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Topological Properties of the Addition Map in Spaces of Borel Measures

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Introduction

Let X be a topological space, $M(X)$ the space of non-negative, finite measures on the Borel σ-field of X endowed with the weak or narrow topology (see [12] or [11]). Let $P(X)$ be the subspace of probability measures. The map

$$\Phi_X : M(X)^2 \ni (\nu_1, \nu_2) \to \nu_1 + \nu_2 \in M(X)$$

is obviously continuous. But since the cone $M(X)$ fails to be a topological vector space, sets of the form $\mu + \Gamma$, apart from exceptional cases, fail to be open for $\mu \in M(X)$ and open Γ. So it is a non-trivial question whether the map Φ_X is open. Eifler has given some partial answers: in [4] he showed for $0 < \lambda < 1$ the maps Φ_X and

$$\Psi_X^\lambda : P(X)^2 \ni (\nu_1, \nu_2) \to \lambda\nu_1 + (1-\lambda)\nu_2 \in P(X)$$

to be open for Polish spaces X and in [5] he extended this result for Ψ_X^λ to Radon spaces X, i.e. those spaces where each finite Borel measure is tight. In fact, however, the mappings Φ_X and Ψ_X^λ are open for completely arbitrary topological spaces.

The proof of this result will be carried out in two steps:

By investigating a suitable neighbourhood base in $M(X)$, in Sect. 2 the original problem will be reduced to a non-topological decomposition problem in the set M of all finite, non-negative measures on the power set of an auxiliary finite space Ω.

This question, which will be answered affirmatively in Sect. 1, reads as follows:

Consider the preordering defined in M by

$$\mu \succ \nu \Leftrightarrow \mu(B) \geqq \nu(B) \text{ for each } B \in \mathscr{B},$$

where $\mathscr{B}$ is a subset of the power set of Ω which is stable with respect to unions and intersections. Moreover, let be given measures $\mu \in M$ and $\nu_i \in M$ satisfying $\mu \succ \nu_1 + \nu_2$. Is it then always possible, to decompose μ into two measures $\mu_i \in M$ such that $\mu_i \succ \nu_i$ for $i = 1, 2$?

Section 3 is devoted to a proof of the closedness of the map Φ_X, which is based again on the decomposition lemma and uses the quasicompactness of the sets $\Phi_X^{-1}(\{\mu\})$.

The results of the present paper play a crucial role in the investigation of topological properties of some measure-valued mappings, that are of interest in probability theory:

(1) Given topological spaces X, Y and a Borel measurable map $f: X \to Y$ consider the map $\bar{f}: M(X) \to M(Y)$, which assigns to each Borel measure on X its image measure under f (see [3, 4]).

(2) Assign to each measure $\mu \in M(X \times Y)$ on the product of two topological spaces X and Y the pair of its marginal measures (see [6]).

(3) Assign to each measure on a space of continuous functions the collection of its one-dimensional marginals (see [4]).

(4) Given topological spaces X, T and a measure $\lambda \in M(T)$ assign to each kernel $\varphi: T \to M(X)$ its integral $\int \varphi d\lambda \in M(X)$ (see [1]).

These mappings will be treated subsequently.

0. Notations

For a topological space X,

(a) $\mathscr{G}(X)$ is the family of all open sets in X,

(b) $\mathscr{B}(X)$ is the family of all Borel sets in X.

The set of all Borel measures on X, which are always assumed to be finite and non-negative, is denoted by $M(X)$, the subset of probability measures by $P(X)$. $M(X)$ is endowed with the weak or narrow topology, i.e. the topology generated by the requirements

$$\mu \to \mu(X) \text{ is continuous},$$

$$\mu \to \mu(G) \text{ is lower semicontinuous for each } G \in \mathscr{G}(X).$$

Suppressing the index X, we denote by $\mathscr{A}$ the collection of all pairs $(\mathscr{G}, \varepsilon)$, where $0 < \varepsilon < 1$ and $\mathscr{G} \subset \mathscr{G}(X)$ is finite and closed under the formation of unions and intersections (so in particular $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset = X$ are elements of $\mathscr{G}$). For $\alpha = (\mathscr{G}, \varepsilon) \in \mathscr{A}$, $t > 0$ and $\mu \in M(X)$ the abbreviations

$$t\alpha = (\mathscr{G}, t\varepsilon),$$

$$A(\alpha, \mu) = \{\varrho \in M(X): \varrho(G) > \mu(G) - \varepsilon \text{ for } G \in \mathscr{G} \text{ and } \varrho(X) < \mu(X) + \varepsilon\},$$

$$n_\alpha = 2^{\#\mathscr{G}}$$

are used. The family $\{A(\alpha, \mu): \alpha \in \mathscr{A}\}$ is obviously a neighbourhood base of μ. By

$$(\mathscr{G}, \varepsilon) \leqq (\tilde{\mathscr{G}}, \tilde{\varepsilon}) \Leftrightarrow \mathscr{G} \subset \tilde{\mathscr{G}} \quad \text{and} \quad \varepsilon \geqq \tilde{\varepsilon}$$

$\mathscr{A}$ becomes a directed set.

By ε_x we denote the Dirac measure in $x \in X$, by $1_B\mu$ we abbreviate for $B \in \mathscr{B}(X)$ and $\mu \in M(X)$ the measure defined by

$$1_B\mu(C) = \mu(B \cap C) \text{ for each } C \in \mathscr{B}(X).$$

1. The Decomposition Lemma

For a natural number n and $\Omega=\{1,\dots,n\}$ let $\mathscr{B}$ be a subset of the power set of Ω, which is closed under the formation of unions and intersections. Then

$$\mu \succ \nu \Leftrightarrow \mu(B) \geqq \nu(B) \text{ for each } B \in \mathscr{B}$$

defines a preordering on the set M of all non-negative and finite measures on Ω equipped with its power set as σ-field.

(1.1) Proposition. *If μ, ν_1 and ν_2 are measures of M, then the following statements are equivalent:*

(i) $\mu \succ \nu_1 + \nu_2$,

(ii) *there are measures $\mu_1, \mu_2 \in M$ such that*

$$\mu = \mu_1 + \mu_2,\ \mu_1 \succ \nu_1,\ \mu_2 \succ \nu_2, \quad \text{and} \quad \mu_2(\Omega) = \nu_2(\Omega).$$

Proof. Only (i) $\Rightarrow$ (ii) requires proof. It is sufficient to show that for $\nu = \nu_1 + \nu_2$ and

$$\mathscr{D} = \{(\mu', \nu') \in M^2 : \nu_1 \leqq \nu' \leqq \nu,\ \mu' \leqq \mu,\ \mu' \succ \nu',\ \mu - \mu' \succ \nu - \nu' \text{ and } (\mu - \mu')(\Omega) = (\nu - \nu')(\Omega)\}$$

there exists an element $(\mu', \nu_1) \in \mathscr{D}$.

Assign to each $(\mu', \nu') \in \mathscr{D}$ a number

$$i(\mu', \nu') = \#\{x \in \Omega : \mu'(\{x\}) > 0\} + \#\{B \in \mathscr{B} : \mu'(B) > \nu'(B)\} + \#\{x \in \Omega : \nu'(\{x\}) > \nu_1(\{x\})\}.$$

The following statement (*) clarifies, that the desired element $(\mu', \nu_1) \in \mathscr{D}$ can be constructed starting with $(\mu, \nu) \in \mathscr{D}$ in a finite number of steps:

(*) For each $(\mu', \nu') \in \mathscr{D}$ with $\nu' \neq \nu_1$ there exists a pair $(\mu'', \nu'') \in \mathscr{D}$ such that $\nu'' \leqq \nu'$, $\mu'' \leqq \mu'$, and $i(\mu'', \nu'') < i(\mu', \nu')$.

Proof of (*). Let $(\mu', \nu') \in \mathscr{D}$ with $\nu' \neq \nu_1$ be given. We may assume $\nu'(\{1\}) > \nu_1(\{1\})$. Consider now the following decomposition of $\mathscr{B}$:

$$\mathscr{B}_1 = \{B \in \mathscr{B} : \nu'(B) < \mu'(B)\},$$
$$\mathscr{B}_2 = \{B \in \mathscr{B} : \nu'(B) = \mu'(B)\}.$$

(1) It suffices to show the existence of a point $x \in \Omega$, such that

(α) $\qquad \mu'(\{x\}) > 0,$

(β) $\qquad 1 \in B \Leftrightarrow x \in B$ for each $B \in \mathscr{B}_2$,

(γ) $\qquad 1 \in B \Rightarrow x \in B$ for each $B \in \mathscr{B}$.

In this case the definitions

$$0 < \delta = \min(\{\mu'(\{x\}),\ \nu'(\{1\}) - \nu_1(\{1\})\} \cup \{\mu'(B) - \nu'(B) : B \in \mathscr{B}_1,\ x \in B,\ 1 \notin B\}),$$
$$(\mu'', \nu'') = (\mu' - \delta\varepsilon_x,\ \nu' - \delta\varepsilon_1)$$

yield the desired element of $\mathscr{D}$ [(β) guarantees $\mu'' \succ \nu''$, (γ) secures $\mu - \mu'' \succ \nu - \nu''$].

(2) If $\mu'(\{1\}) > 0$ nothing has to be shown. Therefore assume $\mu'(\{1\}) = 0$ and abbreviate

$$B_0 = \bigcap \{B \in \mathscr{B} : 1 \in B\}.$$

The statement (γ) is obviously equivalent to "$x \in B_0$".

(3) Given $C, D \in \mathscr{B}_2$ we get

$$\nu'(C \cup D) + \nu'(C \cap D) = \nu'(C) + \nu'(D) = \mu'(C) + \mu'(D) = \mu'(C \cup D) + \mu'(C \cap D).$$

Since $C \cup D$ and $C \cap D$ are elements of $\mathscr{B}$, the above equation yields $C \cup D$, $C \cap D \in \mathscr{B}_2$. So if we denote by $\mathscr{B}_3$ the σ-field generated by $\mathscr{B}_2$, and by B^0 the atom of $\mathscr{B}_3$ containing 1, there exist $B_2 \in \mathscr{B}_2$ and $B \in \mathscr{B}$ such that

$$B^0 = B \setminus B_2.$$

(Choose $B \in \mathscr{B}_2$ if $1 \in \bigcup \mathscr{B}_2$ and $B = \Omega$ else.) The statement (β) is evidently equivalent to "$x \in B^0$".

(4) Since $B_0 \cup B_2 \in \mathscr{B}$

$$\mu'(B_0 \cup B_2) \geqq \nu'(B_0 \cup B_2).$$

It follows

$$\mu'(B_0 \setminus B_2) \geqq \nu'(B_0 \setminus B_2).$$

Combined with $0 = \mu'(\{1\}) < \nu'(\{1\})$ this means

$$\mu'(B_0 \setminus (B_2 \cup \{1\})) > \nu'(B_0 \setminus (B_2 \cup \{1\})).$$

This yields finally an $x \in \Omega$ such that $\mu'(\{x\}) > 0$ and

$$x \in B_0 \setminus (B_2 \cup \{1\}) \subset B_0 \cap (B_0 \setminus B_2) \subset B_0 \cap B^0. \quad \square$$

2. Openness of Addition

(2.1) Theorem. *Let X be a topological space. Then the map*

$$\Phi_X : M(X)^2 \ni (\nu_1, \nu_2) \to \nu_1 + \nu_2 \in M(X)$$

is open.

We consider first the case of finite T_0-spaces X, i.e. those finite spaces where each pair of distinct points can be separated by an open set containing one of the two points. Hence every subset of such a space is a Borel set.

(2.2) Lemma. *If X is a finite T_0-space, $\nu_1, \nu_2 \in M(X)$, $\nu = \nu_1 + \nu_2$, and $\alpha = (\mathscr{G}(X), \varepsilon)$, then*

$$A(\alpha, \nu) \subset A(n_\alpha \alpha, \nu_1) + A(n_\alpha \alpha, \nu_2),$$

which especially means that Φ_X is open.

Proof. Consider

$$\tilde{\nu}_k = \sum_{x \in X} (\nu_k(\{x\}) - \varepsilon)^+ \varepsilon_x, \qquad k = 1, 2.$$

Denoting by "$\succ$" the preordering of (1.1) associated with $\Omega = X$ and $\mathscr{B} = \mathscr{G}(X)$, we show

$$\begin{aligned} A(\alpha, v) &\subset \{\mu \in M(X) : \mu \succ \tilde{v}_1 + \tilde{v}_2 \text{ and } \mu(X) < v(X) + \varepsilon\} \\ &\subset \{\mu_1 \in M(X) : \mu_1 \succ \tilde{v}_1 \text{ and } \mu_1(X) < v(X) + \varepsilon - \tilde{v}_2(X)\} \\ &\quad + \{\mu_2 \in M(X) : \mu_2 \succ \tilde{v}_2 \text{ and } \mu_2(X) = \tilde{v}_2(X)\} \\ &\subset A(n_\alpha \alpha, v_1) + A(n_\alpha \alpha, v_2). \end{aligned}$$

The first inclusion follows by definition of the measures $\tilde{v}_k$, the second one by (1.1) and the third one by the inequalities

$$\begin{gathered} \tilde{v}_k(G) \geqq v_k(G) - \# X \cdot \varepsilon \quad \text{for } k = 1, 2 \text{ and } G \in \mathscr{G}(X), \\ v(X) + \varepsilon - \tilde{v}_2(X) \leqq v_1(X) + (\# X + 1)\varepsilon, \\ \tilde{v}_2(X) \leqq v_2(X), \end{gathered}$$

and the fact $n_\alpha \geqq \# X + 1$, which is due to the already mentioned observation that $\mathscr{B}(X)$ is the power set of X or, equivalently, $\mathscr{G}(X)$ generates a σ-field possessing $\# X$ atoms. □

Proof of the Theorem. Again for $v_1, v_2 \in \mathrm{M}(X)$, $\alpha = (\mathscr{G}, \varepsilon) \in \mathscr{A}$, and $v = v_1 + v_2$ the inclusion

$$(+) \qquad A(\alpha, v) \subset A(n_\alpha \alpha, v_1) + A(n_\alpha \alpha, v_2)$$

will be shown.

Denote by $B_1, \ldots, B_m$, $m \leqq n_\alpha$, the atoms of the σ-field generated by $\mathscr{G}$. Obviously the statement $(+)$ does not depend on the distributions of the mass inside the atoms B_k, but only on the total mass put there by the considered measures. It is therefore natural to consider the space

$$Y = \{1, \ldots, m\}$$

with the topology

$$\mathscr{G}(Y) = \left\{ I \subset Y : \bigcup_{k \in I} B_k \in \mathscr{G} \right\},$$

i.e. Y is the quotient space of X endowed with the topology $\mathscr{G}$, with respect to the equivalence relation

$$x \sim y \Leftrightarrow x, y \in B_k \text{ for some } k.$$

Y is a finite T_0-space and the projection $p : X \to Y$, defined by $p(x) = k$ for $x \in B_k$, is obviously continuous.

Now let $\mu \in A(\alpha, v)$ and denote the image measures of μ, v, v_1, and v_2 under the mapping p by $\tilde{\mu}$, $\tilde{v}$, $\tilde{v}_1$, and $\tilde{v}_2$. Clearly $\tilde{\mu} \in A(\beta, \tilde{v})$ holds for $\beta = (\mathscr{G}(Y), \varepsilon)$. Lemma (2.2) supplies measures $\tilde{\mu}_1 \in A(n_\alpha \beta, \tilde{v}_1)$ and $\tilde{\mu}_2 \in A(n_\alpha \beta, \tilde{v}_2)$ such that $\tilde{\mu}_1 + \tilde{\mu}_2 = \tilde{\mu}$ (observe $n_\alpha = n_\beta$). We transform the measures back to X by setting

$$\mu_l = \sum_{k \in J} \tilde{\mu}_l(\{k\}) (\mu(B_k))^{-1} 1_{B_k} \mu, \qquad l = 1, 2$$

($J=\{k:\mu(B_k)>0\}$). Since $\mu=\mu_1+\mu_2$ and $\mu_l\in A(n_\alpha\alpha,\nu_l)$ for $l=1,2$, the proof is complete. □

Though restrictions of open mappings need not to be open, this is true in the following special case.

(2.3) Corollary. *Let X be a topological space and* $0<\lambda<1$.

(a) *The map*

$$\Psi_X^\lambda: P(X)^2\ni(\nu_1,\nu_2)\to\lambda\nu_1+(1-\lambda)\nu_2\in P(X)$$

is open.

(b) *The restrictions of* Φ_X *and* Ψ_X^λ *to the subspaces consisting of all regular, τ-smooth or tight measures are open (see* [12, p. XII], *for the definitions).*

The results of [4] and [5] are therefore special cases of Corollary (2.3).

Proof. Since multiplying by a real $d>0$ is a homeomorphism in $M(X)$, to get (*a*) it is sufficient to show that the map

$$\Phi_X: P(X)^2\to 2P(X)=\{\mu\in M(X):\mu(X)=2\}$$

is open. This is a consequence of

$$(++)\qquad A(\alpha,\nu)\cap 2P(X)\subset A(2n_\alpha\alpha,\nu_1)\cap P(X)+A(2n_\alpha\alpha,\nu_2)\cap P(X)$$

for $\nu_i\in P(X)$, $\nu=\nu_1+\nu_2$ and $\alpha=(\mathcal{G},\varepsilon)\in\mathcal{A}$.

To prove $(++)$ let $\mu\in A(\alpha,\nu)\cap 2P(X)$. By the inclusion $(+)$ in the proof of (2.1) there are measures $\mu_k\in A(n_\alpha\alpha,\nu_k)$ such that $\mu=\mu_1+\mu_2$. Since

$$\mu_1(X)-1=1-\mu_2(X)\quad\text{and}\quad|\mu_1(X)-1|<n_\alpha\varepsilon$$

a slight modification of the measures μ_k leads to probability measures μ'_k with $\mu=\mu'_1+\mu'_2$ and $\mu'_k\in A(2n_\alpha\alpha,\nu_k)$:

$$\mu'_1=\mu_1-\frac{t}{\mu_1(X)}\mu_1,\quad \mu'_2=\mu_2+\frac{t}{\mu_1(X)}\mu_1\quad\text{if}\quad t\geqq 0,$$

$$\mu'_1=\mu_1-\frac{t}{\mu_2(X)}\mu_2,\quad \mu'_2=\mu_2+\frac{t}{\mu_2(X)}\mu_2\quad\text{if}\quad t<0,$$

where the abbreviation $t=\mu_1(X)-1$ is used.

The statement (*b*) is due to the fact, that the sum of two measures is regular (τ-smooth or tight) if and only if each of the measures is regular (τ-smooth or tight). □

The following easy application of Theorem (2.1) generalizes results of Degens and Eifler ([2] and [4]).

(2.4) Corollary. *Let X be a topological space such that M(X) fulfills the first axiom of countability and* $\mu,\nu\in M(X)$. *Let* ϱ_n, $n\in\mathbb{N}$, *be a sequence in M(X) which converges to* $\mu+\nu$. *Then there exist sequences* μ_n, $n\in\mathbb{N}$ *and* ν_n, $n\in\mathbb{N}$, *in M(X) such that* $\mu_n+\nu_n=\varrho_n$ *for each n and* $\mu_n\to\mu$, $\nu_n\to\nu$.

Proof. Let $\Gamma_i, i \in \mathbb{N}$ and $\Delta_i, i \in \mathbb{N}$, be descending neighbourhood bases of μ and ν respectively. Since $\Gamma_i + \Delta_i$ is a neighbourhood of $\mu + \nu$ there is an increasing sequence of numbers N_i such that $\varrho_n \in \Gamma_i + \Delta_i$ holds for each $n \geqq N_i$. Choose now $\mu_n \in \Gamma_i$ and $\nu_n \in \Delta_i$ with $\mu_n + \nu_n = \varrho_n$ for each $N_i \leqq n < N_{i+1}$, and $\mu_n = \varrho_n$, $\nu_n = 0$ for $n < N_1$ to finish the proof. □

Finally it is worth mentioning, that $M(X)$ fulfills the first axiom of countability if X fulfills the second one (see [12, p. 49]). But there are spaces such that X fulfills the first axiom of countability and $M(X)$ does not: Let X be the space of all countable ordinals endowed with the order topology. Then the Dieudonné measure on X (see [9, p. 231]) has no countable neighbourhood base in $M(X)$ (see [10, (1.8)]).

3. Closedness of Addition

The following result will be used to investigate Φ_X:

A mapping $f: Y \to Z$ is closed if and only if for each $G \in \mathscr{G}(Y)$ the set $\{z \in Z : f^{-1}(\{z\}) \subset G\}$ is open in Z [7, Theorem 1.4.13]. At first we inspect the sets $\Phi_X^{-1}(\{\varrho\})$.

(3.1) Lemma. *Let X be a topological space and $\varrho \in M(X)$. Then the subspaces*

(a) $$D(\varrho) = \{\mu \in M(X) : \mu \leqq \varrho\} \subset M(X)$$

and

(b) $$E(\varrho) = \{(\mu, \nu) \in M(X)^2 : \mu + \nu = \varrho\} \subset M(X)^2$$

are quasicompact.

Proof. To get (a) we show that $D(\varrho)$ is the continuous image of a closed subspace of the compact space

$$Y = \prod_{B \in \mathscr{B}(X)} [0, \varrho(B)].$$

Let p_B, $B \in \mathscr{B}(X)$, be the projections of Y. The subspace

$$Y_0 = \{y \in Y : p_B(y) + p_C(y) = p_{B \cup C}(y) \text{ for each pair of disjoint Borel sets } B, C \in \mathscr{B}(X)\}$$

is closed in Y, the mapping $\Psi: Y_0 \to D(\varrho)$ with

$$\Psi(y)(B) = p_B(y)$$

is well defined since σ-additivity of $\Psi(y)$ follows by

$$\Psi(y)(B) - \sum_{k=1}^{n} \Psi(y)(B_k) = \Psi(y)\left(B \Big\backslash \bigcup_{k=1}^{n} B_k\right) \leqq \varrho\left(B \Big\backslash \bigcup_{k=1}^{n} B_k\right)$$

for each sequence of pairwise disjoint sets $B_k \in \mathscr{B}(X)$ and $B = \bigcup_{k=1}^{\infty} B_k$. Since Ψ is a continuous surjection, the quasi compactness of $D(\varrho)$ is established.

The continuous surjection

$$\Psi' : Y_0 \ni y \to (\Psi(y), \Psi((\varrho(B) - p_B(y))_{B \in \mathscr{B}(X)})) \in E(\varrho)$$

yields finally assertion (b). □

Similar to Theorem (2.1) the reduction to the finite case is the main idea in the proof of the following

(3.2) Theorem. *The mapping Φ_X is closed for any topological space X.*

Proof. As already mentioned, it is sufficient to prove the openness of the set

$$\Gamma_2 = \{\varrho \in M(X) : E(\varrho) \subset \Gamma_1\}$$

for open sets $\Gamma_1 \subset M(X)^2$. To this end let $\varrho \in \Gamma_2$ be given. By Lemma (3.1) there are $\alpha_1, \ldots, \alpha_k \in \mathscr{A}$, $\mu_1, \ldots, \mu_k$, $\nu_1, \ldots, \nu_k \in M(X)$ such that $\mu_i + \nu_i = \varrho$ for each i and

$$\begin{aligned} E(\varrho) &\subset \bigcup \{A(\alpha_i, \mu_i) \times A(\alpha_i, \nu_i) : i \leqq k\} \\ &\subset \bigcup \{A(2\alpha_i, \mu_i) \times A(2\alpha_i, \nu_i) : i \leqq k\} \subset \Gamma_1 . \end{aligned}$$

Choosing an $\alpha = (\mathscr{G}, \varepsilon)$ such that $\alpha \geqq \alpha_i$ for each i, we show for $K = (4n_\alpha)^{-1}$ the inclusion

$$A(K\alpha, \varrho) \subset \Gamma_2 .$$

To this end let μ_0, ν_0 be measures such that $\varrho_0 = \mu_0 + \nu_0 \in A(K\alpha, \varrho)$. To prove $\varrho_0 \in \Gamma_2$ we have to establish $(\mu_0, \nu_0) \in \Gamma_1$. This will be done by constructing measures $\mu, \nu \in M(X)$ such that $\mu + \nu = \varrho$ and $\mu_0 \in A(\alpha, \mu)$, $\nu_0 \in A(\alpha, \nu)$. Indeed, if the index i is chosen such that $(\mu, \nu) \in A(\alpha_i, \mu_i) \times A(\alpha_i, \nu_i)$ holds, this means

$$(\mu_0, \nu_0) \in A(2\alpha_i, \mu_i) \times A(2\alpha_i, \nu_i) \subset \Gamma_1 .$$

In the proof of Theorem (2.1) the complementary problem was solved: The measures μ_0 and ν_0 were constructed for given μ, ν. So to apply Lemma (2.2) the inequality signs have to be converted or, equivalently, open and closed sets have to change roles. Let therefore $B_1, \ldots, B_m$, p, and Y be as in the proof of (2.1). Y is now endowed with the topology

$$\mathscr{G}(Y) = \left\{I \subset Y : X \Big\backslash \bigcup_{k \in I} B_k \in \mathscr{G}\right\}.$$

Since p remains Borel measurable the image measures $\tilde{\mu}_0, \tilde{\nu}_0, \tilde{\varrho}_0$ and $\tilde{\varrho}$ of μ_0, ν_0, ϱ_0 and ϱ under the mapping p can be considered. $\varrho_0 \in A(K\alpha, \varrho)$ yields now $\tilde{\varrho} \in A(2K\beta, \tilde{\varrho}_0)$ for $\beta = (\mathscr{G}(Y), \varepsilon)$ since $X \in \mathscr{G}$ implies

$$\tilde{\varrho}(Y) = \varrho(X) < \varrho_0(X) + K\varepsilon = \tilde{\varrho}_0(Y) + K\varepsilon ,$$

$$\begin{aligned} \tilde{\varrho}(I) &= \tilde{\varrho}(Y) - \tilde{\varrho}(Y \backslash I) \\ &= \varrho(X) - \varrho(X \backslash \bigcup \{B_k : k \in I\}) \\ &> (\varrho_0(X) - K\varepsilon) - (\varrho_0(X \backslash \bigcup \{B_k : k \in I\}) + K\varepsilon) \\ &= \tilde{\varrho}_0(I) - 2K\varepsilon \end{aligned}$$

for each $I \in \mathscr{G}(Y)$. By Lemma (2.2) there exist measures

$$\tilde{\mu} \in A(2^{-1}\beta, \tilde{\mu}_0), \qquad \tilde{\nu} \in A(2^{-1}\beta, \tilde{\nu}_0)$$

such that $\tilde{\mu}+\tilde{\nu}=\tilde{\varrho}$. After putting them back to X as in the proof of (2.1) measures $\mu, \nu \in M(X)$ with image measures $\tilde{\mu}, \tilde{\nu}$ are obtained. As above $\mu_0 \in A(\alpha, \mu)$ and $\nu_0 \in A(\alpha, \nu)$ follows and finishes the proof. □

Let us conclude with some remarks.

(1) Theorem (3.2) can easily be carried over to the spaces of regular, τ-smooth or tight measures as in (2.3).

(2) The following easy application of (3.2) is sometimes useful. Let X be a topological space, ϱ_j, $j \in J$, a net in $M(X)$ which converges to $\varrho \in M(X)$ and Γ an open set in $M(X)$ with $D(\varrho) \subset \Gamma$. Then $D(\varrho_j) \subset \Gamma$ holds eventually.
Indeed, since $E(\varrho) \subset \Gamma \times \Gamma$ and the set $\{\mu \in M(X) : E(\mu) \subset \Gamma \times \Gamma\}$ is open by Theorem (3.2), $E(\varrho_j) \subset \Gamma \times \Gamma$ holds eventually and implies $D(\varrho_j) \subset \Gamma$.

(3) Straightforward considerations provide generalizations of the Theorems (2.1) and (3.2), which yield that the map

$$M(X)^n \ni (\mu_k)_{k \leqq n} \to \sum_{k=1}^{n} \mu_k \in M(X)$$

is open and closed for arbitrary natural numbers $n \geqq 1$. Openness and closedness of countable addition are easily obtained by choosing reasonable domains in $M(X)^{\mathbf{N}}$ and ranges in $M(X)$ (see [10, (1.9) and (2.6)]). The result of Groemig [8] is therefore a consequence of the results of this paper (see [10, (2.7)]).

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