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Sobolev Estimates for the Lewy Operator on Weakly Pseudo-Convex Boundaries

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Let Ω be a bounded domain in $\mathbf{C}^n$ with smooth boundary $b\Omega$, $n \geqq 2$. The Cauchy-Riemann operators $\bar{\partial}$ on $\mathbf{C}^n$ induce in a natural way a complex of differential operators on $b\Omega$, the tangential Cauchy-Riemann complex or $\bar{\partial}_b$ complex. The $\bar{\partial}_b$ complex has played an essential role in the study of boundary values of holomorphic functions and holomorphic extension problems, Kohn and Rossi [16] and Andreotti and Hill [3]. In this paper we shall study the global solvability of the $\bar{\partial}_b$ operator in top degree on weakly pseudo-convex boundaries in L^2 and Sobolev spaces. The $\bar{\partial}_b$ operator in top degree has been of considerable interest since it led Lewy [17] to his famous example of an unsolvable differential equation. When $b\Omega$ is strongly pseudo-convex, using kernel methods Henkin [10] showed that despite the Lewy phenomenon, one can solve the $\bar{\partial}_b$ operator in top degree globally under a suitable compatibility condition. He also obtained the regularity of the solutions in the L^p spaces where $1 \leqq p \leqq \infty$. Recently Kohn [15] has established Sobolev solvability on boundaries of domains of finite type using microlocal analysis and subelliptic estimates. On general weakly pseudo-convex boundaries Rosay [19] has proved C^∞ solvability using results of Kohn [14] and Kohn and Rossi [16]. A simplified proof was given by Shaw [22]. However, the arguments in [19] or [22] do not have good control over the regularity of the solutions (both have a loss of one derivative). In this paper we obtain a solution of the $\bar{\partial}_b$ equation in top degree with sharp Sobolev estimates on weakly pseudo-convex boundaries. Our main result is the following:

Main Theorem. *Let Ω be a bounded domain in $\mathbf{C}^n$ with a smooth weakly pseudo-convex boundary $b\Omega$, $n \geqq 2$. For every $(p, n-1)$ form α on $b\Omega$, $0 \leqq p \leqq n$, with W^s coefficients where s is a nonnegative integer, if α satisfies the compatibility condition*

$$\int_{b\Omega} \alpha \wedge \phi = 0 \tag{0.1}$$

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for every $\bar\partial$-closed $(n-p,0)$ form ϕ on Ω such that ϕ is continuous up to $b\Omega$, then there exists a $(p,n-2)$ form u with W^s coefficients such that

$$\bar\partial_b u = \alpha . \tag{0.2}$$

Furthermore, there exists a constant c_s independent of α such that

$$\|u\|_{s(b\Omega)} \leqq c_s \|\alpha\|_{s(b\Omega)},$$

where $\|\ \|_{s(b\Omega)}$ denotes the norm in the Sobolev space $W^s(b\Omega)$.

Corollary. *Under the same assumption as in the theorem, the $\bar\partial_b$ operator in top degree has closed range in the L^2 as well as W^s sense.*

Note that by Lewy's example, the $\bar\partial_b$ operator in top degree does not have closed range locally. Moreover, it does not always have closed range for abstract CR structures. Burns has observed [6] that for Rossi's example [20] of a non-embeddable strongly pseudo-convex CR structure of real dimension three, the $\bar\partial_b$ operator does not have closed range in either the L^2 sense or the C^∞ sense.

There has also been a great deal of work on the $\bar\partial_b$ complex in lower degrees (i.e., $q<n-1$) when $b\Omega$ is strongly pseudo-convex (see e.g. Kohn [13], Folland and Stein [8], and Folland and Kohn [7]). When $b\Omega$ is weakly pseudo-convex, the L^2 and Sobolev regularity of $\bar\partial_b$ for the lower degrees was proved recently in Shaw [21]. (The case $q=n-2$, not included in the statement of the theorem in [21], follows easily from the same method.) The technique used in this paper could also be used to establish the lower degree case.

The proof of the main theorem consists of two parts: first a priori estimates assuming α is smooth, and then an approximation argument to construct the solution. In Sect. 2 we introduce the jump formula derived from the Bochner-Martinelli-Koppelman kernel. In Sect. 3 we prove the a priori estimates using this jump formula. In order to pass from a priori estimates to actual construction of solutions with estimates, it is necessary to know the regularity of the (weighted) Szegö projection, since condition (0.1) is closely related to the Szegö projection. Section 4 is devoted to proving that the weighted Szegö projection is regular on weakly pseudo-convex boundaries. The method used here is very similar to the one used in Boas [4]. We finish the proof of the theorem in Sect. 5.

With suitable modification our main theorem can be extended to the case of a relatively compact pseudo-convex domain in a Stein manifold. One has to replace the Bochner-Martinelli-Koppelman kernel with its generalization to Stein manifolds (see Theorem 4.5.2 in [25]). The proof then goes through essentially unchanged.

Professor Kohn has kindly informed us that he has obtained a proof of the theorem by using microlocal analysis [24].

1. Notation

We assume that Ω is a bounded pseudo-convex domain in $\mathbf{C}^n$ with defining function r normalized so that the gradient of r has length one on the boundary $b\Omega$. Let B be a fixed large ball containing the closure of Ω in its interior.

Consider the space of $(p, n-1)$ forms $(0 \leqq p \leqq n)$ whose coefficients are smooth functions on $b\Omega$. As usual we identify two such forms f and g when $(f-g) \wedge \bar{\partial} r = 0$. When we wish to be explicit about this identification we use the projection τ onto the subspace of forms that are pointwise orthogonal to the ideal generated by $\bar{\partial} r$. Thus $\bar{\partial}_b f = \tau(\bar{\partial} \tilde{f})$ where $\tilde{f}$ is any smooth extension of f to $\bar{\Omega}$. We extend $\bar{\partial}_b$ to forms with L^2 coefficients by taking its Hilbert space closure. For details see [16].

The norm $\| \ \|_{s(b\Omega)}$ in the Sobolev space $W^s(b\Omega)$ is defined in the usual way via a partition of unity and tangential Fourier transforms. When $s \geqq 0$ the norm $\| \ \|_{s(\Omega)}$ of a function in $W^s(\Omega)$ is the infimum of $W^s(\mathbf{C}^n)$ norms of extensions of the function. When $s<0$ we use $\| \ \|_{s(\Omega)}$ for the norm in the dual space to $W^{|s|}(\Omega)$.

For technical reasons we need special tangential norms in a fixed tubular neighborhood Ω_ε of $b\Omega$. For δ close to zero let $\Gamma_\delta = \{z \in \mathbf{C}^n : r(z) = \delta\}$ be the smooth surface bounding a perturbation of Ω and set

$$|||f|||^2_{s(\Omega_\varepsilon)} = \int_{-\varepsilon}^{\varepsilon} \|f\|^2_{s(\Gamma_r)} \, dr .$$

We also define

$$|||D^m f|||_{s(\Omega_\varepsilon)} = \sum_{0 \leqq k \leqq m} |||D_r^k f|||_{s+m-k(\Omega_\varepsilon)}, \tag{1.1}$$

$$|||DT^m f|||_{s(\Omega_\varepsilon)} = \sum_{0 \leqq k \leqq m} |||Dr^m D_r^k f|||_{s+m-k(\Omega_\varepsilon)}, \tag{1.2}$$

where $D_r = \partial/\partial r$. Norms of forms are computed componentwise. We also write $\Omega_\varepsilon^+ = \Omega_\varepsilon \cap \Omega$, $\Omega_\varepsilon^- = \Omega_\varepsilon \setminus \Omega$.

Kohn's theory of the $\bar{\partial}$-Neumann problem with weights [14] plays a key role in our argument. We use the subscript t (N_t, $\mathbf{S}_t$, etc.) to denote an object (Neumann operator, Szegö projection, etc.) relative to Lebesgue measure weighted by the factor $\chi_t(z) = \exp(-t|z|^2)$.

2. The Jump Formula

Let α be a smooth $(p, n-1)$ form on the boundary $b\Omega$ such that $\alpha = \tau\alpha$. Let K denote the Bochner-Martinelli-Koppelman form:

$$K(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j+1} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} \bigwedge_{k \neq j} d\bar{z}_k \bigwedge_l (d\zeta_l - dz_l) .$$

Define

$$\int_{b\Omega} \alpha(\zeta) \wedge K(\zeta, z) = \begin{cases} \alpha^+(z) & \text{when} \quad z \in \Omega, \\ \alpha^-(z) & \text{when} \quad z \in \mathbf{C}^n \setminus \bar{\Omega}. \end{cases} \tag{2.1}$$

Then α^+ and α^- are $\bar{\partial}$-closed forms which extend smoothly to $b\Omega$. The difference between their tangential parts on the boundary recovers α:

$$\tau(\alpha^+|_{b\Omega} - \alpha^-|_{b\Omega}) = (-1)^p \alpha . \tag{2.2}$$

Indeed the boundary values of α^+ and α^- equal the singular integral on $b\Omega$ corresponding to (2.1) plus and minus one-half of $(-1)^p\alpha$. For details see [9] or [2].

We need the following estimates for $\alpha^{\pm}$ in Sobolev norms. Recall that Ω_ε is a tubular neighborhood of $b\Omega$ and $\Omega_\varepsilon^- = \Omega_\varepsilon \backslash \Omega$.

Lemma. *For each non-negative real number s there are constants C and C_m (independent of α) such that*

$$\|\alpha^+\|_{s+1/2(\Omega)} \leqq C\|\alpha\|_{s(b\Omega)}, \tag{2.3}$$

$$|||D^m\alpha^-|||_{s-m+1/2(\Omega_\varepsilon^-)} \leqq C_m\|\alpha\|_{s(b\Omega)}. \tag{2.4}$$

Proof. We may assume that α has small support contained in a boundary chart with coordinates $t_1, \dots, t_{2n-1}, r$. Except for a smooth perturbation caused by flattening the boundary locally, the coefficients of the forms α^+ and α^- are sums of the Poisson integral of α and the Poisson integrals of the Riesz transforms of α (cf. [23, p. 236]). The Riesz transforms are classical singular integrals that are bounded in $W^s(b\Omega)$ for all s. By elliptic theory the $W^{s+1/2}(\Omega)$ norm of a Poisson integral is bounded by the $W^s(b\Omega)$ norm of its boundary value, so (2.3) holds. Estimate (2.4) is a simple calculation using that the Fourier transform of the Poisson kernel is $\exp(-r|\tau|)$, up to constants. Indeed such an estimate holds for any operator whose Fourier multiplier has the form $\psi(r|\tau|)$ for a smooth function ψ: see [11, Theorem 2.5.7] and Lemma 1 below.

To solve $\bar\partial_b u = \alpha$ it suffices to find forms u^+ and u^- such that $\bar\partial_b u^+ = \alpha^+$ and $\bar\partial_b u^- = \alpha^-$. We solve the first of these problems with estimates in $W^s(b\Omega)$ by applying Kohn's global regularity theorem for the weighted $\bar\partial$-Neumann problem [14]. Put $u^+ = \mathfrak{D}_t N_t \alpha^+$. Then $\bar\partial u^+ = \alpha^+$ in Ω, and for sufficiently large t (depending on s)

$$|||Du^+|||_{s-1/2(\Omega_\varepsilon^+)} \leqq C\|\alpha^+\|_{s+1/2(\Omega)} \leqq C'\|\alpha\|_{s(b\Omega)}.$$

Restricting to the boundary gives $\bar\partial_b u^+ = \alpha^+$, and by the trace theorem

$$\|u^+\|_{s(b\Omega)} \leqq C\|\alpha\|_{s(b\Omega)}. \tag{2.5}$$

It remains to solve $\bar\partial_b u^- = \alpha^-$. Note that α^- inherits the compatibility condition (0.1) from α by the jump formula (2.2) and the $\bar\partial_b$-exactness of α^+.

3. Construction of $\bar\partial$-Closed Extension with Estimates

We shall construct a $\bar\partial$-closed extension of α^- on B with estimates. Our method is similar to the one used in [21]. Thus we will only show the necessary modification and refer the readers to Shaw (Lemma 3 to Lemma 6 in [21]) for details.

Lemma 1. *For arbitrary smooth functions u_j on $b\Omega$, $j=0, 1, \dots, k_0$, there exists a function $Eu \in C_0^\infty(\Omega_\varepsilon)$ such that $D_r^j Eu = u_j$ on $b\Omega$, $j=0, 1, \dots, k_0$. Furthermore, for every real number s and positive integer m, there exists a positive constant c_m depending on s but independent of the u_j's such that*

$$|||D^m Eu|||_{s-|m|+1/2(\Omega_\varepsilon)} \leqq c_m \sum_{j=0}^{k_0} \|u_j\|_{s-j(b\Omega)} \tag{3.1}$$

$$|||DT^m Eu|||_{s-1/2(\Omega_\varepsilon)} \leqq c_m \sum_{j=0}^{k_0} \|u_j\|_{s-j(b\Omega)}. \tag{3.2}$$

Proof. It suffices to prove the lemma assuming u_j is supported in a special boundary coordinate chart with coordinates $t_1, \ldots, t_{2n-1}, r$. Let ψ be a function in $C_0^\infty(\mathbf{R})$ which is equal to 1 in a neighborhood of 0 and let the partial Fourier transform of Eu be

$$(Eu)^\tilde{}(\tau, r) = \psi(\lambda r) \sum_{j=0}^{k_0} \hat{u}_j(\tau) \frac{r^j}{j!},$$

where $\hat{u}_j(\tau)$ is the usual Fourier transform of u_j and $\lambda = (1+|\tau|^2)^{1/2}$. It follows from Theorem 2.5.7 in Hörmander [11] that $D_r^j Eu = u_j$ on $b\Omega$ and that Eu satisfies (3.1).

To prove (3.2), we note that for every nonnegative integer i, by a change of variable, we have

$$\int_{-\infty}^{\infty} |D_r^i(\psi(\lambda r) r^j)|^2 \, dr = \lambda^{2(i-j)-1} \int_{-\infty}^{\infty} |D_r^i(\psi(r) r^j)|^2 \, dr.$$

Thus, from the definition (1.2), we have

$$\begin{aligned} &|||DT^m Eu|||^2_{s-1/2(\Omega_\varepsilon)} \\ &\leqq c \sum_{j=0}^{k_0} \sum_{0 \leqq i \leqq m+1} \int_{R^{2n-1}} (1+|\tau|^2)^{s-j} |\hat{u}_j(\tau)|^2 \, d\tau \int_{-\infty}^{\infty} (D_r^i(\psi(r) r^j))^2 \, dr \\ &\leqq c_m \sum_{j=0}^{k_0} \|u_j\|^2_{s-j(b\Omega)}. \end{aligned}$$

Lemma 2. *Let α be a smooth $(p, n-1)$ form on $b\Omega$ that satisfies condition* (0.1) *and let α^- be defined by* (2.1). *Then for any nonnegative integers k, s, there exists a $(p, n-1)$ form $\tilde{\beta} \in C^k(B)$ such that $\tilde{\beta} = \alpha^-$ on $B \setminus \Omega$ and $\bar{\partial}\tilde{\beta} = 0$ on B. Furthermore, there exists a constant C depending only on s, k but independent of α such that*

$$|||D\tilde{\beta}|||^2_{s-3/2(\Omega_\varepsilon)} \leqq C \|\alpha\|^2_{s(b\Omega)}. \tag{3.3}$$

Proof. By the trace theorem for Sobolev spaces (see e.g. Theorem 2.5.6 in Hörmander [11]) and inequality (2.4), we have

$$\|D_r^j \alpha^-\|_{s-j(b\Omega)} \leqq C |||D^{j+1}\alpha^-|||_{s-j-1/2(\Omega_\varepsilon^-)} \leqq C \|\alpha\|_{s(b\Omega)}. \tag{3.4}$$

Let k_0 be a large positive integer [say $k_0 > 2(k+n)$]. Applying Lemma 1, we can extend α^- componentwise and obtain $\beta \in C^{k_0}(B)$ such that $\beta = \alpha^-$ on $B \setminus \Omega$. Then we have for any integer m

$$|||D^m \beta|||_{s-|m|+1/2(\Omega_\varepsilon^+)} \leqq C \|\alpha\|_{s(b\Omega)}, \tag{3.5}$$

$$|||DT^m \beta|||_{s-1/2(\Omega_\varepsilon^+)} \leqq C \|\alpha\|_{s(b\Omega)}. \tag{3.6}$$

Set $\beta^+ = \beta|_\Omega$. The desired $\tilde{\beta}$ will have the form

$$\tilde{\beta} = \begin{cases} \beta & \text{on} \quad B \setminus \Omega \\ \beta^+ - F_{k+1}\beta^+ & \text{on} \quad \Omega, \end{cases}$$

where $F_{k+1}\beta^+$ vanishes to order $k+1$ on $b\Omega$ and $\bar{\partial} F_{k+1}\beta^+ = \bar{\partial}\beta$ on Ω. Then $\tilde{\beta}$ is $\bar{\partial}$-closed on B and $\tilde{\beta} \in C^k(B)$.

To construct $F_{k+1}\beta^+$ we use the weighted $\bar{\partial}$-Neumann operator N_t^0 on $(p, 0)$ forms, which is defined in terms of the operator N_t^1 on $(p, 1)$ forms by

$$N_t^0 = \mathfrak{D}_t (N_t^1)^2 \bar{\partial}.$$

We first define

$$F\beta^+ = -\overline{*\bar{\partial}N_t^0 * \overline{\bar{\partial}\beta^+}},$$

where $*$ is the Hodge star operator with respect to the weighted metric on Ω. For sufficiently large t, it follows from the regularity theorem for N_t^1 (the Main Theorem in [14]) and the Sobolev embedding theorem that $F\beta^+ \in C^{2k+1}(\bar{\Omega})$. Since β^+ satisfies condition (0.1) by the remark at the end of Sect. 2, it follows from the arguments in [22] that

$$\bar{\partial}F\beta^+ = \bar{\partial}\beta^+ \quad \text{on} \quad \Omega$$

and

$$F\beta^+ \wedge \bar{\partial}r = 0 \qquad \text{on} \quad b\Omega.$$

In order to modify $F\beta^+$ to make it vanish to high order on $b\Omega$, we note that $\bar{\partial}\beta^+$ vanishes to order k_0 on $b\Omega$, so we can repeat the arguments of Lemma 3 in [21] to construct $(p, n-2)$ forms $\beta_0, \beta_1, \ldots, \beta_k$ and a $(p, n-1)$ form η_k such that $F\beta^+ = \bar{\partial}(r\beta_0) + \bar{\partial}(r^2\beta_1) + \ldots + \bar{\partial}(r^{k+1}\beta_k) + r^{k+1}(\eta_k - \bar{\partial}\beta_k)$. Each β_i is obtained from the ith derivatives of the components of $F\beta^+$. Set

$$F_{k+1}\beta^+ = F\beta^+ - \sum_{i=0}^{k} \bar{\partial}(r^{i+1}\beta_i).$$

Then

$$\bar{\partial}F_{k+1}\beta^+ = \bar{\partial}\beta^+ \qquad \text{on} \quad \Omega$$

and

$$F_{k+1}\beta^+ = 0(r^{k+1}) \quad \text{on} \quad b\Omega.$$

Since $F\beta^+ \in C^{2k+1}(\bar{\Omega})$, we have $F_{k+1}\beta^+ \in C^k(\bar{\Omega})$. This completes the construction of $\tilde{\beta}$.

To show that $\tilde{\beta}$ also satisfies (3.3), note that by (2.4), it suffices to estimate $\tilde{\beta}$ on Ω_ε^+. We claim that for any cut-off function $\eta \in C_0^\infty(\Omega_\varepsilon)$ there exists a constant c such that

$$|||D\eta F\beta^+|||_{s-3/2(\Omega_\varepsilon^+)} \leqq c(|||\bar{\partial}\beta^+|||_{s-1/2(\Omega_\varepsilon^+)} + \|\bar{\partial}\beta^+\|_{s-1(\Omega_\varepsilon^+)}), \tag{3.7}$$

$$|||DT^m\eta F\beta^+|||_{s-3/2(\Omega_\varepsilon^+)} \leqq c(|||DT^m\beta^+|||_{s-1/2(\Omega_\varepsilon^+)} + \|\bar{\partial}\beta^+\|_{s-1(\Omega_\varepsilon^+)}). \tag{3.8}$$

Since $\bar{\partial}N_t^0 = N_t^1\bar{\partial}$ on sufficiently smooth $(p,0)$ forms, we have $F\beta^+ = -\overline{*N_t^1\bar{\partial} * \overline{\bar{\partial}\beta^+}}$. By using estimates for N_t^1, inequalities (3.7) and (3.8) can be essentially proved as in Lemmas 4 and 5 in [21]. Combining (3.7), (3.8) and (3.5), (3.6), we have proved that $\tilde{\beta}$ satisfies (3.3) with a slightly smaller ε.

Lemma 3. *Let Ω be a smooth bounded pseudo-convex domain in $\mathbf{C}^n$, $n \geqq 2$, and let α be a smooth $(p, n-1)$ form on $b\Omega$ satisfying condition* (0.1). *Then for every nonnegative integer $s \geqq 0$, one can find a $(p, n-2)$ form $u_s \in W^s(b\Omega)$ such that $\bar{\partial}_b u_s = \alpha$ on $b\Omega$. Furthermore, there exists a constant c_s independent of α such that*

$$\|u_s\|_{s(b\Omega)} \leqq c_s \|\alpha\|_{s(b\Omega)}.$$

Proof. From the remark in Sect. 2, we only need to show that one can solve $\bar{\partial}_b u^- = \alpha^-$ on $b\Omega$. By Lemma 2, we can extend α^- to be a $\bar{\partial}$-closed form $\tilde{\beta}$ on B. We solve the following equations on B:

$$\begin{aligned} \bar{\partial}\tilde{u} &= \tilde{\beta} \\ \mathfrak{D}\tilde{u} &= 0\,. \end{aligned} \tag{3.9}$$

Then for any cut-off function $\xi \in C_0^\infty(\Omega_\varepsilon)$ such that $\xi = 1$ on $b\Omega$, we have

$$|||D\xi\tilde{u}|||_{s-1/2(\Omega_\varepsilon)} \leqq c(|||D\tilde{\beta}|||_{s-3/2(\Omega_\varepsilon)} + \|\tilde{u}\|_{-k-1(B)} + \|\tilde{\beta}\|_{-k-1(B)}) \tag{3.10}$$

by the interior regularity for the elliptic system (3.9). By the global regularity of (3.9) and estimate (3.3), the right-hand side of (3.10) is bounded by a constant times $\|\alpha\|_{s(b\Omega)}$. Denote the restriction of $\tilde{u}$ to $b\Omega$ by u^-. Then the trace theorem for Sobolev spaces gives

$$\|u^-\|_{s(b\Omega)} \leqq c\|\alpha\|_{s(b\Omega)}\,.$$

In view of (2.5), the form $u_s = (-1)^p(u^+ - u^-)$ satisfies the requirements of the lemma.

Remark. Clearly Lemma 3 holds if α is sufficiently smooth (not necessarily C^∞).

4. The Weighted Szegö Projection

To pass from the a priori estimate of Lemma 3 to the final result we need estimates for a boundary projection onto holomorphic functions. In [4] regularity of the Szegö projection was derived from regularity of the $\bar{\partial}$-Neumann operator, but here we have only regularity of the weighted Neumann operator. We modify the proof in [4] to obtain regularity of the weighted Szegö projection, which suffices for the approximation argument in the next section.

The operator of interest is the orthogonal projection $\mathbf{S}_t$ from $L^2(b\Omega, d\sigma_t)$ onto the subspace of boundary values of holomorphic functions, where $d\sigma_t$ is the usual surface measure weighted by the factor $\chi_t(z) = \exp(-t|z|^2)$. It has the following regularity property.

Theorem. *Let Ω be a smooth bounded pseudo-convex domain in $\mathbf{C}^n$, $n \geqq 2$. Fix a positive real number s. There exists t_0 such that for all $t > t_0$ the weighted Szegö projection $\mathbf{S}_t$ is a bounded operator from $W^s(b\Omega)$ into itself [and hence from $W^r(b\Omega)$ into itself when $0 \leqq r \leqq s$].*

Proof. It suffices to consider positive integral s. By Kohn's theory of the $\bar{\partial}$-Neumann problem with weights [14] there is for each positive ε some $t(s,\varepsilon)$ such that when $t > t(s,\varepsilon)$ the weighted Neumann operator N_t admits the estimate

$$\|\mathfrak{D}_t N_t f\|_{s,(t)} \leqq \varepsilon\|f\|_{s,(t)} + C(t)\,\|f\|_0 \tag{4.1}$$

for all $\bar{\partial}$-closed (0, 1) forms f. We will choose ε momentarily, and the corresponding $t(s,\varepsilon)$ will be the t_0 in the theorem. Note that since the weighted Bergman projection $\mathbf{B}_t$ equals $I - \mathfrak{D}_t N_t \bar{\partial}$ the estimate

$$\|\mathbf{B}_t u\|_{s,(t)} \leqq C(t)\,\|u\|_{s,(t)} \tag{4.2}$$

holds on functions u.

If f and h are holomorphic, Green's theorem implies

$$\int_{b\Omega} \bar{f} h \, d\sigma_t = \int_{\Omega} \bar{f} \chi_t L h, \tag{4.3}$$

where, if ϱ is a smooth defining function for Ω and ψ is a smooth function equal to $|\nabla\varrho|^{-1}$ near $b\Omega$,

$$Lh = 4 \sum_{j=1}^{n} \left[\frac{\partial h}{\partial z_j} \frac{\partial \varrho}{\partial \bar{z}_j} \psi + h \chi_t^{-1} \frac{\partial}{\partial z_j} \left(\frac{\partial \varrho}{\partial \bar{z}_j} \psi \chi_t \right) \right].$$

By elliptic theory one can show (see [4]) that

$$\|h\|_{s+1,(t)} \leqq C \|Lh\|_{s,(t)} + C(t) \|h\|_0 .$$

Invoking estimate (4.1) gives

$$\|h\|_{s+1,(t)} \leqq C \|\mathbf{B}_t L h\|_{s,(t)} + C\varepsilon \|h\|_{s+1,(t)} + C(t, \varepsilon) \|h\|_s$$

since $\bar{\partial} h = 0$. We now fix ε equal to $(2C)^{-1}$. If h is sufficiently smooth we can absorb the error terms to obtain

$$\|h\|_{s+1,(t)} \leqq C \|\mathbf{B}_t L h\|_{s,(t)} + C(t) \|h\|_0 . \tag{4.4}$$

We next replace h by $\mathbf{S}_t u$, where u is any smooth harmonic function. Since $\mathbf{S}_t u$ is not a priori smooth, this step needs justification, which we postpone temporarily. By (4.3) and the definition of the weighted Szegö projection

$$\begin{aligned} \mathbf{B}_t L \mathbf{S}_t u &= \int_{b\Omega} B_t(\cdot, z) \, u(z) \, d\sigma_t \\ &= \mathbf{B}_t [\chi_t^{-1} \Delta (u \varrho \psi \chi_t)], \end{aligned}$$

the second step being another use of Green's theorem. By estimate (4.4) and the bound (4.2) on the weighted Bergman projection

$$\|\mathbf{S}_t u\|_{s+1,(t)} \leqq C(t) \left(\|u\|_{s+1,(t)} + \|\mathbf{S}_t u\|_0 \right).$$

This reduces to

$$\|\mathbf{S}_t u\|_{s+1/2(b\Omega)} \leqq C(t) \, \|u\|_{s+1/2(b\Omega)} \tag{4.5}$$

if we restrict to the boundary, observe that the weight χ_t and its reciprocal are bounded on $\bar{\Omega}$, and recall that $\mathbf{S}_t$ is continuous on $L^2(b\Omega, d\sigma_t)$ by definition. By interpolation it follows that $\mathbf{S}_t$ is continuous on $W^r(b\Omega)$ when $0 \leqq r \leqq s + 1/2$.

We have derived (4.5) as an a priori estimate, that is, assuming that $\mathbf{S}_t$ preserves the space $C^\infty(\bar{\Omega})$. To complete the proof we consider the interior approximating domains $\Omega^\delta = \{z \in \Omega : \varrho(z) < -\delta\}$. The weighted Szegö projections for the Ω^δ converge to the weighted Szegö projection for Ω [4], so it is enough to prove (4.5) for the Ω^δ with the constant independent of δ. Since Kohn's global regularity estimates hold uniformly on the Ω^δ, the above argument gives (4.5) as a uniform a priori estimate. Now we choose the defining function ϱ so that the Ω^δ are strictly pseudo-convex. We will be done as soon as we know that the weighted Szegö projection for a strictly pseudo-convex domain D preserves the space of functions smooth up to the boundary. To see this one can take any regularity proof for the

Szegö projection in a strictly pseudo-convex domain [1, 5, 8, 18] and check that it works with weights. Or one can use the trick of Kerzman and Stein [12] to write

$$\mathbf{S}_t = \mathbf{S}(I + \mathbf{S} - \mathbf{S}^*)^{-1},$$

where $\mathbf{S}^*$ is the adjoint of $\mathbf{S}$ in the weighted space $L^2(bD, d\sigma_t)$. Since $\mathbf{S} - \mathbf{S}^*$ is represented on $L^2(bD)$ by the kernel $S(w,z)[1 - \exp(t|w|^2 - t|z|^2)]$, which is a kernel of type 1, it follows that $\mathbf{S}_t$ preserves $C^\infty(\bar{D})$ since $\mathbf{S}$ does. This completes the proof.

5. Proof of the Main Theorem

We shall finish the proof of the main theorem using an approximation argument. Since the holomorphic degree p plays no role in solving the $\bar{\partial}_b$ operators, for simplicity we shall assume α is an $(n, n-1)$ form. We first derive the relation of the condition (0.1) with the weighted Szegö projection $\mathbf{S}_t$.

Any $(n, n-1)$ form α such that $\alpha = \tau\alpha$ can be expressed as $\alpha = f_t(*\partial r)$ for some function f_t on $b\Omega$, where $*$ is the Hodge star operator with respect to the weighted metric on Ω.

Lemma. *The* $(n, n-1)$ *form* α *with* L^2 *coefficients satisfies condition* (0.1) *if and only if* $\mathbf{S}_t \bar{f}_t = 0$.

Proof. Let $\theta = *(\partial r \wedge \bar{\partial} r)$. Since $dr = (\partial + \bar{\partial})r$ vanishes when restricted to $b\Omega$, we have

$$\begin{aligned} *\partial r &= \theta \wedge \partial r \\ &= \theta \wedge dr - \theta \wedge \bar{\partial} r \\ &= \theta \wedge dr + *\bar{\partial} r \\ &= *\bar{\partial} r \end{aligned}$$

on $b\Omega$. If ϕ is a holomorphic function on Ω that is in $C(\bar{\Omega})$, we have

$$\begin{aligned} \int_{b\Omega} \alpha \wedge \phi &= \int_{b\Omega} f_t \phi(*\partial r) \\ &= \int_{b\Omega} f_t \phi(*dr) - \int_{b\Omega} f_t \phi(*\bar{\partial} r) \\ &= \int_{b\Omega} f_t \phi \, d\sigma_t - \int_{b\Omega} f_t \phi(*\partial r). \end{aligned}$$

Therefore

$$\int_{b\Omega} \alpha \wedge \phi = \tfrac{1}{2} \int_{b\Omega} f_t \phi \, d\sigma_t$$

and the lemma is proved since the holomorphic functions that are continuous up to the boundary are dense in the holomorphic functions with L^2 boundary values in pseudo-convex domains.

To prove the main theorem, we approximate α in $W^s(b\Omega)$ by smooth $(n, n-1)$ forms α_j where $\alpha_j = f_t^j(*\partial r)$. Define

$$\begin{aligned} \bar{g}_t^j &= (I - \mathbf{S}_t) \bar{f}_t^j, \\ \alpha_j' &= g_t^j(*\partial r). \end{aligned}$$

Then $S_t\bar{g}_t^j=0$, and from the lemma above we see that α_j' satisfies condition (0.1). By the regularity theorem for S_t proved in Sect. 4, we can make g_t^j as smooth as desired by choosing t large enough. Moreover, $\alpha_j'\to\alpha$ in $W^s(b\Omega)$. We apply Lemma 3 in Sect. 3 to find an $(n, n-2)$ form $u_j'\in W^s(b\Omega)$ such that

$$\bar{\partial}_b u_j'=\alpha_j'$$

and $\|u_j'\|_{s(b\Omega)}\leqq c_s\|\alpha_j'\|_{s(b\Omega)}$.

It is easy to see that u_j' converges in $W^s(b\Omega)$ to an element $u\in W^s(b\Omega)$ satisfying $\bar{\partial}_b u=\alpha$ and $\|u\|_{s(b\Omega)}\leqq c_s\|\alpha\|_{s(b\Omega)}$. The theorem is proved.

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