S) = \varinjlim \check{H}^m(S, D),$$

where D ranges over all cobounded subsets of S. Let $D_0 = S \setminus g^{-1}(0)$, so that from our basic assumption on $(g, f)^{-1}(0)$, D_0 is a cobounded subset of S.

If we let $j_{D_0} : \check{H}^m(S, D_0) \to \check{H}_c^m(S)$ be the natural map into the direct limit, and let $\xi \equiv j_{D_0}(g^*(e_m))$, Proposition 3.1 asserts that $\xi \neq 0$.

We can now invoke Proposition 4.3 of [6] in order to choose a closed connected subset, C, of S, which has dimension at each point at least m, and is such that the class $\xi|_C \in \check{H}_c^m(C)$ is nonzero.

Now let D be any cobounded subset of C such that $D \cap g^{-1}(0) = \emptyset$. From the following commutative diagram,

$$\begin{array}{ccccc} & \overset{g^*}{\nearrow} & \check{H}^m(S, D_0) & \xrightarrow{j_{D_0}} & \check{H}^m_c(S) \\ \check{H}^m(\mathbb{R}^m, \mathbb{R}^m - 0) & & \downarrow \text{inclusion}^* & & \downarrow \text{inclusion}^* \\ & \underset{g^*}{\searrow} & \check{H}^m(C, D) & \xrightarrow[j_D]{} & \check{H}^m_c(C), \end{array}$$

it follows that $g:(C, D) \to (\mathbb{R}^m, \mathbb{R}^m - 0)$ is cohomologically nontrivial. □

4.

The Proof of Theorem 2.3. Theorem 2.1 allows us to choose a closed connected subset, C, od $f = \{(\lambda, x) | (\lambda, x) \in \mathcal{O}, f(\lambda, x) = 0\}$, whose dimension at each point is at least m, and which enjoys property (∗).

Let $\dot{C} = \bar{C} \cap \partial \mathcal{O}$. We will show that $g^*: \check{H}^m(\mathbb{R}^m, \mathbb{R}^m - 0) \to \check{H}^m_c(\bar{C}, \dot{C})$ is nontrivial. To do so, since $\check{H}^m_c(\bar{C}, \dot{C}) = \varinjlim \check{H}^m(\bar{C}, W)$, where W ranges over neighborhoods, in $\bar{C}$, of $\dot{C}$, which are cobounded in $\bar{C}$ and such that $g^{-1}(0) \cap W = \emptyset$, it will suffice to show that for each such W

$$g:(\bar{C}, W) \to (\mathbb{R}^m, \mathbb{R}^m - 0)$$

is cohomologically nontrivial. But $C \cap W$ is a cobounded subset of C, on which g does not vanish, and so

$$g:(C, C \cap W) \to (\mathbb{R}^m, \mathbb{R}^m - 0)$$

is cohomologically nontrivial. Consequently, from the commutativity of the following diagram,

$$\begin{array}{ccc} (C, C \cap W) & \xrightarrow{i} & (\bar{C}, W) \\ & {\scriptstyle g} \searrow \quad \swarrow {\scriptstyle g} & \\ & (\mathbb{R}^m, \mathbb{R}^m - 0) & \end{array}$$

it follows that $g^*: H^m(\mathbb{R}^m, \mathbb{R}^m - 0) \to \check{H}^m_c(\bar{C}, \dot{C})$ is nontrivial.

To verify the first assertion of the theorem we assume that C is bounded, and will prove the claimed properties concerning $\dot{C}$. Note that $\check{H}^m_c(\bar{C}, \dot{C}) = \check{H}^m(\bar{C}, \dot{C})$.

First consider the case when $m = 1$. We claim that $g: \dot{C} \to \mathbb{R} - 0$ is essential; i.e. it assumes both positive and negative values. Indeed, if this were not so, then $g:(\bar{C}, \dot{C}) \to (\mathbb{R}, \mathbb{R} - 0)$ could be deformed, as a map of pairs, via a linear homotopy, to a constant map, in contradiction to its cohomological nontriviality. Thus $g: \dot{C} \to \mathbb{R} - 0$ is essential, and, in particular, $\dot{C}$ has at least two points.

Now consider the case when $m > 1$. We have the following commutative diagram, with the bottom map being an isomorphism:

$$\begin{array}{ccc} \check{H}^{m-1}(\dot{C}) & \longrightarrow & \check{H}^m(\bar{C}, \dot{C}) \\ \uparrow g^* & & \uparrow g^* \\ \check{H}^{m-1}(\mathbb{R}^m - 0) & \xrightarrow{\sim} & \check{H}^m(\mathbb{R}^m, \mathbb{R}^m - 0). \end{array}$$

Consequently, $\check{H}^{m-1}(\dot{C})$ is nontrivial, from which it follows that $\dim(\dot{C}) \geqq m - 1$. (Note also that $g: \dot{C} \to \mathbb{R}^m - 0$ is essential.)

To complete the proof we will assume $g:\bar{C}\to\mathbb{R}^m$ is proper, and that $\dim(\dot{C})<m-1$, and conclude that $g:\bar{C}\to\mathbb{R}^m$ is essential among proper maps.

Since $g:\bar{C}\to\mathbb{R}^m$ is proper it induces a homomorphism $g^*:\check{H}^m_c(\mathbb{R}^m)\to\check{H}^m_c(\bar{C})$. Thus we have the following commutative diagram, where the rows are exact:

$$\begin{array}{ccccccc}
\check{H}^{m-1}_c(\dot{C}) & \longrightarrow & \check{H}^m_c(\bar{C},\dot{C}) & \longrightarrow & \check{H}^m_c(\bar{C}) & \longrightarrow & \check{H}^m_c(\dot{C}) \\
 & & \uparrow g^* & & \uparrow g^* & & \\
0 & \longrightarrow & \check{H}^m(\mathbb{R}^m,\mathbb{R}^m-0) & \xrightarrow{\sim} & \check{H}^m_c(\mathbb{R}^m) & \longrightarrow & 0.
\end{array}$$

On the other hand, since $\dim(\dot{C})<m-1$ we must have $\check{H}^{m-1}_c(\dot{C})=\check{H}^m_c(\dot{C})=0$ (see [8, p. 152]), so that the top center map in the above diagram is an isomorphism.

It follows immediately that $g^*:\check{H}^m_c(\mathbb{R}^m)\to\check{H}^m_c(\bar{C})$ is nontrivial. As a consequence, $g:\bar{C}\to\mathbb{R}^m$ is essential among proper maps, and, in particular, $g(\bar{C})=\mathbb{R}^m$. □

The Proof of Theorem 2.4. The heart of the matter lies in the assertion that there exists a continuous mapping $g_1:\mathbb{R}^m\times X\to\mathbb{R}$ such that the following hold:

$$\text{if } g:\mathbb{R}^m\times X\to\mathbb{R}^m \text{ is defined by } g(\lambda,x)=(g_1(\lambda,x),\ h_1(\lambda),\ \dots,\ h_{m-1}(\lambda)), \text{ then } g \text{ complements } f \text{ on } \mathscr{U}\equiv\mathscr{O}\backslash\{\mathbb{R}^m\times\{0\}\}: \tag{4.1}$$

there exist η,β in $\mathbb{R}$, with $0<\eta<\beta$, and a Γ-neighborhood, Λ, of $[\underline{\lambda},\bar{\lambda}]$, for which

$$g_1(\lambda,x)<0, \text{ when both } \lambda\in[\underline{\lambda},\bar{\lambda}] \text{ and } 0<\|x\|\leqq\eta, \tag{4.2}$$

$$g_1(\lambda,x)>0, \text{ when both } \lambda\in\Gamma\backslash\Lambda \text{ and } x\neq 0 \text{ or when both } \lambda\in\Lambda \text{ and } \|x\|\geqq\beta, \tag{4.3}$$

and

$$f(\lambda,x)\neq 0, \text{ when both } \lambda\in\Lambda\backslash[\underline{\lambda},\bar{\lambda}] \text{ and } 0<\|x\|\leqq\beta. \tag{4.4}$$

Indeed, suppose, for the moment, that a mapping g_1 having the above three properties exists. Then g complements f on $\mathscr{U}$. Hence we may apply Theorem 2.1, with $\mathscr{U}$ playing the role of $\mathscr{O}$, to choose a closed connected subset, C, of $f^{-1}(0)\cap\mathscr{U}$, whose dimension at each point is at least m, and such that property (*) holds. In view of Remark 3.1 and the fact that $\Gamma=\{(\lambda,0)\,|\,g_2(\lambda)=\ldots=g_m(\lambda)=0\}$ it follows that

$$\text{whenever } E \text{ is a compact subset of } C, \text{ with } g^{-1}(0)\cap C\subsetneqq E, \text{ then } g_1 \text{ changes sign on } (C\backslash E)_\Gamma. \tag{4.5}$$

Observe that (4.2)–(4.4) imply that $C\cap g^{-1}(0)\subsetneqq\{(\lambda,x)\,|\,\lambda\in[\underline{\lambda},\bar{\lambda}],\ \eta<\|x\|\leqq\beta\}$.

We first prove that $\bar{C}\cap[\underline{\lambda},\bar{\lambda}]\neq\emptyset$. To do so we suppose the contrary. Let $E=\{(\lambda,x)\,|\,(\lambda,x)\in C_\Gamma,\ \lambda\in[\underline{\lambda},\bar{\lambda}],\ 0<\|x\|\leqq\beta\}$. Since C is closed in $\mathscr{U}$ and $\bar{C}\cap[\underline{\lambda},\bar{\lambda}]=\emptyset$ it follows that E is compact. As noted above, $C\cap g^{-1}(0)\subsetneqq E$. Thus (4.5) implies that g_1 changes sign on $(C\backslash E)_\Gamma$. Since from (4.3) and (4.4) it follows that g_1 is positive on $(C\backslash E)_\Gamma$, we obtain a contradiction. Thus $\bar{C}\cap[\underline{\lambda},\bar{\lambda}]\neq\emptyset$.

It remains to prove the global properties of C. In order to do so, we assume that none of the alternatives listed in (2.7) hold. Now let $E=\{(\lambda,x)\,|\,(\lambda,x)\in C_\Gamma,\ \|x\|\geqq\eta \text{ if } \lambda\in[\underline{\lambda},\bar{\lambda}]\}$. Since (2.7) fails to hold, and Γ is closed, E is compact. Also, $C\cap g^{-1}(0)\subsetneqq E$. Thus g_1 changes sign on $(C\backslash E)_\Gamma$, in contradiction to (4.2).

Thus, at least one of the properties in (2.7) holds and the theorem is proven, once we have verified the existence of the above g_1.

We first prove the existence of the above g_1 when Γ has a simple form. Specifically, we prove the following.

Proposition 4.1. *Suppose, in addition to the assumptions of Theorem 2.4, that* $\Gamma=\{\lambda|\lambda=(\lambda_1,0,\dots,0),\ (\lambda,0)\in\mathcal{O}\}$, *that* $\underline{\lambda}=(-1,0,\dots,0)$ *and* $\bar{\lambda}=(1,0,\dots,0)$. *For* $r>0$, *let* $\varphi:\mathbb{R}\to[0,r^2]$ *be continuous, equal* r^2 *on* $(-\sqrt{1+r},\sqrt{1+r})$, *and satisfy* $\varphi^{-1}(0)=\mathbb{R}\backslash[-1-r,1+r]$. *Then, if r is sufficiently small, the map, g, defined by*

$$g_1(\lambda,x)=\|x\|^2-\varphi(\lambda_1)\,;\qquad g_i(\lambda,x)=\lambda_i,\qquad 1\leqq i\leqq m,$$

is a complement for f on $\mathcal{O}\backslash\{\mathbb{R}^m\times\{0\}\}$.

Proof. Since $\underline{\lambda}$ and $\bar{\lambda}$ are not Γ bifurcation points of f we may choose r such that

$$f(\lambda,x)\neq 0 \quad\text{when}\quad 0<\|x\|\leqq 2r \quad\text{and}\quad \lambda=(\lambda_1,0,\dots,0)\quad\text{with}\quad |\lambda_1-1|\leqq r \text{ or } |\lambda_1+1|\leqq r, \tag{4.6}$$

and we may assume

$$\left\{(\lambda,x)|\lambda=(\lambda_1,0,\dots,0),\frac{|\lambda_1|^2}{1+r}+\frac{\|x\|^2}{r(1+r)}\leqq 1\right\}\subsetneqq\mathcal{O}. \tag{4.7}$$

For such an r we let φ and g be as defined in the statement of the proposition. Let $\mathcal{U}\equiv\mathcal{O}\backslash\{\mathbb{R}^m\times\{0\}\}$.

If we let $K\equiv(g,f)^{-1}(0)\cap\mathcal{U}$, we observe that

$$K=\{(\lambda,x)|f(\lambda,x)=0,\ \|x\|=r,\lambda=(\lambda_1,0,\dots,0),|\lambda_1|\leqq 1-r\}.$$

Thus K is a compact subset of $\mathcal{U}$, making $\deg((g,f),\mathcal{U},0)$ well-defined.

Now, (g,f) is a perturbation of the identity on $\mathbb{R}^m\times X$ by a compact mapping whose range lies in $\{(\lambda,x)|\lambda=(\lambda_1,0,\dots,0)\}$. Consequently, if we identify the latter subspace with $\mathbb{R}\times X$, let $\mathcal{U}'=\mathcal{U}\cap\{\mathbb{R}\times X\}$, and define $h:\mathcal{O}\cap\{\mathbb{R}\times X\}\to\mathbb{R}\times X$ by

$$h(t,x)=(\|x\|^2-\varphi(t),f(t,x)),$$

it follows from the reduction property for Leray-Schauder degree that

$$\deg((g,f),\mathcal{U},0)=\deg(h,\mathcal{U}',0).$$

It follows from (4.7) that

$$W\equiv\left\{(t,x)\in\mathbb{R}\times X|\frac{|t^2|}{1+r}+\frac{\|x\|^2}{r(1+r)}<1\right\}\subsetneqq\mathcal{O}\cap\{\mathbb{R}\times X\},$$

and one easily checks that

$$h^{-1}(0)\cap\mathcal{U}'=h^{-1}(0)\cap W.$$

Hence, from the excision property for degree,

$$\deg(h,\mathcal{U}',0)=\deg(h,W,0).$$

Now observe that $\|x\|^2-\varphi(t)=\|x\|^2-r^2$ on $\bar{W}$. It follows that $\|x\|^2-\varphi(t)=r(1-t^2)$ on ∂W, and since the degree of a mapping is only dependent on its boundary values,

$$\deg(h, W, 0) = \deg(\psi, W, 0),$$

where $\psi(t,x) = (r(1-t^2), f(t,x))$, for $(t,x) \in \bar{W}$.

But the only zeros of ψ in W are $(+1,0)$ and $(-1,0)$. Hence if we let D' and D'' denote two small cubes about $(+1,0)$ and $(-1,0)$, from the excision and additivity properties of degree it follows that

$$\deg(\psi, W, 0) = \deg(\psi, D', 0) + \deg(\psi, D'', 0).$$

One easily sees that ψ is homotopic to $(t,x) \mapsto (r(1-t), f(1,x))$, on D', and to $(t,x) \mapsto (r(1+t), f(1,x))$, on D'', via homotopics which do not vanish on the boundaries.

Finally, from the product formulae for degree, together with the equalities established above, it follows that

$$\deg((g,f), \mathscr{U}, 0) = \mathrm{ind}(f_{\underline{\lambda}}, 0) - \mathrm{ind}(f_{\bar{\lambda}}, 0) \neq 0. \quad \square$$

To complete the proof it remains to translate the function $\|x\|^2 - \varphi(t)$, in the above proposition, onto the curve Γ.

We may choose an $\mathbb{R}^m$ neighborhood, W, of $[\underline{\lambda}, \bar{\lambda}]$, and an $\mathbb{R}^m$ neighborhood, V, of $\{t \in \mathbb{R}^m \,|\, t = (t_1, 0, \ldots, 0), |t_1| \leqq 1\}$, together with a mapping $\eta : \mathbb{R}^m \to \mathbb{R}$ such that the map $\psi : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\psi(\lambda) = (\eta(\lambda), h_1(\lambda), \ldots, h_{m-1}(\lambda))$$

gives a homeomorphism of W onto V. We assume that

$$\psi(\underline{\lambda}) = (-1, 0, \ldots, 0), \qquad \psi(\bar{\lambda}) = (1, 0, \ldots, 0)$$

and

$$W \subsetneqq \mathcal{O} \cap \{\mathbb{R}^m \times \{0\}\}.$$

Choose a $\sigma > 0$ such that $\hat{W} \equiv \{(\lambda, x) \,|\, \lambda \in W, \|x\| < \sigma\} \subsetneqq \mathcal{O}$, and let $\hat{V} \equiv \{(t,x) \,|\, t \in V, \|x\| < \sigma\}$. Now define $\tilde{f} : \hat{V} \to X$ by

$$\tilde{f}(t,x) = f(\psi^{-1}(t), x).$$

It is clear that the assumptions of Proposition 4.1 hold, where $\tilde{f}$ and $\hat{V}$ play the roles of f and $\mathcal{O}$, respectively. Thus we may choose an $r > 0$ such that if $\varphi : \mathbb{R} \to [0, r]$ is continuous, $\varphi^{-1}(0) = \mathbb{R} \backslash [-1-r, 1+r]$, and $\varphi(t) = r^2$ when $|t|^2 \leqq 1 + r$, then

$$\tilde{g}(t,x) = (\|x\|^2 - \varphi(t_1), t_2, \ldots, t_m)$$

defines a mapping, $\tilde{g}$, which is a complement for $\tilde{f}$ on $\hat{V} \backslash \{\mathbb{R}^m \times \{0\}\}$. We also assume that (4.6) holds.

Now define $\tilde{\psi} : \hat{W} \to \hat{V}$ by $\tilde{\psi}(\lambda, x) = (\psi(\lambda), x)$. Observe that $\tilde{\psi}$ is a homeomorphism of $\hat{W} \backslash \{\mathbb{R}^m \times \{0\}\}$ onto $\hat{V} \backslash \{\mathbb{R}^m \times \{0\}\}$, and that $\tilde{\psi}$ is a compact perturbation of the identity on $\mathbb{R}^m \times X$. Consequently, from the composition property of degree, it follows that

$$\deg((\tilde{g}, \tilde{f}) \circ \tilde{\psi}, \hat{W} \backslash \{\mathbb{R}^m \times \{0\}\}, 0) \neq 0. \tag{4.8}$$

But $[(\tilde{g}, \tilde{f}) \circ \tilde{\psi}](\lambda, x) = (g(\lambda, x), f(\lambda, x))$, where $g(\lambda, x) = (\|x\|^2 - \varphi(\eta(\lambda)), h_1(\lambda), \ldots, \ldots, h_{m-1}(\lambda))$, for $(\lambda, x) \in \hat{W} \setminus \{\mathbb{R}^m \times \{0\}\}$.

Now let $q: \mathbb{R}^m \to \mathbb{R}$ be a continuous function which agrees with $\varphi \circ \eta$ in a W-neighborhood of $[\underline{\lambda}, \bar{\lambda}]$, also agrees with $\varphi \circ \eta$ on $W \cap \Gamma$, and which vanishes on $\Gamma \setminus W$. Finally, let $g_1(\lambda, x) = \|x\|^2 - q(\lambda)$, for $(\lambda, x) \in \mathbb{R}^m \times X$. We claim that g_1 satisfies (4.1)–(4.4).

But observe that $\{(\lambda, x) \in \mathcal{O} \setminus \{\mathbb{R}^m \times \{0\}\} \mid (g, f)(\lambda, x) = 0\} = \{(\lambda, x) \mid \lambda \in \{\underline{\lambda}, \bar{\lambda}\}, \|x\| = r, f(\lambda, x) = 0\}$. Hence, from (4.8), together with the excision property for degree, we see that (4.1) holds. Moreover, it is clear that (4.2)–(4.4) hold with $\eta = r/2$, $\beta = 2r$, and $\Lambda = \Gamma \cap \eta^{-1}((-1-r, 1+r))$.

Thus the proof of the theorem is complete. □

Remark 4.1. In his proof of the one-parameter Rabinowitz bifurcation theorem, J. Ize has employed a map similar to the g of Proposition 4.1 (see [12]).

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