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The Uniform Position Principle for Curves in Characteristic p

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Let $\mathbf{P}^n$ be the n-dimensional projective space over an algebraically closed field. A set of points $x_1, \ldots, x_d \in \mathbf{P}^n$ lies in *uniform position*, if for any positive integers k, l any k of these points impose the same number of conditions on hypersurfaces of degree l.

J. Harris proves in [8] that the points of a general hyperplane section of a reduced irreducible curve $C \subset \mathbf{P}^n$ lie in uniform position, if the characteristic of the ground field is 0. He then proceeds (see also [9]) to bound the genus of curves not lying on surfaces of a certain degree. Although one can expect that Harris's bounds hold in arbitrary characteristic, it is not clear how to actually prove them. In view of his methods, however, it suffices to establish his uniform position argument in characteristic $p > 0$.

What can we expect to find?

First of all, in characteristic $p > 0$ there are well known examples of space curves C such that every secant of C is a multisecant [10, IV Ex. 3.8.], i.e., every secant of C intersects C in at least one more point. For these, uniform position certainly fails. Apart from this phenomenon there seem to be no obvious counter-examples. Furthermore, these examples yield so-called strange curves. By Samuel's theorem [20] they are singular except for the conic in characteristic 2 and $\mathbf{P}^1$. Therefore, since our main interest lies in smooth curves, we can lay those aside.

Our main result is

Theorem 0.1. *Let C be a smooth irreducible curve in $\mathbf{P}^n$, $n \geqq 4$ and assume that C is not contained in a hyperplane. Then the points of a general hyperplane section of C lie in uniform position.*

For curves in $\mathbf{P}^3$ we can prove it only under additional assumptions. In general, the problem remains open.

In Sect. 1 we develop the necessary geometric theory for our result. Section 2 proves the main theorem (2.5) using a classification theorem for multiply transitive permutation groups.

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1. The Geometric Theory

Let $C \subset \mathbf{P}^n$ be a nondegenerate (not lying in a hyperplane) reduced irreducible curve. The simplest case where one can check for uniform position arises for $l=1$, i.e., hyperplanes. We discuss this case first. Recall from [10, p. 311] that C is called *strange*, if all of its tangent lines (at regular points) pass through one point.

Lemma 1.1 (General Position Lemma). *Let C be as above, H a general hyperplane. If C is not strange, then every n points of $H \cap C$ are linearly independent.*

Proof. The arguments are well known (see e.g. [1, Proposition 8], [2, p. 110]), however usually a characteristic 0 hypothesis is used:

Let C be as in the assumption such that for every hyperplane H one can find n linearly dependent points in $H \cap C$. We have to show that C is strange.

For $2 \leqq k \leqq n-1$ let $U_k := \{(x_1, \ldots, x_k) \in C^k \mid x_i \text{ pairwise distinct}\}$ and let V_k be the subset of those points $(x_1, \ldots, x_k)$ such that $x_1, \ldots, x_k$ are linearly dependent or there is one more point of C in their linear span. V_k is closed in U_k which is irreducible.

The hyperplanes of $\mathbf{P}^n$ are parametrised by the dual projective space $\mathbf{P}^{n*}$. If $(x_1, \ldots, x_k) \in V_k$ and $x_1, \ldots, x_k$ are linearly independent, then they are responsible for rendering an $(n-k)$-dimensional (linear) subvariety of $\mathbf{P}^{n*}$ undesirable. If those points lie in a $(k-1)$-dimensional subvariety of U_k and, furthermore, $\dim(V_{k-1}) \leqq k-2$, then we can conclude that $\dim(V_k) \leqq k-1$. Now $\dim(V_{n-1}) \leqq n-2$ proves the general position lemma for C; therefore (U_k is irreducible) there exists a (minimally chosen) k such that for every k points $x_1, \ldots, x_k$ of C there is one more point x of C such that $x_1, \ldots, x_k, x$ are linearly dependent. Now either every secant of C is a multisecant ($k=2$), or C can be projected birationally from a point of C into a hyperplane. The image of C now violates the general position lemma in $\mathbf{P}^{n-1}$. Continuing, one arrives at a curve C_0 all of whose secants are multisecants. By [10, IV 3.8] C_0 is strange (all tangent lines pass through one point).

Now C itself is strange: As in the proof of [10, IV 3.8] it is sufficient to show that every two tangent lines meet. So let $P, Q \in C$, choose a $\mathbf{P}^3$ containing the embedded tangent lines t_P, t_Q and choose the projections from above into linear spaces containing that $\mathbf{P}^3$. t_P, t_Q remain fixed under the projections, and, after the last projection, they must intersect.

Example 1.2. Let $C \subset \mathbf{P}^n$ be defined as the complete intersection of $X_0^q - X_1 X_n^{q-1}$, $X_1^q - X_2 X_n^{q-1}, \ldots$ where $\operatorname{char} k = p$ and $q = p^f$ for some $f > 0$. C is reduced and irreducible of degree q^{n-1}. As a configuration of points, the general hyperplane section of C looks like an $(n-1)$-dimensional affine space over a field with q elements.

Proof. Over the affine open subset $\{X_n \neq 0\}$ C can be described by the equations $x_0 = t$, $x_1 = t^q$, $x_2 = t^{q^2}, \ldots$.

If $a_0, \ldots, a_r$ ($r < n$) are linearly independent points of C, then the points of C in their linear span are given by $a = a_0 + \lambda_1(a_1 - a_0) + \ldots + \lambda_r(a_r - a_0)$ where the λ_i satisfy $\lambda_i^q = \lambda_i$ [They are $(q-1)$-th roots of unity or 0.]

Definition 1.3 [7, I]. Let C be a nondegenerate reduced irreducible curve in $\mathbf{P}^n$; let $M=\{(x,H)\in C\times\mathbf{P}^{n*}\mid x\in H\}$ be defined by the point-hyperplane incidence relation. The projection $M\to C$ exhibits M as a $\mathbf{P}^{n-1}$-bundle over C and therefore M is irreducible. The projection $\pi: M\to\mathbf{P}^{n*}$ is a finite separable map of degree d. The induced map $\pi^*: K(\mathbf{P}^{n*})\to K(M)$ on fields of rational functions (the local rings at the generic points) represents $K(M)$ as a finite separable field extension of $K(\mathbf{P}^{n*})$ of degree d. By the primitive element theorem there exists an element $f\in K(M)$ generating $K(M)$ over $K(\mathbf{P}^{n*})$ and satisfying $P(f)=0$ for an irreducible monic polynomial P over $K(\mathbf{P}^{n*})$.

The *monodromy group* G_C of C is defined as the Galois group $\mathrm{Gal}(P, K(\mathbf{P}^{n*}))$ of a splitting field of P over $K(\mathbf{P}^{n*})$. It is independent of the choice of f and it can be regarded as a subgroup of the full permutation group S_d of the d roots of P.

Remark 1.4. For $k=\mathbf{C}$ there is a more geometric description available. Let $U\subset\mathbf{P}^{n*}$ be an open subset such that the induced map $\pi^{-1}(U)\to U$ is étale. Fix a point $H\in U$. Moving H along a path in U gives a bijection of the fibers over different H. Now a loop in H induces a certain permutation in the fiber. The group generated by those permutations is isomorphic to the monodromy group G_C [7, I].

Recall that the action of a permutation group G on a set Ω is called *k-fold transitive*, if for every two sequences $x_1,\ldots,x_k$ and $x'_1,\ldots,x'_k$ of distinct elements of Ω there exists a permutation $\sigma\in G$ mapping x_i on x'_i for $1\leqq i\leqq k$.

Proposition 1.5. *Let $1\leqq k\leqq d$. The monodromy group acts k-fold transitively (on the roots of P) if and only if there exists an irreducible open subset U of*

$$U_k=\{(x_1,\ldots,x_k,H)\in C^k\times\mathbf{P}^{n*}\mid x_1,\ldots,x_k\in H,\ x_i \text{ pairwise distinct}\}$$

such that the induced map $U\to\mathbf{P}^{n*}$ is étale and generically $d!/(d-k)!$-to-one onto its image.

Proof. Recall from [10, I 4.4] that there is an equivalence of categories between the category of varieties and dominant rational maps and the category of finitely generated field extensions over k. Given a variety V, the corresponding field is its field of rational functions $K(V)$.

We want to use this in a slightly different context: There is a bijection between (i) pairs (X,f), where $f: X\to\mathbf{P}^{n*}$ is a rational map such that for every irreducible component X_i of X the restriction $X_i\to\mathbf{P}^{n*}$ is dominant and generically étale, and (ii) direct products of finite field extensions of $K(\mathbf{P}^{n*})$.

We now prove by induction on k that the $K(\mathbf{P}^{n*})$-algebra corresponding to (U_k,π), where $\pi: U_k\to\mathbf{P}^{n*}$ is the projection, is $K(\mathbf{P}^{n*})[f_1,\ldots,f_k]$, where $f_1,\ldots,f_k$ are k different roots of P that are formally adjoined to $K(\mathbf{P}^{n*})$:

The assertion is clear for $k=1$, so let $k>1$; write $K=K(\mathbf{P}^{n*})$. Let V_k be defined as the pullback

$$\begin{array}{ccc} V_k & \to & U_{k-1} \\ \downarrow & & \downarrow \\ U_1 & \to & \mathbf{P}^{n*}, \end{array}$$

$$V_k=\{(x_1,\ldots,x_k,H)\in C^k\times\mathbf{P}^{n*}\mid (x_1,\ldots,x_{k-1},H)\in U_{k-1},\ x_k\in H\}.$$

V_k decomposes into k dominant components, namely those where $x_k = x_j$ for some $1 \leqq j \leqq k-1$ and $x_k \neq x_j$ for all $1 \leqq j \leqq k-1$. The first $k-1$ components are all isomorphic to U_{k-1}, while the last corresponds to U_k. Using the bijection by taking rational functions we get a push-out diagram in the category of finite $K(\mathbf{P}^{n*})$-algebras:

$$\begin{array}{ccc} K(V_k) & \leftarrow & K(U_{k-1}) \\ \uparrow & & \uparrow \\ K(U_1) & \leftarrow & K(\mathbf{P}^{n*}) = K. \end{array}$$

We have $K(U_1) \cong K[f]$, $K(U_{k-1}) \cong K[f_1, \ldots, f_{k-1}]$ by induction. Therefore

$$\begin{aligned} K(V_k) &\cong K[f] \otimes K[f_1, \ldots, f_{k-1}] \\ &\cong K[T]/P(T) \otimes K[f_1, \ldots, f_{k-1}] \\ &\cong K[f_1, \ldots, f_{k-1}][T]/P(T). \end{aligned}$$

Now P decomposes over $K[f_1, \ldots, f_{k-1}]$ as

$$P(T) = (T - f_1) \ldots (T - f_{k-1}) P_k(T),$$

so

$$\begin{aligned} K(V_k) &\cong K[f_1, \ldots, f_{k-1}][T]/(T - f_1) \ldots (T - f_{k-1}) P_k(T) \\ &\overset{(*)}{\cong} K[f_1, \ldots, f_{k-1}][T]/(T - f_1) \oplus \ldots \oplus K[f_1, \ldots, f_{k-1}][T]/P_k(T) \\ &\cong (K[f_1, \ldots, f_{k-1}])^{k-1} \oplus K[f_1, \ldots, f_{k-1}][T]/P_k(T). \end{aligned}$$

(*): The canonical map

$$K[f_1, \ldots, f_{k-1}][T] \to K[f_1, \ldots, f_{k-1}][T]/(T - f_1) \oplus \ldots \oplus K[f_1, \ldots, f_{k-1}][T]/P_k(T)$$

has as its kernel the ideal generated by $(T - f_1) \ldots P_k(T) = P(T)$ and surjectivity follows because we have on both sides of (*) K-vector spaces of the same dimension.

The decompositions of V_k and $K(V_k)$ allow us to identify $K[f_1, \ldots, f_{k-1}][T]/P_k(T)$ as the $K(\mathbf{P}^{n*})$-algebra corresponding to the last component of V_k. Therefore $K[f_1, \ldots, f_{k-1}][T]/P_k[T]$ is a field if and only if that component is irreducible.

The proposition readily follows: $K[f_1, \ldots, f_{k-1}][T]/P_k(T)$ is a field if and only if $K[f_1, \ldots, f_{k-1}]$ is a field and P_k is irreducible over $K[f_1, \ldots, f_{k-1}][T]$. The latter is equivalent to $\mathrm{Gal}(L, K[f_1, \ldots, f_{k-1}])$ acting transitively on the roots of P_k, where L is a splitting field of P over $K(\mathbf{P}^{n*})$. By induction the conclusion follows.

Corollary 1.6. *Let C be a nondegenerate reduced irreducible curve in $\mathbf{P}^n$. If every n points of a general hyperplane section of C are linearly independent, then the monodromy group of C is n-fold transitive.*

Proof. Using (1.5) it suffices to observe that U can be chosen as a $\mathbf{P}^0$-bundle over $C^n - A$, where A is the algebraic subset of C^n consisting of those $(x_1, \ldots, x_n) \in C^n$ such that $x_1, \ldots, x_n$ are linearly dependent. The assumption on the general hyperplane section ensures that A is a proper subset of C^n.

Remark 1.7. Over $\mathbf{C}$ there is an easier argument for (1.6) using the description of the monodromy group given in (1.4). Choose $H, x_1, \ldots, x_n$ and $x_1', \ldots, x_n' \in C \cap H$. First choose a loop γ_1 in $H \in U$ such that x_1 is mapped to x_1', if one moves H along γ_1. Then choose a loop γ_2 such that x_1' is fixed and $\gamma_1(x_2)$ is mapped to x_2'. Inductively one can continue this process n times because the subset of hyperplanes fixing $x_1, \ldots, x_k$ has dimension $n-k$.

Corollary 1.8 [8, Chap. 2]. *If the monodromy group G_C is the whole symmetric group S_d or the alternating group A_d, then the points of a general hyperplane section of C lie in uniform position.*

Proof. $G_C \cong S_d$: The dimension of the linear system of hypersurfaces of degree l passing through $x_1, \ldots, x_k$ is a semi-continuous function on U [U as in (1.5)]. Let a be the minimal dimension. Suppose that for every hyperplane H one can find $x_1, \ldots, x_k \in H \cap C$ such that the linear system has dimension $\geqq a+1$. If $k \geqq n$, then $x_1, \ldots, x_k$ span in general a unique hyperplane, so the closed subset of U that renders hyperplanes undesirable, has dimension $n = \dim(U)$ and must therefore be equal to U contradicting the choice of a. If $k < n$, then the same procedure as in (1.1) gives a contradiction. (Moreover, any $k \leqq n$ points in general position impose independent conditions on hypersurfaces of any degree.)

$G_C \cong A_d$: For $k \leqq d-2$ the same proof as for S_d works; for $k = d-1$ one can regard the condition as a condition on the remaining point.

Example 1.9. Let C be as in (1.2). Then the monodromy group G_C is the affine general linear group $AGL(n-1, q)$.

Proof. The description of the general hyperplane section of C given in (1.2) allows us to identify the irreducible dominant components of U_k [as in (1.5)] for any $1 \leqq k \leqq n$. In particular, we can conclude that every permutation in G_C must respect linear incidence; so G_C is a subgroup of $AGL(n-1, q)$. However, the irreducible dominant component of U_k corresponding to

$$\{(x_1, \ldots, x_k, H) \in C^k \times \mathbf{P}^{n*} \mid x_1, \ldots, x_k \in H, x_1, \ldots, x_k \text{ linearly independent}\}$$

provides a certain subgroup of G_C. These subgroups together generate $AGL(n-1, q)$.

Proposition 1.10. 1. *Let C_0 be the projection of C from a point x of $\mathbf{P}^n - C$ into a hyperplane. Then the monodromy group G_{C_0} can be regarded as a subgroup of G_C. For general x, G_C and G_{C_0} are isomorphic.*

2. *Let C_1 be the projection of C from a nonsingular point $x \in C$ into a hyperplane. Further assume that not every secant through x is a multisecant of C. Then the monodromy group of C_1 can be regarded as a subgroup of the stabilizer of one root under the action of G_C on the d roots of P.*

Proof. 1. G_C is completely described by the irreducible dominant components of U_d, $d = \deg C$, as in (1.5), where $\pi: U \to \mathbf{P}^{n*}$. Now G_{C_0} is described by $\pi^{-1}(P) \to P$ where P is the hyperplane in $\mathbf{P}^{n*}$ corresponding to the center of projection x. The first part follows, and for the second apply [14, 6.3.4].

2. First note that $\deg C_1 = \deg C - 1$. Taking $\pi^{-1}(P) \to P$ as above it is clear that in every $(x_1, \ldots, x_d, H) \in \pi^{-1}(P)$ one of the x_i must be equal to x. Then the same argument works.

For special projections the monodromy group can actually become smaller, see (2.15).

Proposition 1.11. *If there exists a hyperplane H intersecting C in $d-k+1$ nonsingular points of C, where H intersects C transversely at $x_1, \ldots, x_{d-k}$ and has intersection multiplicity k at x_{d-k+1}, then the monodromy group G_C contains a subgroup acting transitively on k roots of P while fixing the other roots.*

Proof. We can assume that all coefficients of P lie in the local ring O_H of $\mathbf{P}^{n*}$ at H. Let $\hat{O}_H$ be the completion of O_H. Complete the structure sheaf of M [in the notation of (1.3)] along $x_1, \ldots, x_{d-k+1}$ to get a finite $\hat{O}_H$-algebra B of rank d. B is isomorphic to $\hat{O}_H[T]/P(T)$. B splits as a product of $d-k+1$ local $\hat{O}_H$-algebras corresponding to the points $x_1, \ldots, x_{d-k+1}$ [18, I 4.2], $B \cong \prod_{i=1}^{d-k+1} B_i$. Then necessarily $B_i \cong \hat{O}_H$ for $1 \leqq i \leqq d-k$ and $B_{d-k+1} \cong \hat{O}_H[T]/Q(T)$ for a polynomial $Q \in \hat{O}_H[T]$ of degree k. Therefore $P(T) = Q(T) \prod_{i=1}^{d-k} (T - a_i)$ over $\hat{O}_H$. Let $\hat{K}$ be the quotient field of $\hat{O}_H$, $K(\mathbf{P}^{n*}) \to \hat{K}$ be induced by restriction. As shown above, P decomposes over $\hat{K}$ as $P = L_1 \ldots L_{d-k} Q$ with $L_i = T - a_i$. Let $K = K(\mathbf{P}^{n*})[a_1, \ldots, a_{d-k}]$. If we can show that Q is irreducible over K, then Galois theory provides the desired subgroup. But this is clear, because Q corresponds to a point of M where M is locally irreducible, even smooth.

2. Applications

Proposition 2.1. *Let C be a reduced irreducible curve in $\mathbf{P}^n$. If there exists a hyperplane H intersecting C in $d-1$ smooth points of C, where H intersects C transversely at $d-2$ points and has intersection multiplicity 2 at the remaining point, then the monodromy group G_C is the whole symmetric group S_d.*

Proof. By (1.11) G_C contains a subgroup acting transitively on 2 elements and fixing the complement, so G_C contains a transposition. For $k=2$ the variety U_2 [defined in (1.5)] is a $\mathbf{P}^{n-2}$-bundle over C and therefore irreducible. By (1.5) G_C acts doubly transitively and contains therefore all transpositions. Now it is well known that S_d is generated by transpositions.

At this point we should mention the implications of the theory of duality for projective varieties as described e.g., in [16]:

Let $V \subset \mathbf{P}^n$ be a reduced irreducible variety. Let $N(V) \subset \mathbf{P}^n \times \mathbf{P}^{n*}$ be the *conormal variety* of V. It is defined as follows: The fiber over a smooth point x of V consists of all those hyperplanes that contain the tangent space at x, and $N(V)$ is the closure. The *dual variety* $V^* \subset \mathbf{P}^{n*}$ is defined as the scheme-theoretic image of $N(V)$ under the projection $\mathbf{P}^n \times \mathbf{P}^{n*} \to \mathbf{P}^{n*}$. It is again reduced and irreducible. Now V is called *reflexive*, if the map $N(V) \to V^*$ is separable.

In order to get a correspondence between the points and tangent hyperplanes of a variety and its dual, one usually restricts oneself to the subset of reflexive varieties.

While in characteristic 0 every curve and in characteristic 2 no curve is reflexive, reflexive curves in characteristic $p>2$ can be characterized as follows: Let $\mu(C)$ be the intersection multiplicity of C with a general hyperplane H at a general point x of C whose tangent line is contained in H. Then C is reflexive if and only if $\mu(C)=2$; and for nonreflexive curves $\mu(C)$ is always a power of p [11, 3.5].

In view of (2.1) we can therefore state using the fact that for reflexive curves the point of contact of a general hyperplane is unique [15, 3.5]:

Corollary 2.2. *Let C be a reduced irreducible curve in $\mathbf{P}^n$. If C is reflexive, then its monodromy group is the symmetric group S_d.*

This result has also been obtained in [3].

Examples of Nonreflexive Curves 2.3. Let k be an algebraically closed field of characteristic $p>0$.

1. Let $q_1, q_2, q_3 \in k[X, Y, Z]$ be homogeneous polynomials of the same degree with no common zero in $\mathbf{P}^2$. Then $f=Xq_1^p+Yq_2^p+Zq_3^p$ describes a smooth nonreflexive plane curve. Moreover, for $p>2$ all smooth nonreflexive plane curves arise in this way [19, II-19].

2. Let $C\subset\mathbf{P}^n$ be a smooth complete intersection curve defined by homogeneous polynomials $f_1,\ldots,f_{n-1}$. Let $g_i=\sum_{j=1}^{n+1} x_j\left(\frac{df_i}{dx_j}\right)^p$. Then $g_1,\ldots,g_{n-1}$ define a smooth nonreflexive complete intersection curve in $\mathbf{P}^n$.

3. Let x_0, x_1 be a basis for $H^0O_{\mathbf{P}^1}(1)$, $q=p^f$. Then x_0^{q+1}, $x_0^q x_1$, $x_0 x_1^q$, $x_1^{q+1}\in H^0O_{\mathbf{P}^1}(q+1)$ embed $\mathbf{P}^1$ as a smooth nonreflexive curve in $\mathbf{P}^3$. C lies on the quadric $X_0X_3-X_1X_2$ and, apart from $p=2$, C is the projection of a reflexive curve, in particular C is not linearly normal.

What does our theory imply for nonreflexive curves? (1.5) and (1.6) still hold and, by choosing special hyperplane sections, we can try to apply (1.11). Now, surprisingly, a faithful doubly transitive action of a group is already a strong condition permitting a classification [4, 5.3]. Since our main interest lies in the question of uniform position, we can even restrict ourselves to nonplane curves. Groups acting faithfully triply transitively on a set of d elements are described by

Theorem 2.4. *Let G be a subgroup of the symmetric group S_d. If G acts triply transitively, then G is contained in the following list (k is the maximal degree of transitivity in each case):*

group	d	k
$AGL(n,2)$, $n\geqq 3$	2^n	3
G_1	16	3
$PSL(2,q)\leqq G\leqq P\Gamma L(2,q)$	$q+1$	3
M_{11}	11	4
M_{11}	12	3
M_{12}	12	5
M_{23}	23	4
M_{24}	24	5
A_d	d	$d-2$
S_d	d	d.

$AGL(n, 2)$ is the group of affine transformations on a vector space of dimension n over a field with 2 elements.

G_1 is a certain subgroup of $AGL(4, 2)$.

$PSL(2, q)$ is the subgroup of even projective linear transformations on the projective line over a field with q elements. If q is even, then $PSL(2, q) = PGL(2, q)$, otherwise it is a subgroup of index 2 in $PGL(2, q)$.

$P\Gamma L(2, q)$ is the automorphism group of $PSL(2, q)$. It can be described as the group of all transformations $x \to (ax^\alpha + b)/(cx^\alpha + d)$ of the projective line, where α is a field automorphism of $\mathbf{F}_q$.

M_i are the Mathieu groups in their usual representations.

Remarks to the Proof. We could not find a reference for this result, so we proceed to outline the main ideas of the proof (see [4]): If G acts doubly transitively on a set Ω, then choose a minimal nontrivial normal subgroup N. Let $C_G(N)$ be the centralizer of N in G. We have to distinguish two cases. If $C_G(N) \neq \{e\}$, then N is isomorphic to $(\mathbf{Z}/p)^n$ for some prime p. Ω can be identified with N and G acts on N by affine linear transformations. All triply transitive groups in this class have been determined in [5, 8.2]. If $C_G(N) = \{e\}$, then N is simple, and, since G acts on N by conjugation, G can be embedded into the automorphism group of N. If N is a simple group of Lie type, all possible G have been determined in [6], while for the sporadic simple groups the results can be found in [4]. ($\mathrm{Aut}(M_i) \cong M_i$ for $i = 11, 23, 24$, M_i has index 2 in $\mathrm{Aut}(M_i)$ for $i = 12, 22$ [12, XII 1.15a].)

Theorem 2.5. *Let $C \subset \mathbf{P}^n$, $n \geqq 4$, be a nondegenerate reduced irreducible curve of degree d. If the points of a general hyperplane section do not lie in uniform position, then C is strange.*

Furthermore one of the following is true:

(1) *every secant of C is a multisecant;*

(2) *every plane spanned by three points of C contains one more point of C;*

(3) *$d \in \{11, 12, 23, 24\}$ and the monodromy group of C is one of the Mathieu groups in its standard representation.*

Proof. Let C be a curve not satisfying (1) or (2). Then G_C acts quadruply transitively, so using (1.6), (2.4) we only have to show that a curve whose monodromy group is one of the Mathieu groups is strange.

First suppose that the general hyperplane containing a tangent line of C has a unique point of contact. Then by (1.11) we have $\mu(C) \geqq l(G)$ where $l(G)$ is the minimal cardinality of a nontrivial subset such that there exists a subgroup of G acting transitively on this subset and fixing the complement. By [12, XII 1] we have $l(M_{11}) = l(M_{12}) = 8$ and $l(M_{23}), l(M_{24}) \geqq 16$. Choosing a hyperplane containing two tangent lines of C we arrive at $d = \deg C \geqq 2\mu(C) \geqq 2l(G)$, an obvious contradiction for each of the Mathieu groups.

If the general hyperplane containing a tangent line of C is tangent to C only at (possibly several) points of this line, then the same argument works. Here we have to use a more general statement than (1.11): The subgroup we find acts no longer transitively, but at least nontrivially.

Now assume that the general hyperplane containing a tangent line of C contains several distinct tangent lines. Varying the hyperplane one sees that every two tangent lines must meet, C is strange.

(1.2) gives examples of curves satisfying (1) ($p \neq 2$) and (2) ($p=2$). We do not know whether there exist curves satisfying (3).

Problem 2.6. Is it possible to give a classification of all curves not satisfying the conclusion of the general position lemma? The standard examples are

(1) as in (1.2),

(2) curves obtained by projecting a curve C as in (1.2) from a point of C into a hyperplane,

(3) projections of curves as in (1), (2).

Corollary 2.7. *Let C be a smooth irreducible curve of degree d in $\mathbf{P}^n$, $n \geqq 4$ not contained in a hyperplane. Then the monodromy group of C contains the alternating group A_d.*

Proof. Suppose that the monodromy group of C does not contain the alternating group.

If the points of a general hyperplane section of C lie in general position, then the monodromy group of C is 4-fold transitive by (1.6). Therefore, by (2.4) this group must be one of the Mathieu groups, and in the proof of (2.5) it is shown that C is strange. All smooth irreducible strange curves are known [20] ($\mathbf{P}^1$ and the plane conic in characteristic 2) and lie in the plane, so C cannot be nondegenerate in $\mathbf{P}^4$. Contradiction!

If the points of a general hyperplane section of C do not lie in general position, then C must be strange by (1.1), and we also get a contradiction.

Proof of (0.1). The result follows immediately from (2.7) and (1.8).

Corollary 2.8. *Let C be a nondegenerate smooth irreducible curve of degree d and genus g in $\mathbf{P}^n$, $n \geqq 4$. Then the bounds given in* [9, 3.7, p. 87], [9, 3.15, p. 99], [9, 3.22, p. 117] *hold regardless of the characteristic of k.*

In particular this establishes

Castelnuovo's Theorem 2.9. *Let C be a smooth irreducible nondegenerate curve in $\mathbf{P}^n$ of degree d and genus g where $\mathbf{P}^n$ is the projective n-space over an algebraically closed field of any characteristic. Let m be the greatest integer $\leqq (d-1)/(n-1)$ and $d = m(n-1)+\lambda$. Then the genus of C satisfies*

$$g \leqq \binom{m}{2}(n-1)+m\lambda.$$

In the literature this result is only stated with a characteristic 0 hypothesis.

Theorem 2.10 ($\mathrm{char} k \neq 2$). *Let C be a nondegenerate reduced irreducible curve of degree d in $\mathbf{P}^n$ with conormal variety $N(C)$ and dual variety C^*. Then the following conditions are equivalent:*

(1) *The monodromy group of C is contained in the alternating group A_d.*

(2) *The field extension $K(C^*) \to K(N(C))$ has even degree.*

If C is not strange, then this is also equivalent with:

(3) *The general tangent line of C is tangent at several points of C, and the number of these points is even.*

Proof. The equivalence of (2) and (3) is clear. For the equivalence of (1) and (2) we want to use the following well known result [13, p. 250]:

Let f be a separable irreducible polynomial over a field K of characteristic $\neq 2$ with discriminant $D = \prod_{i<j} (a_i - a_j)^2$ where $a_1, \ldots, a_d$ are the roots of f in a splitting field. Since D is a symmetric polynomial of the roots of f, it can be expressed as a polynomial in the elementary symmetric functions of $a_1, \ldots, a_d$, i.e., the coefficients of f; in particular, D lies in K. Then the Galois group $\mathrm{Gal}(f, K)$ is contained in the alternating group if and only if D is a square in K.

Let $L = K[a]$ where $f(a) = 0$. Then D is also the discriminant of $1, a, a^2, \ldots, a^{n-1}$ where the discriminant of a basis $u_1, \ldots, u_n$ of L over K is defined as $\det(\mathrm{Trace}(u_i u_j))$ and by [21, II 11] the discriminants of two different bases differ by a square of K.

Now let $R = k[T_1, \ldots, T_d]$ be the coordinate ring of an affine part of $\mathbf{P}^{n*}$, K its quotient field. Let S be the coordinate ring of the corresponding affine part of the normalization of M [M as in (1.3)] with quotient field L. Since M is a projective bundle over a nonsingular curve, all local rings of S are regular. Since R and S are both Cohen-Macaulay, S is locally free as R-module, and Serre's conjecture implies that S is even free.

Let $u_1, \ldots, u_d$ be an R-basis for S.

The prime ideal of the conormal variety inside S is given by the 0-th Fitting ideal of $\Omega_{S/R}$, the module of relative differentials (see [11]) and by [17] it coincides with the Dedekind different $\delta_{S/R}$.

Now by [21, V Theorem 30] the discriminant of $u_1, \ldots, u_d$ is nothing else but the norm of the different $\delta_{S/R}$ (which is a height 1 prime ideal). We can compute the latter ideal locally: If $A = P_1 \ldots P_r$ is a decomposition of an ideal in a Dedekind ring, then its norm is $(P_1 \cap R)^{f_1} \ldots (P_r \cap R)^{f_r}$ where the relative degree f_i is the degree of the field extension $\mathrm{Quot}(R/(P_i \cap R)) \to \mathrm{Quot}(S/P_i)$. The equivalence of (1) and (2) follows.

Note that a curve is reflexive if and only if the map from the conormal variety to its dual variety is birational, and then the field extension $K(C^*) \to K(N(C))$ is an isomorphism.

The situation in characteristic 2 is different, as can be seen from the following example:

Example 2.11. Let C be the strange plane curve defined by $yz^{d-1} = x^d$ in characteristic 2. Define an integer a and an odd integer b by $d = 2^a b$ for d even, $d - 1 = 2^a b$ for d odd.

Then:

1. The monodromy group of C is contained in the alternating group A_d for any $d > 3$.

2. The field extension $K(C^*) \to K(N(C))$ has separable degree b and inseparable degree 2^a.

3. The tangent line at a general point of C is tangent at b different points of C and has intersection multiplicity 2^a at each of them.

Proof. In order to prove 1., we want to use the following result [13, p. 252]: Let f be a separable irreducible polynomial over a field K and $a_1, \ldots, a_d$ its roots in a splitting field. Let $D_1 = \sum_{\sigma \in A_d} a_{\sigma 2}^1 a_{\sigma 3}^2 \ldots a_{\sigma d}^{d-1}$, $D_2 = \sum_{\sigma \in S_d - A_d} a_{\sigma 2}^1 a_{\sigma 3}^2 \ldots a_{\sigma d}^{d-1}$. Then

D_1+D_2 and D_1D_2 can be expressed as polynomials in the elementary symmetric functions of $a_1,\ldots,a_d$, i.e., the coefficients of f. [13, p. 252] now asserts that the Galois group of f is contained in the alternating group A_d if and only if $T^2-(D_1+D_2)T+D_1D_2$ has a root in K.

Choose homogeneous coordinates A, B, C in $\mathbf{P}^{n*}$. M [as in (1.3)] is described by the polynomials $yz^{d-1}=x^d$ and $Ax+By+Cz=0$. Dehomogenize by setting $z=1$, $B=1$. Then $y=x^d$, $Ax+y+C=0$, $y=-Ax-C$, so we have to consider the Galois group of $f(x)=x^d+Ax+C$.

Assume first that d is odd, $d=2s+1$. Denote by σ_i the i-th elementary symmetric function of the roots of f. In our case most of the σ_i vanish, so D_1+D_2 and D_1D_2 depend only on σ_{d-1} and σ_d. Since $\deg\sigma_i=i$ and $\deg D_1+D_2=s(2s+1)$, $\deg D_1D_2=2s(2s+1)$, it follows that

$$D_1+D_2=r_1\sigma_{2s+1}^s,$$

$$D_1D_2=r_2\sigma_{2s+1}^{2s}+r_3\sigma_{2s}^{2s+1}$$

for some constants r_1, r_2, r_3. Since these formulas are reductions of the corresponding formulas over $\mathbf{Z}$, the constants are either 0 or 1.

Now $D_1+D_2=D_1-D_2=\prod_{i<j}(a_i-a_j)$ (use van der Monde's determinant) $\neq 0$ (f is separable), so $r_1=1$.

These formulas must hold for any specialization of A and C. Setting $C=0$, every root of f occurs at least with multiplicity 2, and two double roots (or one root with multiplicity 4) force $D_1=D_2=0$. [If $a_1=a_2$, $a_3=a_4$, then for the summand corresponding to $\sigma\in A_d$ the summand corresponding to $(1\ 2)(3\ 4)\sigma$ contributes the same value.] Therefore $r_3=0$ and the quadratic polynomial above has a root in K for any $s\geqq 2$.

2. and 3. are discussed in [16].

The proof for even d is quite analogous, so we omit it here.

The preceding example provides some evidence for the following

Conjecture 2.12. *Let* $\mathrm{char}\,k=2$, *let C be a nondegenerate reduced irreducible curve of degree d in $\mathbf{P}^n$ with conormal variety $N(C)$ and dual variety C^*. Then the following conditions are equivalent:*

(1) *The monodromy group of C is contained in the alternating group A_d.*

(2) *The field extension $K(C^*)\to K(N(C))$ has degree >2.*

If C is not strange, then this is also equivalent with:

(3) *The general tangent line of C is tangent at several points of C (the separable degree of the field extension in (2) is >1) or the general tangent line has intersection multiplicity >2 with C at that point (the inseparable degree of the field extension in (2) is >2).*

Let C be a curve such that every hyperplane containing a tangent line of C contains the tangent lines of several other points? Then either

(1) H contains a unique tangent line that is tangent at several points, or

(2) H contains several distinct tangent lines that might also be tangent at several points of C.

If (2) is valid, then C is strange.

[16, I-3] gives some plane curves with property (1). Space curves with that property are constructed in the following

Example 2.13. Let $\mathrm{char}k=p>0$, C_1 be an arbitrary curve with field of rational functions K_1. Let K_2 be a separable extension of K_1 of degree a, let K be a purely inseparable extension of K_2 of degree p, let $T\in K$ be a separating transcendental such that $K_1[T]=K$ and set $K_3=k[T]$. If C_2 is a nonsingular curve with function field K, then C_2 admits a separable map f of degree $b=[K:K_3]$ to $\mathbf{P}^1$ and a non-separable map to C_1. These maps together yield a birational map $C_2\to C_1\times\mathbf{P}^1$, and the image of C_2 is in general singular. Now any birational image of $C_1\times\mathbf{P}^1$ in some $\mathbf{P}^n$ that is a scroll maps C_2 in such a way that the fibers of the scroll are exactly the embedded tangent lines of C_2. In particular, the general tangent line at a point of C is tangent to C in exactly a points.

This construction also yields smooth examples by setting $C_1\cong C_2\cong\mathbf{P}^1$, $b=1$ and choosing an embedding of $\mathbf{P}^1\times\mathbf{P}^1$ as a scroll in some $\mathbf{P}^n$.

It can also be used to show that in characteristic >0 any curve has a nonreflexive birational model in some $\mathbf{P}^n$ and a strange model in $\mathbf{P}^2$.

Proposition 2.14. *Let $C\subset\mathbf{P}^3$ be a nonplane smooth irreducible curve of degree d. Assume that the general hyperplane containing a tangent line of C has a unique point of contact and that C is not strange. If the monodromy group of C does not contain the alternating group A_d, then one of the following must hold*

(1) $\mathrm{char}k=2$;

(2) *C is smooth and rational; if x_0, x_1 span $H^0\mathcal{O}_{\mathbf{P}^1}(1)$, then C is embedded into $\mathbf{P}^3$ by x_0^d, $x_0^{d-1}x_1$, $x_0x_1^{d-1}$, $x_1^d\in H^0\mathcal{O}_{\mathbf{P}^1}(d)$ and $d=p^f+1$ for some $f>0$, $\mathrm{char}k=p$.*

Proof. Choose a tangent line l of C, another line l_1 not intersecting l and project C from l to l_1. For a general choice of l the fibres of this map consist of $d-\mu(C)$ points.

Which values can $\mu(C)$ assume, if $G_C\neq A_d, S_d$?

$\mu(C)$ must be a prime power, and (1.11) then gives us a certain subgroup of G_C. The possible cardinalities of a nontrivial subset acting transitively on this subset and fixing the complement are given by

$$G\leqq AGL(n,2)\colon\ d-1 \quad\text{or}\quad d/2 \quad\text{for}\quad \mathrm{char}k=2,$$

$$G\leqq P\Gamma L(2,q),\quad q \text{ odd}\colon\ d-1 \quad\text{or}\quad d-2 \quad(\text{if } d-2 \text{ is a power of } 2),$$

$$G\leqq P\Gamma L(2,q),\quad q \text{ even}\colon\ d-1 \quad(\mathrm{char}k=2) \quad\text{or}\quad d-2.$$

The Mathieu groups are contained in the corresponding alternating groups, so because of (2.10) we do not need to consider them.

By Hurwitz's theorem [10, IV 2.4] we have $2g(C)-2=(d-\mu(C))(-2)+\deg R$, where R is the ramification divisor of the projection.

If $d-\mu(C)=1$, then π is an isomorphism. Let x_0, x_1 be a basis for $H^0\mathcal{O}_{\mathbf{P}^1}(1)$; x_0, x_1 have a zero at p, q. The linear system defining the embedding must contain the divisors $p+\mu(C)q$, $\mu(C)p+q$, $(\mu(C)+1)p$, $(\mu(C)+1)q$. These are linearly independent, and C is as in (2).

If $d-\mu(C)=2$, then $2g(C)-2=-4+\deg R$, $\deg R=2g(C)+2$. For $\mathrm{char}k\neq 2$ the double covering π is only tamely ramified and we must have at least two points of ramification. One of these points can be the point where l is tangent to C, the

other point however must correspond to a point whose tangent line meets l. Choosing the plane spanned by those two tangent lines we get $d=\deg C \geqq 2\mu(C)$, a contradiction.

Example 2.15. The monodromy group of the smooth rational curve in (2.3.3) and (2.14.2) is the group $PGL(2,q)$ in its standard representation.

Proof. The embedding of the curve is defined by $x_0^{q+1}: x_0^q x_1 : x_0 x_1^q : x_1^{q+1}$, so the curve lies on the hypersurfaces $xu=yz$, $y^{q+1}=z^q u$.

Choose homogeneous coordinates A, B, C, D in $\mathbf{P}^{3*}$. M [as in (1.3)] is then described by the ideal of the curve together with $Az+By+Cz+Du=0$. Dehomogenize $D=1$, $x=1$. Then $A+By+Cz+u=0$, $u=yz$, $y^{q+1}=u=yz$, $z=y^q$ and $A+By+Cy^q+y^{q+1}=0$.

We have to find the Galois group of $f(T)=T^{q+1}+CT^q+BT+A$ over $K=k(A,B,C)$.

Choose a root r of f and set $Z=T-r$.

$$\begin{aligned} f(T)=f(Z+r)&=(Z+r)^{q+1}+C(Z+r)^q+B(Z+r)+A \\ &=Z^{q+1}+rZ^q+r^qZ+r^{q+1}+CZ^q+Cr^q+BZ+Br+A \\ &=Z(Z^q+(r+C)Z^{q-1}+(r^q+B)). \end{aligned}$$

Let $g(Z)=Z^q+(r+C)Z^{q-1}+r^q+B$ and set $U=Z^{-1}$.

$$g(Z)=g(U^{-1})=U^{-q}+(r+C)U^{1-q}+r+B=U^{-q}(1+(r+C)U+(r^q+B)U^q).$$

Let $h(U)=1+(r+C)U+(r^q+B)U^q$, choose a root s of h and set $V=U-s$.

$$\begin{aligned} h(U)=h(V+s)&=1+(r+C)(V+s)+(r^q+B)(V+s)^q \\ &=1+(r+C)V+(r+C)s+(r^q+B)V^q+(r^q+B)s^q \\ &=V((r+C)+(r^q+B)V^{q-1}). \end{aligned}$$

Let t be a root of $V^{q-1}(r^q+B)+(r+C)$; all the roots of that polynomial are then given by $\omega t, \omega^2 t, \ldots, \omega^{q-1}t$ with a primitive $(q-1)$-th root of unity ω. Then $U=s$, $s+\omega t, \ldots, s+\omega^{q-1}t$ are the roots of $h(U)$, $Z=s^{-1}, (s+\omega t)^{-1}, \ldots, (s+\omega^{q-1}t)^{-1}$ are the roots of $g(Z)$, and $T=, r+s^{-1}, r+(s+\omega t)^{-1}, \ldots, r+(s+\omega^{q-1}t)^{-1}$ are the roots of $f(T)$. In particular, we can conclude that $\mathrm{Gal}(f,K)$ acts sharply triply transitively on the roots of f.

We now want to show that the map G, defined by $G(\infty)=r$, $G(0)=r+s^{-1}$, $G(x^i)=r+(s+\omega^i t)^{-1}$ where $x=\omega$ is a generator of the multiplicative group of the finite field with q elements, induces an isomorphism of the representations of $PGL(2,q)$ and $\mathrm{Gal}(f,K)$.

Let $f(z)=(az+b)/(cz+d)$ be an arbitrary element of $PGL(2,q)$. We have $f(\infty)=a/c$, $f(0)=b/d$, $f(1)=(a+b)/(c+d)$, and there exists a unique element $\sigma \in \mathrm{Gal}(f,K)$ such that $G(f(z))=\sigma(G(z))$ for $z=\infty, 0, 1$. We have to show that this equation holds for arbitrary z.

Now the given three equations determine $\sigma(r)$, $\sigma(s)$, and $\sigma(t)$. Using this, it is straightforward, but tedious, to verify the above equation for arbitrary z.

We do not know whether the points of the general hyperplane section of the curve in this example lie in uniform position.

Example 2.16. Let C be the smooth nonreflexive plane curve $x^q y + y^q z + z^q x = 0$, $q = p^f$, $p = \mathrm{char}\, k > 0$. Then the monodromy group of C is isomorphic to $PGL(2, q)$.

Proof. M [as in (1.3)] is given in inhomogeneous coordinates by $x^q y + y^q + x = 0$, $Ax + y + C = 0$, so $y = -Ax - C$ and $x^q(-Ax - C) + (-Ax - C)^q + x = 0$, $-Ax^{q+1} - (A^q + C)x^q + x - C^q = 0$. We have to find the Galois group of $f(X) = X^{q+1} + (A^q + C)A^{-1}X^q - A^{-1}X + C^q A^{-1}$. From this form we can already conclude, using the proof of (2.15), that $\mathrm{Gal}(f, K) \subset PGL(2, q)$. Now set $D = CA^{-1}$, $E = A^{-1}$, $F = D + E^{1-q}$, then $K = k(A, C) = k(A, D) = k(E, D) = k(E, F)$ and

$$\begin{aligned} f(X) &= X^{q+1} + (A^q + C)A^{-1}X^q - A^{-1}X + C^q A^{-1} \\ &= X^{q+1} + (A^{q-1} + D)X^q - A^{-1}X + D^q A^{q-1} \\ &= X^{q+1} + (E^{1-q} + D)X^q - EX + D^q E^{1-q} \\ &= X^{q+1} + FX^q - EX + (F - E^{1-q})^q E^{1-q}. \end{aligned}$$

$\mathrm{Gal}(f, K)$ is doubly transitive, and the last form of f shows [by comparing it with the corresponding polynomial in the proof of (2.15)] that after adjoining two roots the remainder is irreducible, so $\mathrm{Gal}(f, K)$ is triply transitive and we must have $\mathrm{Gal}(f, K) \cong PGL(2, q)$.

Example 2.17. Let C be the strange nonreflexive plane curve $yz^{q-1} = x^q$, $q = p^f$, $p = \mathrm{char}\, k > 0$. Then the monodromy group of C is isomorphic to $AGL(1, q)$. [Compare (1.9)!]

Proof. M is given in inhomogeneous coordinates by $y = x^q$, $Ax + y + C = 0$, so we have to find the Galois group of $f(X) = X^q + AX + C$. But this group has been determined in (2.15) as the stabilizer of one element of the standard representation of $PGL(2, q)$, so $\mathrm{Gal}(f, K) \cong AGL(1, q)$.

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