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L^2 Decay for the Navier-Stokes Flow in Halfspaces

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1. Introduction

This paper studies asymptotic behavior in L^2 of weak solutions of the Navier-Stokes equations in halfspaces:

$$\begin{aligned} u_t - \Delta u + u\cdot \nabla u + \nabla p = 0 \quad &\text{in} \quad D^n\times(0,\infty) \\ \nabla\cdot u = 0 \quad &\text{in} \quad D^n\times(0,\infty) \\ u=0 \quad \text{on} \quad \partial D^n\times(0,\infty); \quad u|_{t=0}=a \quad &\text{in} \quad D^n. \end{aligned} \tag{NS}$$

Here $n\geqq 2$, $D^n=\{x\in R^n;\ x=(x',x_n),\ x_n>0\}$ is the upper halfspace of R^n with boundary ∂D^n; $u=(u^i)_{i=1}^n$ and p denote, respectively, unknown velocity and pressure, while $a=(a^i)_{i=1}^n$ is a given initial velocity. In the previous work [7] the second author studied the Cauchy problem for the same equation and proved that for each $a\in L^2(R^n)$ with $\nabla\cdot a=0$ there is a weak solution u which decays in L^2 like the solution of the linear heat equation having the same initial value; see also [14, 15, 23]. In this paper we shall establish the same type of results in the case of halfspaces. More specifically, we show that for each $a\in L^2(D^n)$ such that $\nabla\cdot a=0$ and $a^n=0$ on ∂D^n there is a weak solution u of problem (NS) which tends to zero as $t\to\infty$, and moreover, if $a\in L^2\cap L^r$ for some $1\leqq r<2$, then u decays in L^2 like the solution v of the Stokes system:

$$\begin{aligned} v_t - \Delta v + \nabla p = 0 \quad &\text{in} \quad D^n\times(0,\infty) \\ \nabla\cdot v = 0 \quad &\text{in} \quad D^n\times(0,\infty) \\ v=0 \quad \text{on} \quad \partial D^n\times(0,\infty); \quad &v|_{t=0}=a. \end{aligned} \tag{S}$$

The problem of L^2 decay for weak solutions of the Navier-Stokes equations was first raised by Leray [8] in the case of the Cauchy problem in R^3. Schonbek [14, 15] attacked this problem and succeeded for the first time in showing the

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existence of weak solutions with explicit decay rate. Kajikiya and Miyakawa [7] considered the case of R^n, $n \geqq 2$, and generalized the result in [14] with improvement. Wiegner [23] discussed the same case and obtained the most general results in this direction. However, all of the above-mentioned works rely on the theory of Fourier transform in treating the nonlinear term $u \cdot \nabla u$ and therefore the arguments developed there cannot be directly applied to the case of unbounded domains with boundaries.

In this paper we shall give an approach to the L^2 decay problem in the case of halfspaces which does not use Fourier transform directly. Our main tool is the fractional powers of the Stokes operator in general L^r spaces. The fractional powers of the Stokes operator is studied in detail in [5] in the case where the Stokes operator is boundedly invertible. In our case, however, the Stokes operator is not boundedly invertible, and in such a case it is in general not easy to establish calculus inequalities involving the fractional powers. We can avoid this difficulty in the case of halfspaces by appealing to the fact that the halfspaces D^n are invariant under the action of scaling transformations: $x \to \lambda x$, $\lambda > 0$, and obtain the desired inequalities. Those calculus inequalities will then be applied to prove the so-called (L^p, L^q)-estimates for the solutions of problem (S) as well as to give a good estimate for the nonlinear term of (NS), both of which are needed in establishing our main results.

The existence of fractional powers of the Stokes operator is guaranteed by the fact that the Stokes operator on D^n generates a bounded analytic semigroup in each L^r space, $1 < r < \infty$. This is shown for $n = 3$ by McCracken [9] with the aid of the theory of hydrodynamic potentials. In this paper we extend the result of McCracken to all dimensions $n \geqq 2$, using the formula of Ukai [21] for the solutions of the Stokes system (S).

After showing the boundedness and analyticity of the semigroup, we study the fractional powers of the Stokes operator. Here we again apply Ukai's formula in order to identify the domains of the fractional powers with some complex interpolation spaces. The desired calculus inequalities involving the fractional powers are then easily deduced from the complex interpolation theory and the invariance of D^n under scaling transformations.

Our main results (stated in Sect. 2) are nothing but the restatement of those of [7]. It is also possible to prove the decay results of the form stated in Wiegner [23]. However, we adopt our present version of main results since it is directly connected with the (L^p, L^q)-estimates for the solutions of problem (S) and since the (L^p, L^q)-estimates would be interesting in themselves.

The proof of our main results is carried out for $n \geqq 3$ with the help of a special kind of approximate solutions to problem (NS). We employ the idea of Sohr, von Wahl, and Wiegner [16] to construct the approximate solutions. In Sect. 5 we shall show that for any initial data in L^2 the approximate solutions constructed along the idea of [16] contain a subsequence which converges to a weak solution of (NS).

2. Main Results

We use the following notations: $L^r = L^r(D^n)$, $1 \leqq r \leqq \infty$, denotes the usual Lebesgue space of scalar, as well as vector, functions defined on the halfspace D^n. The norm of

L^r is denoted by $\|\cdot\|_r$. $C_{0,\sigma}^{\infty}(D^n)$ is the set of smooth divergence-free vector fields with compact support in D^n, and $X_r = X_r(D^n)$ denotes the L^r-closure of $C_{0,\sigma}^{\infty}(D^n)$. As will be shown in Sect. 3, the Helmholtz decomposition:

$$L^r = X_r \oplus G_r \text{ (direct sum)}, \qquad G_r = \{f \in L^r;\ f = \nabla p\}$$

holds for $1 < r < \infty$. We write $P = P_r$ the projection onto X_r associated to the above decomposition. $W^{s,r}(D^n)$ denotes the usual L^r Sobolev space with norm $\|\cdot\|_{s,r}$, and $W_0^{s,r}(D^n)$ is the closure in $W^{s,r}(D^n)$ of the set $C_0^{\infty}(D^n)$ of smooth functions with compact support in D^n.

Let a be in X_2. A function u defined on $(0, \infty)$ with values in $X_2 \cap W_0^{1,2}(D^n)$ is called a weak solution of (NS) if

(a) $u \in L^{\infty}(0, T; X_2) \cap L^2(0, T; W_0^{1,2}(D^n))$ for each $0 < T < \infty$; and

(b) $$-\int_0^{\infty} (u, v) g_t\, dt + \int_0^{\infty} (\nabla u, \nabla v) g\, dt + \int_0^{\infty} (u \cdot \nabla u, v) g\, dt = (a, v) g(0)$$

for all $v \in X_2 \cap W_0^{1,2}(D^n) \cap L^n$ and all $g = g(t) \in C^1([0, \infty); R^1)$ vanishing near $t = \infty$. Here and in the following, $(\cdot, \cdot)$ denotes the duality pairing between L^r and $L^{r'}$, $1/r' = 1 - 1/r$. Note that the third integral on the left of *(b)* is finite, due to the requirement $v \in L^n$. It is well known that for each $a \in X_2$ problem (NS) possesses a weak solution. The uniqueness of weak solutions still remains an open problem when $n \geqq 3$. Our main results are now stated as follows.

Theorem 1. *For each $a \in X_2$ there is a weak solution u of* (NS) *such that*

(i) $\|u(t)\|_2 \to 0$ *as* $t \to \infty$;

(ii) *If $a \in X_2 \cap L^r$ for some $1 \leqq r < 2$, then $\|u(t)\|_2 \leqq C t^{-(n/r - n/2)/2}$ for $t > 0$, with $C > 0$ depending only on n, r, and a.*

Theorem 2. *Let $a \in X_2$ and let $u_0(t)$ be the solution of* (S) *with $u_0(0) = a$. Then the weak solution u given in Theorem 1 satisfies*

(iii) $\|u(t) - u_0(t)\|_2 = o(t^{1/2 - n/4})$ *as* $t \to \infty$;

(iv) *If $a \in X_2 \cap L^r$ for some $1 \leqq r < 2$, then*

$$\|u(t) - u_0(t)\|_2 = o(t^{-(n/r - n/2)/2}) \quad \text{as} \quad t \to \infty.$$

As will be shown in Sect. 4, the function $u_0(t)$ satisfies the estimates:

$$\|u_0(t)\|_q \leqq C t^{-(n/r - n/q)/2} \|a\|_r, \qquad a \in X_2 \cap L^r$$

provided either $1 < r \leqq q < \infty$ or $1 \leqq r < q \leqq \infty$. Thus Theorem 2 means that the decay rate given in Theorem 1 is in general determined by the linear part of the weak solution.

We shall prove Theorems 1 and 2 in Sect. 4. In Sect. 3 we define the Stokes operator on halfspaces and prepare some basic properties of fractional powers, which are needed in Sect. 4. Section 5 is devoted to constructing a special kind of approximate solutions to (NS), which we need in showing our main results for $n \geqq 3$. The construction and convergence of the approximate solutions are discussed along the idea of [16].

Throughout this paper C denotes constants which may vary from line to line, while C_j, $j = 1, 2, \ldots$, and M denote fixed constants in each context.

3. The Stokes Operator over the Halfspaces

(I) The Helmholtz Decomposition

We begin by establishing the decomposition of L^r-vector fields on D^n into solenoidal and potential parts, as given in [4] for bounded domains and in [10, 17] for three-dimensional exterior domains. Although the result below is conceptually well known (see [9, Appendix] for an outline of the proof in case $n=3$), no complete proof seems to be available; so we give here a complete proof for the reader's convenience. First recall that X_r is the L^r-closure of $C_{0,\sigma}^{\infty}(D^n)$ and that $G_r=\{f\in L^r;\ f=\nabla p,\ p\in L_{\mathrm{loc}}^r(\bar{D}^n)\}$, where $\bar{D}^n$ is the closure of D^n.

Theorem 3.1. (i) $L^r=X_r\oplus G_r$, $1<r<\infty$ *(direct sum).*

(ii) *$X_r^*=X_{r'}$ and $X_r^{\perp}=G_{r'}$ for $1<r<\infty$, where $r'=r/(r-1)$, X_r^* is the dual space of X_r, and $X_r^{\perp}$ denotes the annihilator of X_r.*

(iii) *If P_r denotes the bounded projection from L^r onto X_r associated to the decomposition (i), then the dual P_r^* equals $P_{r'}$, $r'=r/(r-1)$.*

Proof. We follow the argument in [10]. Consider the space

$$Y_r=\{u\in L^r;\ \nabla\cdot u=0,\ u^n|_{\partial D^n}=0\},\qquad 1<r<\infty.$$

Since $u^n|_{\partial D^n}$ is well-defined in $W^{-1/r,r}(\partial D^n)$ so that the divergence theorem holds, the space Y_r is closed in L^r; see [10] for the details. For $v\in C_0^{\infty}(D^n)$ the argument in [9, Appendix] gives a unique decomposition $v=u+\nabla p$ with $u\in Y_r$, $\nabla p\in L^r$, and $p\in L_{\mathrm{loc}}^r(\bar{D}^n)$ so that

$$\|u\|_r+\|\nabla p\|_r\leqq C\|v\|_r \quad \text{with } C \text{ independent of } v. \tag{3.1}$$

Actually, McCracken [9] shows (3.1) for $n=3$; but her argument applies to all dimensions $n\geqq 2$. Let P_r be the linear operator sending v to u. Since $C_0^{\infty}(D^n)$ is dense in L^r, P_r extends uniquely to a bounded linear operator from L^r to Y_r.

Here we let $G_r=X_{r'}^{\perp}$, $r'=r/(r-1)$, for $1<r<\infty$. Since $C_{0,\sigma}^{\infty}(D^n)$ is dense in X_r, a theorem of De Rham [13, Th. 17′] implies that

$$G_r=\{f\in L^r;\ f=\nabla p,\ p\in L_{\mathrm{loc}}^r(\bar{D}^n)\}. \tag{3.2}$$

We next show that

$$Y_r\cap G_r=0 \quad \text{for} \quad 1<r<\infty. \tag{3.3}$$

Indeed, if $f=\nabla p$ is in $Y_r\cap G_r$, the function $p^n=\partial p/\partial x_n$ is harmonic in D^n, lies in L^r, and vanishes on ∂D^n. By the reflection principle for harmonic functions, p^n extends to a harmonic function on R^n which belongs to $L^r(R^n)$. Hence $p^n=0$, i.e., the function p is independent of x_n. It follows that $f=\nabla p=(\partial p/\partial x_1,\dots,\partial p/\partial x_{n-1})$ is independent of x_n and belongs to $L^r(D^n)$. Thus $f=\nabla p=0$, which shows (3.3).

Consider now the operator P_r. By definition, $1-P_r$ maps L^r into G_r. This, together with (3.3), implies that

$$L^r=Y_r\oplus G_r \text{ (direct sum)},\qquad 1<r<\infty;$$
$$\text{and } P_r \text{ is the associated projection onto } Y_r. \tag{3.4}$$

We can now prove

$$Y_r=X_r,\qquad X_r^*=X_{r'},\qquad r'=r/(r-1),\qquad 1<r<\infty. \tag{3.5}$$

Obviously $X_r \subset Y_r$. To show the reverse inclusion, we observe that $Y_{r'} = L^{r'}/G_{r'}$ by (3.4), and therefore $Y_r^* = G_{r'}^{\perp} = X_r$ by the definition of $G_{r'}$. Now let $u_1, u_2 \in Y_r$ satisfy $(u_1 - u_2, Y_{r'}) = 0$. This implies in particular that $(u_1 - u_2, C_{0,\sigma}^{\infty}(D^n)) = 0$ and so $u_1 - u_2 \in X_{r'}^{\perp} = G_r$. Hence $u_1 - u_2 = 0$ by (3.3). This shows that Y_r can be regarded as a subset of $Y_{r'}^* = G_{r'}^{\perp} = X_r$, and hence we obtain (3.5). We have now proved assertions (i) and (ii). Assertion (iii) is easily verified by direct calculation. The proof is complete.

(II) The Stokes Operator

In this subsection we define the Stokes operator over the halfspaces D^n and discuss its basic properties. The main assertion is that the Stokes operator generates in each X_r, $1 < r < \infty$, a bounded analytic semigroup giving the solution to problem (S) and therefore the fractional powers are defined in the standard manner [19]. This is shown by McCracken [9] for $n = 3$ with the aid of hydrodynamic potentials. We give here a different approach based on the formula of Ukai [21] for solutions to problem (S). As seen below, this approach is simpler than that of [9] and, moreover, enables us to determine the domains of fractional powers by means of interpolation theory of Banach spaces.

To state Ukai's formula we prepare some notations. By $R = (R', R_n)$ with $R' = (R_1, \ldots, R_{n-1})$ we denote the Riesz transforms over R^n (see [18]). $S = (S_1, \ldots, S_{n-1})$ denotes the Riesz transform over R^{n-1}. Each R_j (resp. S_j) is a bounded linear operator on $L^r(R^n)$ [resp. $L^r(R^{n-1})$], $1 < r < \infty$. For a function $f = f(x', x_n)$ we understand that S_j acts as a convolution with respect to the variables x', so S_j is regarded as a bounded operator on both $L^r(R^n)$ and $L^r(D^n)$, $1 < r < \infty$. Let $B = B_r = -\Delta$ be the Laplacian on D^n with domain $D(B_r) = W^{2,r}(D^n) \cap W_0^{1,r}(D^n)$. As is well known, $-B$ generates on each L^r, $1 < r < \infty$, a bounded analytic semigroup $\{e^{-tB};\, t \geqq 0\}$. Moreover, one easily sees that

$$e^{-tB} f = h E_t \tilde{f}, \qquad f \in L^r, \qquad 1 < r < \infty, \tag{3.6}$$

where h is the restriction to D^n; E_t denotes the convolution by the heat kernel; and

$$\tilde{f}(x', x_n) = f(x', x_n) \quad (x_n > 0); \qquad = -f(x', -x_n) \quad (x_n < 0).$$

Finally, we define

$$(ef)(x', x_n) = f(x', x_n) \quad (x_n > 0); \qquad = 0 \quad (x_n < 0).$$

The formula of Ukai [21] is now stated as follows.

Theorem 3.2 [21]. *For $a \in X_r$, $1 < r < \infty$, the solution $u = (u', u^n)$ with $u' = (u^1, \ldots, u^{n-1})$ of problem* (S) *is expressed as*

$$u^n(t) = U e^{-tB} V_1 a; \qquad u'(t) = e^{-tB} V_2 a - S U e^{-tB} V_1 a,$$

where $V_1 a = a^n - S \cdot a'$, $V_2 a = a' + S a^n$, and $U = hS \cdot R'(S \cdot R' + R_n)e$. Moreover, the corresponding pressure is expressed, up to addition of functions of t, as the Poisson integral of $-\partial_n e^{-tB} V_1 a|_{\partial D^n}$ $(\partial_n = \partial/\partial x_n)$.

See [21] for the proof. Here we note that the Fourier transform $\widehat{Uf}$ of Uf with respect to the variables x' is expressed as

$$\widehat{Uf}(\xi', x_n) = |\xi'| \int_0^{x_n} e^{-|\xi'|(x_n - y)} \hat{\tilde{f}}(\xi', y)\, dy, \qquad \xi' \in R^{n-1} \tag{3.7}$$

so that for $\partial_j = \partial/\partial x_j$, $j=1,\dots,n$, and the Laplacian Δ' on R^{n-1} we have

$$\partial_j U = U\partial_j, \quad j=1,\dots,n-1; \qquad \partial_n U = (1-U)(-\Delta')^{1/2}. \tag{3.7'}$$

Now set $T_t a = u(t)$, where u solves (S). Direct calculation using Theorem 3.2 then gives $T_{t+s} = T_t T_s$ for $t, s \geqq 0$, and $T_t a \to a$ in X_r as $t \to 0$. Together with the boundedness and analyticity of $\{e^{-tB}; t \geqq 0\}$, this yields

Corollary 3.3. *$\{T_t; t \geqq 0\}$ defines on each X_r, $1 < r < \infty$, a bounded analytic semigroup of class C_0.*

Let $-A_r$ be the generator of $\{T_t\}$, and consider the resolvent

$$(\lambda + A_r)^{-1} a = L_r(\lambda) a = \int_0^\infty e^{-\lambda t} T_t a\, dt, \qquad \lambda > 0, \qquad a \in X_r.$$

In view of Theorem 3.2, we have

$$\begin{aligned}(L_r(\lambda)a)^n &= U(\lambda + B_r)^{-1} V_1 a;\\ (L_r(\lambda)a)' &= (\lambda + B_r)^{-1} V_2 a - SU(\lambda + B_r)^{-1} V_1 a.\end{aligned} \tag{3.8}$$

Using (3.7) and (3.7'), one finds that $D(A_r) = \mathrm{Range}(L_r(\lambda)) \subset X_r \cap D(B_r)$, $\lambda > 0$. On the other hand, if $a \in X_r \cap D(B_r)$, then $V_1 a$ and $V_2 a$ are in $D(B_r)$; and so $T_t a$ is differentiable at $t=0$. We thus have $a \in D(A_r)$. Hence we obtain

$$D(A_r) = X_r \cap D(B_r) = X_r \cap W^{2,r}(D^n) \cap W_0^{1,r}(D^n). \tag{3.9}$$

We call A_r the Stokes operator. A_r is expressed directly as follows:

$$A_r u = -P_r \Delta u \quad \text{for} \quad u \in D(A_r). \tag{3.10}$$

Indeed, using (3.7) and the relation $u^n = UV_1 u$, we have

$$(A_r u)^n + \Delta u^n = \Delta U V_1 u - U \Delta V_1 u = (\partial_n^2 U V_1 - U \partial_n^2 V_1) u.$$

Since $|\xi'| e^{|\xi'|y} = d e^{|\xi'|y}/dy$, (3.7) and integration by parts applied to the first term on the right-hand side yield

$$(A_r u)^n + \Delta u^n = \partial_n p,$$

where p is the Poisson integral of $-\partial_n V_1 u|_{\partial D^n}$. A similar argument using $u' = V_2 u - SUV_1 u$ gives

$$(A_r u)' + \Delta u' = \nabla' p, \qquad \nabla' = (\partial_1, \dots, \partial_{n-1}).$$

Hence we get (3.10). From now on, we write $T_t a = e^{-tA} a = e^{-tA_r} a$. By analytic continuation, we see that (3.8) holds for every λ in the resolvent set of $-B_r$. We thus obtain

Corollary 3.4. *For each $1 < r < \infty$ and each $\pi/2 < \theta < \pi$, any complex number $\lambda \neq 0$ with $|\arg \lambda| \leqq \theta$ belongs to the resolvent set of $-A_r$ and the estimate*

$$\|(\lambda + A_r)^{-1}\| \leqq M/|\lambda|, \qquad \lambda \neq 0, \qquad |\arg \lambda| \leqq \theta, \tag{3.11}$$

holds with M depending only on r, n, and θ. Here $\|\cdot\|$ denotes the operator norm. Moreover, A_r is defined by (3.10).

By Corollary 3.4, the fractional powers A_r^α, $0<\alpha<1$, are defined as

$$A_r^\alpha v = \lim_{\varepsilon \downarrow 0} (\varepsilon + A_r)^\alpha v, \qquad v \in D(A_r^\alpha) \equiv D((\varepsilon + A_r)^\alpha); \quad \text{see [19]}.$$

Corollary 3.5. (i) *The estimate*

$$\|u\|_{2,r} \leqq C\|(A_r+1)u\|_r, \qquad u \in D(A_r)$$

holds with C independent of u.

(ii) $A_r^* = A_{r'}$, $r' = r/(r-1)$; *in particular,* A_2 *is nonnegative and self-adjoint in the Hilbert space* X_2. *Here* A_r^* *is the dual operator of* A_r.

Proof. (i) follows from the boundedness of $(1+A_r)^{-1}$ from X_r to the Banach space $D(A_r)$ with norm $\|\cdot\|_{2,r}$. The boundedness is an immediate consequence of the closed graph theorem. To show (ii), consider the bilinear form

$$D[u,v] = (\nabla u, \nabla v) + (u,v).$$

The operator $L_r = (1+A_r)^{-1}$ satisfies, by (3.10),

$$D[L_r u, v] = D[u, L_{r'} v] = (u,v), \quad \text{for} \quad u, v \in C_{0,\sigma}^\infty(D^n), \qquad r' = r/(r-1).$$

Since $C_{0,\sigma}^\infty(D^n)$ is dense in both of X_r and $X_{r'}$, we see that

$$(L_r u, v) = (u, L_{r'} v) \quad \text{for} \quad u \in X_r \quad \text{and} \quad v \in X_{r'}.$$

This implies $A_r^* = A_{r'}$ and the nonnegativity of A_2. The proof is complete.

(III) Calculus Inequalities Involving Fractional Powers

This subsection is devoted to the proof of

Theorem 3.6. (i) *The estimate*

$$\|D^2 u\|_r \leqq C\|A_r u\|_r, \qquad u \in D(A_r), \qquad 1<r<\infty,$$

holds with C independent of u.

(ii) $D(A_r^{1/2}) = X_r \cap W_0^{1,r}(D^n)$, $1<r<\infty$, *and we have in particular*

$$\|Du\|_r \leqq C\|A_r^{1/2} u\|_r, \qquad u \in D(A_r^{1/2})$$

with C independent of u.

(iii) *If* $u \in D(A_r^\alpha)$, $1<r<\infty$, $0<\alpha<1$, *and if* $0<1/q=1/r-2\alpha/n<1$, *then* $u \in L^q(D^n)$ *and we have the estimate*

$$\|u\|_q \leqq C\|A_r^\alpha u\|_r, \qquad u \in D(A_r^\alpha)$$

with C independent of u.

Here Du and $D^2 u$ *represent the first and second derivatives of u, respectively.*

Notice that the estimates in Theorem 3.6 are by no means obvious, since A_r is not boundedly invertible. We first prove a simpler version.

Lemma 3.7. *Assertions in Theorem 3.6 are valid with* A_r *replaced by* A_r+1.

Proof. It suffices to show that the Banach space $D(A_r^\alpha)$, $0<\alpha<1$, with norm $\|(A_r+1)^\alpha u\|_r$ is equal to the complex interpolation space $[X_r, D(A_r)]_\alpha$ (see [12] for the definition). Indeed, Lemma 3.7 is then immediately derived from the interpolation theory. For the details we refer to [5] and references therein. To identify $D(A_r^\alpha)$ with complex interpolation spaces, it is enough (see [5]) to show that the complex power $(A_r+1)^{-z}, 0<\operatorname{Re} z\leqq 1$, is well-defined with the property that for each $\varepsilon>0$ there is a constant $C_\varepsilon>0$ satisfying

$$\|(A_r+1)^{-z}v\|_r \leqq C_\varepsilon \exp(\varepsilon|\operatorname{Im} z|)\|v\|_r, \qquad v\in X_r. \tag{3.12}$$

To show (3.12) we recall the formula

$$(A+1)^{-z}=\Gamma(z)^{-1}\int_0^\infty t^{z-1}e^{-t(A+1)}dt, \qquad \operatorname{Re} z>0,$$

where $\Gamma(z)$ is the gamma function. Applying Theorem 3.2 to the right-hand side yields

$$((A_r+1)^{-z}u)''=U(B_r+1)^{-z}V_1u,$$

$$((A_r+1)^{-z}u)'=(B_r+1)^{-z}V_2u-SU(B_r+1)^{-z}V_1u.$$

Our problem is thus reduced to showing an estimate of the form (3.12) for the operator $(B_r+1)^{-z}$. But (3.6) gives

$$(B_r+1)^{-z}f=h(1-\Delta)^{-z}\tilde{f},$$

where Δ is the Laplacian on R^n with domain $D(\Delta)=W^{2,r}(R^n)$. So we have only to show an estimate of the form (3.12) for $(1-\Delta)^{-z}$. The operator $(1-\Delta)^{-z}$ is defined [18] as the convolution on R^n by the function G_z with Fourier transform (on R^n)

$$\hat{G}_z(\xi)=(1+|\xi|^2)^{-z}, \qquad \xi\in R^n.$$

Straightforward calculation yields the following estimates for derivatives $D^m\hat{G}_z$:

$$|D^m\hat{G}_z(\xi)|\leqq C_{\varepsilon,m}\exp(\varepsilon|\operatorname{Im} z|)|\xi|^{-m}, \qquad m=0,1,2,\ldots,$$

where $C_{\varepsilon,m}$ depends on $\varepsilon>0$ and the order m of differentiation. We can therefore apply the multiplier theorem [6, Th. 2.5] to get the desired estimates for $(1-\Delta)^{-z}$. This proves Lemma 3.7.

Proof of Theorem 3.6. First observe that D^n is invariant under the transformation: $x\to\lambda x$, $\lambda>0$. For a function f defined on D^n we set

$$(T_\lambda f)(x)=f(x/\lambda), \qquad \lambda>0.$$

Obviously, T_λ defines an automorphism of X_r and of $D(A_r)$, and so an automorphism of each $D(A_r^\alpha)$, $0<\alpha<1$, as seen from interpolation theory. Moreover, direct calculation using (3.10) gives

$$(\mu+A_r)T_\lambda=T_\lambda(\mu+\lambda^{-2}A_r) \quad \text{for} \quad \lambda>0 \quad \text{and} \quad \mu>0, \tag{3.13}$$

so that

$$T_\lambda(\mu+\lambda^{-2}A_r)^{-1}=(\mu+A_r)^{-1}T_\lambda.$$

This implies that, for $0<\alpha<1$,

$$(1+A_r)^{-\alpha}T_\lambda=\frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(t+1+A_r)^{-1}T_\lambda dt$$

$$=\frac{\sin\pi\alpha}{\pi}T_\lambda\int_0^\infty t^{-\alpha}(t+1+\lambda^{-2}A_r)^{-1}dt=T_\lambda(1+\lambda^{-2}A_r)^{-\alpha}.$$

We thus have

$$(1+A_r)^\alpha T_\lambda = T_\lambda(1+\lambda^{-2}A_r)^\alpha \quad \text{for} \quad \lambda>0. \tag{3.14}$$

We now prove Theorem 3.6, using (3.13) and (3.14). First, Corollary 3.5 gives

$$\|D^2u\|_r \leqq C_1\|(A_r+1)u\|_r, \qquad u\in D(A_r).$$

We set $u=T_\lambda v$, $v\in D(A_r)$, use (3.13), and then evaluate both sides of the resulting inequality via the change of variables: $y=x/\lambda$, to get

$$\|D^2v\|_r \leqq C_1\|(A_r+\lambda^2)v\|_r \quad \text{for all} \quad \lambda>0.$$

Since C_1 is independent of λ, letting $\lambda\downarrow 0$ yields assertion (i). Secondly, Lemma 3.7 gives the estimates

$$\|Du\|_r \leqq C_2\|(A_r+1)^{1/2}u\|_r; \quad \text{and}$$

$$\|u\|_q \leqq C_3\|(A_r+1)^\alpha u\|_r \quad \text{provided that} \quad 0<1/q=1/r-2\alpha/n<1.$$

This time, we use (3.14) and proceed as above to get

$$\|Dv\|_r \leqq C_2\|(A_r+\lambda^2)^{1/2}v\|_r; \qquad \|v\|_q \leqq C_3\|(A_r+\lambda^2)^\alpha v\|_r.$$

Letting $\lambda\downarrow 0$ yields assertions (ii) and (iii). The proof is complete.

Remarks. (i) The use of the transformations T_λ is suggested by McCracken [9]. She used the invariance property of D^3 under T_λ to show the resolvent estimate (3.11). Theorem 3.6 plays a basic role in proving our main results in the next section.

(ii) After the present work was completed, our attention was called to the preprint [24] by Simader and Sohr concerning the Helmholtz decomposition. They give a complete and more elementary proof of Theorem 3.1 without using the theorem of De Rham [13, Th. 17′]. However, we have given our version of proof for the reader's convenience.

4. Proof of Main Results

In this section we prove Theorems 1 and 2 stated in Sect. 2. As in [7] we consider separately the cases $n\geqq 3$ and $n=2$. In case $n\geqq 3$, we employ a special kind of approximate solutions to (NS) and obtain estimates which are uniform in approximation; the desired results are then obtained through passage to the limit. In case $n=2$, we directly treat the (unique) weak solution itself and proceed in the same way as in [7]. In both cases, the basic role is played by the following (L^q, L^r)-estimates for the semigroup $\{e^{-tA};\, t\geqq 0\}$. In what follows we write $A=A_r$ for simplicity in notation.

Proposition 4.1. *Let $a\in X_2\cap L^r$ for some $1\leqq r<\infty$. Then the estimate*

$$\|e^{-tA}a\|_q \leqq Ct^{-(n/r-n/q)/2}\|a\|_r$$

holds with C independent of a and $t>0$, provided either

$$\text{(i) } 1<r\leqq q<\infty; \quad \textit{or} \quad \text{(ii) } 1\leqq r<q\leqq\infty.$$

Proof. Under assumption (i) the conclusion follows immediately from Theorem 3.2, (3.6), and the well-known (L^q, L^r)-estimates for the heat kernel. It suffices therefore to prove the assertion, assuming $1<r<q=\infty$, $1=r<q<\infty$, or $1=r<q=\infty$.

Case 1 ($1<r<q=\infty$). First observe that the Gagliardo-Nirenberg inequality [3, Part I, Chap. 9]:

$$\|f\|_\infty \leqq C\|f\|_s^{1-n/s}\|Df\|_s^{n/s}, \quad n<s<\infty \tag{4.1}$$

still remains valid for functions f on the halfspaces D^n. Indeed, to see this one has only to extend f to R^n by an appropriate reflection and then apply the R^n-version of (4.1) to the extended function.

Applying (4.1) with $s=n+r$ and Theorem 3.6(ii), we have

$$\begin{aligned}\|e^{-tA}a\|_\infty &\leqq C\|e^{-tA}a\|_s^{1-n/s}\|De^{-tA}a\|_s^{n/s}\\ &\leqq C\|e^{-tA}a\|_s^{1-n/s}\|A^{1/2}e^{-tA}a\|_s^{n/s}.\end{aligned}$$

Since $\|e^{-tA}a\|_s \leqq Ct^{-(n/r-n/s)/2}\|a\|_r$ and since the boundedness and analyticity of e^{-tA} gives

$$\begin{aligned}\|A^{1/2}e^{-tA}a\|_s = \|A^{1/2}e^{-tA/2}e^{-tA/2}a\|_s &\leqq Ct^{-1/2}\|e^{-tA/2}a\|_s\\ &\leqq Ct^{-1/2-(n/r-n/s)/2}\|a\|_r,\end{aligned}$$

we get

$$\|e^{-tA}a\|_\infty \leqq Ct^{-n/2r}\|a\|_r.$$

This proves the assertion for $1<r<q=\infty$.

Case 2 ($1=r<q<\infty$). Let $q'=q/(q-1)$ and $v\in X_{q'}$. Since $X_q^*=X_{q'}$ by Theorem 3.1, we have

$$\|e^{-tA}a\|_q = \sup_{\|v\|_{q'}=1}|(e^{-tA}a, v)| = \sup_{\|v\|_{q'}=1}|(a, e^{-tA}v)|.$$

Note that here we have used $A_r^*=A_{r'}$, so that $(e^{-tA_r})^*=e^{-tA_{r'}}$, $r'=r/(r-1)$. By the foregoing result,

$$|(a, e^{-tA}v)| \leqq \|a\|_1\|e^{-tA}v\|_\infty \leqq Ct^{-n/2q'}\|a\|_1\|v\|_{q'}.$$

Hence

$$\|e^{-tA}a\|_q \leqq Ct^{-n(1-1/q)/2}\|a\|_1$$

which shows the result.

Case 3 ($1=r<q=\infty$). By (4.1) with $s=2n$ and Theorem 3.6(ii), we get

$$\|e^{-tA}a\|_\infty \leqq C\|e^{-tA}a\|_{2n}^{1/2}\|A^{1/2}e^{-tA}a\|_{2n}^{1/2}.$$

Applying the foregoing results to the right-hand side yields the assertion. This completes the proof of Proposition 4.1.

Corollary 4.2. *For any* $a\in X_r$, $1<r<\infty$,

$$\|e^{-tA}a\|_r \to 0 \quad as \quad t\to\infty.$$

Proof. The assertion follows from Proposition 4.1 when $a\in C_{0,\sigma}^\infty(D^n)$. Since $C_{0,\sigma}^\infty(D^n)$ is dense in X_r, the result follows for an arbitrary a from the boundedness of the semigroup $\{e^{-tA}\}$. The proof is complete.

We now turn to the proof of Theorems 1 and 2. As approximate solutions of (NS) with $a \in X_2$ and $n \geqq 3$, we take the solution u_k, $k=1,2,\ldots$, of

$$\frac{d}{dt}u_k + A_2 u_k + P_2(w_k \cdot \nabla)u_k = 0, \quad t>0; \quad u_k(0)=a_k. \tag{AP}$$

Here $a_k = J_k a$ with $J_k = (1+k^{-1}A_2)^{-1}$, and $w_k = J_k^N u_k$ with $N = 1+[n/4]$. ($[a]$ is the largest integer in the real number a.) Problem (AP) will be solved in the next section; here we collect a few properties of functions u_k, w_k which are needed in this section.

(a) u_k exist uniquely in $L^2(0,T; D(A_2)) \cap W^{1,2}(0,T; X_2)$ for all $T>0$.

(b) w_k are bounded functions on each $D^n \times [0,T]$, $T>0$, and satisfy $\nabla \cdot w_k = 0$ and

$$\int_0^t \|w_k(s)\|_2^2 ds \leqq \int_0^t \|u_k(s)\|_2^2 ds \quad \text{for all} \quad t>0.$$

(c) There is a subsequence of u_k which converges in $L^2_{\text{loc}}(\bar{D}^n \times [0,\infty))$ to a weak solution of (NS).

From *(a)* we have $(d/dt)\|u_k\|_2^2 = 2(u_k, u_k')$ for a.e. t, where $u_k' = du_k/dt$. Thus, multiplying u_k to (AP) and using $(P_2(w_k \cdot \nabla)u_k, u_k) = 0$, we get

$$\|u_k(t)\|_2^2 + 2\int_0^t \|\nabla u_k(s)\|_2^2 ds = \|a_k\|_2^2 \leqq \|a\|_2^2 \quad \text{for} \quad \text{all} \quad t>0. \tag{4.2}$$

In the following, we write $w = w_k$ and $u = u_k$ for simplicity in notation; for the case $n=2$, we understand that $w=u$ represents the unique weak solution to (NS).

In case $n \geqq 3$, our estimates derived below are uniform in k, and so the desired results are obtained through passage to the limit $k \to \infty$. Let

$$A_2 = \int_0^\infty \lambda dE(\lambda)$$

be the spectral decomposition of the nonnegative self-adjoint operator A_2 in X_2. We first prove the following, which is our key lemma.

Lemma 4.3.

$$\|E(\lambda)P(w \cdot \nabla)u\|_2 \leqq C\|w\|_2 \|u\|_2 \lambda^{(n+2)/4}$$

for all $\lambda > 0$, *where C is independent of* w, u, *and* λ.

Proof. We have

$$\|E(\lambda)P(w \cdot \nabla)u\|_2 = \sup_{\|v\|_2=1} |(E(\lambda)P(w \cdot \nabla)u, v)|, \tag{4.3}$$

where $v \in X_2$. Since $\nabla \cdot w = 0$, the right-hand side is written as

$$|(E(\lambda)P(w \cdot \nabla)u, v)| = |(w \cdot \nabla u, PE(\lambda)v)| = |(u, (w \cdot \nabla)E(\lambda)v)|. \tag{4.4}$$

Applying (4.1) with $s=2n$ and Theorem 3.6, we get

$$\begin{aligned} |(u,(w \cdot \nabla)E(\lambda)v)| &\leqq \|w \cdot u\|_1 \|DE(\lambda)v\|_\infty \\ &\leqq C\|w\|_2\|u\|_2 \|DE(\lambda)v\|_{2n}^{1/2} \|D^2E(\lambda)v\|_{2n}^{1/2} \\ &\leqq C\|w\|_2\|u\|_2 \|A^{1/2}E(\lambda)v\|_{2n}^{1/2} \|AE(\lambda)v\|_{2n}^{1/2}. \end{aligned} \tag{4.5}$$

Here we have used that $E(\lambda)v \in \bigcap_{m=1}^{\infty} D(A_2^m) \subset \bigcap_{m=1}^{\infty} D(A_r^m)$ for any $\lambda > 0$ and $r \geqq 2$, which follows from the regularity theory for the stationary Stokes system [1]. Since $1/2n = 1/2 - (n-1)/2n$, a repeated application of Theorem 3.6*(iii)* yields

$$\begin{aligned} \|A^{1/2}E(\lambda)v\|_{2n} &\leqq C\|A^{1/2+(n-1)/4}E(\lambda)v\|_2 \leqq C\lambda^{(n+1)/4}\|v\|_2 ; \\ \|AE(\lambda)v\|_{2n} &\leqq C\|A^{1+(n-1)/4}E(\lambda)v\|_2 \leqq C\lambda^{(n+3)/4}\|v\|_2 . \end{aligned} \tag{4.6}$$

Combining (4.3)–(4.6) implies the assertion. The proof is complete.

Remark. Lemma 4.3 is proved in [7] for the case of R^n by directly applying the Fourier transform; see also [14], [15], and [23]. However, our proof given above applies also to the case of R^n; indeed, the theory of Riesz potentials as given in [18] provides necessary estimates for fractional powers of the Laplacian in R^n.

Using Proposition 4.1 and Lemma 4.3, we can prove Theorems 1 and 2 in the same way as in [7]. Here we give only an outline of the proof. For the details we refer the reader to [7, pp. 138–145]. We start with

$$\frac{d}{dt}\|u\|_2^2 + 2\|A^{1/2}u\|_2^2 = 0 .$$

For any fixed $\varrho > 0$, the second term is estimated as

$$\|A^{1/2}u\|_2^2 \geqq \int_{\varrho}^{\infty} \lambda d\|E(\lambda)u\|_2^2 \geqq \varrho(\|u\|_2^2 - \|E(\varrho)u\|_2^2) \geqq (\varrho/2)(\|u\|_2^2 - \|E(\varrho)u\|_2^2) .$$

This gives

$$\frac{d}{dt}\|u\|_2^2 + \varrho\|u\|_2^2 \leqq \varrho\|E(\varrho)u\|_2^2 . \tag{4.7}$$

To estimate the right-hand side we use the integral equation

$$u(t) = e^{-tA}a_k + \int_0^t e^{-(t-s)A}F(w,u)(s)ds ; \qquad F(w,u) = -P(w\cdot\nabla)u ,$$

and obtain, after an integration by parts,

$$\begin{aligned} E(\varrho)u(t) &= E(\varrho)e^{-tA}a_k + \int_0^t \left[\int_0^{\varrho} e^{-\lambda(t-s)}d(E(\lambda)F(w,u)(s))\right]ds \\ &= E(\varrho)e^{-tA}a_k + \int_0^t e^{-\varrho(t-s)}E(\varrho)F(w,u)(s)ds \\ &\quad + \int_0^t (t-s)\left[\int_0^{\varrho} e^{-\lambda(t-s)}E(\lambda)F(w,u)(s)d\lambda\right]ds . \end{aligned}$$

By Lemma 4.3 [and property (b) for $n \geqq 3$] the last two terms are estimated in X_2 as

$$\leqq \varrho^{(n+2)/4}C\int_0^t \|u(s)\|_2^2 ds .$$

Since $\|E(\varrho)e^{-tA}a_k\|_2 \leqq \|e^{-tA}a_k\|_2 \leqq \|e^{-tA}a\|_2$, this gives

$$\|E(\varrho)u(t)\|_2 \leqq \|e^{-tA}a\|_2 + \varrho^{(n+2)/4}C\int_0^t \|u(s)\|_2^2 ds . \tag{4.8}$$

We substitute (4.8) into (4.7), use the estimate $\|u(s)\|_2 \leqq \|a_k\|_2 \leqq \|a\|_2$ [see (4.2)], to get

$$\frac{d}{dt}\|u\|_2^2 + \varrho\|u\|_2^2 \leqq \varrho C[\|e^{-tA}a\|_2^2 + t^2\varrho^{(n+2)/2}] \tag{4.9}$$

for all $\varrho > 0$. Now take $\varrho = \alpha t^{-1}$ with $\alpha - n/2 > -1$ and multiply both sides above by t^α, and proceed in exactly the same way (using Corollary 4.2 with $r=2$) as in [7] to get Assertion *(i)*. The proof of Assertion *(ii)* for $n \geqq 3$ is carried out in the same way as in [7] if we apply Proposition 4.1 to estimate the term $\|e^{-tA}a\|_2$ in (4.9). To treat the case $n=2$ as in [7], we need the following

Lemma 4.4.

$$\|e^{-tA}P(w\cdot\nabla)u\|_r \leqq C\|w\|_2\|\nabla u\|_2 t^{-n(1-1/r)/2}, \qquad t>0, \qquad 1<r<\infty.$$

We use Lemma 4.4 with $n=2$ and $1<r\leqq 2$ and proceed in exactly the same way as in [7, pp. 142–143] to conclude Assertion *(ii)* for $n=2$. As for Theorem 2, the argument given in [7, pp. 143–145] works with no change, and so the details are omitted here.

Proof of Lemma 4.4. By duality we have

$$\|e^{-tA}P(w\cdot\nabla)u\|_r = \sup_{\|v\|_{r'}=1} |(e^{-tA}P(w\cdot\nabla)u, v)| = \sup_{\|v\|_{r'}=1} |(w\cdot\nabla u, e^{-tA}v)|,$$

where $v \in X_{r'}$ and $r' = r/(r-1)$. By Proposition 4.1, we have

$$|(w\cdot\nabla u, e^{-tA}v)| \leqq \|w\cdot\nabla u\|_1 \|e^{-tA}v\|_\infty \leqq C\|w\|_2\|\nabla u\|_2 t^{-n/2r'}\|v\|_{r'}$$

which gives the result.

5. Construction of Approximate Solutions

We now prove that for each $k=1,2,\ldots$, and each $a \in X_2$ the problem

$$\frac{d}{dt}u_k + A_2u_k + P_2(w_k\cdot\nabla)u_k = 0, \qquad t>0; \qquad u_k(0) = a_k \tag{AP}$$

admits a unique solution u_k and that a subsequence of u_k converges to a weak solution of problem (NS). Here $a_k = J_k a$, $w_k = J_k^N u_k$, $N = 1+[n/4]$, $n \geqq 3$, and $J_k = (1+k^{-1}A_2)^{-1}$. By the regularity theory for the stationary Stokes system [1] we have the estimate

$$\|w_k\|_\infty \leqq C\|u_k\|_2\,. \tag{5.1}$$

Also, since A_2 is nonnegative and selfadjoint, we have $\|J_k\| \leqq 1$ as a bounded operator on X_2. Thus property *(b)* stated in Sect. 4 follows immediately. As in [16] we consider problem (AP) in the form of the integral equation

$$u_k(t) = e^{-tA}a_k + \int_0^t e^{-(t-s)A}F(w_k, u_k)(s)ds; \qquad F(w,u) = -P_2(w\cdot\nabla)u \tag{IE}$$

and solve (IE) applying the contraction mapping principle.

Proposition 5.1. *For any fixed* $T>0$ *and* $k\geqq 1$, *Eq.* (IE) *has a unique solution in* $C([0,T];D(A_2^{1/2}))$. *The solution belongs to* $L^2(0,T;D(A_2))\cap W^{1,2}(0,T;X_2)$ *and satisfies* (AP) *for a.e. t.*

Proof. We write $w=w_k$, $u=u_k$, and $a=a_k$ for simplicity in notation. Note that $a\in D(A_2)$ by this convention. Also, the norm of the Banach space $C([0,T];D(A_2^{1/2}))$ is denoted by

$$|||v|||_T=\sup_{0\leqq t\leqq T}(\|v(t)\|_2+\|A^{1/2}v(t)\|_2).$$

Now define the closed set $S(M,T,a)$ of $C([0,T];D(A_2^{1/2}))$ by

$$S(M,T,a)=\{v;\,v(0)=a,\ |||v|||_T\leqq M\},$$

and consider on $S(M,T,a)$ the nonlinear operator

$$Gv(t)=e^{-tA}a+\int_0^t e^{-(t-s)A}F(w,v)(s)ds,\qquad w=J_k^N v.$$

By Theorem 3.6 and (5.1),

$$\|F(w,v)\|_2\leqq C_1\|w\|_\infty\|\nabla v\|_2\leqq C_2\|v\|_2\|A^{1/2}v\|_2.$$

This, together with the bound $\|A^{1/2}e^{-tA}\|\leqq Ct^{-1/2}$ on X_2, gives

$$|||Gv|||_T\leqq\|a\|_2+\|A^{1/2}a\|_2+C_3M^2(T+T^{1/2})\tag{5.2}$$

for $v\in S(M,T,a)$. Similarly, using

$$\|F(w_1,v_1)-F(w_2,v_2)\|_2\leqq C_2(\|v_1-v_2\|_2\|A^{1/2}v_1\|_2+\|v_2\|_2\|A^{1/2}(v_1-v_2)\|_2)$$

with $w_j=J_k^N v_j$, $j=1,2$, we get

$$|||Gv_1-Gv_2|||_T\leqq C_4M(T+T^{1/2})|||v_1-v_2|||_T\tag{5.3}$$

for $v_j\in S(M,T,a)$, $j=1,2$. Here we fix M so that $\|a\|_2+\|A^{1/2}a\|_2\leqq M/2$ and then choose T so that $C_3M(T+T^{1/2})\leqq 1/2$ and $C_4M(T+T^{1/2})\leqq 1/2$. The estimates (5.2) and (5.3) then show that G is a strict contraction from $S(M,T,a)$ into itself with respect to the metric $|||v_1-v_2|||_T$. By the contraction mapping principle, there is a unique u in $S(M,T,a)$ which solves (IE) on the interval $[0,T]$. Further, since $F(w,u)\in L^\infty(0,T;X_2)\subset L^2(0,T;X_2)$, we easily see that u is in $L^2(0,T;D(A_2))\cap W^{1,2}(0,T;X_2)$ and solves (AP) a.e. on $[0,T]$. To show the existence for arbitrary T, as well as the uniqueness in $C([0,T];D(A_2^{1/2}))$, it suffices therefore to derive a priori bounds for $|||u|||_T$. Since u, Au, and du/dt are all in $L^2(0,T;X_2)$, we get

$$\frac{d}{dt}\|u\|_2^2+2\|A^{1/2}u\|_2^2=0\tag{5.4}$$

and by (5.1)

$$\begin{aligned}\frac{d}{dt}\|A^{1/2}u\|_2^2+2\|Au\|_2^2&=2(F(w,u),Au)\\&\leqq C\|u\|_2\|A^{1/2}u\|_2\|Au\|_2\leqq C\|u\|_2^2\|A^{1/2}u\|_2^2+\|Au\|_2^2.\end{aligned}\tag{5.5}$$

From (5.4) we get

$$\|u(t)\|_2\leqq\|a\|_2.\tag{5.6}$$

Using this in the right-hand side of (5.5) we have

$$\frac{d}{dt}\|A^{1/2}u\|_2^2 \leqq C\|a\|_2^2\|A^{1/2}u\|_2^2,$$

which gives

$$\|A^{1/2}u(t)\|_2^2 \leqq \|A^{1/2}a\|_2^2 \exp(C\|a\|_2^2 t). \tag{5.7}$$

By (5.6) and (5.7) the proof is complete.

Proposition 5.2. *Let u_k be the solution of* (AP) *given in Proposition 5.1. Then there exists a subsequence of u_k which converges in $L^2_{\mathrm{loc}}(\bar{D}^n \times [0,\infty))$ to a weak solution of* (NS).

Proof. First we observe that (5.4) with $u=u_k$ and the estimate $\|a_k\|_2 \leqq \|a\|_2$ together imply the boundedness of u_k in $L^\infty(0,T;X_2)\cap L^2(0,T;D(A_2^{1/2}))$. Consider next the functions $u_k^0(t)=e^{-tA}a_k$; since $a_k \to a$ in X_2, we get by direct calculation,

$$\begin{aligned}\|u_k^0(t)-u_l^0(t)\|_2^2 + 2\int_0^t \|A^{1/2}(u_l^0(s)-u_l^0(s))\|_2^2 ds \\ = \|a_k - a_l\|_2^2 \to 0 \quad \text{as} \quad k,l\to\infty.\end{aligned} \tag{5.8}$$

This implies in particular that

$$v_k = u_k - e^{-tA}a_k \quad \text{are bounded in} \quad L^\infty(0,T;X_2)\cap L^2(0,T;D(A_2^{1/2})). \tag{5.9}$$

Now let $q=(n+2)/(n+1)$. Since $\|J_k\| \leqq M_r$ as a bounded operator on X_r with M_r independent of k, we have

$$\begin{aligned}\|F(w_k,u_k)\|_q &\leqq C\|w_k\cdot\nabla u_k\|_q \leqq C\|w_k\|_{2(n+2)/n}\|\nabla u_k\|_2 \\ &\leqq C\|u_k\|_{2(n+2)/n}\|A^{1/2}u_k\|_2 \\ &\leqq C\|u_k\|_2^{2/(n+2)}\|u_k\|_{2n/(n-2)}^{n/(n+2)}\|A^{1/2}u_k\|_2 \\ &\leqq C\|a\|_2^{2/(n+2)}\|A^{1/2}u_k\|_2^{2/q}.\end{aligned}$$

Here we have used (5.6) for $u=u_k$, the estimate $\|a_k\|_2 \leqq \|a\|_2$ and Theorem 3.6*(ii)*, *(iii)*. This implies that

$$\int_0^T \|F(w_k,u_k\|_q^q dt < C\int_0^T \|A^{1/2}u_k\|_2^2 dt \leqq C\|a\|_2^2. \tag{5.10}$$

Since v_k satisfies

$$\frac{d}{dt}v_k + Av_k = F(w_k,u_k), \qquad t>0; \qquad v_k(0)=0,$$

we can apply Solonnikov's estimate ([17] for $n=3$ and [22] for $n\geqq 4$) to conclude that, with $v_k'=dv_k/dt$,

$$\int_0^T \|v_k'\|_q^q dt + \int_0^T \|Av_k\|_q^q dt \leqq C\int_0^T \|F(w_k,u_k)\|_q^q dt. \tag{5.11}$$

Combining (5.10) and (5.11) gives

$$v_k \text{ are bounded in} \quad L^q(0,T;D(A_q))\cap W^{1,q}(0,T;X_q). \tag{5.12}$$

By (5.9) and (5.12), we can apply Theorem 2.1 in [20, Chap. III] to conclude that a subsequence of v_k converges in $L^2_{\mathrm{loc}}(\bar{D}^n \times [0, \infty))$ to a function v belonging to

$$L^\infty(0, T; X_2) \cap L^2(0, T; D(A_2^{1/2})) \cap L^q(0, T; D(A_q)) \cap W^{1,q}(0, T; X_q) \quad \text{for each} \quad T > 0.$$

This and (5.8) together imply the existence of a subsequence of $u_k = v_k + e^{-tA}a_k$ which converges in $L^2_{\mathrm{loc}}(\bar{D}^n \times [0, \infty))$ to the function $u = v + e^{-tA}a$. It is easily verified that the function u is a weak solution of (NS). This completes the proof of Proposition 5.2.

Remarks. Problem (AP) is obtained from the original problem (NS) by replacing the coefficient of the convective term $u \cdot \nabla u$ by a regularization $w = J_k^N u$. The idea of using equations with regularized convective term to construct weak solutions goes back to Leray [8]. A modified version of Leray's construction is applied by Caffarelli, Kohn, and Nirenberg [2] to the analysis of singularities of weak solutions. The idea of using the operator J_k is due to Sohr, von Wahl, and Wiegner [16].

In [2] and [16] the weak solutions are constructed in the case of bounded domains [2] and exterior domains [16] in R^3 for the initial data satisfying more stringent conditions than ours. This is because both works aim at constructing weak solutions with an additional regularity property of the corresponding pressure. In the forthcoming paper [11] it will be shown that our proof of Proposition 5.2 can be modified in case $n = 3, 4$, so as to provide weak solutions as treated in [2] and [16] under no other conditions than $a \in X_2$. It turns out therefore that when $n = 3, 4$, our weak solutions obtained in this section are the same kind of solutions as treated in [2] and [16].

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