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L-Indistinguishability for Real Groups

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1. Introduction

The purpose of this paper is to complete our study of L-indistinguishability for the tempered spectrum of a real reductive algebraic group. We follow closely Langlands' formulations described in [13] for SL_2; see (1.1) and (1.2) below for a discussion of the topic. Most of the necessary properties of orbital integrals have already been established in [14–16]. However, because we find it useful to modify some notions associated with L-distinguishability [Sect. 2, (1.2)], we have several additional lemmas to prove. There also remains a question from Sect. 11 of [15] (Theorem 3.5.4). With the properties of orbital integrals established (Sect. 3), we will turn to the lifting of tempered characters (cf. [13, Sect. 7]). Existence of the lifting is quite immediate but the formalism of [13] requires explicit information about the coefficients in a lift, and about the characters themselves. This is included in Theorem 4.1.1. As a consequence, we will obtain a result helpful in identifying L-packets (Theorem 4.3.2). In Sect. 5, we invert the lifting (cf. Theorem 5.4.27), at the same time demonstrating a structure on L-packets of tempered representations proposed by Langlands (cf. [13, Sect. 13]).

(1.1) *L-Distinguishability and Orbital Integrals*

Throughout this paper $\mathbf{G}$ will denote a connected reductive linear algebraic group defined over $\mathbb{R}$, and G the group of $\mathbb{R}$-rational points on $\mathbf{G}$. In (1.1) and (1.2) we assume that $\mathbf{G}$ is semisimple and simply connected. Then G is a connected semisimple Lie group and ${}^LG^0$, the connected component of the identity in the L-group ${}^LG$ for $\mathbf{G}$, is of adjoint type. Some simplification of definitions will result.

We use the same notation for a Langlands parameter (equivalence class of admissible homomorphisms of the Weil group W of $\mathbb{R}$ into ${}^LG$ relevant to G) and a homomorphism $\varphi: W \to {}^LG$ representing it. Two irreducible admissible representations of G are L-indistinguishable, or belong to the same L-packet, if they are attached to the same parameter or homomorphism. Then they have the same

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attached L-factors (cf. [2]). The members of Π_φ, the L-packet attached to φ, are (each) tempered if and only if φ is bounded. Let $\Phi_0(G)$ denote the set of such parameters. We set S_φ equal to the centralizer of $\varphi(W)$ in ${}^LG^0$; S_φ^0 will be the connected component of the identity in S_φ.

The group S_φ has been found useful (cf. [7]). For example, since the constituents of a *unitary* principal series representation π belong to the same L-packet, we may speak of the parameter φ attached to π. If π belongs to the minimal principal series, then π is irreducible if and only if S_φ is connected; in general, π is irreducible if and only if S_φ/S_φ^0 contains only the "obvious" elements (cf. [7, Theorem 3.4], Lemma 5.4.1). Set $\mathbb{S}_\varphi = S_\varphi/S_\varphi^0$. Then, in a certain precise sense (implicit in [6, 7]; cf. (5.4)), $\mathbb{S}_\varphi$ controls the reducibility of π. Since this notion is central to the discussion of (5.4), in (5.3) we will give some relevant unpublished arguments of Langlands. These describe the stability group of a discrete series representation of a Levi component of a parabolic subgroup of G in terms of S_φ (see [7], for a statement) and identify the root system of the connected reductive complex group S_φ^0 with a system of Harish-Chandra's [4, Sect. 40] (or the indivisible roots in a system of Knapp-Stein (cf. [6])).

If $\varphi \in \Phi_0(G)$ then by definition, Π_φ consists of the constituents of several principal series. One consequence of the results just described is that the cardinality of Π_φ is bounded by the order of $\mathbb{S}_\varphi$; it is exactly this order if $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbf{G}) = 1$ (recall that $\mathbf{G}$ is simply-connected). We will add formal objects ("ghosts") to Π_φ, to obtain $\bar{\Pi}_\varphi$ in $1-1$ correspondence with $\mathbb{S}_\varphi$.

We consider now a quite different problem, that of matching combinations of orbital integrals of a Schwartz function on G with certain combinations of orbital integrals of a Schwartz function on a lower dimensional group H (at first, only orbital integrals with respect to regular semisimple elements). There are two motivations for this. The first, and that which provides the formalism described in (1.2), is the manipulation of a term in the trace formula for SL_2 (cf. [8]). While the trace formula in general is not so tractable, the manipulation of Labesse and Langlands is based on the "regular elliptic" term, and has a proposed general analogue due to Langlands. Our results will support the analogue. A second motivation comes from choosing the linear combinations on H as symmetric ("stable") and thus, in a sense, as smooth as possible (for example, the unnormalized stable orbital integrals of a cusp form are smooth). A matching then reveals something of the singularities of orbital integrals. Heuristic attempts to define the group H lead essentially to that prescribed by the global trace formula considerations.

As examples show, all (allowed) combinations of orbital integrals on G are stable, so that there are only trivial matchings, precisely when there is no L-indistinguishability for G, by which we mean that each L-packet is a singleton. More generally, we will see that matching of orbital integrals plays the following role in L-indistinguishability. First we place some conditions on the normalization factors for the integrals. Then, dual to a matching, there will be a lifting of stable tempered characters on H to tempered virtual characters on G. It is not evident from definitions that this lifting follows the principle of functoriality in the L-group (that is, preserves L-packets); but that will turn out to be the case, due to a simple analogue of *coherent continuation to the wall* for admissible homomorphisms $\varphi: W \to {}^L G$. It appears difficult to divorce the lifting from orbital integrals, although coherent continuation extends it to some untempered distributions. The heart of the matter is, however, the inverse of lifting, as it will be seen to identify a representation within its L-packet [cf. (1.2)]. We find that there are dual pairings between $\mathbb{S}_\varphi$ and $\tilde{\Pi}_\varphi$ which control the inversion. Labesse and Langlands have shown this for SL_2, more significantly in the p-adic case, where Π_φ may have four elements. The pairings intervene in the multiplicity formula for certain representations in the space of cusp forms for SL_2, and thence in applications of the trace formula (cf. [12]). On restriction to tempered representations, we have then a general analogue for the component at infinity of this multiplicity formula.

(1.2) *Summary of Definitions and Results*

Again $\mathbf{G}$ is assumed simply-connected. The groups $\mathbf{H}$ have been defined [11] by means of pairs (T, κ), where T is a Cartan subgroup of G and κ is a quasicharacter on $X_*(\mathbf{T})$, the lattice generated by the coroots of $(\mathbf{G}, \mathbf{T})$, invariant under the Galois action. By Tate-Nakayama duality, etc., κ determines a function on $\mathfrak{D}(T)$, and we may form the combination

$$\Phi_f^{(T,\kappa)}(\gamma, dt, dg) = \sum_{\omega \in \mathfrak{D}(T)} \kappa(\omega) \int_{G/T^\omega} f(g\gamma^\omega g^{-1}) \frac{dg}{dt}$$

(cf. [15]). Once $\mathbf{H}$ has been attached to (T, κ), we must take account of other pairs (T', κ') determining this same group, as $\Phi_f^{(T',\kappa')}$, suitably normalized, is also to be matched. With this in mind, and looking ahead to inversion of the lifting on characters, we formulate the definition of H differently. Thus we will specify the L-group of H and distinguish an element in the center of ${}^L H$ (this is an intermediate step in the original construction). If G is not quasi-split, then new groups appear, but each is attached in the original manner to some inner form of $\mathbf{G}$. From the distinguished central element we recover pairs (T, κ). Details, and a number of simple lemmas, are given in Sect. 2. We call the groups $\mathbf{H}$ *endoscopic groups* for $\mathbf{G}$.

Let $\mathbf{H}$ be an endoscopic group for $\mathbf{G}$; ${}^L H$ is to be in "standard position" [cf. (3.2)]. The means for comparing points on H and G will be pairs (T, η), where T is a Cartan subgroup of G and η a pseudo-diagonalization of $\mathbf{T}$, or map of $\mathbf{T}$ to the distinguished maximal torus in the quasi-split form for $\mathbf{G}$ [cf. (1.3)]. A natural condition on these pairs (2.4.1) provides both (T, κ) attached to H and embeddings of T in H. We then described certain elements of H as originating in G via (T, η).

To an embedding $\xi: {}^L H \hookrightarrow {}^L G$ [see (3.3) for technical assumptions] we will attach a family $\{\Delta_{(T,\eta)}\}$, where $\Delta_{(T,\eta)}$ is a function on $T \cap G_{\text{reg}}$, with the following property:

for each $f \in \mathscr{C}(G)$ there exists $f_H \in \mathscr{C}(H)$ such that

$$\Phi_{f_H}^{(T',1)}(\gamma', dt', dh) = \begin{cases} \Delta_{(T,\eta)}(\gamma)\Phi_f^{(T,\kappa)}(\gamma, dt, dg) \text{ if } \gamma' \\ \text{originates from } \gamma \in G_{\text{reg}} \text{ via } (T, \eta), \\ 0 \text{ if } T' \text{ does not originate in } G. \end{cases}$$

Here $\mathscr{C}(\ )$ denotes the Schwartz space; see (2.4) and (3.1) concerning measures, etc. The conditions we place on $\{\Delta_{(T,\eta)}\}$ allow only for replacement by $\{-\Delta_{(T,\eta)}\}$. The factors $\Delta_{(T,\eta)}$ have been defined (essentially) up to a sign in [15, 16]; Theorem 3.5.4 takes care of the signs.

Dual to the correspondence (f, f_H) we have a well-defined lifting of stable tempered distributions (cf. [14]) to invariant tempered distributions on G; Lift $\Theta(f)=\Theta(f_H)$, $f\in\mathscr{C}(G)$. If Θ is an eigendistribution then so is Lift Θ (Lemma 4.2.1); in general, there is a shift in infinitesimal character which cannot be avoided by changing $\xi: {}^LH \hookrightarrow {}^LG$. If $\varphi'\in\Phi_0(H)$ then $\chi_{\varphi'}=\sum_{\pi\in\Pi_{\varphi'}}\chi_\pi$ is stable (cf. [14]). Note that $\varphi=\xi\circ\varphi': W\to{}^LG$ need not be relevant to G. Theorem 4.1.1 will show that

$$\operatorname{Lift}(\chi_{\varphi'})=\begin{cases}0 & \textit{if } \varphi \textit{ is not relevant to } G\\ \sum_{\pi\in\Pi_\varphi}\varepsilon(\pi)\chi_\pi & \textit{otherwise},\end{cases}$$

where each $\varepsilon(\pi)=\pm 1$ is given explicitly.

For the inversion, we state the result and then remark on our solution to a problem that was resolved differently in [8]. For each endoscopic group H we fix $\xi: {}^LH\hookrightarrow{}^LG$ and pick one of the two families $\{\Delta_{(T,\eta)}\}$ attached. Let $\varphi\in\Phi_0(G)$ and $\mathfrak{x}\in\mathbb{S}_\varphi$. Then $\mathfrak{x}$ points to some groups H and parameters $\varphi'\in\Phi_0(H)$ which lift to φ under ξ [this is illustrated in (5.2), and shown in general in (5.4)]. For a suitably chosen sign ε, $\varepsilon\chi_{\varphi'}(f_H)$ is independent of all choices except φ, f, $\mathfrak{x}$ and that described in the next paragraph. We set $\varepsilon\chi_{\varphi'}(f_H)=\hat{\chi}_{(\varphi,\mathfrak{x})}(f)$. Then there is a dual pairing $\langle\,,\rangle:\mathbb{S}_\varphi\times\bar{\Pi}_\varphi\to\{\pm1\}$ such that

$$\chi_\pi(f)=\frac{1}{[\mathbb{S}_\varphi]}\sum_{\mathfrak{x}\in\mathbb{S}_\varphi}\langle\mathfrak{x},\pi\rangle\hat{\chi}_{(\varphi,\mathfrak{x})}(f),$$

$f\in\mathscr{C}(G)$, $\pi\in\bar{\Pi}_\varphi$.

If $\pi\in\bar{\Pi}_\varphi-\Pi_\varphi$ then $\chi_\pi\equiv 0$, by definition; thus the right hand side is zero also.

The problem indicated is that of choices. As always, we fix L-group data for G [cf. (1.3)], but this does not determine the pairings or, equivalently, the inversion, as is easily seen in the case of SL_2. Our pairings will be prescribed by the choice of a set of representatives for the conjugacy classes of Cartan subgroups of G together with a pseudo-diagonalization of each group in this set, and the choice of a certain Weyl chamber depending on φ [cf. (5.4.18)].

Finally we remark that when $\mathbf{G}$ is not simply-connected, the definition of $\mathbb{S}_\varphi$ must be modified [cf. (5.1)], the lifting from H to G applies only when ${}^LH$ embeds in ${}^LG$, and the inversion when ${}^LH$ embeds in ${}^LG$, for all endoscopic groups H. The few cases where these conditions fail can be dealt with (in applications) following the suggestions of [11].

(1.3) *Notation*

We fix L-group data as usual: a quasi-split inner form $\mathbf{G}^*$ of $\mathbf{G}$ and inner twist $\psi:\mathbf{G}\to\mathbf{G}^*$, a Borel subgroup $\mathbf{B}^*$ of $\mathbf{G}^*$ defined over $\mathbb{R}$ and containing maximal torus $\mathbf{T}^*$, dual complex group ${}^LG^0$ with Borel subgroup ${}^LB^0$ containing maximal torus ${}^LT^0$ such that $X^*({}^LT^0)=X_*(\mathbf{T}^*)$. We fix a root vector $X_{\alpha^\vee}$ for each root of $({}^LG^0, {}^LT^0)$, requiring that $[X_{\alpha^\vee}, X_{-\alpha^\vee}]=H_{\alpha^\vee}$, where $H_{\alpha^\vee}$ denotes the coroot of $\alpha^\vee$ as element of the Lie algebra ${}^L\mathfrak{t}$ of ${}^LT^0$. Then σ_G denotes both the Galois action on $\mathbf{G}$

and the dual automorphism of $({}^{\mathrm{L}}G^0, {}^{\mathrm{L}}B^0, {}^{\mathrm{L}}T^0, \{X_{\alpha^\vee} : \alpha^\vee\ {}^{\mathrm{L}}B^0\text{-simple}\})$. Also ${}^{\mathrm{L}}G = {}^{\mathrm{L}}G^0 \rtimes W$, with the Weil group W acting through $\mathfrak{G} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, under the natural projection. By an abstract L-group we will mean any extension ${}^{\mathrm{L}}G^0 \rtimes W$, with ${}^{\mathrm{L}}G^0$ a connected complex algebraic group, ${}^{\mathrm{L}}B^0$, ${}^{\mathrm{L}}T^0$, $\{X_{\alpha^\vee} : \alpha^\vee\ B^0\text{-simple}\}$ as above, and W acting through an action of $\mathfrak{G}$ on $({}^{\mathrm{L}}G^0, {}^{\mathrm{L}}B^0, {}^{\mathrm{L}}T^0, \{X_{\alpha^\vee} : \alpha^\vee\ {}^{\mathrm{L}}B^0\text{-simple}\})$.

For the L-group of a Levi group (= Levi component of a parabolic subgroup of $\mathbf{G}$ defined over $\mathbb{R}$), there is a small, but crucial, point to consider. Let $\mathbf{T}$ be a maximal torus in $\mathbf{G}^*$ defined over $\mathbb{R}$, and let $\mathbf{S}_T$ denote the maximal $\mathbb{R}$-split torus in $\mathbf{T}$. We call $\mathbf{M}_T$, the centralizer of $\mathbf{S}_T$ in $\mathbf{G}^*$, and $\mathbf{T}$ *standard* if $\mathbf{S}_T \subseteqq \mathbf{T}^*$. Then $\mathbf{M}_T \supseteqq \mathbf{T}^*$ and ${}^{\mathrm{L}}M_T$ is defined in the obvious way (cf. [10, 16]). Consider, however, a torus $\mathbf{T}$ in $\mathbf{G}$; then there are several realizations for ${}^{\mathrm{L}}M_T$. We accommodate this as follows. A *pseudo-diagonalization* (p.d.) η of $\mathbf{T}$ will be a composition of maps $\mathrm{ad}\,x_1 \circ \psi : \mathbf{T} \to \mathbf{T}_1$ defined over $\mathbb{R}$, with $\mathbf{T}_1$ standard in $\mathbf{G}^*$ ($x_1 \in \mathbf{G}^*$), and $\mathrm{ad}\,m_1$, $m_1 \in \mathbf{M}_{T_1}$, mapping $\mathbf{T}_1$ to $\mathbf{T}^*$; thus η maps $\mathbf{T}$ to $\mathbf{T}^*$, the distinguished torus, and specifies a standard Levi group $\mathbf{M}_{T_1}$ in $\mathbf{G}^*$, together with an inner twist $(\mathrm{ad}\,x_1 \circ \psi)$ from $\mathbf{M}_T$ to $\mathbf{M}_{T_1}$. Given η, we set ${}^{\mathrm{L}}M_T = {}^{\mathrm{L}}M_{T_1}$. Also, $\mathbf{P}_{T_1}$ will denote the parabolic subgroup of $\mathbf{G}^*$ with Levi component $\mathbf{M}_{T_1}$ and containing $\mathbf{B}^*$, and $\mathbf{P}_T$ the parabolic subgroup (over $\mathbb{R}$) of $\mathbf{G}$ equal to $(\mathrm{ad}\,x_1 \circ \psi)^{-1}(\mathbf{P}_{T_1})$.

Let η be a p.d. of $\mathbf{T}$. When transferring data from $\mathbf{T}$ to $\mathbf{T}^*$ or ${}^{\mathrm{L}}T^0$ "via η" we use the induced maps $X^*(\mathbf{T}) \to X^*(\mathbf{T}^*) = X_*({}^{\mathrm{L}}T^0)$, $X_*(\mathbf{T}) \to X_*(\mathbf{T}^*) = X^*({}^{\mathrm{L}}T^0)$, $X^*(\mathbf{T}) \otimes \mathbb{C} \to X_*({}^{\mathrm{L}}T^0) \otimes \mathbb{C} = {}^{\mathrm{L}}T^0$, etc., as necessary and without additional comment.

In (5.4) our pairings will be based on the choice of a *skeleton* for $\mathbf{G}$: that is, a fixed set of representatives for the conjugacy classes of Cartan subgroups of G, together with a p.d. of each group in this set. *Until* (5.4), *this choice will be ignored.*

Finally, $\mathbf{G}_{\mathrm{sc}}$ will denote the simply-connected covering of the derived group of $\mathbf{G}$, and $\mathbf{T}_{\mathrm{sc}}$ the preimage in $\mathbf{G}_{\mathrm{sc}}$ of the maximal torus $\mathbf{T}$ in $\mathbf{G}$; $\Delta(\mathbf{G}, \mathbf{T})$ will denote the set of roots for $(\mathbf{G}, \mathbf{T})$ and $\Omega(\mathbf{G}, \mathbf{T})$ the Weyl group of $(\mathbf{G}, \mathbf{T})$; σ_T will denote the Galois action on $\mathbf{T}$ and $\sigma_{T,\eta}$ the transfer of σ_T to ${}^{\mathrm{L}}T^0$ via the p.d. η.

2. Endoscopic Groups

(2.1) *Data for Endoscopic Groups*

We introduce a set $\mathfrak{S}$, with the purpose of parametrizing the endoscopic groups for G.

Let s be a semi-simple element of ${}^{\mathrm{L}}G^0$. Then ${}^{\mathrm{L}}H^0$, the identity component of the centralizer in ${}^{\mathrm{L}}G^0$ of s, is a connected reductive group. We are interested in pairs (s, ϱ_s), where ϱ_s is an action of W on ${}^{\mathrm{L}}H^0$. This action is to be "realized in ${}^{\mathrm{L}}G$", and to be such that ${}^{\mathrm{L}}H^0 \rtimes W$ is an L-group. In order to make precise definitions we introduce sextuples as follows.

First, in place of a single element s, it is convenient to use a coset of Z^W, the group of W-invariant elements in the center Z of ${}^{\mathrm{L}}G^0$. Thus s now denotes a coset of Z^W in ${}^{\mathrm{L}}G^0$ consisting of semi-simple elements. We consider objects

$$(s, {}^{\mathrm{L}}H^0, {}^{\mathrm{L}}B^0_H, {}^{\mathrm{L}}T^0_H, \{Y_{\alpha^\vee}\}, \varrho_s),$$

where

(2.1.1) ${}^LH^0$ is the identity component of the centralizer in ${}^LG^0$ of any element of s, denoted $\mathrm{Cent}(s)^0$ for convenience,

(2.1.2) ${}^LB_H^0$ is a Borel subgroup of ${}^LH^0$,

(2.1.3) ${}^LT_H^0$ is the maximal torus in ${}^LB_H^0$,

(2.1.4) $\{Y_{\alpha^\vee}\}$ is a set of root vectors for the simple roots of $({}^LB_H^0, {}^LT_H^0)$, and

(2.1.5) ϱ_s is a homomorphism of W into $\mathrm{Aut}({}^LH^0, {}^LB_H^0, {}^LT^0, \{Y_{\alpha^\vee}\})$ which factors through $W \to \mathfrak{G}$ and satisfies $\varrho_s(w) = \mathrm{ad}\, n(w)|_{{}^LH^0}$, for some $n(w) \in {}^LG^0 \times w$ which fixes each element of s.

We will usually write w_H, or σ_H if $w \to \sigma$ under $W \to \mathfrak{G}$, for $\varrho_s(w)$. The following result is useful in calculations. We omit the (straightforward) proof.

Lemma 2.1.6. *If x is a semi-simple element of ${}^LG^0$ then we may build an object $(xZ^W, \ldots)$ of the form above, if and only if the conjugacy class of x in ${}^LG^0$ is invariant under W.*

A further result, which we will use repeatedly, indicates some possibilities for ϱ_s once $(s, {}^LH^0, {}^LB_H^0, {}^LT_H^0, \{Y_{\alpha^\vee}\})$ has been given. Note that to specify ϱ_s we need only give an automorphism σ_H of ${}^LH^0$ such that $\sigma_H^2 = 1$ and σ_H is of the form $\mathrm{ad}\, g|_{{}^LH_0}$, where $g \in {}^LG^0 \times (1 \times \sigma)$ fixes each element of s. For then we set $\varrho_s(w) = \sigma_H$ if w maps to σ under $W \to \mathfrak{G}$, and $\varrho_s(w) = 1$ otherwise.

Lemma 2.1.7. *Suppose that τ is an automorphism of ${}^LT^0$ such that $\tau^2 = 1$. Suppose also that τ fixes each element of s and is of the form* $\mathrm{ad}\, g|_{{}^LT_0}$, $g \in {}^LG^0 \times (1 \times \sigma)$. *Then there is a unique automorphism τ_H of $({}^LH^0, {}^LB_H^0, {}^LT_H^0, \{Y_\alpha\})$ satisfying $\tau = (\tau_H|_{{}^LT_0}) \cdot \omega$, for some $\omega \in \Omega({}^LH^0, {}^LT^0)$. Further, $\tau_H^2 = 1$ and τ_H is of the form* $\mathrm{ad}\, g'|_{{}^LH^0}$ *for some $g' \in {}^LG^0 \times (1 \times \sigma)$ fixing s* pointwise.

Again the proof is straightforward, and details are omitted.

We call $(s_1, {}^LH_1^0, {}^LB_{H_1}^0, {}^LT_{H_1}^0, \{Y_{\alpha^\vee}\}, \varrho_{s_1})$ and $(s_2, {}^LH_2^0, {}^LB_{H_2}^0, {}^LT_{H_2}^0, \{Z_{\beta^\vee}\}, \varrho_{s_2})$ *equivalent* if there exists $g \in {}^LG^0$ such that ${}^LH_2^0 = g\,{}^LH_1^0 g^{-1}$, ${}^LB_{H_2}^0 = g\,{}^LB_{H_1}^0 g^{-1}$, ${}^LT_{H_2}^0 = g\,{}^LT_{H_1}^0 g^{-1}$, $Z_{\mathrm{Ad}\, g(\alpha^\vee)} = \mathrm{Ad}\, g(Y_{\alpha^\vee})$ and $\varrho_{s_2}(w) = \mathrm{ad}\, g \circ \varrho_{s_1}(w) \circ \mathrm{ad}\, g^{-1}$, $w \in W$. It is *not* required that $s_2 = g s_1 g^{-1}$. A typical equivalence class will be denoted $\mathfrak{s}$; later [cf. (2.4)] we will also use $\mathfrak{s}$ to denote a representative.

Finally, $\mathfrak{S}$ is the set of all equivalence classes of objects $(s, {}^LH^0, {}^LB_H^0, {}^LT_H^0, \{Y_{\alpha^\vee}\}, \varrho_s)$. Clearly $\mathfrak{S}$ depends only on ${}^LG$; that is, $\mathfrak{S}$ is attached to the family of inner forms of **G**, rather than to **G** alone. We write $\mathfrak{S} = \mathfrak{S}({}^LG)$.

(2.2) *Endoscopic Groups*

Let $\mathfrak{s} \in \mathfrak{S}$. The last five entries in a representative for a $\mathfrak{s}$ define an abstract L-group. Clearly then, $\mathfrak{s}$ determines a unique isomorphism class of L-groups. Any L-group in this class is said to be attached to $\mathfrak{s}$.

Definition 2.2.1. *Let* **H** *be a quasi-split group over* $\mathbb{R}$. *Then* **H** *is an endoscopic group for* **G** *if ${}^LH$ is attached to some $\mathfrak{s} \in \mathfrak{S}({}^LG)$.*

Let $\mathbf{H}$ be an endoscopic group for $\mathbf{G}$. We fix $\mathbf{B}_H$, a Borel subgroup of $\mathbf{H}$ defined over $\mathbb{R}$, and let $\mathbf{T}_H$ denote the maximal torus in $\mathbf{B}_H$. Let $\mathfrak{s}_H$ be the element of $\mathfrak{S}(^LG)$ defined by $\mathbf{H}$. We fix a representative $(s_H, {}^LH)$ for $\mathfrak{s}_H$, denoted also by $\mathfrak{s}_H$. The group ${}^LH$ will serve as fixed L-group for $\mathbf{H}$. Later [cf. (3.3)] we will place restrictions on the choice of pair $(s_H, {}^LH)$ but for the present we require only that

(2.2.2) $$ {}^LT_H^0 = {}^LT^0 . $$

Then $s_H \subset {}^LT^0$ and

(2.2.3) $$ X_*(\mathbf{T}_H) = X^*({}^LT^0) = X_*(\mathbf{T}^*) . $$

(2.3) *(T,κ)-Pairs*

Let $\mathscr{K}(G)$ be the set of all pairs (T,κ), where $\mathbf{T}$ is a maximal torus in $\mathbf{G}$ defined over $\mathbb{R}$ and κ is a quasicharacter on

$$ X_*(\mathbf{T}_{sc})/\{\sigma_T\mu^\vee - \mu^\vee : \mu^\vee \in X_*(\mathbf{T})\} \cap X_*(\mathbf{T}_{sc}) . $$

The construction of "$H(T,\kappa,\ )$" in [11] provides a map $\mathscr{K}(G) \to \mathfrak{S}(^LG)$ as follows.

Fix a pseudo-diagonalization η of $\mathbf{T}$. Then η transfers κ to a quasicharacter on $X_*(\mathbf{T}^*_{sc})$; this new quasicharacter lifts, in several ways, to a quasicharacter on $X_*(\mathbf{T}^*) = X^*({}^LT^0)$ invariant under the action of $\mathfrak{G}$ obtained by transferring the Galois action on $\mathbf{T}$ to $\mathbf{T}^*$ via η. These various extension may be viewed as elements of ${}^LT^0$; as such, they form a coset of Z^W (cf. [16, Sect. 2.1]), say s. We now form $(s, {}^LH^0, {}^LB^0 \cap H^0, {}^LT^0, \{Y_{\alpha^\vee}\}, \varrho_s)$, where ${}^LH^0 = \mathrm{Cent}(s)^0$ [cf. (2.1.1)], $Y_{\alpha^\vee}$ is the root vector for $\alpha^\vee$ already chosen [cf. (1.3)], and ϱ_s remains to be defined. Let $\sigma_{T,\eta}$ denote the automorphism of ${}^LT^0$ obtained by transferring the Galois action on $\mathbf{T}$ by η. Then $\sigma_{T,\eta}$ fixes each element of s and is of the form $\mathrm{ad}\, g|_{{}^LT_0}$, $g \in {}^LG^0 \times (1 \times \sigma)$. Since $\sigma_{T,\eta}^2 = 1$, we may now apply Lemma 2.1.7 to define σ_H and hence ϱ_s.

Thus to each $(T,\kappa) \in \mathscr{K}(G)$ and p.d. η of $\mathbf{T}$ there is attached a representative for a class in $\mathfrak{S}$. If η is replaced by another p.d. then this representative is replaced by an equivalent one (cf. [11]). Therefore to each $(T,\kappa) \in \mathscr{K}(G)$ we have attached a well-defined element $\mathfrak{s} = \mathfrak{s}(T,\kappa)$ of $\mathfrak{S}(^LG)$.

Proposition 2.3.1. $\mathfrak{s}(T,\kappa) = \mathfrak{s}(T',\kappa')$ *if and only if there exists* $g \in \mathbf{G}$ *such that* $\mathrm{ad}\, g : \mathbf{T} \to \mathbf{T}'$ *and*

(i) $\kappa(\alpha^\vee) = 1$ *if and only if* $\kappa'(g\alpha^\vee) = 1$, $\alpha^\vee \in \Delta^\vee(\mathbf{G},\mathbf{T})$,

(ii) $\sigma_G(g^{-1})g \in \Omega^{(\kappa)}(\mathbf{G},\mathbf{T})$, *the subgroup of* $\Omega(\mathbf{G},\mathbf{T})$ *generated by the reflections with respect to those roots* α *for which* $\kappa(\alpha^\vee) = 1$.

Proof. This is an easy application of the definitions.

Proposition 2.3.2. *If* $\mathbf{G}$ *is quasi-split over* $\mathbb{R}$ *then the map* $(T,\kappa) \to \mathfrak{S}(T,\kappa)$ *of* $\mathscr{K}(G)$ *to* $\mathfrak{S}(^LG)$ *is surjective.*

Proof. This follows from Theorem 1.7 of [18].

Lemma 2.3.3. $\mathfrak{S}(^LG)$ *is a finite set.*

Proof. We write $\mathfrak{s} \in \mathfrak{S}(^LG)$ as $\mathfrak{s}(T,\kappa)$, for some $(T,\kappa) \in \mathscr{K}(G^*)$. Proposition 2.3.1 shows that there are only a finite number of possibilities for $\mathfrak{s}(T,\kappa)$.

For any group $\mathbf{G}$, we define (T,κ) and (T',κ') to be stably equivalent if there exists $\omega \in \mathfrak{A}(T)$ such that $\omega : \mathbf{T} \to \mathbf{T}'$ and $\kappa' = \kappa^\omega$ (cf. [15, Sect. 3]). We denote by

$\mathscr{K}_{st}(G)$ the set of equivalence classes under this relation, and by $\langle T,\kappa\rangle$ the equivalence class of (T,κ). By Proposition 2.3.1, we may set $\mathfrak{s}(\langle T,\kappa\rangle)=\mathfrak{s}((T,\kappa))$ to obtain a well-defined map of $\mathscr{K}_{st}(G)$ into $\mathfrak{S}({}^{L}G)$.

On the other hand, the twist $\psi:\mathbf{G}\to\mathbf{G}^*$ provides embeddings over $\mathbb{R}$ into $\mathbf{G}^*$ of the maximal tori over $\mathbb{R}$ in $\mathbf{G}$ (cf. [10]). Thus ψ induces a map of $\mathscr{K}_{st}(G)$ into $\mathscr{K}_{st}(G^*)$. Clearly this map is injective.

As supplement to Proposition 2.3.2, we observe that

$$\begin{array}{ccc} \mathscr{K}_{st}(G) & \hookrightarrow & \mathscr{K}_{st}(G^*) \\ & \searrow \quad \swarrow & \\ & \mathfrak{S}({}^{L}G) & \end{array}$$

is commutative.

Finally, if, as usual, $\mathbf{G}_{sc}$ is the simply-connected covering group of the derived group of $\mathbf{G}$, then there are natural inclusions $\mathscr{K}(G)\hookrightarrow\mathscr{K}(G_{sc})$ and $\mathfrak{S}({}^{L}G)\hookrightarrow\mathfrak{S}({}^{L}(G_{sc}))$. Moreover, the diagram

$$\begin{array}{ccc} \mathscr{K}(G) & \hookrightarrow & \mathscr{K}(G_{sc}) \\ \downarrow & & \downarrow \\ \mathfrak{S}({}^{L}G) & \hookrightarrow & \mathfrak{S}({}^{L}(G_{sc})) \end{array}$$

is commutative.

(2.4) *Elements of H Originating in G*

Let $\mathbf{H}$ be an endoscopic group for $\mathbf{G}$, with $\mathfrak{s}_H=(s_H,{}^{L}H)$ fixed as in (2.2). Let $\mathscr{K}_H(G)$ be the set of all pairs (T,κ) which map to (the equivalence class of) $\mathfrak{s}_H$ under $\mathscr{K}(G)\to\mathfrak{S}({}^{L}G)$. We recover some elements of $\mathscr{K}_H(G)$ as follows. Let $\mathscr{T}_H(G)$ be the set of pairs (T,η), where T is a Cartan subgroup of G and η is a p.d. of T with the property that the transfer $\sigma_{T,\eta}$ of the Galois action on $\mathbf{T}$ to ${}^{L}T^0$ via η satisfies

(2.4.1) $$\sigma_{T,\eta}=\omega\sigma_H,\quad \text{some}\quad \omega\in\Omega({}^{L}H^0,{}^{L}T^0).$$

Then for $(T,\eta)\in\mathscr{T}_H(G)$, the transfer of s_H to $\mathbf{T}$ via η defines a quasicharacter κ on

$$X_*(\mathbf{T}_{sc})/\{\sigma_T\mu^\vee-\mu^\vee:\mu^\vee\in X_*(\mathbf{T})\}\cap X_*(\mathbf{T}_{sc}),\quad\text{and}\quad (T,\kappa)\in\mathscr{K}_H(G).$$

In this manner, we obtain sufficiently many elements of $\mathscr{K}_H(G)$ for our purposes, as the next lemma will show. In the statement of the lemma we allow "p.d. η of $\mathbf{T}$" to mean *any* map $\eta:\mathbf{T}\to\mathbf{T}^*$ of the form $\operatorname{ad}x\circ\psi$, $x\in\mathbf{G}^*$. [In a final version of the matching theorem for orbital integrals (cf. Sect. 3) we may omit further assumptions on η.]

Lemma 2.4.2. *Suppose that $(T,\kappa)\in\mathscr{K}_H(G)$. Then there exists a p.d. η of T such that $(T,\eta)\in\mathscr{T}_H(G)$ and the attached quasicharacter agrees with κ on*

$$\mathscr{E}(T)=\{\lambda^\vee\in X_*(\mathbf{T}_{sc}):\lambda^\vee+\sigma_T\lambda^\vee=0\}/X_*(\mathbf{T}_{sc})\cap\{\mu^\vee-\sigma_T\mu^\vee:\mu^\vee\in X_*(\mathbf{T})\}.$$

Proof. We may choose η so that $(T,\eta)\in\mathscr{T}_H(G)$ and the representative attached to T,η and κ is $(s',{}^{L}H)$. Then both s' and s_H are fixed by σ_H and $\sigma_{T,\eta}$. Let $\alpha^\vee$ be a root of $({}^{L}G^0,{}^{L}T^0)$ such that $\sigma_{T,\eta}\alpha^\vee=-\alpha^\vee$. Then:

(*) $$\alpha^\vee(s')=\alpha^\vee(s_H)=\pm1\,.$$

By construction κ is the quasicharacter attached to T, η and s'. Let κ_H be the quasicharacter attached to T, η and s_H. Then $(*)$ implies that κ and κ_H coincide on the span of the coroots $\alpha^\vee$ of $(\mathbf{G}, \mathbf{T})$ for which $\sigma_T \alpha^\vee = -\alpha^\vee$. The lemma then follows.

Notation. If $(T, \eta) \in \mathscr{T}_H(G)$ then κ denotes the attached quasicharacter, unless otherwise stated.

Suppose that $(T, \eta) \in \mathscr{T}_H(G)$. Then by [11] (cf. [15, Sect. 6]), there exists $h \in \mathbf{H}$ such that

(2.4.3) $$\mathbf{T}' = h^{-1} \mathbf{T}_H h \quad \text{is defined over } \mathbb{R},$$

and

(2.4.4) $$X_*(\mathbf{T}') \xrightarrow{\operatorname{ad} h} X_*(\mathbf{T}_H) = X_*(\mathbf{T}^*) \xrightarrow{\eta^{-1}} X_*(\mathbf{T})$$

commutes with Galois action and so lifts to an isomorphism $i(h, \eta): \mathbf{T}' \to \mathbf{T}$ defined over $\mathbb{R}$.

Let G_{reg} be the set of regular elements in G. Then $\gamma' \in H$ *originates from* $\gamma \in G_{\text{reg}}$ *via* $(T, \eta) \in \mathscr{T}_H(G)$ if γ' is the preimage of γ under some such map $i(h, \eta)$.

Proposition 2.4.5. *Suppose that $\gamma' \in H$ originates from $\gamma \in G_{\text{reg}}$ via (T, η). Then:*

(i) $\gamma' \in H_{\text{reg}}$,

(ii) *$\gamma'' \in H$ also originates from γ via (T, η) if and only if $\gamma'' = (\gamma')^{\omega'}$, $\omega' \in \mathfrak{A}_H(T')$, and*

(iii) *γ' originates from $\bar{\gamma}$ via (T, η) if and only if $\bar{\gamma} = \gamma^\omega$, with $\omega \in \Omega_0(\mathbf{G}, \mathbf{T}) \cap \Omega^{(\kappa)}(\mathbf{G}, \mathbf{T})$. As usual, $\Omega_0(\mathbf{G}, \mathbf{T}) = \{\omega \in \Omega(\mathbf{G}, \mathbf{T}) : \omega\sigma = \sigma\omega\}$; $\Omega^{(\kappa)}(\mathbf{G}, \mathbf{T})$ was defined in Proposition 2.3.1.*

Again the proof is straightforward (cf. [15, Sect. 6]).

Finally, we will say that a Cartan subgroup T' of H originates from a Cartan subgroup T of G if $\mathbf{T}'$ is the preimage of $\mathbf{T}$ under some $i(h, \eta)$. Note that then to each Haar measure dt on T there is then attached a unique Haar measure dt' on T'.

3. Orbital Integrals

(3.1) *(G, H)-Orbital-Integral-Transfer Factors*

We continue with G and endoscopic group H; $\mathfrak{s}_H = (s_H, {}^L H)$ is fixed as in (2.2). The Schwartz spaces of G, H will be denoted $\mathscr{C}(G)$, $\mathscr{C}(H)$ respectively. We continue also with $\mathscr{T}_H(G)$.

Suppose that for each $(T, \eta) \in \mathscr{T}_H(G)$ we are given a function $\Delta_{(T, \eta)}$ on $T \cap G_{\text{reg}}$. Then we call the family $\{\Delta_{(T, \eta)}\}$ of these functions *a set of (G, H)-orbital-integral transfer factors* (or, more briefly, "transfer factors") if for each $f \in \mathscr{C}(G)$ there exists $f' \in \mathscr{C}(H)$ such that

(3.1.1) $$\Phi_{f'}^{(T', 1)}(\gamma', dt', dh) = \begin{cases} \Delta_{(T, \eta)}(\gamma) \Phi_f^{(T, \kappa)}(\gamma, dt, dg) \text{ if } \gamma' \\ \text{originates from } \gamma \in G_{\text{reg}} \text{ via } (T, \eta) \\ 0 \text{ if } T' \text{ does not originate in } G. \end{cases}$$

The measures dh, dg on H, G respectively are subject only to the conditions of [15, Sect. 9]; dt is arbitrary and dt' as in (2.4); $\Phi_f^{(T,\kappa)}(\ ,\ ,\ )$ and $\Phi_{f'}^{(T',1)}(\ ,\ ,\ )$ are as in [15], although T, T' are now inserted in the notation.

Because of our proposed application (cf. Sect. 4), we place some restrictions on our candidates for transfer factors. First, for each (T,η) we define a function $\grave{\Delta}_{(T,\eta)}$ on T by

$$\grave{\Delta}_{(T,\eta)}(\gamma)=\prod_{\substack{\alpha>0\\ \sigma\alpha=-\alpha\\ \kappa(\alpha^\vee)\neq 1}}(1-\alpha(\gamma^{-1}))\prod_{\substack{\alpha>0\\ \sigma\alpha\neq-\alpha\\ \kappa(\alpha^\vee)\neq 1}}|\alpha(\gamma)^{1/2}-\alpha(\gamma)^{-1/2}|$$

where $\alpha>0$ means that $\alpha\in\mathbf{\Delta}(\mathbf{G},\mathbf{T})$ is positive with respect to the order determined by η and the Borel subgroup $\mathbf{B}^*$ of $\mathbf{G}^*$; $|\alpha(\gamma)^{1/2}-\alpha(\gamma)^{-1/2}|$ is to be interpreted as $|1-\alpha(\gamma)|^{1/2}|1-\alpha(\gamma^{-1})|^{1/2}$.

Proposition 3.1.2. *Let* $\omega\in\Omega_0(\mathbf{G},\mathbf{T})\cap\Omega^{(\kappa)}(\mathbf{G},\mathbf{T})$. *Then*

(i) $\Phi_f^{(T,\kappa)}(\gamma^\omega,dt,dg)=\kappa(\omega)\Phi_f^{(T,\kappa)}(\gamma,dt,dg)$, *and*

(ii) $\grave{\Delta}_{(T,\eta)}(\gamma^\omega)=\kappa(\omega)(\iota_*-\omega^{-1}\iota_*)(\gamma)\grave{\Delta}_{(T,\eta)}(\gamma)$, *where*

$$\iota_*=1/2\sum_{\substack{\alpha>0\\ \sigma\alpha=-\alpha\\ \kappa(\alpha^\vee)\neq 1}}\alpha.$$

Proof. Note that $\iota_*-\omega^{-1}\iota_*\in X^*(\mathbf{T})$, so that $(\iota_*-\omega^{-1}\iota_*)(\gamma)$ is well-defined. The proof of (i) is immediate (cf. [15, Sect. 4]). The proof of (ii) is similar to that of Lemma 8.2 of [15], and we omit the details.

We will call a family $\{\Delta_{(T,\eta)}\}$ of (G,H)-orbital-integral transfer factors *admissible* if each $\Delta_{(T,\eta)}$ is of the form

$$(3.1.3)\qquad (-1)^{q(G,H)}\varepsilon(T,\eta)A_{(T,\eta)}\grave{\Delta}_{(T,\eta)}$$

where $q(G,H)$ is a constant [defined in (3.7) and inserted only for convenience], $\varepsilon(T,\eta)=\pm 1$, and $A_{(T,\eta)}$ is a character on T satisfying

$$(3.1.4)\qquad A_{(T,\eta)}(\gamma^\omega)=(\omega^{-1}\iota_*-\iota_*)(\gamma)A_{(T,\eta)}(\gamma),$$

$\omega\in\Omega_0(\mathbf{G},\mathbf{T})\cap\Omega^{(\kappa)}(\mathbf{G},\mathbf{T})$.

If $\{A_{(T,\eta)}\}$ is any family of characters satisfying (3.1.4) and $\{\varepsilon(T,\eta)\}$ any choice of signs, then the family $\{\Delta_{(T,\eta)}\}$ defined by (3.1.3) has the property that

$$\gamma'\to\Delta_{(T,\eta)}(\gamma)\Phi_f^{(T,\kappa)}(\gamma,dt,dg),$$

for γ' *originating from*

$$\gamma\in G_{\mathrm{reg}}\,via\,(T,\eta),$$

is well-defined and invariant under $\mathfrak{A}(T'_{\gamma'})$, where $T'_{\gamma'}$ is the Cartan subgroup of H containing γ' (cf. Propositions 2.4.5 and 3.1.2). This function depends, however, on the choice of (T,η). Suppose that γ' originates from $\gamma\in G_{\mathrm{reg}}$ via (T,η) and from $\bar{\gamma}\in G_{\mathrm{reg}}$ via $(\bar{T},\bar{\eta})$. Then clearly there is $\omega\in\mathfrak{A}(T)$ mapping T to $\bar{T}$ and γ to $\bar{\gamma}$, and satisfying $\bar{\eta}=\omega_H\circ\eta\circ\omega^{-1}$ for some $\omega_H\in\Omega(\mathbf{H},\mathbf{T}_H)\subset\Omega(\mathbf{G}^*,\mathbf{T}^*)$. Thus $\bar{T}=T^\omega$; $\bar{\kappa}$, the quasicharacter attached to $(\bar{T},\bar{\eta})$, is κ^ω and $\Phi_f^{(\bar{T},\bar{\kappa})}(\bar{\gamma},d\bar{t},dg)=\kappa(\omega)\Phi_f^{(T,\kappa)}(\gamma,dt,dg)$, where $d\bar{t}$ is the measure $(dt)^\omega$. Note that if dt' is attached to dt on T, then dt' is attached to $(dt)^\omega$ on $\bar{T}$.

On the other hand,

$$\begin{aligned}{}^{`}\Delta_{(\bar{T},\bar{\eta})}(\bar{\gamma}) &= {}^{`}\Delta_{(T^{\omega},\,\omega_H\circ\eta\circ\omega^{-1})}(\gamma^{\omega}) \\ &= {}^{`}\Delta_{(T,\,\omega_H\circ\eta)}(\gamma) \\ &= (-1)^{[R(\omega_H)]}\Big(\sum_{\alpha\in R(\omega_H)}\alpha\Big)(\gamma)\,{}^{`}\Delta_{(T,\eta)}(\gamma),\end{aligned}$$

where $R(\omega_H)=\{\alpha\in\mathbf{\Delta}(\mathbf{G},\mathbf{T}):\sigma_T\alpha=-\alpha,\quad \kappa(\alpha^{\vee})\neq 1,\quad \tilde{\alpha}=\eta\alpha\in\mathbf{\Delta}(\mathbf{B}^*,\mathbf{T}^*),\ \omega_H\tilde{\alpha}\in-\mathbf{\Delta}(\mathbf{B}^*,\mathbf{T}^*)\}$. Hence for $\{\Delta_{(T,\eta)}\}$ to be a family of transfer factors we must have

$$(3.1.5)\qquad \begin{cases} \varepsilon(T,\omega_H\circ\eta\circ\omega^{-1})=\kappa(\omega)(-1)^{[R(\omega_H)]}\varepsilon(T,\eta) \\ \textit{and} \\ \Delta_{(T,\,\omega_H\circ\eta\circ\omega^{-1})}=\Big(\Big(\sum_{\alpha\in R(\omega_H)}\alpha\Big)^{-1}\Delta_{(T,\eta)}\Big)^{\omega}\end{cases}$$

for all $\omega_H\in\Omega(\mathbf{H},\mathbf{T}_H)$ and $\omega\in\mathfrak{A}(T)$.

Conversely, we may use these conditions to calculate within a fixed "framework of Cartan subgroups" (cf. [15, 16]). This is the only motivation for all but the first paragraph of (3.2).

(3.2) *Standard Position, Framework of Cartan Subgroups, etc.*

Let $\mathbf{H}$ be an endoscopic group for $\mathbf{G}$. Fix a standard Cartan subgroup of G^*, say T_N, among those from which T_H originates. Let $\mathbf{M}_N=\mathbf{M}_{T_N}$ and σ_N denote the (canonical) transfer of σ_{T_N} to ${}^{\mathrm{L}}T^0$ by an element of $\mathrm{ad}\,\mathbf{M}_N$. We now insist that $\mathfrak{s}_H=(s_H,{}^{\mathrm{L}}H)$ have the following properties:

$$(3.2.1)\qquad {}^{\mathrm{L}}T_H^0={}^{\mathrm{L}}T^0$$

and

$$(3.2.2)\qquad \sigma_H \text{ acts on } {}^{\mathrm{L}}T^0 \text{ as } \sigma_N.$$

Then ${}^{\mathrm{L}}H$ is "in standard position with respect to T_N" (cf. [16, Sect. 2]). That such a representative $\mathfrak{s}_H$ exists is shown in [16].

Next we build "a framework of Cartan subgroups around T_N" (cf. [16]). Thus we select standard Cartan subgroups $T_0',\dots,T_N'=T_H$ of H representing all conjugacy classes, and standard Cartan subgroups $T_0,\dots,T_N$ *of* G^* such that T_n' originates from T_n, $n=0,1,\dots,N$ (in definitions concerning G^* we take the inner twist ψ to be the identity). We assume that $\mathbf{S}(T_n)\subsetneqq\mathbf{S}(T_N)$ for all n and that $T_n=T_p$ if T_n is conjugate to T_p. Set $M_n'=M_{T_n'}$ and $M_n=M_{T_n}$. Then we choose $m_n'\in\mathbf{M}_n'$, $m_n\in\mathbf{M}_n$ such that $\mathrm{ad}\,m_n':\mathbf{T}_n'\to\mathbf{T}_H=\mathbf{T}^*$ (over $\mathbb{C}$), $\mathrm{ad}\,m_n:\mathbf{T}_n\to\mathbf{T}^*$ (over $\mathbb{C}$) and $i_n=\mathrm{ad}\,m_n^{-1}\circ\mathrm{ad}\,m_n':\mathbf{T}_n'\to\mathbf{T}_n$ over $\mathbb{R}$. Let σ_n be the transfer to ${}^{\mathrm{L}}T^0$, via $\mathrm{ad}\,m_n$, of the Galois action on $\mathbf{T}_n$ (equivalently, the transfer of the Galois action on $\mathbf{T}_n'$ via $\mathrm{ad}\,m_n'$). If T_n' originates in G we select T_n^G in G and a map $\psi_n=\mathrm{ad}\,x_n\circ\psi:\mathbf{T}_n^G\to\mathbf{T}_n$ over $\mathbb{R}$. Set $\mathbf{M}_n^G=\mathbf{M}_{T_n^G}$, $\eta_n^G=\mathrm{ad}\,m_n\circ\psi_n:\mathbf{T}_n^G\to\mathbf{T}^*$ and $i_n^G=\psi_n\circ i_n:\mathbf{T}_n'\to\mathbf{T}_n^G$. We refer to [16, Sect. 2] for a proof that all these choices are possible. Note that (2.3.3, d) of [16] is omitted; this is because we now have the element $\mathfrak{s}_H$.

Since $\sigma_n \in \Omega({}^L H^0, {}^L T^0)\sigma_H$, we have $(T_n, \mathrm{ad}\, m_n) \in \mathcal{T}_H(G^*)$; let κ_n be the attached quasicharacter. Also $(T_n^G, \eta_n^G) \in \mathcal{T}_H(G)$; κ_n^G will be the attached quasicharacter.

Proposition 3.2.3. *Suppose that $\gamma' \in H$ originates from $\gamma \in G_{\mathrm{reg}}$ via (T, η). Then $T = (T_n^G)^\omega$, and $\eta = \omega_H \circ \eta_n^G \circ \omega^{-1}$, for some n, $\omega_H \in \Omega(\mathbf{H}, \mathrm{T}_H)$ and $\omega \in \mathfrak{A}(T^{G_n})$.*

Proof. This is immediate.

If $\{\Delta_{(T,\eta)}\}$ is a family of functions satisfying (3.1.3) and (3.1.5) (that is, $\{\Delta_{(T,\eta)}\}$ is a proposed admissible family of transfer factors) set

$$\varepsilon_n^G = \varepsilon(T_n^G, \eta_n^G),$$

$$\Lambda_n^G = \Lambda_{(T_n^G, \eta_n^G)},$$

$$`\Delta_n^G = `\Delta_{(T_n^G, \eta_n^G)}.$$

Proposition 3.2.4. $\{\Delta_{(T,\eta)}\}$, *satisfying* (3.1.3) *and* (3.1.5), *is an admissible family of (G, H)-orbital-integral-transfer factors if and only if for each $f \in \mathscr{C}(G)$ there exists $f' \in \mathscr{C}(H)$ such that*

$$\Phi_{f'}^{(T_n', 1)}(\gamma', dt', dh) = (-1)^{q(G,H)} \varepsilon_n^G \Lambda_n^G(\gamma) `\Delta_n^G(\gamma) \Phi_f^{(T_n^G, \kappa_n^G)}(\gamma, dt, dg)$$

if $\gamma \in G_{\mathrm{reg}}$ and $\gamma' = (i_n^G)^{-1}(\gamma)$, and $\Phi_{f'}^{(T', 1)}(\,,) \equiv 0$ if T' does not originate in G.

Proof. This follows from Proposition 2.3.3 and the discussion in (3.1).

Our next step is to recall some candidates for $\{\Lambda_n^G\}$ (cf. [16]).

(3.3) *Correction Characters*

We assume now that $\mathfrak{s}_H = (s_H, {}^L H)$, with ${}^L H$ in standard position with respect to some standard Cartan subgroup T_N of G^*; we also fix a framework of Cartan subgroups around T_N. By an *admissible embedding of ${}^L H$ in ${}^L G$* we mean a homomorphism $\xi: {}^L H \to {}^L G$ such that

(3.3.1) $\quad \xi|_{{}^L H^0 \times 1}$ is the inclusion mapping and

(3.3.2) $\quad \xi(1 \times w) = \xi_0(w) \times w, w \in W, \quad$ where $\quad \xi_0(w) \in {}^L G^0$.

These maps have been studied in [16].

Let ξ be an admissible embedding of ${}^L H$ in ${}^L G$. Then $\xi_0(1 \times \sigma) \in {}^L M_N^0$ since ${}^L H$ is in standard position with respect to T_N (cf. [16]), and $\xi_0(\mathbb{C}^\times \times 1) \subset {}^L T^0$. As in [16], we attach to ξ the pair (μ^*, λ^*), where

$$\lambda^\vee(\xi_0(z \times 1)) = z^{\langle \mu^*, \lambda^\vee \rangle} \bar{z}^{\langle \sigma_H \mu^*, \lambda^\vee \rangle}, \qquad \lambda^\vee \in X^*({}^L T^0)$$

and

$$\lambda^\vee(\xi_0(1 \times \sigma)) = e^{2\pi i \langle \lambda^*, \lambda^\vee \rangle}, \qquad \lambda^\vee \in X^*({}^L M_N^0).$$

We write $\xi = \xi(\mu^*, \lambda^*)$. If ι_n denotes one-half the sum of the coroots for $({}^L B^0 \cap {}^L M_n^0, {}^L T^0)$ and ι_n' one-half the sum of the coroots for $({}^L B^0 \cap {}^L (M_n')^0, {}^L T^0)$ then

$$1/2(\mu^* - \sigma_n \mu^*) + \iota_n - \iota_n' \equiv (\lambda^* + \sigma_n \lambda^*)(\mathrm{mod}\, X_*({}^L T^0))$$

and $\mu^* - \sigma_n \mu^* \in \mathrm{X}_*({}^L T^0)$, $n = 0, 1, \ldots, N$ (Theorem 3.4.1 of [16]). Thus, after transfer by m_n, we have a well-defined quasicharacter $\chi(\mu^* + \iota_n - \iota_n', \lambda^*)$ on T_n (cf. [16, Sect. 4.1]). If T_n' originates in G we have further a quasicharacter $\chi^G(\mu^* + \iota_n - \iota_n', \lambda^*)$ on T_n^G.

We say that ξ is *of unitary type* if each $\chi(\mu^* + \iota_n - \iota_n', \lambda^*)$ is a character (that is, is unitary). If $\xi = \xi(\mu^*, \lambda^*)$ is arbitrary then $1/2(\mu^* - \sigma_H \mu^*)$, λ^* are also parameters for an admissible embedding; clearly this embedding is of unitary type.

We assume now that ${}^L H$ embeds admissibly in ${}^L G$ and fix an embedding $\xi = \xi(\mu^*, \lambda^*) : {}^L H \hookrightarrow {}^L G$ of unitary type. We set

$$\Lambda_n^G = \chi^G(\mu^* + \iota_n - \iota_n', \lambda^*). \tag{3.3.3}$$

Then $\{\Lambda_n^G\}$ is a "set of correction characters" in the sense of [16] (this is the main result of [16]). Hence if we set

$$\Delta_n^G = (-1)^{q(G,H)} \varepsilon_n^G \Lambda_n^G {}' \Delta_n^G$$

and define $\{\Delta_{(T,\eta)}\}$ by setting

$$\Delta_{(T_n^G, \eta_n^G)} = \Delta_n^G$$

and requiring that (3.1.5) be satisfied then $\{\Delta_{(T,\eta)}\}$ will be a set of (G,H)-orbital-integral-transfer factors if and only if $\{\varepsilon_n^G\}$ satisfies the conditions of (10.1) in [15] (with ε_n^G replacing ε_n). Before discussing these conditions, and showing that they can be satisfied, we introduce the set-up for some reduction arguments.

(3.4) *Endoscopic Groups for the Levi Components of a Parabolic Subgroup*

Suppose that $\mathbf{H}$ is an endoscopic group for $\mathbf{G}$, with $\mathfrak{s}_H = (s_H, {}^L H)$ chosen so that ${}^L H$ is in standard position with respect to a (standard) Cartan subgroup $\bar{T}$ of G^*.

It will be sufficient for our purpose to consider cuspidal Levi groups. Thus we take $\mathbf{M} = \mathbf{M}_T$, where T is a Cartan subgroup of G. We fix a p.d. η of $\mathbf{T}$ [cf. (1.3)]. Then $\mathbf{M}^*$ will denote the attached Levi group in $\mathbf{G}^*$, and $\mathbf{P}^*$ in $\mathbf{G}^*$, $\mathbf{P}$ in $\mathbf{G}$ the attached parabolic subgroups; ${}^L M = {}^L M^*$, by definition. Our basic assumption will be:

$$\sigma_H = \omega \sigma_{T,\eta}, \quad \text{some} \quad \omega \in \Omega({}^L M^0 \cap {}^L H^0, {}^L T^0). \tag{3.4.1}$$

We may then construct a subgroup $\mathbf{M}'$ of $\mathbf{H}$ as follows. Let ${}^L(M')^0 = {}^L H^0 \cap {}^L M^0$. Then (3.4.1) implies that ${}^L(M')^0$ is invariant under σ_H. Set ${}^L M' = {}^L(M')^0 \rtimes W$. Then ${}^L M'$ is the L-group of a subgroup $\mathbf{M}'$ of $\mathbf{H}$ defined over $\mathbb{R}$; $\mathbf{M}'$ contains $\mathbf{T}_H$ and is a Levi component of some parabolic subgroup $\mathbf{P}'$ of $\mathbf{H}$ defined over $\mathbb{R}$ and containing $\mathbf{B}_H$. A simple argument shows that there exists $m' \in \mathbf{M}'$ such that $\mathbf{T}' = (m')^{-1} \mathbf{T}_H m'$ and $i(m', \eta) : \mathbf{T}' \to \mathbf{T}$ are defined over $\mathbb{R}$. Thus $\mathbf{M}' = \mathbf{M}_{T'}$.

If, in the notation of the last section, we have $\bar{T} = T_N$ then we may take $T = T_n^G$ and $\eta = \eta_n^G$, so that $\mathbf{M} = \mathbf{M}_n^G$ and $\mathbf{M}^* = \mathbf{M}_n$. Then $\mathbf{M}' = \mathbf{M}_n'$.

Let s_H^M be the coset of s_H with respect to the group of W-invariants in the center of ${}^L M^0$. Then $\mathfrak{s}_H^M = (s_H^M, {}^L M')$ is a representative for some element of $\mathfrak{s}({}^L M)$. Thus $\mathbf{M}'$ is an endoscopic group for $\mathbf{M}$. Further, (3.4.1) and a simple argument show that $\bar{\mathbf{T}} \subset \mathbf{M}^*$; ${}^L M'$ is in standard position with respect to $\bar{T}$. Note that if $\xi = \xi(\mu^*, \lambda^*) : {}^L H \hookrightarrow {}^L G$ is an admissible embedding of unitary type then $\xi^M = \xi|_{{}^L M'}$ is an admissible embedding of ${}^L M'$ in ${}^L M$, again of unitary type and with parameters (μ^*, λ^*).

Next, let $(U, \eta_U) \in \mathscr{T}_H(G)$, with attached quasicharacter κ. We say that (U, η_U) *is subordinate to* M if $U \subset M$ and η_U is of the form $\operatorname{ad} m \circ \eta$, $m \in \mathbf{M}^*$. Then clearly

$(U,\eta_U)\in\mathscr{T}_{M'}(M)$ [relative to the twist from $\mathbf{M}$ to $\mathbf{M}^*$ provided by η, cf. (1.3)]. If κ_M denotes the restriction of κ to $\{\mu^\vee\in X_*(\mathbf{T}_{sc}):\mathrm{Nm}\,\mu^\vee=\mu^\vee+\sigma_T\mu^\vee=0\}$, then κ_M is the quasicharacter attached to (U,η_U) as an element of $\mathscr{T}_{M'}(M)$. Conversely, if $(U,\eta_U)\in\mathscr{T}_{M'}(M)$ then as an element of $\mathscr{T}_H(G)$, (U,η_U) is subordinate to M.

If $f\in\mathscr{C}(G)$ then we define $f_M\in\mathscr{C}(M)$ by

$$f_M(m)=(\delta_P(m))^{1/2}\int_N \bar{f}(mn)dn \text{ and } \bar{f}(x)=\int_K f(kxk^{-1})dk,$$

$x\in G$. Here K is a suitably chosen maximal compact subgroup of G (cf. [3, Sect. 3]), $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$ and δ_P is the modular function of P... note that f_M is the function "$f^{(P)}$" of [3]. The measures dk and dn are fixed (arbitrarily). Let dm be the measure on M for which $dg=dk\,dn\,dm$. Then

$$\Phi_f^{(U,\kappa)}(\gamma,du,dg)=\prod_{\substack{\alpha>0\\ \sigma\alpha\neq-\alpha}}|\alpha(\gamma)^{1/2}-\alpha(\gamma)^{-1/2}|^{-1}\Phi_{f_M}^{(U,\kappa_M)}(\gamma,du,dm)$$

for $\gamma\in U\cap M_{\mathrm{reg}}$ and (U,η_U) subordinate to M. Similarly we may define $f'_{M'}\in\mathscr{C}(M')$, given $f'\in\mathscr{C}(H)$. Then

$$\Phi_{f'}^{(U',1)}(\gamma',du',dh)=\prod_{\substack{\alpha'>0\\ \sigma\alpha'\neq-\alpha'}}|\alpha'(\gamma')^{1/2}-\alpha'(\gamma')^{-1/2}|^{-1}\Phi_{f'_{M'}}^{(U',1)}(\gamma',du',dm')$$

for $U'\subset M'$ and $\gamma'\in U'\cap H_{\mathrm{reg}}$.

If (U,η_U) is subordinate to M and $\{\Delta_{(\,,\,)}\}$ satisfies (3.1.3) we set

$$\varepsilon_M(U,\eta_U)=(-1)^{q(G,H)-q(M,M')}\varepsilon(U,\eta_U)$$

and

$$\Delta^M_{(U,\eta_U)}=\frac{\varepsilon_M(U,\eta_U)\Lambda_{(U,\eta_U)}\,{}'\Delta_{(U,\eta_U)}}{\prod\limits_{\substack{\alpha>0\\ \sigma\alpha\neq-\alpha\\ \kappa(\alpha^\vee)\neq1}}|\alpha^{1/2}-\alpha^{-1/2}|}.$$

Then the following result is immediate.

Proposition 3.4.2. *If $\{\Delta_{(\,,\,)}\}$ is a set of (G,H)-orbital-integral-transfer factors then $\{\Delta^M_{(U,\eta_U)}:(U,\eta_U)$ is subordinate to $M\}$ defines a set of (M,M')-orbital-integral-transfer factors. Moreover, if $f'\in\mathscr{C}(H)$ corresponds to $f\in\mathscr{C}(G)$ relative to $\{\Delta_{(\,,\,)}\}$ then $f_{M'}\in\mathscr{C}(M')$ corresponds to $f_M\in\mathscr{C}(M)$ relative to $\{\Delta^M_{(\,,\,)}\}$.*

(3.5) *Choice of Signs*

We continue the discussion of (3.3). We wish to choose $\{\varepsilon_n^G\}$ so that the set $\{\Delta_{(T,\eta)}\}$ defined in (3.3) is a family of (G,H)-orbital-integral-transfer factors. Our first step will be to review the conditions this places on $\{\varepsilon_n^G\}$ (cf. [15]).

Let T'_m and T'_n be adjacent Cartan subgroups of H, within our framework (cf. [15]). Let I_m^+ and I_n^+ denote the (chosen) systems of positive imaginary roots for $T_m^G=i_m^G(T'_m)$ and $T_n^G=i_n^G(T'_n)$. Then in [15] there is attached to $(T'_m,T'_n,i_m^G,i_n^G,I_m^+,I_n^+)$ a sign $\varepsilon(m,n)=\varepsilon_+(m,n)\varepsilon_\kappa(m,n)$ (cf. proof of Proposition 3.5.3 below). In order that $\{\Delta_{(T,\eta)}\}$ be a family of (G,H)-orbital-integral-transfer factors it is necessary and sufficient that

(3.5.1) $$\varepsilon_m^G\varepsilon_n^G=\varepsilon(m,n)$$

for each adjacent pair (T'_m, T'_n) (cf. Theorem 10.2 of [15]). If $\{\varepsilon_n^G\}$ satisfies these conditions then so does $\{-\varepsilon_n^G\}$, but there are no further possibilities.

Suppose that both T'_{n_1}, T'_{n_2} succeed T'_m in H, and that T'_p succeeds both T'_{n_1} and T'_{n_2} (in the sense of [15]). Then in order to construct $\{\varepsilon_n^G\}$ satisfying (3.5.1) we need:

(3.5.2) $$\varepsilon(m, n_1)\varepsilon(n_1, p) = \varepsilon(m, n_2)\varepsilon(n_2, p).$$

Of more interest is the sufficiency of these conditions; that is, it is sufficient that we verify (3.5.2) for all "diamonds" $(T'_m, T'_{n_1}, T'_{n_2}, T'_p)$ (cf. [15, Sect. 11]).

Let α'_i be a noncompact root of T'_m for which there is a Cayley transform s'_i; $\mathbf{T}'_m \to \mathbf{T}'_{n_i}$ $(i=1,2)$, and β'_i be a noncompact root of $\mathbf{T}'_{n_i}$ for which there is a Cayley transform $t'_i : \mathbf{T}'_{n_i} \to \mathbf{T}'_p$. Let α_i, β_i, s_i, t_i be the corresponding "images in $\mathbf{G}$" [15]. We may change our choices for T_m, T_{n_i}, T_p (within respective conjugacy classes), α'_i, s'_i, β'_i and t'_i without changing $\varepsilon(m, n)$, $\varepsilon(n_i, p)$ [15, Sect. 10]. It will also be convenient to change i_m^G, $i_{n_i}^G$, $i_{n_2}^G$ and i_p^G; that is, to adjust the various *images in* $\mathbf{G}$.

Proposition 3.5.3. *$\varepsilon(m, n_1)\varepsilon(n_1, p)\varepsilon(m, n_2)\varepsilon(n_2, p)$ does not depend on the choices for $i_m^G, i_{n_1}^G, i_{n_2}^G, i_p^G$.*

Proof. We may replace i_m^G only by ad $\omega_m \circ i_m^G$, $\omega_m \in \mathfrak{A}(T_m^G)$, $i_{n_1}^G$ by ad $\omega_{n_1} \circ i_{n_1}^G$, etc. By definition, $\varepsilon(m, n) = \varepsilon_\kappa(m, n)\varepsilon_+(m, n)$, and $\varepsilon_\kappa(m, n) = \varepsilon_{\kappa_m^G}(s_1)$, the κ_m^G*-signature* of s_1. The change in i_m^G, $i_{n_1}^G$ replaces s_1 by $\omega_{n_1} s_1 \omega_m^{-1}$. A simple calculation shows that $\varepsilon_{\kappa_m^G}(s_1)$ is replaced by $\kappa_{n_1}^G(\omega_{n_1})\varepsilon_{\kappa_m^G}(s_1)\kappa_m^G(\omega_m)$. Arguing similarly for s_2, t_1, and t_2 we find that $\varepsilon_\kappa(m, n_1)\varepsilon_\kappa(n_1, p)\varepsilon_\kappa(m, n_2)\varepsilon_\kappa(n_2, p)$ is multiplied by

$$\kappa_p^G(\omega_p)\kappa_{n_1}^G(\omega_{n_1})\kappa_{n_1}^G(\omega_{n_1})\kappa_m^G(\omega_m)\kappa_p^G(\omega_p)\kappa_{n_2}^G(\omega_{n_2})\kappa_{n_2}^G(\omega_{n_2})\kappa_m^G(\omega_m)$$

which equals one. It is immediate from the definition of $\varepsilon_+(,)$ (cf. [15, Sect. 10]) that the above change in i_m and i_{n_1} multiplies $\varepsilon_+(m, n_1)$ by $\mathrm{sgn}(\omega_m)$ $\mathrm{sgn}(\omega_{n_1})$, where sgn denotes the signature with respect to (the appropriate system of) imaginary roots. Hence the proposition follows.

Theorem 3.5.4. *For all $(T'_m, T'_{n_1}, T'_{n_2}, T'_p)$ we have*

(3.5.5) $$\varepsilon(m, n_1)\varepsilon(n_1, p) = \varepsilon(m, n_2)\varepsilon(n_2, p).$$

Proof. We observe first that it is sufficient to prove this in the case that $\mathbf{G}$ is simply-connected and simple and T_m^G is anisotopic over $\mathbb{R}$. Indeed, given any diamond $(T'_m, T'_{n_1}, T'_{n_2}, T'_p)$ we may assume that $(T_{n_1}^G, \eta_{n_1}^G)$, (T_p^G, η_p^G) are subordinate to M_m^G. We may also assume that T'_{n_i}, T'_p lie in M'_m. We see that the terms we have to compute are then the same, whether we compute in (G, H) or in (M_m^G, M'_m). Thus it is sufficient to verify (3.5.5) in the case that T_m^G is compact modulo the center of G. Now we argue as in [15, Sect. 11] to reduce to the case that $\mathbf{G}$ is simply-connected and simple.

From now on, $\mathbf{G}$ is simply-connected and simple, and T_m^G is compact. We will omit the superscript "G" in notation.

We consider first the case that $T_{n_1} = T_{n_2}$. Then $\alpha_2 = \omega\alpha_1$, $\omega \in \Omega(\mathbf{G}, \mathbf{T}_m)$. Hence α'_1, α'_2 have the same length. Because T'_{n_1} is not conjugate to T'_{n_2}, α'_1 and α'_2 lie in different components of $\Delta(\mathbf{H}, \mathbf{T}'_m)$. Thus $\langle\langle \alpha'_1, \alpha'_2 \rangle\rangle$, the set of roots in $\Delta(\mathbf{H}, \mathbf{T}'_m)$ belonging to the $\mathbb{Q}$-span of α'_1 and α'_2, is of type $A_1 \times A_1$. On the other hand,

$\langle\!\langle\alpha_1,\alpha_2\rangle\!\rangle$, the set of roots of $\mathbf{\Delta}(\mathbf{G},\mathbf{T}_m)$ belonging to the $\mathbb{Q}$-span of α_1 and α_2, is either of type C_2 or of type $A_1\times A_1$.

Suppose that $\langle\!\langle\alpha_1,\alpha_2\rangle\!\rangle$ is of type C_2. If α_1 and α_2 are long, then $\alpha_1^\vee$ and $\alpha_2^\vee$ are short and $\alpha_1^\vee+\alpha_2^\vee\in\mathbf{\Delta}^\vee(\mathbf{G},\mathbf{T}_m)$. Since $\kappa_m(\alpha_1^\vee+\alpha_2^\vee)=\kappa_m(\alpha_1^\vee)\kappa_m(\alpha_2^\vee)=1$, we have that $(\alpha_1^\vee+\alpha_2^\vee)^\vee\in\mathbf{\Delta}(\mathbf{G},\mathbf{T}_m)$ "comes from $\mathbf{H}$" (cf. [16, Sect. 2.4]), so that $1/2(\alpha'_1+\alpha'_2)$ is a root of $(\mathbf{H},\mathbf{T}'_m)$, a contradiction. We conclude then that α_1, α_2 are short. In $\mathbf{H}$ we may arrange that s'_1, s'_2 are standard, $\beta'_2=s'_2\alpha'_1$, $\beta'_1=s'_1\alpha'_2$, t'_1 and t'_2 are standard, and $t'_1s'_1=t'_2s'_2$. Then we adjust i_m so that α_1 is noncompact. Suppose that α_2 is compact. Then adjust i_{n_1} so that s_1 is standard and i_{n_2} so that $s_2=s_1\omega_{\alpha_1-\alpha_2}$. Finally, adjust i_p so that t_1 is standard. Then $t_2=\omega t_1$, $\omega\in\Omega_0(\mathbf{G},\mathbf{T}_p)$. Since $t_1s_1=t_2s_2$, we have that $\omega=t_1s_1\omega_{\alpha_1-\alpha_2}(t_1s_1)^{-1}$. Thus

$$\varepsilon_\kappa(m,n_1)\varepsilon_\kappa(n_1,p)\varepsilon_\kappa(m,n_2)\varepsilon_\kappa(n_2,p)=\kappa_p(\omega)\kappa_m(\omega_{\alpha_1-\alpha_2}).$$

Since $t_1s_1(\alpha_1-\alpha_2)$ is a real root we have $\kappa_p(\omega)=1$, and since $\alpha_1-\alpha_2$ is a noncompact root of $(\mathbf{G},\mathbf{T}_m)$ "not from $\mathbf{H}$", we have $\kappa_m(\omega_{\alpha_1-\alpha_2})=\kappa_m((\alpha_1-\alpha_2)^\vee)=-1$. On the other hand.

$$\varepsilon_+(m,n_1)\varepsilon_+(n_1,p)\varepsilon_+(m,n_2)\varepsilon_+(n_2,p)=\operatorname{sgn}\omega\operatorname{sgn}\omega_{\alpha_1-\alpha_2}.$$

Since ω is the reflection with respect to a real root, ω fixes the imaginary roots of $\mathbf{T}_p$ and $\operatorname{sgn}\omega=1$; $\operatorname{sgn}\omega_{\alpha_1-\alpha_2}=-1$. Thus (3.5.5) holds if α_2 is compact. Suppose now that α_2 is noncompact. Perform a standard Cayley transform with respect to α_1. Then the image $\tilde{\alpha}_2$ of α_2 is compact, since the subgroup of $\mathbf{G}$ generated by $\mathbf{T}_m$ and the roots α_1, α_2 must be of type Sp(2,2). There is $\tilde{\omega}$ in the imaginary Weyl group of the image of $\mathbf{T}$ such that $\tilde{\omega}\tilde{\alpha}$ is noncompact, because there exist Cayley transforms with respect to $\tilde{\alpha}_1$ (cf. [14, Lemma 4.11]). Let ω be the preimage of $\tilde{\omega}$ in $\Omega(\mathbf{G},\mathbf{T}_m)$. Then $\omega\alpha_1=\alpha_1$ and $\omega\alpha_2$ is compact. Thus for the case α_2 noncompact (after we have adjusted i_m to obtain α_1 noncompact) we may further adjust i_m to obtain α_1 noncompact and α_2 compact. Then our previous argument applies, and the case $\langle\!\langle\alpha_1,\alpha_2\rangle\!\rangle$ of type C_2 is disposed of.

Suppose now that $\langle\!\langle\alpha_1,\alpha_2\rangle\!\rangle$ is of type $A_1\times A_1$. An argument similar to that of the last paragraph shows that we may adjust i_m so that both α_1 and α_2 are noncompact. Then we may adjust i_{n_1},i_{n_2},i_p so that s_1,s_2,t_1 and $t_2=t_1s_1s_2^{-1}$ are each standard. The product of all terms $\varepsilon_\kappa(,)$ is one. To compute the product of all terms $\varepsilon_+(,)$ we observe that if $\omega\in\Omega(\mathbf{G},\mathbf{T}_m)$ permutes α_1 and α_2 then the product coincides with $\operatorname{sgn}\omega\operatorname{sgn}\tilde{\omega}$ where $\tilde{\omega}$ is the automorphism of $\{\alpha\in\mathbf{\Delta}(\mathbf{G},\mathbf{T}_m):\langle\alpha,\alpha_1^\vee\rangle=\langle\alpha,\alpha_2^\vee\rangle=0\}$ obtained by restricting ω to this system (cf. [15, paragraph following Lemma 11.3]). If we inspect the various irreducible reduced root systems we find that such an ω exists and that $\operatorname{sgn}\omega\operatorname{sgn}\tilde{\omega}$, which does not depend on the choice of ω, is one. Thus (3.5.5) follows. This completes the proof for the case $T_{n_1}=T_{n_2}$.

Suppose now that T_{n_1} and T_{n_2} are not conjugate. Recall that T_m is compact and $\mathbf{G}$ simple. Thus α_1,α_2 must be of different length. We may therefore assume that $\mathbf{G}$ is of type B_n, C_n, F_4 or G_2. The case of type G_2 has been dealt with in [15, Sect. 11]. Thus we will assume that $\mathbf{G}$ is of type B_n, C_n or F_4. Also, α'_2 will be the long root.

Suppose that α'_1, α'_2 are not perpendicular. We may assume that $\alpha'_2=\alpha'_1+(s'_1)^{-1}\beta'_1$, $\beta'_2=s'_2(\alpha'_1-(s'_1)^{-1}\beta'_1)$ and that $t'_1s'_1=t'_2s'_2$. Then adjust i_m so that α_2 is noncompact; if α_1 is then compact we may further adjust i_m to obtain both α_1,

α_2 noncompact. Then we adjust i_{n_1} so that s_1 is standard. This implies that β_1 is noncompact (arguing in Sp_4). Next we adjust i_p so that t_1 is standard, and i_{n_2} so that s_2 is standard. Then β_2 is noncompact (arguing again in Sp_4) and $t_2 = t_1 s_1 s_2^{-1}$ is of the form ωt_2^{st}, where t_2^{st} is standard and ω belongs to the subgroup of $\Omega(\mathbf{G}, \mathbf{T}_p)$ generated by the Weyl reflections with respect to the (real) roots α_1, α_2. Hence

$$\varepsilon_\kappa(m, n_1)\varepsilon_\kappa(n_1, p)\varepsilon_\kappa(m, n_2)\varepsilon_\kappa(n_2, p) = 1 .$$

On the other hand, we compute the product of all terms $\varepsilon_+(,)$ as in the paragraph following Lemma 11.3 of [15]. Explicit calculation shows that the product is one.

Finally, suppose that α'_1, α'_2 are perpendicular. Then both $\langle\langle \alpha'_1, \alpha'_2 \rangle\rangle$ and $\langle\langle \alpha_1, \alpha_2 \rangle\rangle$ are of type $A_1 \times A_1$. We can argue as earlier (case α'_1, α'_2 of same length) to show that the product of all terms $\varepsilon_\kappa(,)$ is one. Explicit calculation shows that the product of all terms $\varepsilon_+(,)$ is one also.

This completes the proof of the theorem.

(3.6) *Conclusions*

We have now attached to each admissible embedding $\xi : {}^L H \hookrightarrow {}^L G$ of unitary type, two admissible families of (G, H)-orbital-integral-transfer factors. If $\{\Delta_{(\Gamma, \eta)}\}$ is one family, then $\{-\Delta_{(T, \eta)}\}$ is the other.

Note that, by [16], if $\mathbf{G}$ is quasi-split then $\xi, \eta : {}^L H \hookrightarrow {}^L G$ have attached the same two families of transfer factors if and only if ξ and η are Φ-equivalent, that is, if and only if ξ, η induce the same map $\Phi(H) \to \Phi(G)$ on L-packet parameters (cf. Sect. 4).

We remark that the condition that ξ be of unitary type is only for convenience. An arbitrary admissible embedding $\xi : {}^L H \hookrightarrow {}^L G$ provides quasicharacters Λ_n^G [cf. (3.3.3)], in place of characters. However, by [16, Sect. 9], we may find a quasicharacter on H, say Λ_ξ, so that $\Lambda_n^G = \Lambda_\xi \Lambda_n^*$ for each n, where we have transferred Λ_ξ to T_n^G without change in notation and $\{\Lambda_n^*\}$ is a set of unitary correction characters. Thus the families attached to $\{\Lambda_n^G\}$ as in (3.3) will match the κ-orbital integrals of $f \in \mathscr{C}(G)$ with the stable orbital integrals of a function f' on H such that $\Lambda_\xi^{-1} f' \in \mathscr{C}(H)$ (clearly such a function has well-defined orbital integrals).

(3.7) *Definition of* $q(G, H)$

We begin with the assumption that the fundamental Cartan subgroups of H originate in G; then the set of Cartan subgroups "shared by G and H" is nonempty. Suppose that the fundamental Cartan subgroup T' of H originates from T in G. Let τ' be the dimension of the maximal compact subgroups of $M_{T'}$, and τ be the dimension of the maximal compact subgroups of M_T. Then we set

$$q(G, H) = 1/2(\tau - \tau') .$$

Clearly $q(G, H)$ does not depend on the choice of T' and T.

Proposition 3.7.1. (i) *$q(G, H)$ is an integer.*

(ii) *If $H = G^*$ then $q(G, H) = 1/2(\tau_G - \tau_H) = q_G - q_H$, where τ_G is the dimension of the maximal compact subgroups of G and $2q_G$ is the dimension of the symmetric space attached to G_{sc}.*

Note that (ii) reconciles $q(G, H)$ with the choice in [14].

Proof. For (i), let a be the dimension of the derived group of $\mathbf{M}_T$, b be the number of positive roots of $(\mathbf{M}_T, \mathbf{T})$ and c be the dimension of the center of $\mathbf{M}$; define a', b', and c' similarly, with respect to $(\mathbf{M}_{T'}, \mathbf{T}')$. Then $a=2b-c+\dim\mathbf{T}$ and $a'=2b'-c'+\dim\mathbf{T}$. Hence $b-b'=1/2((a+c)-(a'+c'))$ and so $(b-b')-1/2(\tau-\tau')=q_{M_T}-q_{M_{T'}}$ in the notation of (ii), since the split components of the centers of $\mathbf{M}_T$ and $\mathbf{M}_{T'}$ are isomorphic over $\mathbb{R}$. But q_{M_T} and $q_{M_{T'}}$ are integers since $\mathbf{M}_T$ and $\mathbf{M}_{T'}$ are cuspidal. Hence (i) follows. Also, (ii) follows from this discussion and Lemma 2.8 of [14].

Corollary 3.7.2. $q(G,H)=q(G,G^*)+q(G^*,H)$.

If the fundamental Cartan subgroups of H do not originate in G we set

$$q(G,H)=q(G,G^*)+q(G^*,H),$$

both numbers on the right-hand side of this equation being well-defined.

4. Lifting Characters

Throughout this section $\mathbf{H}$ will be an endoscopic group for $\mathbf{G}$, with $\mathfrak{s}_H=(s_H, {}^{\mathrm{L}}H)$ fixed so that ${}^{\mathrm{L}}H$ is in standard position (with respect to some standard Cartan subgroup of G^*). We assume that ${}^{\mathrm{L}}H$ embeds admissibly in ${}^{\mathrm{L}}G$ and fix $\xi=\xi(\mu^*,\lambda^*):{}^{\mathrm{L}}H\hookrightarrow{}^{\mathrm{L}}G$ of unitary type. Then $\{\Delta_{(T,\eta)}\}$ will be one of the two admissible families of (G,H)-orbital-integral-transfer factors attached to ξ. As in Sect. 3, we write $\Delta_{(T,\eta)}$ as

$$(-1)^{q(G,H)}\varepsilon(T,\eta)\Lambda_{(T,\eta)}{}^{`}\Delta_{(T,\eta)}$$

where $\varepsilon(T,\eta)=\pm 1$, $\Lambda_{(T,\eta)}$ is a character on T and ${}^{`}\Delta_{(T,\eta)}$ is the "discriminant" function of (3.1).

The family $\{\Delta_{(T,\eta)}\}$ determines a correspondence (f,f') between the Schwartz spaces $\mathscr{C}(G)$ and $\mathscr{C}(H)$. Let Θ' be a stable tempered distribution (cf. [14, Sect. 5]) on H and set

$$\Theta(f)=\Theta'(f'), \quad f\in\mathscr{C}(G).$$

Lemma 4.0.1. *Θ is a well-defined invariant tempered distribution on G.*

The proof is straightforward (cf. Proposition 6.1 of [14]); we omit the details.

We call Θ *the lift of Θ' to G.*

(4.1) *The Lift of a Stable Tempered Character*

Let $\Phi_0(H)$ be the set of parameters for the L-packets of infinitesimal equivalence classes of tempered irreducible admissible representations of H; if $\varphi'\in\Phi_0(H)$ then $\Pi_{\varphi'}$ will denote the attached L-packet. In [14] we have established that the tempered distribution

$$\chi_{\varphi'}=\sum_{\pi\in\Pi_{\varphi'}}\chi_\pi$$

is stable. Our main purpose now is to compute the lift of $\chi_{\varphi'}$ to G.

Let φ be the image of φ' under the map $\xi^\Phi:\Phi_0(H)\to\Phi_0(G)$ induced by our fixed embedding $\xi:{}^{\mathrm{L}}H\hookrightarrow{}^{\mathrm{L}}G$ (cf. [16]). We will say that φ *is relevant to* G if φ lies in the subset $\Phi_0(G)$ of $\Phi_0(G^*)$.

Theorem 4.1.1. (i) *If φ is relevant to G then the lift of $\chi_{\varphi'}$ is $\sum_{\pi\in\Pi_\varphi} \varepsilon(\pi)\chi_\pi$, where each $\varepsilon(\pi)$ is a number, either ± 1 or -1, as defined below* [*cf.* (4.4.3), (4.4.10), *and* (4.5.1)].

(ii) *If φ is not relevant to G then the lift of $\chi_{\varphi'}$ is zero.*

Corollary 4.1.2. *In the notation of the theorem, $\sum_{\pi\in\Pi} \varepsilon(\pi)\chi_\pi$ vanishes on the Cartan subgroups of G not originating in H.*

The proof of the theorem will occupy the next several subsections.

(4.2) *The Lift of a Stable Tempered Eigendistribution*

It will be useful to know that the lift of an eigendistribution is an eigendistribution. More precisely, let $\mathscr{Z}'$ be the center of the universal enveloping algebra of $\mathfrak{h}$, and $\mathscr{Z}$ be that for $\mathfrak{g}$. Then we define a homomorphism $\mathscr{Z}\to\mathscr{Z}'$ as follows. Identify the symmetric algebra on the Lie algebra $\mathfrak{t}$ of a torus $\mathbf{T}$ with the universal enveloping algebra $\mathfrak{U}(\mathfrak{t})$ of $\mathfrak{t}$. Let $\mathfrak{U}(\mathfrak{t})^\Omega$ denote the set of invariants in $\mathfrak{U}(\mathfrak{t})$ of the (appropriate) Weyl group Ω. There is a canonical isomorphism $\mathfrak{U}(\mathfrak{t}^*)\to\mathfrak{U}(\mathfrak{t}_H)$ inducing a map $\mathfrak{U}(\mathfrak{t}^*)^{\Omega(\mathbf{G}^*,\mathbf{T}^*)}\to\mathfrak{U}(\mathfrak{t}_H)^{\Omega(\mathbf{H},\mathbf{T}_H)}$, and canonical isomorphisms $\mathfrak{U}(\mathfrak{t})^{\Omega(\mathbf{G},\mathbf{T})}\to\mathfrak{U}(\mathfrak{t}^*)^{\Omega(\mathbf{G}^*,\mathbf{T}^*)}$, $\mathfrak{U}(\mathfrak{t}')^{\Omega(\mathbf{H},\mathbf{T}')}\to\mathfrak{U}(\mathfrak{t}_H)^{\Omega(\mathbf{H},\mathbf{T}_H)}$, for any maximal tori $\mathbf{T}$ in $\mathbf{G}$, $\mathbf{T}'$ in $\mathbf{H}$. Let $\Gamma:\mathscr{Z}\to\mathfrak{U}(\mathfrak{t}^*)^{\Omega(\mathbf{G}^*,\mathbf{T}^*)}$ and $\Gamma':\mathscr{Z}'\to\mathfrak{U}(\mathfrak{t}_H)^{\Omega(\mathbf{H},\mathbf{T}_H)}$ be the Harish-Chandra isomorphisms. The element $\mu^*\in X_*({}^LT^0)\otimes\mathbb{C}$ provided by $\xi=\xi(\mu^*,\lambda^*)$ is naturally an element of $\mathrm{Hom}(\mathfrak{t}^*,\mathbb{C})$ and so induces an isomorphism I_{μ^*} of $\mathfrak{U}(\mathfrak{t}^*)$; $I_{\mu^*}(X)=X+\mu^*(X)I$, $X\in\mathfrak{t}^*$. Since $\langle\mu^*,\alpha^\vee\rangle=0$, $\alpha^\vee\in\Delta({}^LH^0,{}^LT^0)$, I_{μ^*} preserves $\mathfrak{U}(\mathfrak{t}_H)^{\Omega(\mathbf{H},\mathbf{T}_H)}$. We will use $z\to z'$ to denote the homomorphism $(\Gamma')^{-1}\circ I_{\mu^*}\circ\Gamma$ of $\mathscr{Z}$ into $\mathscr{Z}'$. If χ' is a character on $\mathscr{Z}'$ then we may define a character χ on by

$$\chi(z)=\chi'(z'),\qquad z\in\mathscr{Z}.$$

If χ' is attached to the $\Omega(\mathbf{H},\mathbf{T}_H)$-orbit of the linear functional μ' on $\mathfrak{t}_H$ then χ is attached to the $\Omega(\mathbf{G}^*,\mathbf{T}^*)$-orbit of $\mu=\mu'+\mu^*$, on our identifying $\mathfrak{t}_H$ with $\mathfrak{t}^*$.

Lemma 4.2.1. *If $z'\Theta'=\chi'(z')\Theta'$, $z'\in\mathscr{Z}'$, then $z\Theta=\chi(z)\Theta$, $z\in\mathscr{Z}$.*

Proof. Let $f\in\mathscr{C}(G)$ and suppose that $f'\in\mathscr{C}(H)$ corresponds to f under our fixed (G,H)-orbital-integral transfer. Then it is sufficient to prove that $(z')^*f'$ corresponds to z^*f. Here * denotes the adjoint; note that $z\to z'$ does not respect this operation. The correspondence is verified by a simple modification of the argument in the proof of Lemma 6.2 in [14]. We omit the details.

There is then an analytic class function F_Θ on G_{reg} such that

$$\Theta(f)=\sum_T\frac{1}{[\Omega(G,T)]}\int_{T\cap G_{\mathrm{reg}}} F_\Theta(\gamma)\Phi_f(\gamma,dt,dg)J_G^T(\gamma)\,dt \tag{4.2.2}$$

for $f\in\mathscr{C}(G)$, where $J_G^T(\gamma)=\prod_{\alpha\in\Delta(\mathbf{G},\mathbf{T})}|1-\alpha(\gamma^{-1})|$ and the summation extends over a set of representatives for the conjugacy classes of Cartan subgroups of G (cf. [3, Sect. 13]). The integrals on the right-hand side of (4.2.2) are absolutely convergent. To describe F_Θ in terms of $F_{\Theta'}$ it is sufficient to give a formula for F_Θ on $T\cap G_{\mathrm{reg}}$.

Proposition 4.2.3. *If no Cartan subgroup of H originates from T then the restriction of F_Θ to $T\cap G_{\mathrm{reg}}$ is zero.*

The proof will be included with that of the next lemma.

Suppose now that ${}^{(1)}T', \ldots, {}^{(R)}T'$ represent the conjugacy classes of Cartan subgroups of H originating from T. Fix $(T, \eta_v) \in \mathscr{T}_H(G)$ with attached κ_v, and $i_v = i(h_v, \eta_v): {}^{(v)}\mathbf{T}' \to \mathbf{T}$ defined over $\mathbb{R}$, $v = 1, \ldots, R$. We set

$$F_{\Theta'}^{(v)}(\gamma) = (F_{\Theta'} \circ i_v^{-1})(\gamma), \qquad \gamma \in T \cap G_{\text{reg}},$$

and

$$\Delta_v^{H/G}(\gamma) = \chi(\mu^* - \iota_*^{(v)}, \lambda^*)(\gamma) \prod_{\substack{\alpha > 0 \\ \sigma\alpha = -\alpha \\ \kappa_v(\alpha^\vee) \neq 1}} (1 - \alpha(\gamma^{-1}))^{-1} \prod_{\substack{\alpha > 0 \\ \sigma\alpha \neq -\alpha \\ \kappa_v(\alpha^\vee) \neq 1}} |\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}|^{-1}$$

where $\alpha > 0$ means that α belongs to the positive system for $\Delta(\mathbf{G}, \mathbf{T})$ determined by (η_v, B^*), $\iota_*^{(v)} = 1/2 \sum_{\substack{\alpha > 0 \\ \sigma\alpha = -\alpha \\ \kappa_v(\alpha^\vee) \neq 1}} \alpha$ and μ^*, λ^* denote the transfer to $X^*(\mathbf{T}) \otimes \mathbb{C}$ by η_v of the data μ^*, λ^* provided by our fixed embedding $\xi: {}^L H \hookrightarrow {}^L G$. Finally, we fix v and set

$$n(T) = \#\{\alpha \in \Delta(\mathbf{G}, \mathbf{T}): \alpha > 0, \sigma\alpha = -\alpha, \kappa_v(\alpha^\vee) \neq 1\};$$

$n(T)$ does not depend on the choice of v among $1, \ldots, R$.

Lemma 4.2.4. *For $\gamma \in T \cap G_{\text{reg}}$ we have*

$$F_\Theta(\gamma) = (-1)^{q(G,H) + n(T)} \sum_{v=1}^{R} \varepsilon(T, \eta_v) \sum_{\omega \in \Omega_0(\mathbf{G}, \mathbf{T})/\Omega^{(\kappa_v)}(\mathbf{G}, \mathbf{T})} \kappa_v(\omega) \Delta_v^{H/G}(\gamma^\omega) F_{\Theta'}^{(v)}(\gamma^\omega).$$

Recall that $\Omega_0(\mathbf{G}, \mathbf{T}) = \{\omega \in \Omega(\mathbf{G}, \mathbf{T}): \omega\sigma = \sigma\omega\}$ and that $\Omega_0^{(\kappa_v)}(\mathbf{G}, \mathbf{T}) = \Omega_0(\mathbf{G}, \mathbf{T}) \cap \Omega^{(\kappa_v)}(\mathbf{G}, \mathbf{T})$, where $\Omega^{(\kappa_v)}(\mathbf{G}, \mathbf{T})$ is the subgroup of $\Omega(\mathbf{G}, \mathbf{T})$ generated by the reflections with respect to the roots α of $(\mathbf{G}, \mathbf{T})$ for which $\kappa_v(\alpha^\vee) = 1$.

Proof. Let $f \in C_0^\infty((T \cap G_{\text{reg}})^G)$. We may calculate $\Theta'(f')$ as

$$\sum_{T'} \frac{1}{[\Omega_0(\mathbf{H}, \mathbf{T}')]} \int_{T' \cap H_{\text{reg}}} F_{\Theta'}(\gamma') \Phi_{f'}^{(T',1)}(\gamma', dt', dh) J_H^{T'}(\gamma') dt',$$

where the summation extends over a set of representatives for the conjugacy classes of Cartan subgroups of H. If T' does not originate in G then the integral over $T' \cap H_{\text{reg}}$ vanishes by our definition of (G, H)-orbital integral transfer. If T' does originate in G, but from a Cartan subgroup not conjugate to T, then the integral over $T' \cap H_{\text{reg}}$ again is zero, by definition of f. Thus Proposition 4.2.3 is proved. For Lemma 4.2.4, we may replace $\sum_{T'}$ by $\sum_{(v)T'}$; we then use $\{i_v\}$ to obtain an integral over $T \cap G_{\text{reg}}$. Our (G, H)-orbital-integral transfer and some elementary manipulations give the formula asserted. We omit the details, except to note that

$$(-1)^{q(G,H) + n(T)} \varepsilon(T, \eta_v) \Delta_v^{H/G} = \Delta_{(T, \eta_v)} J_H^{(v)T'} (J_G^T)^{-1},$$

[cf. (3.1.3), (3.3.3), and (3.1.5)].

(4.3) *Some Parameters for Tempered L-Packets*

We recall some definitions concerning L-packets, and state a theorem (Theorem 4.3.2), a special case of which we will need for the proof of

Theorem 4.1.1. We will then prove that special case. In (4.6) we will use Theorem 4.1.1 to complete the proof of Theorem 4.3.2.

A homomorphism $\varphi: W \to {}^L G$ is *admissible* if, for each $w \in W$, $\varphi(w)$ is of the form $\varphi_0(w) \times w$, where $\varphi_0(w)$ is a semisimple element of ${}^L G^0$. Two such homomorphisms are *equivalent* if they are conjugate under ${}^L G^0$; the set of equivalence classes is denoted $\Phi(G^*)$. The subset of classes of homomorphisms φ for which $\varphi(W)$ lies only in parabolic subgroups of ${}^L G$ relevant to G is denoted $\Phi(G)$; then φ, or its class, is *relevant to* G. We write $\{\varphi\}$ for the equivalence class of φ when there is need to distinguish between a homomorphism and its class; at other times, both will be denoted by φ, as in (4.1).

A homomorphism $\varphi: W \to {}^L G$ is *tempered* if $\varphi(W)$ is bounded; $\Phi_0(G^*)$ will denote the classes of tempered homomorphisms and $\Phi_0(G) = \Phi(G) \cap \Phi_0(G^*)$.

For the rest of (4.3) we will assume that there exist irreducible unitary representations of G square-integrable modulo the center of G (that is, discrete series representations of G). This is equivalent to the assumption that there exists an element g of ${}^L G^0$ such that $g \times (1 \times \sigma)$ normalizes ${}^L T^0$ and acts as -1 on the roots of $({}^L G^0, {}^L T^0)$. Two such elements $g \times (1 \times \sigma)$, $g' \times (1 \times \sigma)$ have the same action on ${}^L T^0$; this action will be denoted by $\bar{\sigma}$.

Let $\mathfrak{X} = \{(\mu, \underline{\lambda}) : \mu \in X_*({}^L T^0) \otimes \mathbb{C},\ \underline{\lambda} \in X_*({}^L T^0) \otimes \mathbb{C} / X_*({}^L T^0) + \{\nu - \bar{\sigma}\nu : \nu \in X_* \cdot ({}^L T^0) \otimes \mathbb{C}\}$, *and* $\iota + 1/2(\mu - \bar{\sigma}\mu) + (\lambda + \bar{\sigma}\lambda) \in X_*({}^L T^0)$, $\lambda \in \underline{\lambda}\}$. Here ι denotes one-half the sum of the roots of $({}^L B^0, {}^L T^0)$. If $\mathbf{G}$ is semisimple then $\bar{\sigma}$ acts as -1 on $X_*({}^L T^0) \otimes \mathbb{C}$, so that $\mathfrak{X}$ may be identified as $\iota + X_*({}^L T^0)$. The group $\Omega({}^L G^0, {}^L T^0)$ acts on $\mathfrak{X}$ by the natural action on each coordinate; the action on the second coordinate is trivial. By an *orbit* in $\mathfrak{X}$ we will mean an orbit of $\Omega({}^L G^0, {}^L T^0)$; an orbit is *regular* if it has no fixed points, and *singular* otherwise.

For $\mu, \lambda \in X_*({}^L T^0) \otimes \mathbb{C}$ satisfying $\iota + 1/2(\mu - \bar{\sigma}\mu) + (\lambda + \bar{\sigma}\lambda) \in X_*({}^L T^0)$, we define admissible homomorphisms $\varphi: W \to {}^L G$ as follows. First, set $\varphi(z \times 1) = z(\mu, \lambda) \times (z \times 1)$, $z \in \mathbb{C}^\times$, where $z(\mu, \lambda) \in {}^L T^0$ and $\lambda^\vee(z(\mu, \lambda)) = z^{\langle \mu, \lambda^\vee \rangle} \bar{z}^{\langle \bar{\sigma}\mu, \lambda^\vee \rangle}$, $\lambda^\vee \in X^*({}^L T^0)$. Then choose n in the normalizer of ${}^L T^0$ in ${}^L G^0$ such that $n \times (1 \times \sigma)$ acts on ${}^L T^0$ as $\bar{\sigma}$ and $\lambda^\vee(n) = e^{2\pi i \langle \lambda, \lambda^\vee \rangle}$, $\lambda^\vee \in X_*({}^L G^0) \subset X_*({}^L T^0)$; set $\varphi(1 \times \sigma) = \mathrm{n} \times (1 \times \sigma)$. That φ is a homomorphism follows from Lemma 3.2 of [10] (cf. [16, Appendix]). We write $\varphi = \varphi(\mu, \lambda)$.

Lemma 4.3.1. *If $\varphi = \varphi(\mu, \lambda)$ and $\varphi' = \varphi(\mu', \lambda')$ then φ' is equivalent to φ if and only if $(\mu, \underline{\lambda})$, $(\mu', \underline{\lambda}')$ belong to the same orbit.*

Proof. Let $\varphi' = \mathrm{ad}\, g \circ \varphi$, $g \in {}^L G^0$. By modifying g by a suitable element of the centralizer of $\varphi'(\mathbb{C}^\times \times 1)$ in $({}^L G^0)_{\mathrm{der}}$ we may assume that g normalizes ${}^L T^0$. The rest is immediate.

Since we are interested only in homomorphisms with bounded image we set $\mathfrak{X}_0 = \{(\mu, \underline{\lambda}) \in \mathfrak{X}:\ \mathrm{Re}\,\mu = 0\}$; $\mathfrak{X}_0$ is invariant under $\Omega({}^L G^0, {}^L T^0)$. Clearly we have established a $1-1$ correspondence between orbits of $\mathfrak{X}_0$ and a subset of $\Phi_0(G^*)$. If $\mathcal{O}$ is an orbit we denote by $\{\varphi\}_{\mathcal{O}}$ the corresponding parameter. The parameters $\{\varphi\}_{\mathcal{O}}$, $\mathcal{O}$ regular, are precisely the classes of those homomorphisms $\varphi: W \to {}^L G$ for which $\varphi(W)$ is (bounded and) contained in no proper parabolic subgroup of G... the *discrete* parameters.

Let $\mathcal{O}$ be regular. Then to $\{\varphi\}_{\mathcal{O}}$ there is attached an L-packet of discrete series representations of G, as follows (cf. [10]). Choose (μ, λ) such that $(\mu, \underline{\lambda}) \in \mathcal{O}$ and let Ψ

denote the (unique) system of positive roots for $({}^LG^0, {}^LT^0)$ with respect to which μ is dominant. Fix a Cartan subgroup T of G, compact modulo the center of G, and a p.d. η of $\mathbf{T}$. Via η, transfer μ, λ, $\bar{\sigma}$ to $X^*(\mathbf{T})\otimes\mathbb{C}$ and Ψ to a positive system for $\Delta(\mathbf{G},\mathbf{T})$, without change in notation. Then $\bar{\sigma}$ acts as the Galois automorphism of $\mathbf{T}$. Also if $\iota_\Psi = 1/2 \sum_{\alpha\in\Psi} \alpha$ then the characters $\chi(\omega\mu - \iota_\Psi, \lambda)$, $\omega\in\Omega(\mathbf{G},\mathbf{T})$, are well-defined. To $\omega\in\Omega(\mathbf{G},\mathbf{T})$ we attach the discrete series representation $\pi(\omega,\Psi)$ with character $\Theta(\omega\mu,\lambda,\omega\Psi)$, where

$$\Theta(\omega\mu,\lambda,\omega\Psi)|_{T\cap G_{\mathrm{reg}}} = \frac{(-1)^{q_G}\det\omega \sum_{\omega_0\in\Omega(G,T)} \det\omega_0\chi(\omega_0\omega\mu - \iota_\Psi,\lambda)}{\prod_{\alpha\in\Psi}(1-\alpha^{-1})}$$

and q_G is as defined in (3.7). The set $\{\pi(\omega,\Psi) : \omega\in\Omega(\mathbf{G},\mathbf{T})\}$ does not depend on the choice of μ, λ, T or η, and forms the L-packet attached to $\{\varphi\}_{\mathcal{O}}$.

Suppose now that $\mathcal{O}$ is singular. Again choose (μ,λ) such that $(\mu,\underline{\lambda})\in\mathcal{O}$, and *some* positive system Ψ for $\Delta({}^LG^0, {}^LT^0)$ with respect to which μ is dominant. Fix T, η and transfer μ, λ, Ψ as before. Then distributions $\Theta(\omega\mu,\lambda,\omega\Psi)$, $\omega\in\Omega(\mathbf{G},\mathbf{T})$, are again defined, by coherent continuation of the characters $\Theta(\ ,\ ,\ )$ attached to regular orbits (cf. [5], also [17] since G may be disconnected; our notation is easily reconciled with that of [5, 17]). The collection $\{\Theta(\omega\mu,\lambda,\omega\Psi) : \omega\in\Omega(\mathbf{G},\mathbf{T})\}$ is independent of the choice of μ, λ, Ψ, T, and η.

On the other hand, $\varphi=\varphi(\mu,\lambda)$ may factor through parabolic subgroups of ${}^LG$ not relevant to G. Then φ is not relevant to G; that is, $\{\varphi\}=\{\varphi\}_{\mathcal{O}}\notin\Phi(G)$.

Theorem 4.3.2. (4.3.3) *If φ is not relevant to G then each of the distributions $\Theta(\omega\mu,\lambda,\omega\Psi)$, $\omega\in\Omega(\mathbf{G},\mathbf{T})$, is zero.*

(4.3.4) *If φ is relevant to G then the non-zero distributions in $\{\Theta(\omega\mu,\lambda,\omega\Psi)$: $\omega\in\Omega(\mathbf{G},\mathbf{T})\}$ are exactly the characters of the representations in the L-packet attached to $\{\varphi\}$.*

Regarding (4.3.4), the non-zero distributions among the $\Theta(\omega\mu,\lambda,\omega\Psi)$ are well known to be tempered irreducible characters; it is only the assignment of (the underlying representations of) these characters to L-packets that we have to check.

A simple argument with K-types shows that if $\Theta(\omega\mu,\lambda,\omega\Psi)\neq 0$ then $\Theta(\omega'\mu,\lambda,\omega'\Psi)=\Theta(\omega\mu,\lambda,\omega\Psi)$ if and only if $\omega'\in\Omega(G,T)\omega$ (cf. [19, Sect. 7] for similar arguments).

As indicated earlier, we postpone the proof of Theorem 4.3.2 until (4.6), apart from a special case.

Set $\Delta_\mu^\vee=\{\alpha^\vee\in\Delta({}^LG^0, {}^LT^0) : \langle\mu,\alpha^\vee\rangle=0\}$.

Lemma 4.3.5. *Suppose that $\Delta_\mu^\vee$ is of type $A_1\times\ldots\times A_1$. Then* (4.3.3) *is true.*

Proof. Let $\Delta_\mu^\vee=\{\pm\alpha_1^\vee,\ldots,\pm\alpha_r^\vee\}$, where each $\alpha_j^\vee$ belongs to Ψ. Clearly the coroots $\alpha_1,\alpha_2,\ldots$ are Ψ-simple. In the next paragraph we will fix (T,η), transfer μ, λ, Ψ, α_1 etc. to $\mathbf{T}$ and define $\Theta(\omega\mu,\lambda,\omega\Psi)$ accordingly. If $\omega\in\Omega(\mathbf{G},\mathbf{T})$ then $\omega\alpha_1,\omega\alpha_2,\ldots,\omega\alpha_r$ are Ψ-simple. Therefore to show that $\Theta(\omega\mu,\lambda,\omega\Psi)$ is zero we have only to show that there is a compact root among $\omega\alpha_1,\omega\alpha_2,\ldots,\omega\alpha_r$ (cf. [5, Proposition 3.6], also

[19, Lemma 7.3]). Returning now to the L-group, set

$$\mathbf{s}_j = \begin{cases} \exp \pi/4(X_{\alpha_j^\vee} - X_{-\alpha_j^\vee}) & \text{if} \quad \varphi(1\times\sigma)X_{\alpha_j^\vee} = X_{-\alpha_j^\vee} \\ \exp i\pi/4(X_{\alpha_j^\vee} + X_{-\alpha_j^\vee}) & \text{if} \quad \varphi(1\times\sigma)X_{\alpha_j^\vee} = -X_{-\alpha_j^\vee} \end{cases}$$

and $\mathbf{s} = \mathbf{s}_1 \dots \mathbf{s}_r$. Recall that the root vectors $X_{\alpha_j^\vee}$ were fixed in (1.3). Define φ_1 by $\varphi_1(w) = \mathbf{s}\varphi(w)\mathbf{s}^{-1}$, $w \in W$, i.e. $\varphi_1 = \mathrm{ad}\,\mathbf{s} \circ \varphi$. Then $\varphi_1(z\times 1) = \varphi(z\times 1)$, $z \in \mathbb{C}^\times$, and $\varphi_1(1\times\sigma)$ normalizes ${}^LT^0$, acting on ${}^LT^0$ as $\omega_{\alpha_1^\vee} \dots \omega_{\alpha_r^\vee} \cdot \varphi(1\times\sigma)|_{{}^LT^0}$. By replacing (μ, λ) by some suitable $(\omega\mu, \lambda)$, $\omega \in \Omega({}^LG^0, {}^LT^0)$, we may assume that $\omega_{\alpha_1^\vee} \dots \omega_{\alpha_r^\vee} \varphi(1\times\sigma)|_{{}^LT^0} = \sigma_{\bar{T}^*}$, for some standard maximal torus $\bar{\mathbf{T}}^*$ in $\mathbf{G}^*$; here we have used $\sigma_{\bar{T}^*}$ to denote the canonical transfer of the Galois action on $\bar{\mathbf{T}}^*$ to ${}^LT^0$ by an element of $\mathrm{ad}\,\mathbf{M}_{\bar{T}^*}$. Let $\bar{\mathbf{M}}^* = \mathbf{M}_{\bar{T}^*}$ and ${}^L\bar{M}$ be the L-group of $\bar{\mathbf{M}}^*$. Then $\Delta({}^L\bar{M}^0, {}^LT^0)$ consists of the roots of $({}^LG^0, {}^LT^0)$ perpendicular to $\Delta_\mu^\vee$. Clearly $\varphi_1(W) \subset {}^L\bar{M}$ and φ_1 is a discrete parameter for $\bar{\mathbf{M}}^*$. By the assumption of (4.3.3), ${}^L\bar{M}$ is not relevant to G. Hence $\bar{T}^*$ does not originate in G.

The pair (T, η), used to define $\Theta(\omega\mu, \lambda, \omega\Psi)$, will be chosen as follows. In $\mathbf{G}^*$ we have fixed $\bar{\mathbf{T}}^*$; fix $\bar{m} \in \bar{\mathbf{M}}^*$ such that $\mathrm{ad}\,\bar{m}: \bar{\mathbf{T}}^* \to \mathbf{T}^*$. The roots $\alpha_1^\vee, \dots, \alpha_r^\vee$ of $({}^LG^0, {}^LT^0)$ are transferred to coroots of $(\mathbf{G}^*, \bar{\mathbf{T}}^*)$ by $\mathrm{ad}\,\bar{m}$. Taking coroots, we obtain roots $\alpha_1, \dots, \alpha_r$ of $(\mathbf{G}^*, \bar{\mathbf{T}}^*)$. These roots are real. We let s_v^* be a standard inverse Cayley transform with respect to α_v, and set $\mathbf{T}^{G^*} = (\bar{\mathbf{T}}^*)^{s_1^* \dots s_r^*}$, $\bar{y} = \mathrm{ad}\,\bar{m}(s_1^* \dots s_r^*)^{-1}$. Clearly T^{G^*} is compact modulo the center of G^* and $\bar{y}: \mathbf{T}^{G^*} \to \mathbf{T}^*$. We then fix $\mathrm{ad}\,a \circ \psi^{-1}$: $\mathbf{T}^{G^*} \hookrightarrow \mathbf{G}$ over $\mathbb{R}$, $a \in \mathbf{G}$, and let $\mathbf{T}$ be the image of $\mathbf{T}^{G^*}$; for η we take the now obvious *p.d.*

We use η to transfer $\alpha_1^\vee, \dots, \alpha_r^\vee$ to coroots for $\mathbf{T}$, and again denote the dual roots by $\alpha_1, \dots, \alpha_r$. We claim that after some sequence of Cayley transform with respect to roots among $\alpha_1, \dots, \alpha_r$, at least one α_j becomes *totally compact* in the following sense.

Definition 4.3.6. *Let T be a Cartan subgroup of G and α be an imaginary root of* $(\mathbf{G}, \mathbf{T})$. *Then α is totally compact if and only if each $\omega\alpha$ is compact, for ω in the imaginary Weyl group of* $(\mathbf{G}, \mathbf{T})$ *or, equivalently, for $\omega \in \mathfrak{A}(T)$.*

A necessary and sufficient condition that there exist a Cayley transform with respect to α (in the sense of [16]) is that α not be totally compact (cf. [14]).

If our claim were false, then a sequence of Cayley transforms, using all roots among $\alpha_1, \dots, \alpha_r$, would produce a Cartan subgroup of G from which $\bar{T}^*$ originates, a contradiction.

We claim next that each set $\{\omega\alpha_1, \dots, \omega\alpha_r\}$, $\omega \in \Omega(\mathbf{G}, \mathbf{T})$, contains a compact root. If not, then some $\{\omega_0\alpha_1, \dots, \omega_0\alpha_r\}$ contains only noncompact roots. Because the roots are superorthogonal (that is, $\{\pm\omega_0\alpha_1, \dots, \pm\omega_0\alpha_r\}$ exhausts the roots in the $\mathbb{Q}$-span of $\omega_0\alpha_1, \dots, \omega_0\alpha_r$), no root in this set can become totally compact after a sequence of Cayley transforms, a contradiction. Hence Lemma 4.3.5 is proved.

Lemma 4.3.7. *Suppose that $\Delta_\mu^\vee$ is of type $A_1 \times \dots \times A_1$. Then* (4.3.4) *is true.*

Proof. The Levi group ${}^L\bar{M}$ of the last proof is relevant to G. Thus $\bar{T}^*$ originates in G. Let $\mathrm{ad}\,b \circ \psi$: $\bar{\mathbf{T}} \to \bar{\mathbf{T}}^*$ be defined over $\mathbb{R}$. We transfer $\alpha_1, \dots, \alpha_r$ from $\bar{\mathbf{T}}^*$ to $\bar{\mathbf{T}}$ via $\mathrm{ad}\,b \circ \psi$ and without change in notation; $\alpha_1, \dots, \alpha_r$ remain real. We let s_v be a standard inverse Cayley transform with respect to α_v, $v = 1 \dots, r$. We choose

$\mathbf{T}=(\bar{\mathbf{T}})^{s_1 s_2 \dots s_r}$ and set $\eta=\bar{\eta}(s_1 \dots s_r)^{-1}$, where $\bar{\eta}$ is the p.d. of $\bar{\mathbf{T}}$ implicit above; $\alpha_1, \dots, \alpha_r$ are noncompact on $\mathbf{T}$.

The L-packet attached to $\{\varphi\}$ is described in terms of data attached to the homomorphism $\varphi_1 : W \to {}^L\bar{M}$ defined in the last proof. Let $\varphi_1=\varphi(\mu_1, \lambda_1)$ relative to ${}^L\bar{M}$. Then $\mu_1=\mu$ and $\lambda_1 \equiv \lambda \bmod (X_*({}^LT^0)+\{v-\bar{\sigma}v\colon\ v \in X_*({}^LT^0)\otimes\mathbb{C}\})$. Let $\bar{\Psi}=\Psi \cap \Delta({}^L\bar{M}^0, {}^LT^0)$ and $\bar{P}$ be the parabolic subgroup of G with Levi component $\bar{M}=M_{\bar{T}}$ defined in (1.3). Then the L-packet in question consists of the constituents of $\operatorname{Ind}\left(\bigoplus_{\omega\in\Omega(\bar{M},\bar{T})\backslash\Omega(\bar{\mathbf{M}},\bar{\mathbf{T}})} \pi(\omega,\bar{\Psi})\otimes 1_{\bar{N}}, \bar{P}, G\right)$. To decompose the the representations $\operatorname{Ind}(\pi(\omega,\bar{\Psi})\otimes 1_{\bar{N}}, \bar{P}, G)$ we appeal directly to the Hecht-Schmid character identities (cf. [6, 17]). Note that the representation $\pi(\omega,\bar{\Psi})$ of $\bar{M}$ has character $\Theta(\omega\mu, \lambda_1, \omega\bar{\Psi})$.

Let $\bar{\mathbf{T}}_1=s_1\bar{\mathbf{T}}s_1^{-1}$ and $\bar{\eta}_1=\bar{\eta}\circ \operatorname{ad} s_1^{-1}$. Set $\bar{M}_1=M_{\bar{T}_1}$ and $\bar{\Psi}_1=\Psi\cap\Delta({}^L\bar{M}_1^0, {}^LT^0)$; α_1 is $\bar{\Psi}_1$-simple. Then on $\bar{M}_1$ we have

$$\text{(4.3.8)}\quad \Theta(\omega\mu, \lambda_1, \omega\bar{\Psi}_1)+\Theta(\omega\mu, \lambda_1, \omega_{\alpha_1}\omega\bar{\Psi}_1) = \operatorname{Ind}(\Theta(\omega\mu, \lambda_1, \omega\bar{\Psi})\otimes 1_{\bar{N}\cap\bar{M}_1}, \bar{M}(\bar{N}\cap\bar{M}_1), \bar{M}_1),$$

$\omega\in\Omega(\bar{\mathbf{M}}, \bar{\mathbf{T}})$, if ω_{α_1} is not realized in M_1, and

$$\text{(4.3.9)}\quad \Theta(\omega\mu, \lambda_1, \omega\bar{\Psi}_1)=\operatorname{Ind}(\Theta(\omega\mu, \lambda_1, \omega\bar{\Psi})\otimes 1_{\bar{N}\cap\bar{M}_1}, \bar{M}(\bar{N}\cap\bar{M}_1), \bar{M}_1),$$

$\omega\in\Omega(\bar{\mathbf{M}}, \bar{\mathbf{T}})$, if ω_{α_1} is realized in M_1. On the left-hand side, $\Omega(\mathbf{M},\mathbf{T})$ has been embedded in $\Omega(\bar{\mathbf{M}}_1, \bar{\mathbf{T}}_1)$ via s_1; ω_{α_1}, μ, λ_1, $\bar{\Psi}_1$ and α_1 have been transferred also, without change in notation. To justify (4.3.8) and (4.3.9) we must check a property of λ_1. Calculation shows that $\varphi_1(1\times\sigma)X_{\alpha_1^\vee}=-X_{\alpha_1^\vee}$, but, on the other hand, a lemma of Langlands (cf. [1, Lemma 2.3]) implies that $\varphi_1(1\times\sigma)X_{\alpha_1^\vee} = -(-1)^{\langle 2\lambda_1+\varrho_1, \alpha_1\rangle}X_{\alpha_1^\vee}$, where ϱ_1 denotes one half the sum of the coroots for the roots in $\bar{\Psi}_1$. Thus $\langle 2\lambda_1, \alpha_1^\vee\rangle \equiv \langle \varrho_1, \alpha_1^\vee\rangle \pmod{2\mathbb{Z}}$. This ensures that (4.3.8) and (4.3.9) are well-defined Hecht-Schmid character identities (cf. [6], also [17]).

To continue our argument, we set $\bar{\mathbf{T}}_2=s_2\bar{\mathbf{T}}_1 s_2^{-1}$; etc. After r such steps we conclude, by induction in stages, that $\operatorname{Ind}\left(\bigoplus_{\omega\in\Omega(\bar{M},\bar{T})\backslash\Omega(\bar{\mathbf{M}},\bar{\mathbf{T}})} \Theta(\omega\mu, \lambda_1, \omega\bar{\Psi})\otimes 1_{\bar{N}}, \bar{P}, G\right)$ is a sum of characters $\Theta(\omega\mu, \lambda, \omega\Psi)$, where ω lies in the subgroup $\Omega_{(\mu)}(\mathbf{G},\mathbf{T})$ generated by the reflections with respect to $\alpha_1, \dots, \alpha_r$, together with the reflections with respect to the roots perpendicular to each of $\alpha_1, \dots, \alpha_r$. For each $\omega\in\Omega_{(\mu)}(\mathbf{G},\mathbf{T})$, $\Theta(\omega\mu, \lambda, \omega\Psi)$ appears in the decomposition of the induced representation. On the other hand, the irreducible constituents of this induced representation appear with multiplicity one (Multiplicity One Theorem for unitary principal series and Lemma 3.2 of [14]). Thus we obtain

$$\operatorname{Ind}\left(\bigoplus_{\omega\in\Omega(\bar{M},\bar{T})\backslash\Omega(\bar{\mathbf{M}},\bar{\mathbf{T}})} \Theta(\omega\mu, \lambda_1, \omega\bar{\Psi})\otimes 1\right) = \sum_{\omega\in\Omega(G,T)\backslash\Omega(G,T)\Omega_{(\mu)}(\mathbf{G},\mathbf{T})} \Theta(\omega\mu, \lambda, \omega\Psi).$$

To complete the proof of Lemma 4.3.7 we have only to show that if $\omega\notin\Omega(G,T)\Omega_{(\mu)}(\mathbf{G},\mathbf{T})$ then $\Theta(\omega\mu, \lambda, \omega\Psi)=0$.

We have arranged that $\alpha_1, \dots, \alpha_r$ are all noncompact on T. Suppose that also all of $\omega\alpha_1, \dots, \omega\alpha_r$ are noncompact. Then we have just to show that $\omega\in\Omega(G,T)\Omega_{(\mu)}(\mathbf{G},\mathbf{T})$. There is $\omega_1\in\Omega(G,T)\langle 1, \omega_{\alpha_1}\rangle$ such that $\omega\alpha_1=\omega_1\alpha_1$ [14, Lemma 4.2]. Then both α_2, $\omega_1^{-1}\omega\alpha_2$ are noncompact and superorthogonal to α_1.

Now apply a standard Cayley transform with respect to α_1 to $\mathbf{T}$. Then both α_2, $\omega_1^{-1}\omega\alpha_2$ are noncompact on $\mathbf{T}^{(1)}$, the image of $\mathbf{T}$, and $\omega_1^{-1}\omega$ belongs to the imaginary Weyl group of $\mathbf{T}^{(1)}$. Choose ω_2' in the real imaginary Weyl group of $T^{(1)}$ so that $(\omega_2')^{-1}\omega_1^{-1}\omega\alpha_2=\alpha_2$ and let ω_2 be the preimage of ω_2' in $\Omega(\mathbf{G},\mathbf{T})$. Then $\omega_2\in\Omega(G,T)\langle 1,\omega_{\alpha_2}\rangle$ [14], and $\omega_2^{-1}\omega_1^{-1}\omega\alpha_1=\omega_1^{-1}\omega\alpha_1=\alpha_1$ so that $\omega\alpha_1=\omega_1\omega_2\alpha_1$; $\omega_2^{-1}\omega_1^{-1}\omega\alpha_2=\alpha_2$, so that $\omega\alpha_2=\omega_1\omega_2\alpha_2$. Also, $\omega_1\omega_2\in\Omega(G,T)\langle 1,\omega_{\alpha_1},\omega_{\alpha_2}\rangle$. Proceeding by induction, we thus find $\omega_3,\ldots,\omega_r$ such that $\omega\alpha_i=\omega_1\omega_2\ldots\omega_r\alpha_i$, $i=1,\ldots,r$, and $\omega_1\omega_2\ldots\omega_r\in\Omega(G,T)\Omega_{(\mu)}(\mathbf{G},\mathbf{T})$. Lemma 4.3.7 now follows.

(4.4) *Proof of Theorem 4.1.1* (*Two Cases*)

Suppose that $\varphi': W\to{}^LH$ is a tempered admissible homomorphism. Then the image of $\{\varphi'\}$ under ξ^Φ: $\Phi_0(H)\to\Phi_0(G^*)$ is simply the class of $\varphi=\xi\circ\varphi'$.

We assume first that both $\{\varphi'\}$ and $\{\varphi\}$ are discrete. We may take $\varphi'=\varphi(\mu',\lambda')$, where $\langle\mu',\alpha'\rangle>0$ for all roots $\alpha^\vee$ of $({}^LH^0\cap{}^LB^0,{}^LT^0)$. Then φ has parameters (μ,λ), where $\mu=\mu'+\mu^*$ and $\lambda=\lambda'+\lambda^*$. Since we have assumed φ discrete we must have:

(4.4.1) the action of $\varphi'(1\times\sigma)$ on ${}^LT^0$, which coincides with that of $\varphi(1\times\sigma)$, is by -1 on all roots of $({}^LG^0,{}^LT^0)$, and

(4.4.2) $\langle\mu,\alpha^\vee\rangle\neq 0$ for all roots of $({}^LG^0,{}^LT^0)$.

Our first task is to define the numbers $\varepsilon(\pi)$, $\pi\in\Pi_{\{\varphi\}}$. First, there exists a unique element ω_* of $\Omega({}^LG^0,{}^LT^0)$ such that $\langle\omega_*\mu,\alpha^\vee\rangle>0$ for all roots $\alpha^\vee$ of $({}^LB^0,{}^LT^0)$. Let T be a Cartan subgroup of G, compact modulo the center of G. Since φ is discrete such a Cartan subgroup can be found. We pick η such that $(T,\eta)\in\mathcal{T}_H(G)$; this is possible, by (4.4.1) and Lemma 2.4.2. Set $\Psi=\Delta({}^LB^0,{}^LT^0)$. For $\omega\in\Omega(\mathbf{G},\mathbf{T})$ we define $\pi(\omega,\Psi)$ as in (4.3), and set

(4.4.3) $\varepsilon(\pi(\omega,\Psi))=\det\omega_*\varepsilon(T,\eta)\kappa(\omega\omega_*)$.

Here we have transferred ω_* to $\mathbf{T}$ via η, without change in notation.

Proposition 4.4.4. $\varepsilon(\pi)$ *is well-defined*, $\pi\in\Pi_{\{\varphi\}}$.

Proof. Each $\pi\in\Pi_{\{\varphi\}}$ is of the form $\pi(\omega,\Psi)$, some $\omega\in\Omega(\mathbf{G},\mathbf{T})$. Further $\pi(\omega,\Psi)=\pi(\omega',\Psi)$ if and only if $\omega'=\omega_0\omega$, some $\omega_0\in\Omega(G,T)$. Since $\kappa(\omega_0\omega\omega_*)=\kappa^{\omega\omega_*}(\omega_0)\kappa(\omega\omega_*)=\kappa(\omega\omega_*)$, the assertion is immediate.

The next result can be deduced from our proof of Theorem 4.1.1, for we will show that if $\varepsilon(\pi)$ is defined as in (4.4.3) then the lift of $\chi_{\varphi'}$ is $\sum_{\pi\in\Pi_\varphi}\varepsilon(\pi)\chi_\pi$. Nevertheless a direct proof is of some interest.

Lemma 4.4.5. $\varepsilon(\pi)$ *is independent of the choice of* (T,η).

Proof. Suppose that $\bar{T}$ is a Cartan subgroup of G, compact modulo the center of G, and $(\bar{T},\bar{\eta})\in\mathcal{T}_H(G)$, with attached character $\bar{\kappa}$. Then $\eta=\bar{\eta}\circ\omega_0$ where $\omega_0\in\mathfrak{A}(T)$ maps T to $\bar{T}$. Hence $\bar{\kappa}=\kappa^{\omega_0}$ and $\varepsilon(\bar{T},\bar{\eta})=\kappa(\omega_0)\varepsilon(T,\eta)$. Now replace T by $\bar{T}$ in the definition of $\varepsilon(\pi)$. It is sufficient to consider two cases: $\omega_0\in\Omega(\mathbf{G},\mathbf{T})$ and $\omega_0\in G$. In the former, $\pi=\pi(\omega,\Psi)$ becomes $\pi(\omega\omega_0^{-1},\Psi)$ and ω_* is replaced by $\omega_0\omega_*\omega_0^{-1}$.

Hence $\varepsilon(\pi)$ is replaced by

$$\frac{\kappa(\omega_0)\kappa^{\omega_0}(\omega\omega_0^{-1}\omega_0\omega_*\omega_0^{-1})}{\kappa(\omega\omega_*)}\varepsilon(\pi)=\frac{\kappa(\omega\omega_0^{-1}\omega_0\omega_*\omega_0^{-1}\omega_0)}{\kappa(\omega\omega_*)}\varepsilon(\pi)=\varepsilon(\pi).$$

In the latter case, $\pi=\pi(\omega,\Psi)$ becomes $\pi(\omega_0\omega\omega_0^{-1},\Psi)$ and $\varepsilon(\pi)$ is replaced by

$$\frac{\kappa(\omega_0)\kappa^{\omega_0}(\omega_0\omega\omega_0^{-1}\omega_0\omega_*\omega_0^{-1})}{\kappa(\omega\omega_*)}\varepsilon(\pi)=\varepsilon(\pi)$$

again, and the lemma is proved.

It remains now to verify that the lift Θ of $\chi_{\varphi'}$ to G is $\sum_{\pi\in\Pi_\varphi}\varepsilon(\pi)\chi_\pi$.

Lemma 4.4.6. (i) *Θ is a tempered invariant eigendistribution with infinitesimal character μ. Regard Θ as a function on G_{reg}. Then:*

(ii) *on $T\cap G_{\text{reg}}$, Θ coincides with $\sum_{\pi\in\Pi_\varphi}\varepsilon(\pi)\chi_\pi$ and*

(iii) *Θ vanishes off $G_{\text{reg}}\cap Z(G)G^0$, where G^0 is the identity component of G, and $Z(G)$ the center of G.*

Proof. (i) follows from Lemma 4.2.1. For (ii), fix $i(h,\eta)$: $\mathbf{T}'\to\mathbf{T}$ over $\mathbb{R}$; this data includes $\operatorname{ad}h:\mathbf{T}'\to\mathbf{T}_H$, which serves as p.d. for $\mathbf{T}'$ in $\mathbf{H}$. By definition, $\chi_{\varphi'}=\sum_{\omega\in\Omega(H,T')\backslash\Omega(\mathbf{H},\mathbf{T}')}\Theta(\omega\mu',\lambda',\omega\Psi')$, where $\Psi'=\Psi\cap\Delta({}^LH^0,{}^LT^0)=\Delta({}^LH^0\cap{}^LB^0,{}^LT^0)$. Lemma 4.2.4 then implies that

$$\Theta(\gamma)=(-1)^{q(G,H)+n(T)}\varepsilon(T,\eta)\sum_{\omega\in\Omega(\mathbf{G},\mathbf{T})/\Omega^{(\kappa)}(\mathbf{G},\mathbf{T})}\kappa(\omega)\Delta^{H/G}(\gamma^\omega)$$
$$\cdot\sum_{\omega'\in\Omega(H,T')\backslash\Omega(\mathbf{H},\mathbf{T}')}\Theta(\omega'\mu',\lambda',\omega'\Psi')(\gamma^\omega)$$

for $\gamma\in T\cap G_{\text{reg}}$, where $\Delta^{H/G}$ is as in (4.2). Explicit calculation shows that this expression coincides with

$$\det\omega_*\varepsilon(T,\eta)\sum_{\omega\in\Omega(G,T)\backslash\Omega(\mathbf{G},\mathbf{T})}\kappa(\omega\omega_*)\Theta(\omega\omega_*\mu,\lambda,\omega\Psi)(\gamma),$$

and (ii) follows.

For (iii), it is sufficient to show that if U is a Cartan subgroup of G from which Cartan subgroups of H originate then $\Theta|_{U\cap G_{\text{reg}}}$ vanishes off $U\cap Z(G)G^0\cap G_{\text{reg}}$. Suppose that U' is a Cartan subgroup of H originating from U. Then since $\chi_{\varphi'}|_{U'}$ vanishes off $U'\cap Z(H)H^0\cap H_{\text{reg}}$, it is enough to show that each of our maps $i(\ ,\ )$: $\mathbf{U}'\to\mathbf{U}$ defined over $\mathbb{R}$ sends $U'\cap Z(H)H^0$ into $U\cap Z(G)G^0$.

First, $(U'\cap H^0_{\text{der}})^0$ is mapped into $U\cap G^0_{\text{der}}$. Representatives for the cosets of $(U'\cap H^0_{\text{der}})^0$ in $U'\cap H^0_{\text{der}}$ are given by $\{\exp i\pi\lambda^\vee:\sigma_{U'}\lambda^\vee=\lambda^\vee,\ \lambda^\vee\in\mathbb{Z}[\Delta^\vee(\mathbf{H},\mathbf{U}')]\}$, since $U'\cap H^0_{\text{der}}$ is the image of the $\mathbb{R}$-rational points in the preimage of $\mathbf{U}'$ in $\mathbf{H}_{\text{sc}}$ (cf. [16, Sect. 4.1]). Under $i(\ ,\ )$ these elements are mapped into $U\cap Z(G)G^0$. Recall that T is compact modulo the center of G. We have that $T=Z(G)(T\cap G^0_{\text{der}})$. If $z\in Z(H)$ maps into $Z(G)$ under $i(h,\eta)$: $\mathbf{T}'\to\mathbf{T}$ then clearly z maps into $Z(G)$ under $i(\ ,\ )$: $\mathbf{U}'\to\mathbf{U}$. Thus suppose that z maps into $T\cap G^0_{\text{der}}$ under $i(h,\eta)$. Suppose also that $\mathbf{U}'$ is obtained from $\mathbf{T}'$ by a Cayley transform s' with respect to the root α' of $(\mathbf{H},\mathbf{T}')$.

Then **U** is obtained from **T** by the Cayley transform $s=i(\ ,\ )\circ s'\circ i(h,\eta)^{-1}$ with respect to α, the transfer of α' to **T** via $i(h,\eta)$. Also $\alpha(i(h,\eta)(z))=1$. A simple argument, using passage to $\mathbf{G}_{sc}$, shows that if $\alpha(\gamma)=1$, $\gamma\in T\cap G^0_{reg}$, then $\gamma^s\in U\cap G^0_{der}$. Since $(i(h,\eta)(z))^s$ is the image of z under $i(\ ,\ )$ we are done. For a general Cartan subgroup U' in H originating in G, we may pick a suitable sequence of Cayley transforms starting from T', and argue similarly. This completes the proof of Lemma 4.4.6.

We conclude now that Θ coincides with $\sum_{\pi\in\Pi_\varphi}\varepsilon(\pi)\chi_\pi$, by lifting the eigendistribution $\Theta-\sum\varepsilon(\pi)\chi_\pi$ on $Z(G)G^0$ to G_{sc}, and applying Harish-Chandra's uniqueness theorem. Hence Theorem 4.1.1 is proved for the case that both φ' and φ are discrete.

Next we retain the assumptions on φ', so that $\varphi'=\varphi(\mu',\lambda')$ is discrete and $\varphi'(1\times\sigma)$ *acts on all roots of* $({}^LG^0,{}^LT^0)$ *by* -1 [cf. (4.4.1)], but no longer assume that $\varphi=\xi\circ\varphi'$ is discrete. Recall that $\varphi=\varphi(\mu,\lambda)$, where $\mu=\mu'+\mu^*$, $\lambda=\lambda'+\lambda^*$.

Proposition 4.4.7. $\Delta^\vee_\mu=\{\alpha^\vee\in\Delta({}^LG^0,{}^LT^0)\colon\langle\mu,\alpha^\vee\rangle=0\}$ *is of type* $A_1\times A_1\times\ldots\times A_1$.

Proof. It is sufficient to show that if $\alpha^\vee$, $\beta^\vee\in\Delta^\vee_\mu$ then $\alpha^\vee\pm\beta^\vee\notin\Delta({}^LG^0,{}^LT^0)$. Recall that $\mathfrak{s}_H=(s_H,{}^LH)$, and s_H is a coset of Z^W in ${}^LT^0$. We argue in ${}^LG^0/Z^W$. Clearly $s_H^2=1$, $\alpha^\vee\in\Delta({}^LH^0,{}^LT^0)$, if and only if $\alpha^\vee(s_H)=1$. Because φ' is discrete, $\alpha^\vee\in\Delta^\vee_\mu$ implies that $\alpha^\vee\notin\Delta({}^LH^0,{}^LT^0)$. Then $\alpha^\vee(s_H)=-1$, and the proposition follows.

We will compute Θ, the lift of $\chi_{\varphi'}$ to G, in terms of the distributions $\Theta(\omega\mu,\lambda,\omega\Psi)$, and then apply Lemmas 4.3.5 and 4.3.8.

Because of (4.4.1) there is a Cartan subgroup T of G compact modulo the center of G; also we may choose η so that $(T,\eta)\in\mathcal{T}_H(G)$. Let $\Psi=\Delta({}^LB^0,{}^LT^0)$. As in the case φ discrete, we choose ω_* such that $\omega_*\mu$ is Ψ-dominant; $\omega_*\mu$ is uniquely determined but ω_* is not. Again we transfer ω_*, μ, λ, Ψ to **T** without change in notation.

Lemma 4.4.8. $\Theta=\det\omega_*\varepsilon(T,\eta)\sum_{\omega\in\Omega(G,T)\backslash\Omega(\mathbf{G},\mathbf{T})}\kappa(\omega\omega_*)\Theta(\omega\omega_*\mu,\lambda,\omega\Psi)$.

Proof. We have already proved this in the case that $\omega_*\mu$ is strictly Ψ-dominant (that is, φ discrete). If $\omega_*\mu$ is singular, but still Ψ-dominant, then the right hand side of (4.4.8) is obtained by coherent continuation to the wall (cf. [5, 17]). Thus it is sufficient to prove coherency of the lifts to G of the characters $\sum_{\omega\in\Omega(H,T')\backslash\Omega(\mathbf{H},\mathbf{T}')}\Theta(\omega'\mu',\lambda',\omega'\Psi')$, where $\Psi'=\Psi\cap\Delta({}^LH^0,{}^LT^0)$, μ' is (strictly) Ψ'-dominant, T' is some Cartan subgroup of H originating from T, and μ', λ', Ψ' have been transferred to T' by some suitable p.d. Certainly the characters to be lifted form a coherent family. Coherency of the lifts then follows from Lemma 4.2.4 (cf. [15, 16]) concerning roots etc. "from H").

Corollary 4.4.9. *If* $\Theta(\omega\omega_*\mu,\lambda,\omega\Psi)\neq 0$ *then* $\det\omega_*\kappa(\omega\omega_*)$ *does not depend on the choice of* ω_*.

A simple direct argument using the explicit form of ω [cf. (4.3)] also shows this.

If φ is not relevant to G then Lemma 4.3.5 shows that the lift of $\chi_{\varphi'}$ to G is zero and Theorem 4.1.1 is proved in that case.

If φ is relevant to G we choose (T,η) so that if $\alpha^\vee\in\Delta_\mu^\vee$ then α is noncompact on T (cf. our earlier discussion). If $\pi\in\Pi_\varphi$ has character $\Theta(\omega\omega_*\mu,\lambda,\omega\Psi)$ (cf. Lemma 4.3.7), we set

(4.4.10) $$\varepsilon(\pi)=\det\omega_*\varepsilon(T,\eta)\kappa(\omega\omega_*).$$

Then $\varepsilon(\pi)$ is well-defined and does not depend on the choices for (T,η) and ω_*. Finally, the lift of $\chi_{\varphi'}$ to G is, clearly, $\sum_{\pi\in\Pi_\varphi}\varepsilon(\pi)\chi_\pi$.

(4.5) *Proof of Theorem 4.1.1* (*Conclusion*)

Recall that ${}^LH$ is in standard position with respect to some standard Cartan subgroup of $\mathbf{G}^*$. It is convenient to fix a framework of Cartan subgroups around this group, and argue with the Levi groups so provided [cf. (3.2) and (3.4)].

Suppose that $\varphi': W\to{}^LH$ is any tempered admissible homomorphism. Replacing φ' by an equivalent homomorphism, we may assume that for some n, $\varphi'(W)\subset{}^LM'_n$ and $\varphi'(W)$ lies in no proper parabolic subgroup of ${}^LM'_n$. Then $\varphi(W)\subset{}^LM_n$ since ${}^LH$ is in standard position. The L-packet $\Pi_{\varphi'}$ for H consists of the irreducible constituents of the principal series representations defined by the discrete series representations of M'_n attached to φ'. Hence $\chi_{\varphi'}$ vanishes on any Cartan subgroup of H not conjugate to a Cartan subgroup of M'_n. Then the lift of $\chi_{\varphi'}$ to G vanishes on each Cartan subgroup of G from which no Cartan subgroup of M'_n originates (cf. Lemma 4.2.3).

Suppose that φ is not relevant to G. Then T_n (in G^*) does not originate in G. Hence no Cartan subgroup of M'_n originates in G. We conclude then that the lift of $\chi_{\varphi'}$ to G is zero.

It remains now to consider the case that φ is relevant to G. Then M_n is relevant to G and M_n^G is defined. To our fixed family $\{\Delta_{(T,\eta)}\}$ of (G,H)-orbital-integral-transfer factors we have attached a family $\{\Delta_{(T,\eta)}^{(n)}=\Delta_{(T,\eta)}^{M_n^G}\}$ for (M_n^G, M'_n)-transfer [cf. (3.4)]. The factor $\Delta_{(T,\eta)}^{(n)}$ is defined only for (T,η) subordinate to M_n^G, and is chosen so that if $f'\in\mathscr{C}(H)$ corresponds to $f\in\mathscr{C}(G)$ under (G,H)-transfer then $f'_{(n)}=f'_{M'_n}$, as defined in (3.4), corresponds to $f_{(n)}=f_{M_n^G}$ under (M_n^G,M'_n)-transfer.

Let $\Pi_{\varphi'}^{(n)}$ be the L-packet attached to φ' as element of $\Phi(M'_n)$, and $\chi_{\varphi'}^{(n)}=\sum_{\pi\in\Pi_{\varphi'}^{(n)}}\chi_\pi$. Then $\chi_{\varphi'}(f')=\chi_{\varphi'}^{(n)}(f'_{(n)})$, by a standard argument for induced (tempered) characters. From (4.4) we have that

$$\chi_{\varphi'}^{(n)}(f'_{(n)})=\sum_{\pi\in\Pi_\varphi^{(n)}}\varepsilon_{(n)}(\pi)\chi_\pi(f_{(n)}).$$

Here $\Pi_\varphi^{(n)}$ is the L-packet attached to φ as element of $\Phi(M_n^G)$ and

$$\varepsilon_{(n)}(\pi)=(-1)^{q(G,H)-q(M_n^G,M'_n)}\varepsilon(T,\eta)\det\omega_*\kappa(\omega\omega_*),$$

where (T,η) is subordinate to M_n^G, with T fundamental in M_n^G, and $\pi=\Theta(\omega\omega_*\mu,\lambda,\omega\Psi_{(n)})$, with $\Psi_{(n)}=\Delta({}^LM_n^0\cap{}^LB^0,{}^LT^0)$ and $\omega_*\mu$ $\Psi_{(n)}$-dominant; implicit is the assumption that $\varphi'=\varphi(\mu',\lambda')$, with μ' strictly $\Delta({}^L(M'_n)^0\cap{}^LB^0,{}^LT^0)$-dominant.

On the other hand,

$$\chi_\pi(f_{(n)})=\chi_{\mathrm{Ind}(\pi\otimes 1,P_n^G,G)}(f),$$

where P_n^G is the parabolic subgroup used to define $f_{(n)}$ [cf. (3.4), also (1.3)]. Thus the lift of $\chi_{\varphi'}$ to G is

$$\sum_{\pi\in\Pi_{\varphi}^{(n)}} \varepsilon_{(n)}(\pi)\chi_{\mathrm{Ind}(\pi\otimes 1)}\,.$$

In (4.3) we expressed $\bigoplus_{\omega\in\Omega(M_n^G, T)\backslash\Omega(\mathbf{M}_n^G,\mathbf{T})} \Theta(\omega\mu,\lambda,\omega\Psi_{(n)})$ explicitly as a sum of principal series characters on M_n^G (cf. proof of Lemma 4.3.7). We may now argue by induction in stages to conclude that the irreducible constituents of $\Pi=\mathrm{Ind}\left(\bigoplus_{\omega}\Theta(\omega\mu,\lambda,\omega\Psi_{(n)}\otimes 1_{N_n^G}, P_n^G, G\right)$ form the L-packet attached to φ as element of $\Phi(G)$ and, moreover, that these constituents appear in Π with multiplicity one (cf. Lemma 3.2 of [14], and the Multiplicity One Theorem for unitary principal series). Hence we have that the lift of $\chi_{\varphi'}$ to G is $\sum_{\pi\in\Pi_\varphi}\varepsilon(\pi)\chi_\pi$, where

$$\varepsilon(\pi)=\varepsilon_{(n)}(\pi^{(n)})\,, \tag{4.5.1}$$

if π is a constituent of $\mathrm{Ind}(\pi^{(n)}\otimes 1, P_n^G, G)$, $\pi^{(n)}\in\Pi_\varphi^{(n)}$.

This completes the proof of Theorem 4.1.1.

(4.6) *Proof of Theorem 4.3.2*

Recall that we have defined admissible homomorphisms $\varphi(\mu,\lambda)\colon W\to{}^{\mathrm{L}}G$, with μ singular. Suppose that μ is Ψ-dominant. Then Theorem 4.3.2 states that

(4.6.1) *if φ is not relevant to G then all distributions $\Theta(\omega\mu,\lambda,\omega\Psi)$, $\omega\in\Omega(\mathbf{G},\mathbf{T})$, are zero and*

(4.6.2) *if φ is relevant to G then the non-zero distributions among $\Theta(\omega\mu,\lambda,\omega\Psi)$, $\omega\in\Omega(\mathbf{G},\mathbf{T})$, are exactly the characters of the representations in the L-packet Π_φ.*

The notation has been explained in (4.3). We may as well assume that $\Psi=\Delta({}^{\mathrm{L}}B^0,{}^{\mathrm{L}}T^0)$.

Our proof will be by induction on $\dim G$. Both (4.6.1) and (4.6.2) are immediate if $\dim G\leqq 3$. Given now G arbitrary, we choose (H,ξ,φ'), where H is an endoscopic group for G such that $\dim H<\dim G$, $\xi\colon{}^{\mathrm{L}}H\hookrightarrow{}^{\mathrm{L}}G$ is an admissible embedding, and $\varphi=\xi\circ\varphi'$. Such a choice is possible (cf. [16, Sect. 9]). If $\xi=\xi(\mu^*,\lambda^*)$ then $\varphi'=\varphi(\mu',\lambda')$, where $\mu'=\mu-\mu^*$, $\lambda'=\lambda-\lambda^*$; μ' is $\Psi'=\Delta({}^{\mathrm{L}}H^0,{}^{\mathrm{L}}T^0)\cap\Psi$-dominant. In lifting characters from H to G we assume (G,H)-orbital-integral transfer via one of the two admissible families of factors attached to ξ.

By the inductive hypothesis, (4.6.2) is true for H. Thus

$$\chi_{\varphi'}=\sum_{\omega'\in\Omega(H,T')\backslash\Omega(\mathbf{H},\mathbf{T}')}\Theta(\omega'\mu',\lambda',\omega'\Psi)\,.$$

Suppose that φ is relevant to G. Then Theorem 4.1.1 implies that the lift of $\chi_{\varphi'}$ to G is $\sum_{\pi\in\Pi_\varphi}\varepsilon(\pi)\chi_\pi$, with each $\varepsilon(\pi)=\pm 1$. On the other hand, the coherent continuation argument of the proof of Lemma 4.4.8 shows that the lift of $\chi_{\varphi'}$ is

$$\varepsilon(T,\eta)\sum_{\omega\in\Omega(G,T)\backslash\Omega(\mathbf{G},\mathbf{T})}\kappa(\omega)\Theta(\omega\mu,\lambda,\omega\Psi)\,.$$

Hence (4.6.2) follows for G. Similarly (4.6.1) is true, and Theorem 4.3.2 is proved.

(4.7) *Theorem 4.1.1 as a Set of Character Identities*

If we combine Theorem 4.1.1 with Lemmas 4.2.3 and 4.2.4 we obtain the following identities.

Let φ' be a tempered parameter for H and φ be its lift to G^*. Then:

(4.7.1) *if φ is not relevant to G then*

$$\sum_{\pi\in\Pi_\varphi} \varepsilon(\pi)\chi_\pi \equiv 0$$

on all Cartan subgroups of G, and

(4.7.2) *if φ is relevant to G then*

(a) *for $\gamma\in T\cap G_{\mathrm{reg}}$*

$$\sum_{\pi\in\Pi_\varphi} \varepsilon(\pi)\chi_\pi(\gamma) = (-1)^{q(G,H)+n(T)} \sum_{R}^{\nu=1} \varepsilon(T,\eta_\nu)$$

$$\cdot \sum_{\omega\in\Omega(\mathbf{G},\mathbf{T})/\Omega^{(\kappa_0^\nu)}(\mathbf{G},\mathbf{T})} \kappa_\nu(\omega)\,\Delta_\nu^{H/G}(\gamma^\omega)\,\chi_{\varphi'}^{(\nu)}(\gamma^\omega)$$

if ${}^{(\nu)}T'$, $\nu=1,\ldots,R$, represent the conjugacy classes of Cartan subgroups of H originating from T, and

(b) $\sum_{\pi\in\Pi_\varphi} \varepsilon(\pi)\chi_\pi \equiv 0$ *on those Cartan subgroups of G from which no Cartan subgroup of H originates.*

Here we have, of course, regarded characters as (analytic) functions on G_{reg}. In Sect. 5 we will return to the stronger distribution-theoretic version of these identities.

5. Inversion of Character Identities and the Structure of Tempered L-Packets

Our last task is to invert the identities of Theorem 4.1.1, following the procedure outlined in the introduction to this paper. We will assume that for each endoscopic group H for G, ${}^LH$ embeds admissibly in ${}^LG$ (cf. [11, 16] and comments in (1.3)). Our fixed data will be that of Sect. 1: the usual ψ, $\mathbf{G}^*$, …, ${}^LG^0$, … etc., together with a *skeleton* for G. To formulate our results further choices are needed, but we will show that all but one of these choices are of no consequence. Thus, we fix a complete set of representatives for $\mathfrak{S}({}^LG)$. If $\mathbf{H}$ is an endoscopic group for $\mathbf{G}$ we denote by $\mathfrak{s}_H=(s_H, {}^LH)$ the attached representative. We assume that ${}^LH$ is in standard position with respect to some subgroup in our skeleton. We choose an admissible embedding $\xi: {}^LH \hookrightarrow {}^LG$ of unitary type and let

$$\{\Delta_{(T,\eta)} = (-1)^{q(G,H)}\varepsilon(T,\eta)\,\Delta_{(T,\eta)}\,\grave{\Delta}_{(T,\eta)}\}$$

be one of the two attached admissible families of (G,H)-orbital-integral-transfer factors. If $f\in\mathscr{C}(G)$ then f_H will denote some function in $\mathscr{C}(H)$ corresponding to f

under the transfer so prescribed. Recall that Theorem 4.1.1 states that

$$\chi_{\varphi'}(f_H)=\begin{cases}\sum_{\pi\in\Pi_\varphi}\varepsilon(\pi)\chi_\pi(f) & \textit{if } \varphi=\xi\circ\varphi' \textit{ is relevant to } G\\ 0 & \textit{otherwise.}\end{cases}$$

For most of this section we will find it convenient to distinguish in notation between a tempered admissible homomorphism $\varphi: W\to {}^LG$ and its class in $\Phi_0(G)$; as earlier, $\{\varphi\}$ will then denote the class of φ.

Definition 5.0.1. *Let* $\{\varphi\}\in\Phi_0(G)$. *Then* $\{\varphi\}$ *factors through* $\{\varphi_H\}\in\Phi_0(H)$ *or, equivalently,* $\{\varphi_H\}$ *lifts to* $\{\varphi\}$, *if* $\varphi'=\xi\circ\varphi'_H$ *for some* $\varphi'\in\{\varphi\}$, $\varphi'_H\in\{\varphi_H\}$.

(5.1) *The Group* $\mathbb{S}_{\{\varphi\}}$

Let $\varphi: W\to {}^LG$ be a tempered admissible homomorphism. Denote by S_φ the centralizer of $\varphi(W)$ in ${}^LG^0$, by S_φ^0 the connected component of the identity in S_φ, and by Z^W the set of W-invariants in the center of ${}^LG^0$ [recall the role of Z^W in (2.1)]. We define

(5.1.1) $$\mathbb{S}_\varphi=S_\varphi/Z^W S_\varphi^0.$$

If $\mathbf{G}$ is simply-connected then ${}^LG^0$ is adjoint, so that $\mathbb{S}_\varphi=S_\varphi/S_\varphi^0$. In general, let $p: {}^LG\to({}^LG^0)_{\mathrm{ad}}\rtimes W$ denote the natural projection (cf. [2]) and $\varphi_*=p\circ\varphi$. Then $\mathbb{S}_\varphi$ is naturally a subgroup of $\mathbb{S}_{\varphi_*}=S_{\varphi_*}/S_{\varphi_*}^0$.

Suppose that $\varphi': W\to {}^LG$ is equivalent to φ. If $\varphi'=\mathrm{ad}\,g\circ\varphi$, $g\in {}^LG^0$, then $\mathrm{ad}\,g$ induces an isomorphism $\mathbb{S}_\varphi\to\mathbb{S}_{\varphi'}$. Langlands has shown that $\mathbb{S}_\varphi$ is abelian and, in fact, a sum of groups of order two [by arguments similar to those reported in (5.3); see also Corollary 5.4.10]. Hence the isomorphism $\mathbb{S}_\varphi\to\mathbb{S}_{\varphi'}$ is canonical. We will often write $\mathbb{S}_{\{\varphi\}}$ in place of $\mathbb{S}_\varphi$.

(5.2) *An Example: The L-Packets of Discrete Series Representations*

We include this discussion only as motivation for the more general arguments of (5.4).

Suppose that $\{\varphi\}\in\Phi_0(G)$ is discrete and choose a representative $\varphi=\varphi(\mu,\lambda)$ for $\{\varphi\}$, with μ strictly $\Delta({}^LB^0, {}^LT^0)$-dominant.

On the other hand, let T be a Cartan subgroup of G compact modulo the center of G, and η be a p.d. of T. The group $\mathscr{E}(T)$ has been defined in [11] (cf. [15]). By Tate-Nakayama duality, it may be identified with

$$X_*(\mathbf{T}_{\mathrm{sc}})/X_*(\mathbf{T}_{\mathrm{sc}})\cap\{\nu^\vee-\sigma_T\nu^\vee:\nu^\vee\in X_*(\mathbf{T})\}.$$

Proposition 5.2.1. η *induces an isomorphism between* $\mathbb{S}_\varphi$ *and* $\mathscr{E}(T)^\vee$, *the dual of* $\mathscr{E}(T)$.

Proof. By [10] (cf. [2 (10.5)]), S_φ is contained in ${}^LT^0$. Thus S_φ is simply the set of $\bar\sigma$-invariants in ${}^LT^0$ [$\bar\sigma$ was defined in (4.3)]. The isomorphism between $\mathbb{S}_\varphi$ and $\mathscr{E}(T)^\vee$ is then produced as in [16]. In particular, $\mathbb{S}_{\{\varphi\}}$ is a sum of groups of order two.

To each $\mathfrak{x}\in\mathbb{S}_{\{\varphi\}}$ we now attach a representative $\mathfrak{s}(\mathfrak{x})$ for an element of $\mathfrak{S}({}^LG)$. By definition, $\mathfrak{x}$ is a coset of Z^W in ${}^LT^0$. Thus we may form $(\mathfrak{x}, \mathrm{Cent}(\mathfrak{x})^0, {}^LB^0\cap\mathrm{Cent}(\mathfrak{x})^0, {}^LT^0, \{Y_{\alpha^\vee}\})$ as usual [cf. (2.3)]. To define the action of W on $\mathrm{Cent}(\mathfrak{x})^0$, we have that

$\varphi(1\times\sigma)$ fixes $\mathfrak{x}$, normalizes ${}^LT^0$ and acts on ${}^LT^0$ as $\bar{\sigma}$, an automorphism of order two. We thus extract the action by the method of Lemma 2.1.7.

There is exactly one of our fixed representatives for $\mathfrak{S}({}^LG)$, say $\mathfrak{s}_H=(s_H,{}^LH)$, equivalent to $\mathfrak{s}(\mathfrak{x})$. Suppose that $g\in{}^LG^0$ and g maps $\mathfrak{s}(\mathfrak{x})$ to $\mathfrak{s}_H$ in the sense of (2.1).

Proposition 5.2.2. *$g\in\Omega({}^LG^0,{}^LT^0)$ and $\mathfrak{x}^g=s_H$.*

Proof. That $g\in\Omega({}^LG^0,{}^LT^0)$ follows from definitions. Next, we argue in ${}^LG^0/Z^W$. It is clear that $\alpha^\vee(s_H)=1$ if and only if $\alpha^\vee(\mathfrak{x}^g)=1$, $\alpha^\vee\in\Delta({}^LG^0,{}^LT^0)$. To complete the proof we have only to observe that either s_H and $\mathfrak{x}^g$ are of order two or both are trivial.

It now follows that g may be replaced only by τg, where $\tau\in\mathrm{Cent}(s_H)$ induces an automorphism of $({}^LH^0,\ {}^LH^0\cap{}^LB^0,\ {}^LT^0,\ \{Y_{\alpha^\vee}\},\ \{w_H\})$.

We use g to construct a homomorphism $\varphi': W\to{}^LH$ such that $\{\varphi'\}$ lifts to $\{\varphi\}$. Indeed, for each $w\in W$, $\varphi(w)$ acts on $\mathrm{Cent}(\mathfrak{x})^0$ as an element of ${}^LH_{\mathfrak{x}}$, the L-group defined by $\mathfrak{s}(\mathfrak{x})$. Hence $g\varphi(w)g^{-1}$ acts on ${}^LH^0$ as an element of ${}^LH$. We conclude then that $g\varphi(W)g^{-1}$ lies in the image of $\xi:{}^LH\hookrightarrow{}^LG$. Thus $\mathrm{ad}\,g\circ\varphi$, which has parameters $(g\mu,\lambda)$ [or $(g\mu,g\lambda)$], is the lift of some $\varphi':W\to{}^LH$ with parameters $(g\mu-\mu^*,\ \lambda-\lambda^*)$. If g is replaced by τg, τ as above, then φ' is replaced by φ'_τ with parameters $(\tau g\mu-\mu^*,\ \lambda-\lambda^*)$.

In conclusion, to each $\mathfrak{x}\in\mathbb{S}_{\{\varphi\}}$ we have attached $\mathfrak{s}(\mathfrak{x})$ equivalent to some $\mathfrak{s}_H=(s_H,{}^LH)$ among our chosen representatives for $\mathfrak{S}({}^LG)$, and a parameter $\{\varphi'\}$ for H which lifts to $\{\varphi\}$ (or more precisely, a family $\{\varphi'_\tau\}$ of parameters which lift to $\{\varphi\}$).

We have $\chi_{\{\varphi'\}}(f_H)=\sum\limits_{\pi\in\Pi_{\{\varphi\}}}\varepsilon(\pi)\chi_\pi(f)$. Recall the definition of $\varepsilon(\pi)$. We choose any $(T,\eta)\in\mathcal{T}_H(G)$ with T compact modulo the center of G. Note that $g\mu-\mu^*$ is $\Delta({}^LH^0\cap{}^LB^0,\ {}^LT^0)$-dominant, and let $\Psi=\Delta({}^LB^0,{}^LT^0)$. Then

$$\varepsilon(\pi(\omega,\Psi))=\varepsilon(T,\eta)\det g\ \ \kappa(\omega g^{-1}),\ \omega\in\Omega(\mathbf{G},\mathbf{T}),$$

where, as usual, g has been transferred to $\mathbf{T}$ via η without change in notation.

Our fixed skeleton provides a (unique) suitable pair (T,η) for computing $\varepsilon(\pi)$. We use this pair, and then extract from $\varepsilon(\pi)$ a term which would otherwise fail to be well defined. Thus, we write

$$\varepsilon(\pi(\omega,\Psi))=\varepsilon(T,\eta)\det g\ \ \kappa(g^{-1})\kappa^{g^{-1}}(\omega),\ \omega\in\Omega(\mathbf{G},\mathbf{T}).$$

Note that if $\{\varphi'\}$ is replaced by $\{\varphi'_\tau\}$, so that g is replaced by τg, then $\kappa^{g^{-1}}(\omega)$ is replaced by $(\kappa^{\tau^{-1}})^{g^{-1}}(\omega)=\kappa^{g^{-1}}(\omega)$, since $s_H^\tau=s_H$; that is, $\kappa^{g^{-1}}(\omega)$ is unchanged. Hence we may set

$$\langle\mathfrak{x},\pi(\omega,\Psi)\rangle=\kappa^{g^{-1}}(\omega),\ \omega\in\Omega(\mathbf{G},\mathbf{T}).$$

We emphasize that $\langle\mathfrak{x},\pi(\omega,\Psi)\rangle$ is well-defined only because (T,η) has been prescribed.

Clearly $\varepsilon(T,\eta)\det g\kappa(g^{-1})\chi_{\{\varphi'\}}(f_H)$ must also be independent of the choice for $\{\varphi'\}$ in the family attached to $\mathfrak{x}$. We set

$$\hat{\chi}_{\{\varphi'\}}(f_H)=\varepsilon(T,\eta)\det g\ \ \kappa(g^{-1})\chi_{\{\varphi'\}}(f_H).$$

Our identity then becomes:

(5.2.3) $$\sum_{\pi\in\Pi_{\{\varphi\}}} \langle\mathfrak{x},\pi\rangle\chi_\pi(f)=\hat{\chi}_{\{\varphi'\}}(f_H).$$

A more natural way to compute $\langle\mathfrak{x},\pi\rangle$ is the following. Regard $\mathfrak{x}$ as a character on $X_*(\mathbf{T}^*_{sc})/X_*(\mathbf{T}^*_{sc})\cap\{v^\vee-\bar{\sigma}v^\vee: v^\vee\in X_*(\mathbf{T}^*)\otimes\mathbb{C}\}$ (cf. [16, Sect. 2.1]) and let $\tilde{\omega}_\sigma$ be the element of this quotient determined by ω and η (cf. [15, Sect. 2]).

Proposition 5.2.4. $\langle\mathfrak{x},\pi(\omega,\Psi)\rangle=\mathfrak{x}(\tilde{\omega}_\sigma)$.

The proof is immediate.

Corollary 5.2.5. (i) $\langle\mathfrak{x},\pi\rangle$ *is independent of the choice for* $\mathfrak{s}_H=(s_H,{}^LH)$, $\xi:{}^LH\hookrightarrow{}^LG$ *and the family* $\{\Delta_{(T,\eta)}\}$.

(ii) $\hat{\chi}_{\{\varphi'\}}(f_H)$ *depends only on* $\{\varphi\}$, $\mathfrak{x}$ *and* f.

We write $\hat{\chi}_{(\{\varphi\},\mathfrak{x})}(f)$ in place of $\hat{\chi}_{\{\varphi'\}}(f_H)$.

As an immediate consequence of Propositions 5.2.1 and 5.2.4, we have:

Lemma 5.2.6. (i) $\langle\mathfrak{x}\mathfrak{y},\pi\rangle=\langle\mathfrak{x},\pi\rangle\langle\mathfrak{y},\pi\rangle$ *for* $\mathfrak{x},\mathfrak{y}\in\mathbb{S}_{\{\varphi\}}$, $\pi\in\Pi_{\{\varphi\}}$.

(ii) $\langle\mathfrak{x},\pi\rangle=\langle\mathfrak{x},\pi'\rangle$ *for all* $\mathfrak{x}\in\mathbb{S}_{\{\varphi\}}$ *if and only if* $\pi=\pi'$.

We now omit $\{\ \}$ in notation. We have identified Π_φ as a subset of $\mathbb{S}_\varphi^\vee$, the dual of $\mathbb{S}_\varphi$. This subset may be proper. Let $\bar{\Pi}_\varphi$ be some fixed set containing Π_φ and in $1-1$ correspondence with $\mathbb{S}_\varphi^\vee$. If $\bar{\pi}\in\bar{\Pi}_\varphi-\Pi_\varphi$, let $\mathfrak{x}\to\langle\mathfrak{x},\bar{\pi}\rangle$ denote the character on $\mathbb{S}_\varphi$ corresponding to $\bar{\pi}$ and define the character $\chi_{\bar{\pi}}$ of $\bar{\pi}$ to be the zero distribution on G. We will call $\bar{\pi}$ a *ghost*. The identity (5.2.3) is then replaced by:

Theorem 5.2.7. $\sum_{\pi\in\bar{\Pi}_\varphi}\langle\mathfrak{x},\pi\rangle\chi_\pi(f)=\hat{\chi}_{(\varphi,\mathfrak{x})}(f)$, $f\in\mathscr{C}(G)$.

We are now prepared for inversion. Let $\pi_0\in\bar{\Pi}_\varphi$. Then

$$\sum_{\mathfrak{x}\in\mathbb{S}_\varphi}\sum_{\pi\in\bar{\Pi}_\varphi}\langle\mathfrak{x},\pi\rangle\langle\mathfrak{x},\pi_0\rangle\chi_\pi(f)=\sum_{\mathfrak{x}\in\mathbb{S}_\varphi}\langle\mathfrak{x},\pi_0\rangle\hat{\chi}_{(\varphi,\mathfrak{x})}(f).$$

We therefore conclude:

Theorem 5.2.8. $\chi_{\pi_0}(f)=\dfrac{1}{[\mathbb{S}_\varphi]}\sum_{\mathfrak{x}\in\mathbb{S}_\varphi}\langle\mathfrak{x},\pi_0\rangle\hat{\chi}_{(\varphi,\mathfrak{x})}(f)$, $\pi_0\in\bar{\Pi}_\varphi$.

Corollary 5.2.9. *If* π_0 *is a ghost then*

$$\sum_{\mathfrak{x}\in\mathbb{S}_\varphi}\langle\mathfrak{x},\pi_0\rangle\hat{\chi}_{(\varphi,\mathfrak{x})}(f)=0,\ f\in\mathscr{C}(G).$$

(5.3) *Preliminaries*

In this subsection we recall Langlands' (unpublished) L-group description of the stability group of a discrete series representation of a Levi group in G (cf. [7], for a summary). With his permission we have included his arguments along with the statements we need. One proof (cf. Lemma 5.3.13) is a transcription of arguments of Knapp and Zuckerman for Theorem 2.3 of [7].

Our main conclusion will be that a lemma of Harish-Chandra shows that the order of the group R_φ introduced by Langlands is an upper bound for the number of irreducible constituents of an attached principal series representation. We do not make explicit the connection with the results of [6], since in (5.4) it will be clear from Harish-Chandra's lemma and a necessary and sufficient condition for the existence of a Hecht-Schmid character identity (cf. [6], also [17]), how to decompose unitary principal series representations directly in terms of L-group data.

Let M be a cuspidal Levi group in G, and consider a tempered homomorphism $\varphi: W \to {}^L G$ which factors through ${}^L M$, but through no proper parabolic subgroup of ${}^L M$. We assume that $\varphi = \varphi(\mu, \lambda)$, the parameters being defined relative to ${}^L M$ [cf. (4.3)]; μ is $\Delta({}^L M^0, {}^L T^0)$-regular.

As before, S_φ is the centralizer of $\varphi(W)$ in ${}^L G^0$. We say that $\omega \in \Omega({}^L G^0, {}^L T^0)$ is realized in S_φ if there is $s \in S_\varphi$ such that $\omega = \operatorname{ad} s|_{{}^L T^0}$; $\Omega_\varphi({}^L G^0, {}^L T^0)$ will denote the subgroup of $\Omega({}^L G^0, {}^L T^0)$ consisting of elements realized in S_φ.

Proposition 5.3.1. *$\omega \in \Omega({}^L G^0, {}^L T^0)$ is realized in S_φ if and only if ω commutes with the action of $\varphi(1 \times \sigma)$ on ${}^L T^0$ and $\omega\mu = \mu, \omega\lambda \equiv \lambda \operatorname{mod}(X_*({}^L T^0) + \{v - \varphi(1 \times \sigma)v : v \in X_*({}^L T^0) \otimes \mathbb{C}\}$.*

Proof. If ω is realized in S_φ then only the congruence requires an argument. By definition, $\varphi(1 \times \sigma) = \varphi_0(\sigma) \times (1 \times \sigma)$, where $\varphi_0(\sigma) \in {}^L M^0$ and $\lambda^\vee(\varphi_0(\sigma)) = e^{2\pi i \langle \lambda, \lambda^\vee \rangle}$, $\lambda^\vee \in X^*({}^L M^0)$. If $w \in S_\varphi$ realizes ω then w normalizes ${}^L M^0$. Thus $e^{2\pi i \langle \omega\lambda, \lambda^\vee \rangle} = e^{2\pi i \langle \lambda, \lambda^\vee \rangle}$, $\lambda^\vee \in X^*({}^L M^0)$. An argument as in [10] (cf. proof of Proposition 3.4.2 of [16]) the yields the congruence.

For the converse we note first that we may write any $\omega \in \Omega({}^L G^0, {}^L T^0)$ which commutes with the action of $\varphi(1 \times \sigma)$ on ${}^L T^0$ as a product $\omega_1 \omega_2$, where $\omega_1 \in \Omega({}^L M^0, {}^L T^0)$ and ω_2 commutes with the action of σ_G (... we argue in $\mathbf{G}$, using [14, Theorem 2.1] and [3, Corollary 3.5]). Then if $\omega\mu = \mu$ and $\omega\lambda \equiv \lambda \operatorname{mod}(X_*({}^L T^0) + \{v - \varphi(1 \times \sigma)v : v \in X_*({}^L T^0) \otimes \mathbb{C}\})$ also, we have that ω_1 commutes with σ_G too (argument in $\mathbf{G}$, using Lemma 3.2 of [14]). We conclude then that $\omega\sigma_G = \sigma_G\omega$. By [10], (cf. also [9]) there is an element w of ${}^L G^0$ which realizes ω and is fixed by σ_G. Let $\varphi(1 \times \sigma) = z_1 m_1 \times (1 \times \sigma)$, $z_1 \in Z({}^L M^0)$, $m_1 \in ({}^L M^0)_{\mathrm{der}}$. Then $w\varphi(1 \times \sigma)w^{-1}\varphi(1 \times \sigma)^{-1} = w z_1 w^{-1} z_1^{-1} t$, where $t \in {}^L T^0 \cap ({}^L M^0)_{\mathrm{der}}$. Because of the congruence and the fact that $\varphi(1 \times \sigma) t \varphi(1 \times \sigma)^{-1} = t^{-1}$, $w z_1 w^{-1} z_1^{-1} t$ may be written as $u^{-1}\varphi(1 \times \sigma) u \varphi(1 \times \sigma)^{-1}$, for some $u \in {}^L T^0$. Then uw realizes ω and lies in S_φ. This completes the proof.

We will identify $\Omega_\varphi({}^L G^0, {}^L T^0)$ with $\operatorname{Norm}({}^L G^0, {}^L T^0) \cap S_\varphi / {}^L T^0 \cap S_\varphi$ when convenient (cf. Proposition 5.2.3). It is easier now to work with Lie algebras: ${}^L\mathfrak{g}$, ${}^L\mathfrak{b}$, ${}^L\mathfrak{t}$, $\mathfrak{s}_\varphi$ will denote the Lie algebras of ${}^L G^0$, ${}^L B^0$, ${}^L T^0$, S_φ respectively. Note that $\mathfrak{s}_\varphi$ is reductive. Let

$$ {}^L\mathfrak{t}_\varphi = \{X \in {}^L\mathfrak{t} : \varphi(1 \times \sigma)X = X\}. $$

Proposition 5.3.2. *${}^L\mathfrak{t}_\varphi$ is a Cartan subalgebra of $\mathfrak{s}_\varphi$.*

Proof. See [7, Lemma 3.2].

Let $\mathrm{A}_{S_\varphi}(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi)$ denote the group of automorphisms of $\mathfrak{s}_\varphi$ which preserve ${}^L\mathfrak{t}_\varphi$ and are of the form $\operatorname{Ad} s$, $s \in S_\varphi$, modulo $\operatorname{Ad}({}^L T^0 \cap S_\varphi) = \operatorname{Ad}({}^L T^0 \cap S_\varphi^0)$.

Proposition 5.3.3. $\Omega_\varphi({}^LG^0, {}^LT^0) = \mathrm{A}_{S_\varphi}(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi)$.

Proof. We have only to show that if $s \in S_\varphi$ fixes ${}^L\mathfrak{t}_\varphi$ then $s \in {}^LT^0$. But if s fixes ${}^L\mathfrak{t}_\varphi$ then $s \in {}^LM^0$. Since φ is discrete relative to ${}^LM$, $S_\varphi \cap {}^LM^0 = S_\varphi \cap {}^LT^0$, and the proposition is proved.

We will need detailed information about the root spaces for $(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi)$. If $\alpha^\vee \in \Delta({}^L\mathfrak{g}, {}^L\mathfrak{t})$ we set $\mathfrak{X}_{\alpha^\vee} = \{\beta^\vee \in \Delta({}^L\mathfrak{g}, {}^L\mathfrak{t}) : \beta^\vee|_{{}^L\mathfrak{t}_\varphi} = c\alpha^\vee|_{{}^L\mathfrak{t}_\varphi}$, some $c > 0\}$ and $\mathfrak{s}_{\alpha^\vee} = \mathfrak{s} \cap \sum_{\beta^\vee \in \mathfrak{X}_{\alpha^\vee}} \mathbb{C}X_{\beta^\vee}$. Clearly each weight space for the adjoint action of ${}^L\mathfrak{t}_\varphi$ on ${}^L\mathfrak{g}$ is contained in some $\sum_{\beta^\vee \in \mathfrak{X}_{\alpha^\vee}} \mathbb{C}X_{\beta^\vee}$. Thus each root space for $(\mathfrak{g}_\varphi, {}^L\mathfrak{t}_\varphi)$ is contained in some $\mathfrak{s}_{\alpha^\vee}$.

Proposition 5.3.4. *Suppose that* $\varphi(1 \times \sigma)\alpha^\vee = \alpha^\vee$. *Then*

$$\mathfrak{s}_{\alpha^\vee} = \begin{cases} \mathbb{C}X_{\alpha^\vee} & \text{if} \quad \varphi(1\times\sigma)X_{\alpha^\vee} = X_{\alpha^\vee}, \langle \mu, \alpha^\vee \rangle = 0 & (5.3.5) \\ \mathbb{C}(X_{\beta^\vee} + \varphi(1\times\sigma)X_{\beta^\vee}) & \text{if} \quad \alpha^\vee = \beta^\vee + \varphi(1\times\sigma)\beta^\vee, \text{ where } \langle \mu, \beta^\vee \rangle = 0 & (5.3.6) \\ 0 & \text{otherwise.} & (5.3.7) \end{cases}$$

In case of (5.3.5) *and* (5.3.7), $\langle \mu, \gamma^\vee \rangle \neq 0$ *for all* $\gamma^\vee \in \mathfrak{X}_{\alpha^\vee}$ *distinct from* $\alpha^\vee$; *in case of* (5.3.6), $\varphi(1\times\sigma)X_{\alpha^\vee} = -X_{\alpha^\vee}$, $\beta^\vee$ *is uniquely determined and* $\langle \mu, \gamma^\vee \rangle \neq 0$ *for all* $\gamma^\vee \in \mathfrak{X}_{\alpha^\vee}$ *distinct from* $\alpha^\vee$, $\beta^\vee$, $\varphi(1\times\sigma)\beta^\vee$. *Suppose that* $\varphi(1\times\sigma)\beta^\vee \neq \beta^\vee$ *for all* $\beta^\vee \in \mathfrak{X}_{\alpha^\vee}$. *Then there exists at most one root* $\beta^\vee \in \mathfrak{X}_{\alpha^\vee}$ *such that* $\langle \mu, \beta^\vee \rangle = 0$, *and*

$$\mathfrak{s}_{\alpha^\vee} = \begin{cases} \mathbb{C}(X_{\beta^\vee} + \varphi(1\times\sigma)X_{\beta^\vee}) & \text{if} \quad \beta^\vee \in \mathfrak{X}_{\alpha^\vee} \quad \text{and} \quad \langle \mu, \beta^\vee \rangle = 0 & (5.3.8) \\ 0 & \text{if } \langle \mu, \beta^\vee \rangle \neq 0, \quad \beta^\vee \in \mathfrak{X}_{\alpha^\vee}. & (5.3.9) \end{cases}$$

Proof. Note that in the definition of $\mathfrak{s}_{\alpha^\vee}$ we may replace "$\beta^\vee \in \mathfrak{X}_{\alpha^\vee}$" by "$\beta^\vee \in \mathfrak{X}_{\alpha^\vee}, \langle \mu, \beta^\vee \rangle = 0$", by arguing with $({}^L\mathfrak{h}, {}^L\mathfrak{t})$, where ${}^L\mathfrak{h}$ is the centralizer of $\varphi(\mathbb{C}^\times)$ in ${}^L\mathfrak{g}$. In either case we obtain a set of roots preserved by $\varphi(1\times\sigma)$.

Suppose that $\varphi(1\times\sigma)\alpha^\vee = \alpha^\vee$, $\varphi(1\times\sigma)X_{\alpha^\vee} = X_{\alpha^\vee}$ and $\langle \mu, \alpha^\vee \rangle = 0$. Then to prove (5.3.5) it is sufficient to show that if $\beta^\vee \in \mathfrak{X}_{\alpha^\vee}$, $\beta^\vee \neq \alpha^\vee$, then $\langle \mu, \beta^\vee \rangle \neq 0$. Suppose on the contrary that $\langle \mu, \beta^\vee \rangle = 0$. Clearly $\varphi(1\times\sigma)\beta^\vee \neq \beta^\vee$; $\alpha^\vee$, $\beta^\vee$, $\varphi(1\times\sigma)\beta^\vee$ generate a root system invariant under $\varphi(1\times\sigma)$. Also $\langle \mu, \gamma^\vee \rangle = 0$ for each root γ in this system. If $\langle \alpha, \gamma^\vee \rangle = 0$ then $\varphi(1\times\sigma)\gamma^\vee = -\gamma^\vee$ so that $\langle \mu, \gamma^\vee \rangle \neq 0$. Thus the system is of type A_2 and $\alpha^\vee = \beta^\vee + \varphi(1\times\sigma)\beta^\vee$. This implies that $X_{\alpha^\vee} = c[X_{\beta^\vee}, \varphi(1\times\sigma)X_{\beta^\vee}]$, some $c \neq 0$, so that $\varphi(1\times\sigma)X_{\alpha^\vee} = -X_{\alpha^\vee}$, a contradiction. Hence (5.3.5) follows.

Suppose next that $\alpha^\vee = \beta^\vee + \varphi(1\times\sigma)\beta^\vee$, where $\langle \mu, \beta^\vee \rangle = 0$. Then $\beta^\vee \in \mathfrak{X}_{\alpha^\vee}$ and $\langle \mu, \beta^\vee - \varphi(1\times\sigma)\beta^\vee \rangle = 0$, so that $\beta^\vee - \varphi(1\times\sigma)\beta^\vee$ is not a root. Hence $\alpha^\vee$, $\beta^\vee$, $\varphi(1\times\sigma)\beta^\vee$ generate a root system of type A_2; clearly $\varphi(1\times\sigma)X_{\alpha^\vee} = -X_{\alpha^\vee}$. To prove (5.3.6) and that $\beta^\vee$ is the only root with the property that $\langle \mu, \beta^\vee \rangle = 0$ and $\alpha^\vee = \beta^\vee + \varphi(1\times\sigma)\beta^\vee$, we have only to show that if $\gamma^\vee \in \mathfrak{X}_{\alpha^\vee}$, $\gamma^\vee \neq \alpha^\vee$, $\beta^\vee$, $\varphi(1\times\sigma)\beta^\vee$ then $\langle \mu, \gamma^\vee \rangle \neq 0$. Suppose that $\gamma^\vee \in \mathfrak{X}_{\alpha^\vee}$, $\gamma^\vee \neq \alpha^\vee$, $\beta^\vee$, $\varphi(1\times\sigma)\beta^\vee$ and that $\langle \mu, \gamma^\vee \rangle = 0$. Then $\gamma^\vee - \sigma\gamma^\vee$ is not a root. A check of two-dimensional diagrams shows that $\gamma^\vee$ cannot lie in the $\mathbb{Q}$-span of $\alpha^\vee$ and $\beta^\vee$. Thus the root system R generated by $\alpha^\vee$, $\beta^\vee$, $\gamma^\vee$ is three dimensional. Since $\langle \mu, \delta^\vee \rangle = 0$, $\delta^\vee \in R$, we have that $\varphi(1\times\sigma)\delta^\vee \neq -\delta^\vee$, $\delta^\vee \in R$. Thus $\delta^\vee + \varphi(1\times\sigma)\delta^\vee = c\alpha^\vee$, some $c \neq 0$, and $\{\delta^\vee \in R : \langle \alpha, \delta^\vee \rangle > 0\}$ is a positive system for R invariant under $\varphi(1\times\sigma)$. There are three simple roots for this system; one is $\alpha^\vee$ and the other two are $\delta^\vee$ and $\varphi(1\times\sigma)\delta^\vee$, for some $\delta^\vee \in R$. Since $\delta^\vee + \varphi(1\times\sigma)\delta^\vee = a\alpha^\vee$, $a \neq 0$, we have a contradiction.

To complete the arguments for the case $\varphi(1\times\sigma)\alpha^\vee=\alpha^\vee$, assume that $\varphi(1\times\sigma)X_{\alpha^\vee}=X_{\alpha^\vee}$. We have proved the assertions if $\langle\mu,\alpha^\vee\rangle=0$. If $\langle\mu,\alpha^\vee\rangle\neq 0$ then we claim that $\langle\mu,\beta^\vee\rangle\neq 0$ for all $\beta^\vee\in\mathfrak{X}_{\alpha^\vee}$. Indeed, suppose that $\langle\mu,\beta^\vee\rangle=0$. Then $\beta^\vee+\varphi(1\times\sigma)\beta^\vee=c\alpha^\vee$, some $c\neq 0$ and $\langle\mu,\alpha^\vee\rangle=0$, a contradiction. On the other hand, if $\varphi(1\times\sigma)X_{\alpha^\vee}=-X_{\alpha^\vee}$ we have proved the assertions when $\alpha^\vee=\beta^\vee+\varphi(1\times\sigma)\beta^\vee$, where $\langle\mu,\beta^\vee\rangle=0$. If $\alpha^\vee$ is not of this form then arguments as in the second paragraph show that $\langle\mu,\beta^\vee\rangle\neq 0$ for any $\beta^\vee\in\mathfrak{X}_\alpha$ different from $\alpha^\vee$, and we are done.

The arguments for (5.3.8) and (5.3.9) follow the same lines. We omit the details.

Corollary 5.3.10. *The non-zero spaces among the* $\mathfrak{s}_{\alpha^\vee}$*'s are precisely the root spaces for* $(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi)$.

Clearly the positive system $\Delta({}^LB^0, {}^LT^0)$ determines a (unique) Borel subalgebra $\mathfrak{s}_\varphi^+$ of $\mathfrak{s}_\varphi$ containing ${}^L\mathfrak{t}_\varphi$. We may write

(5.3.11) $$A_{S_\varphi}(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi)=R_\varphi\Omega(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi),$$

where $R_\varphi=A_{S_\varphi}(\mathfrak{s}_\varphi, \mathfrak{s}_\varphi^+, {}^L\mathfrak{t}_\varphi)$, the group of elements in $A_{S_\varphi}(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi)$ preserving $\mathfrak{s}_\varphi^+$. There is an exact sequence:

(5.3.12) $$1\to S_\varphi\cap {}^LT^0/S_\varphi^0\cap {}^LT^0\to S_\varphi/S_\varphi^0\to R_\varphi\to 1.$$

The next result is a transcription of Theorem 2.3 in [7], (although we leave implicit the connection between $\Delta_\varphi^\vee$ and the superorthogonal set of [7]). We regard R_φ now as a subgroup of $\Omega({}^L\mathfrak{g}, {}^L\mathfrak{t})=\Omega({}^LG^0, {}^LT^0)$. Let Q be the operator on $\operatorname{Hom}({}^L\mathfrak{t}, \mathbb{C})$ defined by $Q\lambda=\frac{1}{[R_\varphi]}\sum_{r\in R_\varphi} r\lambda$, $\lambda\in\operatorname{Hom}({}^L\mathfrak{t},\mathbb{C})$. The analogue of Q on $\mathbf{G}$ was introduced by Knapp and Zuckerman, and indeed the following argument is essentially theirs; we include details only for the sake of completeness.

Lemma 5.3.13. *Let* $\Delta_\varphi^\vee=\{\alpha^\vee\in\Delta({}^L\mathfrak{g}, {}^L\mathfrak{t})\langle\mu,\alpha^\vee\rangle=0,\ Q\alpha^\vee=0\}$. *Then*

(i) *if* $\alpha^\vee\in\Delta_\varphi^\vee$ *then* $\varphi(1\times\sigma)\alpha^\vee=\alpha^\vee$ *and* $\varphi(1\times\sigma)X_{\alpha^\vee}=-X_{\alpha^\vee}$,

(ii) $\Delta_\varphi^\vee$ *is of type* $A_1\times A_1\times\ldots\times A_1$,

(iii) R_φ *is contained in* $\Omega(\Delta_\varphi^\vee)$, *the Weyl group generated by* $\Delta_\varphi^\vee$, *and*

(iv) *each* $\alpha^\vee\in\Delta_\varphi^\vee$ *appears in the expression of some* $r\in R_\varphi$ *as a product of distinct reflections in* $\Omega(\Delta_\varphi^\vee)$.

Proof. For (i) assume that $\alpha^\vee\in\Delta_\varphi^\vee$. Then $\langle\mu,\alpha^\vee\rangle=0$ implies that $\varphi(1\times\sigma)\alpha^\vee\neq-\alpha^\vee$. Suppose that $\varphi(1\times\sigma)\alpha^\vee\neq\alpha^\vee$ also. Then $X_{\alpha^\vee}+\varphi(1\times\sigma)X_{\alpha^\vee}$ is a root vector for $(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi)$. The corresponding root is $\alpha_\dagger^\vee=\alpha^\vee|_{{}^L\mathfrak{t}_\varphi}$. We claim that $Q(\alpha^\vee)=0$ is impossible. To prove this we may assume that $\alpha^\vee\in\Delta({}^L\mathfrak{b}, {}^L\mathfrak{t})$. Then $\alpha_\dagger^\vee\in\Delta(\mathfrak{s}_\varphi^+, {}^L\mathfrak{t}_\varphi)$. Then also $r\alpha_\dagger^\vee\in\Delta(\mathfrak{s}_\varphi^+, {}^L\mathfrak{t}_\varphi)$, $r\in R_\varphi$. But $r\alpha_\dagger^\vee=r\alpha^\vee|_{{}^L\mathfrak{t}_\varphi}$. Thus $r\alpha^\vee\in\Delta({}^L\mathfrak{b}, {}^L\mathfrak{t})$, $r\in R_\varphi$, and $Q(\alpha^\vee)\neq 0$. We conclude that $\varphi(1\times\sigma)\alpha^\vee=\alpha^\vee$. If $\varphi(1\times\sigma)X_{\alpha^\vee}=X_{\alpha^\vee}$ then $\alpha^\vee\in\Delta(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi)$ and again $Q\alpha^\vee=0$ is contradicted. Thus $\varphi(1\times\sigma)X_{\alpha^\vee}=-X_{\alpha^\vee}$.

To prove (ii) we have only to show that if $\alpha^\vee, \beta^\vee\in\Delta_\varphi^\vee$ then $\alpha^\vee+\beta^\vee$ is not a root. But $\alpha^\vee, \beta\in\Delta_\varphi^\vee$ and $\alpha^\vee+\beta^\vee$ a root together imply that $X_{\alpha^\vee+\beta^\vee}=c[X_{\alpha^\vee}, X_{\beta^\vee}]$, some $c\neq 0$, and $\varphi(1\times\sigma)X_{\alpha^\vee+\beta^\vee}=X_{\alpha^\vee+\beta^\vee}$; at the same time $Q(\alpha^\vee+\beta^\vee)=Q\alpha^\vee+Q\beta^\vee=0$, a contradiction as before. Hence (ii) follows.

For (iii), R_φ is contained in $\Omega(\mathfrak{h}, {}^L\mathfrak{t})$, $\mathfrak{h}$ again denoting the centralizer of $\varphi(\mathbb{C}^\times)$ in ${}^L\mathfrak{g}$. Further, each element of R_φ fixes the image of Q, and so may be written as a product of reflections in $\Omega(\mathfrak{h}, {}^L\mathfrak{t})$, each fixing the image of Q. These are exactly the reflections with respect to the roots in $\Delta_\varphi^\vee$.

If $\alpha^\vee \in \Delta_\varphi^\vee$ and $\alpha^\vee$ does not appear in the expansion of some element of R_φ as a product of distinct reflections in $\Omega(\Delta_\varphi^\vee)$, then clearly $Q\alpha^\vee = \alpha^\vee$, a contradiction. Thus (iv) and the lemma are proved.

Suppose that $\varphi(\mu, \lambda)$ is replaced by $\varphi' = \operatorname{ad} w \circ \varphi(\mu, \lambda)$, $w \in \operatorname{Norm}({}^LM^0, {}^LT^0)$. Then φ' has parameters $(w\mu, \lambda)$. Clearly ${}^L\mathfrak{t}_{\varphi'} = {}^L\mathfrak{t}_\varphi$; examination of 5.3.4 shows that $\mathfrak{s}_{\varphi'} = \mathfrak{s}_\varphi$ and hence $S_{\varphi'}^0 = S_\varphi^0$. While $S_{\varphi'} = wS_\varphi w^{-1}$ and $\Omega_{\varphi'}({}^LG^0, {}^LT^0) = w\Omega_\varphi({}^LG^0, {}^LT^0)w^{-1}$, which may be distinct from S_φ, $\Omega_\varphi({}^LG^0, {}^LT^0)$ respectively, we have $\Delta_{\varphi'}^\vee = \Delta_\varphi^\vee$ and $R_{\varphi'} = R_\varphi$, by Lemma 5.3.13 or a direct argument with the definitions.

We turn now to representations. Suppose that ${}^LM$ is the L-group of a cuspidal Levi group $\mathbf{M} = \mathbf{M}_T$ in $\mathbf{G}$, attached in the manner of (1.3); implicit is the choice of p.d. η of $\mathbf{T}$; η will be used to transfer data from $\mathbf{T}$ to ${}^LT^0$, and vise-versa, without indication in notation. We continue with $\varphi = \varphi(\mu, \lambda)$. Let Ψ_M be that positive system for $\Delta({}^LM^0, {}^LT^0)$ with respect to which μ is dominant. The L-packet Π_φ^M consists of discrete series representations $\pi(\omega, \Psi_M)$, $\omega \in \Omega({}^LM^0, {}^LT^0)$, of M as described in (4.3); $\pi(\omega, \Psi_M)$ has character $\Theta(\omega\mu, \lambda, \omega\Psi_M)$. In (1.3) a parabolic subgroup $P = MN$ was defined. We set $\pi(\omega) = \operatorname{Ind}(\pi(\omega, \Psi_M) \otimes 1_N, P, G)$, $\omega \in \Omega({}^LM^0, {}^LT^0)$.

The restricted Weyl group $\mathscr{W}_T$ of $\mathbf{T}$ acts on (infinitesimal equivalence classes of) irreducible admissible representations of M. Let $\pi \in \Pi_\varphi^M$. We set

$$\mathscr{W} = \{\tilde{\omega} \in \mathscr{W}_T : \pi^{\tilde{\omega}} = \pi\}.$$

Then $\mathscr{W}$ is independent of the choice for π (cf. [14, Lemma 3.2]). Let $\Omega_0(\mathbf{G}, \mathbf{T}) = \{\omega \in \Omega(\mathbf{G}, \mathbf{T}) : \omega\sigma_T = \sigma_T\omega\}$. There is a natural projection $\Omega_0(\mathbf{G}, \mathbf{T}) \to \mathscr{W}_T$ (cf. [16, Sect. 5.1]). If $\tilde{\omega} \in \mathscr{W}$ there is a unique element ω in the preimage of $\tilde{\omega}$ satisfying $\omega\mu = \mu$ and $\omega\lambda \equiv \lambda \bmod (X^*(\mathbf{T}) + \{\nu - \sigma_T\nu : \nu \in X^*(\mathbf{T}) \otimes \mathbb{C}\}$. We thus have $\mathscr{W} \hookrightarrow \Omega_0(\mathbf{G}, \mathbf{T})$. On the other hand, η allows us to identify $\Omega_0(\mathbf{G}, \mathbf{T})$ with the subgroup of $\Omega({}^LG^0, {}^LT^0)$ consisting of the elements which commute with the action of $\varphi(1 \times \sigma)$ on ${}^LT^0$. We conclude the following from Proposition 5.3.1.

Lema 5.3.14. *η induces an isomorphism between $\mathscr{W}$ and $\Omega_\varphi({}^LG^0, {}^LT^0)$.*

By means of η we regard the coroot α of a root $\alpha^\vee$ of $({}^L\mathfrak{g}, {}^L\mathfrak{t})$ as a root of $(\mathfrak{g}, \mathfrak{t})$. Let $\mathfrak{a}$ be the split part of $\mathfrak{t}$. If $\tilde{\alpha}$ is a root of $(\mathfrak{g}, \mathfrak{a})$ we denote by $\mu_{\tilde{\alpha}}$ the Plancherel factor attached to $\tilde{\alpha}$ and (any one of) the representations in Π_φ^M (cf. [4, Sect. 13, 24, 36]).

Lemma 5.3.15. *$\alpha^\vee|_{{}^L\mathfrak{t}_\varphi}$ is a root of $(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi)$ if and only if $\mu_{\alpha|_\mathfrak{a}} = 0$ and $\alpha|_\mathfrak{a}$ is indivisible.*

Proof. Let $\alpha^\vee \in \Delta({}^L\mathfrak{g}, {}^L\mathfrak{t})$. Proposition 5.3.4 describes whether or not $\alpha^\vee|_{{}^L\mathfrak{t}_\varphi}$ belongs to $\Delta(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi)$. We need a further result. Suppose that $\varphi(1 \times \sigma)\alpha^\vee = \alpha^\vee$, and set ϱ_α equal to one-half the sum of the roots β of $(\mathfrak{g}, \mathfrak{t})$ such that $\beta|_\mathfrak{a} = c\alpha|_\mathfrak{a}$, some $c > 0$ (that is, such that $\beta^\vee \in \mathfrak{X}_{\alpha^\vee}$). Then

$$\varphi(1 \times \sigma)X_{\alpha^\vee} = -(-1)^{\langle 2\lambda + \varrho_\alpha, \alpha^\vee \rangle} X_{\alpha^\vee}. \tag{5.3.16}$$

We used this result in Sect. 4. The proof has been given in [1]. We conclude then that if $\varphi(1\times\sigma)\alpha^\vee=\alpha^\vee$ then $\alpha^\vee\in\Delta(\mathfrak{s}_\varphi, {}^L\mathfrak{t}_\varphi)$ if and only if either

(5.3.17) $$(-1)^{\langle 2\lambda+\varrho_\alpha,\alpha^\vee\rangle}=-1 \quad \text{and} \quad \langle\mu,\alpha^\vee\rangle=0$$

or

(5.3.18) $$(-1)^{\langle 2\lambda+\varrho_\alpha,\alpha^\vee\rangle}=1, \quad \langle\mu,\alpha^\vee\rangle=0$$

and $\langle\mu,\beta^\vee\rangle=0$ for some $\beta^\vee\in\mathfrak{X}_{\alpha^\vee}$ different from $\alpha^\vee$. Moreover, if $(-1)^{\langle 2\lambda+\varrho_\alpha,\alpha^\vee\rangle}=-1$ and $\langle\mu,\alpha^\vee\rangle\neq 0$ then $\langle\mu,\beta^\vee\rangle\neq 0$ for all $\beta^\vee\in\mathfrak{X}_{\chi^\vee}$.

We have now only to examine the explicit formulas for $\mu_{\tilde{\alpha}}$ in Sect. 24 and 36 of [4] to complete the proof of the lemma.

Corollary 5.3.16. *Each representation $\pi(\omega)$ has at most $[R_\varphi]$ constituents.*

In view of (5.3.3), (5.3.11), (5.3.14) and (5.3.15) this is a restatement of the corollary to Lemma 3 in Sect. 40 of [4]. That $\pi(\omega)$ has exactly $[R_\varphi]$ constituents is, of course, known (cf. [6]).

(5.4) *L-packets of Tempered Representations*

Suppose now that $\{\varphi\}\in\Phi(G)$ is any tempered parameter. We will prove analogues of the results of (5.2).

As usual, let φ be a representative for $\{\varphi\}$ such that $\varphi(W)\subset {}^LM$, for some cuspidal Levi group M in G. We assume that $\varphi=\varphi(\mu,\lambda)$ relative to ${}^LM$, where μ is $\Delta({}^LM^0\cap{}^LB^0, {}^LT^0)$-dominant. Let $Z({}^LM^0)^W$ denote the set of W-invariants in the center of ${}^LM^0$. The following result, due essentially to Langlands, shows that our present definition of $\mathbb{S}_\varphi$ coincides with the more cumbersome one suggested by (5.3.12) which may be reinterpreted as

$$1\to\mathscr{E}(T)^\vee\to S_\varphi/Z({}^LM^0)^W S_\varphi^0\to R_\varphi\to 1$$

(see Proposition 5.4.11).

Lemma 5.4.1. $Z({}^LM^0)^W=Z^W(S_\varphi^0\cap{}^LT^0)$

Proof. Because φ is discrete relative to ${}^LM$, $S_\varphi^0\cap{}^LT^0$ is the connected component of the identity in $Z({}^LM^0)^W$. Thus to prove the lemma we have just to show that if ${}^LG^0$ is of adjoint type then $Z({}^LM^0)^W$ is connected, for each cuspidal Levi group **M** in **G**. Let Y be the submodule of $X^*({}^LT^0)$ generated by $\Delta({}^LM^0, {}^LT^0)$ and the elements $\lambda^\vee-\sigma_G\lambda^\vee$, $\lambda^\vee\in X^*({}^LT^0)$. Then $X^*(Z({}^LM^0)^W)=X^*({}^LT^0)/Y$, so that we have just to show that $X^*({}^LT^0)/Y$ is torsion-free; that is, that if $\lambda^\vee\in X^*({}^LT^0)$ and $n\lambda^\vee\in Y$ for some $n\geqq 1$ then $\lambda^\vee\in Y$. Our system of simple roots for $({}^LB^0, {}^LT^0)$ has the property that if $\alpha^\vee$ is simple and not a root of $({}^LM^0, {}^LT^0)$ then $\varphi(1\times\sigma)\alpha^\vee$ is positive. Recall that $\Delta({}^LM^0, {}^LT^0)=\{\alpha^\vee\in\Delta({}^LG^0, {}^LT^0): \varphi(1\times\sigma)\alpha^\vee=-\alpha^\vee\}$. We may write $\lambda^\vee$ as $\sum n(\alpha^\vee)\alpha^\vee$, where $n(\alpha^\vee)$ is an integer and the summation is over simple roots $\alpha^\vee$. It is then a straightforward calculation to show that if $n\lambda^\vee\in Y$ then $\lambda^\vee\in Y$, so that the lemma is proved.

To obtain information about $\mathbb{S}_{\{\varphi\}}$ and produce parameters which lift to $\{\varphi\}$ we introduce another representative $\tilde{\varphi}$ for $\{\varphi\}$, essentially by inversion of the process in the proof of Lemmas 4.3.5 and 4.3.7. Recall the root system $\Delta_\varphi^\vee$ attached to $\varphi:\Delta_\varphi^\vee$

$=\{\alpha^\vee \in \Delta({}^LG^0, {}^LT^0) : \langle \mu, \alpha^\vee \rangle = 0$ and $Q\alpha^\vee = 0\}$, where Q is as in Lemma 5.2.13. Note that since φ is discrete relative to ${}^LM$ we have

$$Z({}^LM^0)^W = Z_\varphi(S_\varphi^0 \cap {}^LT^0)$$

where $Z_\varphi = \{z \in Z({}^LM^0)^W : \alpha^\vee(z) = 1, \alpha^\vee \in \Delta_\varphi^\vee\}$. It is convenient to write $\mathbb{S}_\varphi$ as $S_\varphi/Z_\varphi S_\varphi^0$, as Lemma 5.4.1 allows.

If $\alpha^\vee \in \Delta_\varphi^\vee$ we set $\mathbf{s}_{\alpha^\vee} = \exp i\pi/4(X_{\alpha^\vee} + X_{-\alpha^\vee})$. Then $\mathbf{s}_{\alpha^\vee}\varphi(1\times\sigma)\mathbf{s}_{\alpha^\vee}^{-1} = \mathbf{s}_{\alpha^\vee}^2 = (1\times\sigma), \mathbf{s}_{\alpha^\vee}^2$ realizing the Weyl reflection with respect to $\alpha^\vee$ in $\Omega({}^LG^0, {}^LT^0)$. If $\beta^\vee \in \Delta_\varphi^\vee$ then $\mathbf{s}_{\alpha^\vee}\mathbf{s}_{\beta^\vee} = \mathbf{s}_{\beta^\vee}\mathbf{s}_{\alpha^\vee}$ since $\alpha^\vee, \beta^\vee$ are either superorthogonal or equal. Let

$$\mathbf{s} = \prod \mathbf{s}_{\alpha^\vee} \quad \text{and} \quad \boldsymbol{\omega} = \prod \omega_{\alpha^\vee},$$

the products to be taken over positive roots $\alpha^\vee$ in $\Delta_\varphi^\vee$.

We denote by ${}^L\tilde{M}^0$ that subgroup of ${}^LG^0$ containing ${}^LT^0$, which has $\{\alpha^\vee \in \Delta({}^LG^0, {}^LT^0) : \varphi(1\times\sigma)\alpha^\vee = -\boldsymbol{\omega}\alpha^\vee\}$ as root system relative to ${}^LT^0$. A simple argument shows that ${}^L\tilde{M}^0$ is invariant under σ_G and ${}^L\tilde{M} = {}^L\tilde{M}^0 \rtimes W$ is the L-group of a cuspidal Levi group $\tilde{M}$ in G.

Proposition 5.4.2. (i) $\Delta({}^L\tilde{M}^0, {}^LT^0)$ *contains* $\Delta_\varphi^\vee$.

(ii) *If* $\alpha^\vee \in \Delta({}^L\tilde{M}^0, {}^LT^0)$ *and* $\langle \mu, \alpha^\vee \rangle = 0$ *then* $\alpha^\vee \in \Delta_\varphi^\vee$.

Proof. (i) is immediate. For (ii) take $\alpha^\vee$ as in the statement. If $\varphi(1\times\sigma)\alpha^\vee = \alpha^\vee$ then $\alpha^\vee$ lies in the $\mathbb{Z}$-span of $\Delta_\varphi^\vee$ and hence by 5.3.13 (ii) in $\Delta_\varphi^\vee$ itself. Clearly $\varphi(1\times\sigma)\alpha^\vee = -\alpha^\vee$ is impossible. Suppose that $\varphi(1\times\sigma)\alpha^\vee \neq \pm\alpha^\vee$. Then by the argument of 5.3.13 (i) $Q(\alpha^\vee)$ is a linear combination, with positive coefficients, of simple roots:

$$Q(\varphi(1\times\sigma)\alpha^\vee) = \varphi(1\times\sigma)(Q(\alpha^\vee))$$

is also a positive combination of simple roots. But the condition that $\alpha^\vee$ be a root of ${}^L\tilde{M}^0$ implies that $Q(\varphi(1\times\sigma)\alpha^\vee) = -Q(\alpha^\vee)$, a contradiction. Hence (ii) follows.

Proposition 5.4.3. $\tilde{\varphi} = \operatorname{ad}\mathbf{s}\circ\varphi$ *has the following properties:*

(i) $\tilde{\varphi}(W) \subset {}^L\tilde{M}$,

(ii) $\tilde{\varphi}(\mathbb{C}^\times) \subset {}^LT^0 \times \mathbb{C}^\times$ *and* $\tilde{\varphi}(1\times\sigma)$ *normalizes* ${}^LT^0$, *acting as* -1 *on the roots of* $({}^L\tilde{M}^0, {}^LT^0)$, *and*

(iii) *if* $\tilde{\varphi} = \varphi(\tilde{\mu}, \tilde{\lambda})$ *relative to* $\tilde{M}$ *then* $\tilde{\mu} = \mu$ *and*

$$\tilde{\lambda} \equiv \lambda \bmod \{X_*({}^LT^0)\} + \{\nu - \tilde{\varphi}(1\times\sigma)\nu : \nu \in X_*({}^LT^0)\otimes\mathbb{C}\}.$$

Proof. This is immediate.

Theorem 5.4.4. $\mathbb{S}_\varphi \cong S_{\tilde{\varphi}} \cap {}^LT^0/Z_\varphi$.

Proof. It is clear that $\mathbf{s}$ centralizes Z_φ. Thus $\operatorname{ad}\mathbf{s} : S_\varphi \to S_{\tilde{\varphi}}$ induces an isomorphism $\mathbb{S}_\varphi \to S_{\tilde{\varphi}}/Z_\varphi S_{\tilde{\varphi}}^0$. We claim that

(5.4.5) $S_{\tilde{\varphi}}/S_{\tilde{\varphi}}^0$ has a complete set of representatives in ${}^LT^0$, implying

$$S_{\tilde{\varphi}}/Z_\varphi S_{\tilde{\varphi}}^0 = (S_{\tilde{\varphi}} \cap {}^LT^0)Z_\varphi S_{\tilde{\varphi}}^0/Z_\varphi S_{\tilde{\varphi}}^0 = S_{\tilde{\varphi}} \cap {}^LT^0/Z_\varphi(S_{\tilde{\varphi}}^0 \cap {}^LT^0),$$

and that

(5.4.6) $${}^LS_{\tilde{\varphi}}^0 \cap {}^LT^0 \subsetneqq Z_\varphi.$$

An isomorphism $\mathbb{S}_\varphi \to S_{\tilde\varphi} \cap {}^L T^0 / Z_\varphi$ has then been produced.

Note also that Lemma 5.4.1 implies that $Z_\varphi \subseteq Z^W(S^0_{\tilde\varphi} \cap {}^L T^0)$, so that (5.4.6) will show that $Z_\varphi = (S^0_{\tilde\varphi} \cap {}^L T^0) Z^W$.

To prove (5.4.5), we first describe a complete set of representatives for $S_\varphi / S^0_\varphi({}^L T^0 \cap S_\varphi)$. Recall the exact sequence $1 \to S_\varphi \cap {}^L T^0 / S^0_\varphi \cap {}^L T^0 \to S_\varphi / S^0_\varphi \to R_\varphi \to 1$ (cf. 5.3.12). Thus we have just to describe a representative in S_φ for each element of R_φ. If $r \in R_\varphi$ then $r = \omega_{\alpha_1^\vee} \dots \omega_{\alpha_m^\vee}$ for some $\alpha_v^\vee \in \Delta^\vee_\varphi$. Let $w_{\alpha_v^\vee} = \exp \pi/2 (X_{-\alpha_v^\vee} - X_{\alpha_v^\vee})$. Then $\mathrm{s} w_{\alpha_v^\vee} \mathrm{s}^{-1} = \exp i\pi/2 \alpha_v \in {}^L T^0$; here α_v denotes $(\alpha_v^\vee)^\vee$. However, $\varphi(1 \times \sigma) w_{\alpha_v^\vee} \varphi(1 \times \sigma)^{-1} = \exp i\pi\alpha_v) w_{\alpha_v^\vee}$, so that $w_{\alpha_1^\vee} \dots w_{\alpha_m^\vee}$, a representative for r, need not lie in S_φ. By definition, there is some representative $w_{\alpha_1^\vee} \dots w_{\alpha_m^\vee} t$ for r in S_φ. Write t as $\exp(x_1\alpha_1 + \dots + x_n\alpha_n) \exp X$, where $x_v \in \mathbb{C}$, $\alpha_v^\vee \in \Delta^\vee_\varphi$ and $X \in {}^L\mathfrak{t}$ is such that $\langle X, \alpha^\vee \rangle = 0$, $\alpha^\vee \in \Delta^\vee_\varphi$. Since $\exp(x_1\alpha_1 + \dots + x_n\alpha_n) \in S^0_\varphi$, we may replace the representative $w_{\alpha_1^\vee} \dots w_{\alpha_m^\vee} t$ by $w_{\alpha_1^\vee} \dots w_{\alpha_m^\vee} \exp X$. Note that

$$\mathrm{s}(w_{\alpha_1^\vee} \dots w_{\alpha_m^\vee} \exp X)\mathrm{s}^{-1} = \exp i\pi/2(\alpha_1 + \dots + \alpha_m) \exp X \in {}^L T^0 .$$

Thus turning to $\tilde\varphi = \mathrm{ad\,s} \circ \varphi$, we have found in ${}^L T^0$ a complete set of representatives for $S_{\tilde\varphi} / S^0_{\tilde\varphi} \mathrm{s}(S_\varphi \cap {}^L T^0)\mathrm{s}^{-1}$. Next, we have $\mathrm{s}(S_\varphi \cap {}^L T^0)\mathrm{s}^{-1} \subseteq S^0_{\tilde\varphi}(S_{\tilde\varphi} \cap {}^L T^0)$ since, if $t \in S_\varphi \cap {}^L T^0$ and we write t as $\exp(x_1\alpha_1 + \dots + x_n\alpha_n) \exp X$ as above, then $t \in S^0_\varphi \exp X$ and $\mathrm{s}t\mathrm{s}^{-1} \in S^0_{\tilde\varphi} \exp X$. But $\mathrm{s}t\mathrm{s}^{-1} \in S_{\tilde\varphi}$ and so we obtain $\mathrm{s}t\mathrm{s}^{-1} \in S^0_{\tilde\varphi}(S_{\tilde\varphi} \cap {}^L T^0)$. (5.4.5) now follows.

For (5.4.6), let $t \in S^0_{\tilde\varphi} \cap {}^L T^0$. We write t as $\exp(x_1\alpha_1 + \dots + x_n\alpha_n) \exp X$, with $x_v \in \mathbb{C}$, $\alpha_v^\vee \in \Delta^\vee_\varphi$ and $\langle X, \alpha^\vee \rangle = 0$, $\alpha^\vee \in \Delta^\vee_\varphi$. Then $t \in S_{\tilde\varphi}$ implies that $t = \exp(-x_1\alpha_1 - \dots - x_n\alpha_n) \exp(\tilde\varphi(1 \times \sigma) X)$. Thus $\exp(2x_1\alpha_1 + \dots + 2x_n\alpha_n) = \exp(\tilde\varphi(1 \times \sigma) X - X)$. A straightforward argument shows that $2x_1, \dots, 2x_n$ belong to $\pi i \mathbb{Z}$. If $x_1, \dots, x_n \in \pi i \mathbb{Z}$ then we are done. Suppose that notation is arranged so that exactly $x_1, \dots, x_l \notin \pi i \mathbb{Z}$. Thus $\mathrm{s}^{-1} t \mathrm{s}$ is a representative for $\omega_{\alpha_1^\vee} \dots \omega_{\alpha_l^\vee}$ in S_φ. In the next paragraph we will show that:

(5.4.7) $$\Omega(\Delta^\vee_\varphi) \cap \Omega_\varphi({}^L G^0, {}^L T^0) = R_\varphi .$$

We conclude then that $\omega_{\alpha_1^\vee} \dots \omega_{\alpha_l^\vee} \in R_\varphi$. Thus $\mathrm{s}^{-1} t \mathrm{s} \notin S^0_\varphi$ so that $t \notin S^0_{\tilde\varphi}$ a contradiction. Hence if $t \in S^0_{\tilde\varphi} \cap {}^L T^0$ then all x_v belong to $\pi i \mathbb{Z}$ and so $\alpha^\vee(t) = 1$, $\alpha^\vee \in \Delta^\vee_\varphi$. Recall $t = \exp(x_1\alpha_1 + \dots + x_n\alpha_n) \exp X$. Clearly $\exp X \in S^0_\varphi \cap {}^L T^0$. Since φ is a discrete parameter for M, $S^0_\varphi \cap {}^L T^0$ lies in the center of ${}^L M^0$. We conclude that t lies in the center of ${}^L M^0$ also, and thus $t \in Z_\varphi$.

The proof of (5.4.7) is another argument with the operator Q. We have to show that if $\omega \in \Omega(\Delta^\vee_\varphi)$ can be realized in S_φ by, say, s then $\mathrm{Ad}\, s$ preserves $\mathfrak{s}^+_\varphi$. Let $\alpha^\vee_+ \in \Delta(\mathfrak{s}^+_\varphi, {}^L\mathfrak{t}_\varphi)$. We choose $\alpha^\vee \in \Delta({}^L\mathfrak{b}, {}^L\mathfrak{t})$ such that $\alpha^\vee_+ = \alpha^\vee|_{{}^L\mathfrak{t}_\varphi}$. Then $\mathrm{Ad}\, s(\alpha^\vee_+) = \omega\alpha^\vee|_{{}^L\mathfrak{t}_\varphi}$. Clearly $[R_\varphi] Q(\alpha^\vee)$ is a sum of roots from $({}^L\mathfrak{b}, {}^L\mathfrak{t})$. Since $Q(\omega\alpha^\vee) = Q(\alpha^\vee)$ we must have that $\omega\alpha^\vee \in \Delta({}^L\mathfrak{b}, {}^L\mathfrak{t})$, and so $\mathrm{Ad}\, s(\alpha^\vee_+) \in \Delta(\mathfrak{s}^+_\varphi, {}^L\mathfrak{t}_\varphi)$. Thus (5.4.7) follows, and the theorem is proved.

Corollary 5.4.8. $R_\varphi \cong S_{\tilde\varphi} \cap {}^L T^0 / S_{\tilde\varphi} \cap \{t \in {}^L T^0 : \alpha^\vee(t) = 1, \alpha^\vee \in \Delta^\vee_\varphi\}$.

Proof. We use the argument for (5.4.6) to establish a map $S_{\tilde\varphi} \cap {}^L T^0 \to R_\varphi$. This map is surjective by the argument for (5.4.5). A simple computation verifies that the kernel is $S_{\tilde\varphi} \cap \{t \in {}^L T^0 ; \alpha^\vee(t) = 1,\ \alpha^\vee \in \Delta^\vee_\varphi\}$.

We return now to the setting of the last part of (5.3). Thus $\mathbf{M}=\mathbf{M}_T$ and η is a p.d. of $\mathbf{T}$. When regarded as a root of $\mathbf{T}$, the coroot α of a root $\alpha^\vee$ in $\Delta_\varphi^\vee$ is real. We perform, in turn, a standard inverse Cayley transform with respect to each root α such that $\alpha^\vee\in\Delta_\varphi^\vee\cap\Delta({}^LB^0,{}^LT^0)$. This determines a Cartan subgroup $\tilde{T}$ of G and a p.d. $\tilde{\eta}$ of $\tilde{\mathbf{T}}$ such that ${}^L\tilde{M}={}^LM_{\tilde{T}}$. Recall that $\tilde{\eta}$ induces an isomorphism

$$\mathscr{E}(\tilde{T})^\vee \xrightarrow{\sim} S_{\tilde{\varphi}}\cap{}^LT^0/Z({}^L\tilde{M}^0)^W$$

(cf. Proposition 5.2.1). The following is then a immediate consequence of Theorem 5.4.4.

Proposition 5.4.9. *$\tilde{\eta}$ induces a surjective homomorphism $\mathscr{E}(\tilde{T})^\vee\to\mathbb{S}_{\{\varphi\}}$.*

Corollary 5.4.10. *$\mathbb{S}_{\{\varphi\}}$ is a sum of groups of order two.*

On the other hand, η induces a isomorphism $\mathscr{E}(T)^\vee\to S_\varphi\cap{}^LT^0/Z({}^LM^0)^W$.

Proposition 5.4.11. *η induces an exact sequence*

$$1\to\mathscr{E}(T)^\vee\to\mathbb{S}_{\{\varphi\}}\to R_\varphi\to 1\,.$$

Proof. This follows immediately from (5.3.12) and Lemma 5.4.1.

Corollary 5.4.8 also has an interpretation in terms of $\mathbf{G}$-data. Via $\tilde{\eta}$ we regard each $\alpha^\vee$ in $\Delta_\varphi^\vee$ as a coroot for $\tilde{\mathbf{T}}$. We define a submodule $\langle\Delta_\varphi^\vee\rangle$ of $\mathscr{E}(\tilde{T})$ as follows. By Tate-Nakayama duality, $\mathscr{E}(\tilde{T})$ may be identified as

$$\{\lambda^\vee\in X_*(\tilde{\mathbf{T}}_{sc}):\lambda^\vee+\sigma_{\tilde{T}}\lambda^\vee=0\}/\{\mu^\vee-\sigma_{\tilde{T}}\mu^\vee:\mu^\vee\in X_*(\tilde{\mathbf{T}})\}\cap X_*(\tilde{\mathbf{T}}_{sc});$$

$\langle\Delta_\varphi^\vee\rangle$ is then the submodule generated by the cosets of the coroots $\alpha^\vee$ in $\Delta_\varphi^\vee$. We will realize R_φ as the quotient in Corollary 5.4.8.

Proposition 5.4.12. *$\tilde{\eta}$ induces a surjective homomorphism $\mathscr{E}(\tilde{T})^\vee\to R_\varphi$. The kernel of this homomorphism is the annihilator of $\langle\Delta_\varphi^\vee\rangle$ in $\mathscr{E}(\tilde{T})^\vee$.*

We identify $\Omega(\Delta_\varphi^\vee)$, the subgroup of $\Omega({}^LG^0,{}^LT^0)$ generated by reflections with respect to the roots in $\Delta_\varphi^\vee$, as a subgroup of $\Omega(\mathbf{M},\tilde{\mathbf{T}})$, again via $\tilde{\eta}$.

Corollary 5.4.13. *The dual of the homomorphism $\mathscr{E}(\tilde{T})^\vee\to R_\varphi$ embeds $R_\varphi^\vee$ in $\mathscr{E}(\tilde{T})$. Under Tate-Nakayama duality the image of $R_\varphi^\vee$ is identified with*

$$\Omega(\tilde{M},\tilde{T})\cap\Omega(\Delta_\varphi^\vee)\backslash\Omega(\Delta_\varphi^\vee)\,.$$

Proof. We have just to check that the image of $\Omega(\tilde{M},\tilde{T})\cap\Omega(\Delta_\varphi^\vee)\backslash\Omega(\Delta_\varphi^\vee)$ under $T-N$ (cf. [15, Sect. 2]) is $\langle\Delta_\varphi^\vee\rangle$. This is immediate from Proposition 2.1 of [15] since $\Delta_\varphi^\vee$ is a set of superorthogonal noncompact roots.

Turning to L-packets, we consider first the packet $\Pi_\varphi^{\tilde{M}}$ of representations of $\tilde{M}$ attached to φ or $\tilde{\varphi}$. Let $\Psi_{\tilde{M}}$ be a positive system for $\Delta({}^L\tilde{M}^0,{}^LT^0)$ with respect to which μ is dominant. Then it is easily seen that the roots in $\Psi_{\tilde{M}}\cap\Delta_\varphi^\vee$ are $\Psi_{\tilde{M}}$-simple and that $\Psi_{\tilde{M}}\supset\Psi_M=\Delta({}^LM^0\cap{}^LB^0,{}^LT^0)$. In view of Proposition 5.4.2 (ii) we can now argue as in the proof of Lemmas 4.3.5 and 4.3.7 to show that for each $\omega_0\in\Omega(\mathbf{M},\mathbf{T})$,

$$\mathrm{Ind}(\pi(\omega_0,\Psi_M)\otimes 1_{N\cap\tilde{M}},M(N\cap\tilde{M}),\tilde{M})=\sum_{\omega\in\Omega(\Delta_\varphi^\vee)\cap\Omega(\tilde{M},\tilde{T})\backslash\Omega(\Delta_\varphi^\vee)}\Theta(\omega\omega_0\mu,\lambda,\omega\omega_0\Psi_{\tilde{M}}),$$

each distribution in this sum being non-zero. Thus

$$\{\Theta(\omega\mu, \lambda, \omega\Psi_{\tilde{M}}), \omega \in \Omega(\tilde{M}, \tilde{T})\Omega_{(\mu)}(\tilde{\mathbf{M}}, \tilde{\mathbf{T}})\}$$

is exactly the set of (characters of) representations in $\Pi_\varphi^{\tilde{M}}$. Recall that $\Omega_{(\mu)}(\tilde{\mathbf{M}}, \tilde{\mathbf{T}})$ is the subgroup of $\Omega(\tilde{\mathbf{M}}, \tilde{\mathbf{T}})$ generated by $\Omega(\Delta_\varphi^\vee)$ and $\Omega({}^L M^0, {}^L T^0)$, with the usual identifications. Let $\Theta^G(\omega\mu, \lambda, \omega\Psi_{\tilde{M}}) = \mathrm{Ind}(\Theta(\omega\mu, \lambda, \omega\Psi_{\tilde{M}}) \otimes 1_{\tilde{N}}, \tilde{M}\tilde{N}, G)$. Then by induction in stages we have

$$\mathrm{Ind}(\pi(\omega_0, \Psi_M) \otimes 1_N, P, G) = \sum_{\omega \in \Omega(\Delta_\varphi^\vee) \cap \Omega(M, T) \backslash (\Delta_\varphi^\vee)} \Theta^G(\omega\omega_0\mu, \lambda, \omega\omega_0\Psi_M).$$

By Corollary 5.4.13 there are $[R_\varphi]$ (non-zero) distributions in this sum. Hence, by Corollary 5.3.16, the representations (characters) $\Theta^G(\omega\mu, \lambda, \omega\Psi_{\tilde{M}})$ are irreducible, $\omega \in \Omega(\tilde{M}, \tilde{T})\Omega_{(\mu)}(\tilde{\mathbf{M}}, \tilde{\mathbf{T}})$. We conclude then that $\Pi_{\{\varphi\}} = \Pi_\varphi^G$ consists exactly of the representations $\Theta^G(\omega\mu, \lambda, \omega\Psi_{\tilde{M}})$, $\omega \in \Omega(\tilde{M}, \tilde{T})\Omega_{(\mu)}(\tilde{\mathbf{M}}, \tilde{\mathbf{T}})$, and that $\Theta^G(\omega\mu, \lambda, \omega\Psi_{\tilde{M}}) = \Theta^G(\omega'\mu, \lambda, \omega'\Psi_{\tilde{M}})$ if and only if $\omega' \in \Omega(\tilde{\mathbf{M}}, \tilde{\mathbf{T}})\omega$ (cf. Lemma 3.2 of [14]; note that it is not really necessary to use the Multiplicity One Theorem for unitary principal series).

Suppose now that:

(5.4.14) $(\bar{T}, \bar{\eta})$ *belongs to the fixed skeleton,* $\bar{M} = M_{\bar{T}}$, $\bar{P} = \bar{M}\bar{N}$, *as in* (1.3), *and* $\bar{\varphi} = \varphi(\bar{\mu}, \bar{\lambda}): W \to {}^L M$ *belongs to* $\{\varphi\}$.

Then $\Pi_{\{\varphi\}}$ consists of the irreducible constituents of the representations $\Theta^G(\bar{\omega}\bar{\mu}, \bar{\lambda}, \bar{\omega}\bar{\Psi}) = \mathrm{Ind}(\Theta(\bar{\omega}\bar{\mu}, \bar{\lambda}, \bar{\omega}\bar{\Psi}) \otimes 1_{\bar{N}}, \bar{P}, \bar{G})$, where $\bar{\Psi}$ is some positive system for $\Delta({}^L\bar{M}, {}^L\bar{T})$ with respect to which $\bar{\mu}$ is dominant, and $\bar{\omega} \in \Omega(\bar{\mathbf{M}}, \bar{\mathbf{T}})$ is such that $\Theta(\bar{\omega}\bar{\mu}, \bar{\lambda}, \bar{\omega}\bar{\Psi})$ is nonzero (cf. Theorem 4.3.2 and induction in stages).

Suppose also that each non-zero $\Theta^G(\bar{\omega}\bar{\mu}, \bar{\lambda}, \bar{\omega}\bar{\Phi})$ is irreducible and:

(5.4.15) $\bar{\mu}$ *is nondegenerate in the sense that, for each root* α *of* $(\bar{\mathbf{M}}, \bar{\mathbf{T}})$, $\langle \bar{\mu}, \alpha^\vee \rangle = 0$ *implies that* α *is noncompact and* $\alpha^\vee \notin (1 - \sigma_T) X_*(\bar{T})$ (cf. [6, 7], [15, 2.1]).

Then, according to [7, Theorem 4.1(b)], there exists $g \in G$ such that $\mathrm{ad}\, g: \bar{T} \to \tilde{T}$ and $g\bar{\mu} = \omega\tilde{\mu}$, $g\bar{\lambda} = \tilde{\lambda}$, $g\bar{\Psi} = \omega\Psi_{\tilde{M}}$, for some $\omega \in \Omega_{(\mu)}(\mathbf{G}, \tilde{\mathbf{T}})$.

The following is then immediate:

(5.4.16) $\Delta_1^\vee = \{\alpha^\vee \in \Delta({}^L\bar{M}^0, {}^L\bar{T}^0);\ \langle \bar{\mu}, \alpha^\vee \rangle = 0\}$ *is of type* $A_1 \times A_1 \times \ldots \times A_1$,

and if $\Delta_2^\vee = \{\beta^\vee \in \Delta({}^L\bar{M}^0, {}^L\bar{T}^0): \langle \beta, \alpha^\vee \rangle = 0, \alpha^\vee \in \Delta_1^\vee\}$ and $\Omega_{(\bar{\mu})}(\bar{\mathbf{M}}, \bar{\mathbf{T}})$ is the subgroup of $\Omega(\bar{\mathbf{M}}, \bar{\mathbf{T}})$ generated by the reflections with respect to roots in $\Delta_1^\vee \cup \Delta_2^\vee$, then

(5.4.17) $\Theta(\bar{\omega}\bar{\mu}, \bar{\lambda}, \bar{\omega}\bar{\Psi}) \neq 0$ *if and only if* $\bar{\omega} \in \Omega(\bar{M}, \bar{T})\Omega_{(\mu)}(\bar{\mathbf{M}}, \bar{\mathbf{T}})$; *for such* $\bar{\omega}$, *if* $\Theta^G(\bar{\omega}'\bar{\mu}, \bar{\lambda}, \bar{\omega}'\bar{\Psi}) = \Theta^G(\bar{\omega}\bar{\mu}, \bar{\lambda}, \bar{\omega}\bar{\Psi})$ *then* $\bar{\omega}' \in \Omega(G, \bar{T})\bar{\omega}$.

Suppose that $(\bar{\bar{\mu}}, \bar{\bar{\lambda}})$ also satisfies (5.4.14) and (5.4.15), and that $\bar{\bar{\Psi}}$ is some positive system with respect to which $\bar{\bar{\mu}}$ is dominant. Then by [7, Theorem 4.1(b)] again, there exists $g \in G$ normalizing $\bar{T}$ and $\bar{\omega} \in \Omega_{(\bar{\mu})}(\bar{\mathbf{M}}, \bar{\mathbf{T}})$ such that $g\bar{\bar{\mu}} = \bar{\omega}\bar{\mu}$, $g\bar{\bar{\lambda}} = \bar{\lambda}$ and $g\bar{\bar{\Psi}} = \bar{\omega}\bar{\Psi}$.

We will fix:

(5.4.18) *a positive system* $\bar{\Psi}$ *for* $\Delta({}^L\bar{M}^0, {}^L\bar{T}^0)$ *with respect to which some* $\bar{\mu}$ *satisfying* (5.4.14) *and* (5.4.15) *is dominant.*

Then $(\bar{\mu}, \bar{\lambda})$ is determined up to conjugation by an element of G normalizing $\bar{T}$, for the element $\bar{\omega}$ above lies in $\Omega(\bar{M}, \bar{T})$ when $\bar{\bar{\Psi}} = \bar{\Psi}$ (cf. proof of Lemma 3.2 in [14]). We write $\pi(\bar{\omega})$ in place of $\Theta^G(\bar{\omega}\bar{\mu}, \bar{\lambda}, \bar{\omega}\bar{\Psi})$. The group $\mathbb{S}_{\{\varphi\}}$ may be realized as $S_{\bar{\varphi}} \cap {}^L T^0/(S^0_{\bar{\varphi}} \cap {}^L T^0)Z^W$ (cf. proof of Theorem 5.4.4). Note that if $t \in (S^0_{\bar{\varphi}} \cap {}^L T^0)Z^W$ then $\alpha^\vee(t) = 1$ for $\alpha^\vee \in \Delta_1^\vee \cup \Delta_2^\vee$.

Let $\mathfrak{x} \in \mathbb{S}_{\{\varphi\}}$. Choose $x \in S_{\bar{\varphi}} \cap {}^L T^0/Z^W$ mapping to $\mathfrak{x}$ under the natural projection. By means of $\bar{\eta}$ and natural identifications, we may regard x as a character on

$$X_*(\bar{T}_{sc})/X_*(\bar{T}_{sc}) \cap \{v^\vee - \sigma_{\bar{T}} v^\vee : v^\vee \in X_*(\bar{T})\}$$

(cf. [16, Sect. 2.1]). On the other hand, if $\bar{\omega} \in \Omega(G, \bar{T})\Omega_{(\bar{\mu})}(\mathbf{G}, \bar{\mathbf{T}})$ then $\bar{\omega}$ determines an element of $H^1(\bar{\mathbf{T}})$ and hence, by the Tate-Nakayama isomorphism, an element $\bar{\omega}_\sigma$ of the above quotient (cf. [15, Sect. 2]).

Lemma 5.4.19. *$x(\bar{\omega}_\sigma)$ is independent of the choice for x.*

Proof. Suppose that x is replaced by xt, $t \in (S^0_{\bar{\varphi}} \cap {}^L T^0)Z^W$. Then we have only to show that $\bar{t}(\bar{\omega}_\sigma) = 1$, where $\bar{t}$ is the coset of t in $S_{\bar{\varphi}} \cap {}^L T^0/Z^W$. Since $\bar{\omega} \in \Omega(G, \bar{T})\Omega_{(\bar{\mu})}(\bar{\mathbf{M}}, \bar{\mathbf{T}})$ we have that $\bar{\omega}_\sigma$ belongs to the $\mathbb{Z}$-span of $\Delta_1^\vee \cup \Delta_2^\vee$ (cf. [15, Sect. 2]). The lemma then follows.

Let $\mathfrak{x} \in \mathbb{S}_{\{\varphi\}}$, $\pi \in \Pi_{\{\varphi\}}$. Choose $x \in S_{\bar{\varphi}} \cap {}^L T^0/Z^W$ mapping to $\mathfrak{x}$ under the natural projection and $\bar{\omega} \in \Omega(\bar{\mathbf{M}}, \bar{\mathbf{T}})$ such that $\pi = \pi(\bar{\omega})$ with respect to some choice of $\bar{\mu}$ as above. Then we set

$$\langle \mathfrak{x}, \pi \rangle = x(\bar{\omega}_\sigma).$$

For given $\bar{\mu}$, $\langle \mathfrak{x}, \pi \rangle$ is independent of the choice of x and the choice of $\bar{\omega}$ since $\pi(\bar{\omega}) = \pi(\bar{\omega}')$ if and only if $\bar{\omega}' = \Omega(G, \bar{T})\bar{\omega}$. Secondly, $\bar{\mu}$ may be replaced only by $g\bar{\mu}$ where $g \in G$ normalizes $\bar{T}$ and the chosen $\bar{\Psi}$. Then x^g is a replacement for x, and $g\omega g^{-1}$ a replacement for ω. It is easily shown that $x^g((g\bar{\omega}g^{-1})_\sigma) = x(\bar{\omega}_\sigma)$. Hence the following is immediate.

Lemma 5.4.20. *$\langle\ ,\ \rangle : \mathbb{S}_{\{\varphi\}} \times \Pi_{\{\varphi\}} \to \{\pm 1\}$ depends only on $\{\varphi\}$ and the data of* (1.3) *and* (5.4.18).

Lemma 5.4.21. (i) *$\langle \mathfrak{x}\mathfrak{y}, \pi \rangle = \langle \mathfrak{x}, \pi \rangle \langle \mathfrak{y}, \pi \rangle$ for $\mathfrak{x}, \mathfrak{y} \in \mathbb{S}_{\{\varphi\}}$, $\pi \in \Pi_{\{\varphi\}}$, and*

(ii) *$\langle \mathfrak{x}, \pi \rangle = \langle \mathfrak{x}, \pi' \rangle$ for all $\mathfrak{x} \in \mathbb{S}_{\{\varphi\}}$ if and only if $\pi = \pi'$.*

Proof. (i) is immediate. For (ii) we have only to note that $S_{\bar{\varphi}} \cap {}^L T^0/Z^W \cong \mathscr{E}(\bar{T})^\vee$.

Therefore, as in the discrete case, we have identified Π_φ as a subset of $\mathbb{S}_\varphi^\vee$, the dual of $\mathbb{S}_\varphi$. Again this subset may be proper, and so we adjoin some "ghost" representations with zero character and form $\bar{\Pi}_\varphi$ in full duality with $\mathbb{S}_\varphi^\vee$ (cf. paragraph following Lemma 5.1.6). Note that $[\Pi_\varphi] = [\mathfrak{D}(T)][R_\varphi]$, where $\mathfrak{D}(T) = \Omega(M, T)\backslash\Omega(\mathbf{M}, \mathbf{T})$ (cf. [15, Sect. 2]) and $[\mathbb{S}_\varphi^\vee] = [\mathscr{E}(T)][R_\varphi]$ by Lemma 5.4.11. On the other hand, the number of ghosts in the discrete packet $\bar{\Pi}_\varphi^M$ is $[\mathscr{E}(T)] - [\mathfrak{D}(T)]$. We have therefore added exactly $[R_\varphi]$ ghosts to Π_φ for each ghost in $\bar{\Pi}_\varphi^M$.

It is convenient now to divide the discussion into two parts. In the first, we assume that ${}^L\bar{M} = {}^L G$. Let $\mathfrak{x} \in \mathbb{S}_{\{\varphi\}}$, and choose x as above. To x we attach the representative $\mathfrak{s}(x)$ for an element of $\mathfrak{S}({}^L G)$ given by

$$\mathfrak{s}(x) = (x, \mathrm{Cent}(x)^0, {}^L B_x^0, {}^L T^0, \{Y_{\alpha^\vee}\}, \{w_x\}),$$

where ${}^{\mathrm{L}}B_x^0 \supseteq {}^{\mathrm{L}}T^0$ and $\Delta({}^{\mathrm{L}}B_x^0, {}^{\mathrm{L}}T^0) \subsetneq \bar{\Psi}$, and the action of W on $\mathrm{Cent}(x)^0$ is extracted from that of $\bar{\varphi}(1 \times \sigma)$ in the usual way. Clearly, different choices of x may give inequivalent elements of $\mathfrak{S}({}^{\mathrm{L}}G)$.

We proceed now as in the discrete case: $\mathfrak{s}(x)$ is equivalent to exactly one of our fixed representatives for $\mathfrak{S}({}^{\mathrm{L}}G)$, say $\mathfrak{s}_H = (s_H, {}^{\mathrm{L}}H)$. If $g \in {}^{\mathrm{L}}G^0$ is such that $\mathrm{ad}\, g: \mathfrak{s}(x) \to \mathfrak{s}_H$ then $g x g^{-1} = s_H$ (cf. Proposition 5.2.2). If also g' maps $\mathfrak{s}(x)$ to $\mathfrak{s}_H$ then $g' = \tau g$, where $\tau \in \mathrm{Cent}(s_H)$ induces an automorphism of $({}^{\mathrm{L}}H^0, {}^{\mathrm{L}}H^0 \cap {}^{\mathrm{L}}B^0, {}^{\mathrm{L}}T^0, \{Y_{\alpha^\vee}\})$. As before, $g\bar{\varphi}(W)g^{-1}$ lies in the image of ${}^{\mathrm{L}}H$ under $\xi: {}^{\mathrm{L}}H \hookrightarrow {}^{\mathrm{L}}G$. Thus $\mathrm{ad}\, g \circ \bar{\varphi} = \xi \circ \varphi'$ for some admissible homomorphism $\varphi': W \to {}^{\mathrm{L}}H$. We have then that $\{\varphi'\} \in \Phi(H)$ lifts to $\{\varphi\}$. Recall that $\bar{\varphi}$ has parameters $(\bar{\mu}, \bar{\lambda})$, $\bar{\mu}$ is $\bar{\Psi}$-dominant, and $\xi = \xi(\mu^*, \lambda^*)$. Thus φ' has parameters $(g\bar{\mu} - \mu^*,\ \lambda - \lambda^*)$, $g\bar{\mu} - \mu^*$ being $\Delta({}^{\mathrm{L}}B^0 \cap {}^{\mathrm{L}}H^0, {}^{\mathrm{L}}T^0)$-dominant.

The character identity of Theorem 4.1.1 states that $\sum_{\pi \in \Pi_{\{\varphi\}}} \varepsilon(\pi)\chi_\pi(f) = \chi_{\{\varphi'\}}(f_H)$. We need a description of $\varepsilon(\pi(\bar{\omega}))$ in terms of $\bar{\omega}$; that was not provided in Sect. 4.

Lemma 5.4.22. $\varepsilon(\pi(\bar{\omega})) = \varepsilon(\bar{\Psi})\varepsilon(\bar{T}, \bar{\eta}) \det g\ \bar{\kappa}(g^{-1})\langle \mathfrak{x}, \pi(\omega)\rangle$, *where* $\varepsilon(\bar{\Psi})$ *is the signature of* $\bar{\Psi}$ *with respect to* $\Delta({}^{\mathrm{L}}B^0, {}^{\mathrm{L}}T^0)$, $g: \mathfrak{s}(x) \to \mathfrak{s}_H$, $\varphi' = \varphi(g\bar{\mu} - \mu^*, \bar{\lambda} - \lambda^*)$, $\bar{\kappa}$ *is obtained from* s_H *via* $(\bar{T}, \bar{\eta})$, $\varepsilon(\bar{T}, \bar{\eta})$ *is from our chosen family of* (G, H)*-orbital-integral-transfer factors, and in the terms* $\det g$ *and* $\bar{\kappa}(g^{-1})$, *g has been transferred to* $\bar{T}$ *via* $\bar{\eta}$.

Proof. First we observe that $\langle \mathfrak{x}, \pi(\bar{\omega})\rangle = \bar{\kappa}^{g^{-1}}(\bar{\omega})$, so that $\bar{\kappa}(g^{-1})\langle \mathfrak{x}, \pi(\bar{\omega})\rangle = \bar{\kappa}(\bar{\omega} g^{-1})$. Thus the assertion is true if $\bar{\mu}$ is ${}^{\mathrm{L}}G^0$-regular (cf. Sect. 4; the term $\varepsilon(\bar{\Psi})$ is now inserted because $\bar{\mu}$ is $\bar{\Psi}$-dominant, rather than $\Delta({}^{\mathrm{L}}B^0, {}^{\mathrm{L}}T^0)$-dominant as in Sect. 4). In general, we argue by coherent continuation as in the proof of Theorem 4.3.2.

Corollary 5.4.23. $\varepsilon(\bar{\Psi})\varepsilon(\bar{T}, \bar{\eta}) \det g\ \bar{\kappa}(g^{-1})\chi_{\{\varphi'\}}(f_H)$ *depends only on* $\{\varphi\}$, $\mathfrak{x}$ *and* f.

We set

$$\hat{\chi}_{(\{\varphi\}, \mathfrak{x})}(f) = \varepsilon(\bar{\Psi})\varepsilon(\bar{T}, \bar{\eta}) \det g\ \bar{\kappa}(g^{-1})\chi_{\{\varphi'\}}(f_H).$$

Then the analogues of Theorems 5.2.7 and 5.2.8 and Corollary 5.2.9 are true. We defer the statements until the end of our discussion for the general case.

We will no longer assume that ${}^{\mathrm{L}}\bar{M} = {}^{\mathrm{L}}G$. Let $\mathfrak{x} \in S_{\{\varphi\}}$ and $x \in S_{\bar{\varphi}} \cap {}^{\mathrm{L}}T^0 | Z({}^{\mathrm{L}}\bar{M}^0)^W$ map to $\mathfrak{x}$ under the natural projection. Then $\mathfrak{s}(x)$, a representative for an element of $\mathfrak{S}({}^{\mathrm{L}}\bar{M})$, has been defined. Suppose that $g \in G$, $\hat{T} = g\bar{T}g^{-1}$, and that $\hat{\eta}$ is a p.d. of $\hat{T}$. Let $\hat{M} = M_{\hat{T}}$. Then a representative $\mathfrak{s}(x)^g$ for an element of $\mathfrak{S}({}^{\mathrm{L}}\hat{M})$ is defined in the obvious manner.

Proposition 5.4.24. *There exist* $\mathfrak{s}_H$ *among the fixed representatives for* $\mathfrak{S}({}^{\mathrm{L}}G)$ *and Cartan subgroup* $\hat{T} = g\bar{T}g^{-1}$, $g \in G$, *together with a p.d.* $\hat{\eta}$ *of* $\hat{\mathbf{T}}$, *such that*

(i) σ_H *satisfies* (3.4.1) *relative to* $(\hat{T}, \hat{\eta})$, *and*

(ii) $\mathfrak{s}(\bar{x})^g$ *is equivalent to* $\mathfrak{s}_H^{\hat{M}}$ *relative to* $\hat{M}$.

The element $\mathfrak{s}_H^{\hat{M}}$ was defined in (3.4).

Proof. We exhibit a choice for $\mathfrak{s}_H$, g and $\hat{\eta}$, as follows. First, we may choose $x \in S_{\bar{\varphi}}^0 \cap {}^{\mathrm{L}}T^0/Z^W$ mapping to $\bar{x}$ under the natural projection, such that $\mathrm{Cent}(x)^0 \subset {}^{\mathrm{L}}\bar{M}^0$; this is immediate if x is realized as a quasicharacter on $\langle \Delta^\vee \rangle / \langle \Delta^\vee \rangle \cap \{\nu^\vee - \bar{\varphi}(1 \times \sigma)\nu^\vee,\ \nu^\vee \in X^*({}^{\mathrm{L}}T^0)\}$, where $\langle \Delta^\vee \rangle$ denotes the $\mathbb{Z}$-span of

$\Delta({}^L G^0, {}^L T^0)$, and $\bar{x}$ as a character on

$$\{\lambda^\vee \in \langle \Delta^\vee \rangle : \bar{\varphi}(1 \times \sigma)\lambda^\vee = -\lambda^\vee\}/\langle \Delta^\vee \rangle \cap \{v^\vee - \bar{\varphi}(1 \times \sigma)v^\vee : v^\vee \in X^*({}^L T^0)\}.$$

We form $\mathfrak{s}(x)$ in the usual way; $\mathfrak{s}(x)$ is equivalent to some chosen representative $\mathfrak{s}_H$. There exists $\bar{\kappa}$ such that $(\bar{T}, \bar{\kappa}) \in \mathscr{K}(G)$ and $\mathfrak{s}(x)$ is the 6-tuple attached to $(\bar{T}, \bar{\kappa}, \bar{\eta})$ as in (2.3). On the other hand, by arguing first with the quasisplit group dual to $\mathfrak{s}(x)$, we see that the conjugacy class of Cartan subgroups of G from which the fundamental Cartan subgroups of H originate is that of $\bar{T}$; moreover, the fundamental Cartan subgroups of H are compact modulo the center of H. There exist a Cartan subgroup $\hat{T}$ conjugate to $\bar{T}$, say $\hat{T} = g\bar{T}g^{-1}$, $g \in G$, and p.d. $\hat{\eta}$ of $\hat{\mathbf{T}}$ such that $(\hat{T}, \hat{\eta}) \in \mathscr{T}_H(G)$ [for example, $\hat{T}$ may be chosen from a fixed framework for H (cf. (3.3))]. In particular, ${}^L H^0 \subset {}^L \hat{M}^0$ and σ_H satisfies (3.4.1) relative to $(\hat{T}, \hat{\eta})$. Because $\mathfrak{s}(x)$ and $\mathfrak{s}_H$ are equivalent we may conclude that on $\mathscr{E}(\hat{T})$, the quasicharacter $\hat{\kappa}$ attached to $(\hat{T}, \hat{\eta})$ coincides with $(\bar{\kappa}^g)^\omega$ for some $g \in G$ mapping $\bar{T}$ to $\hat{T}$ and $\omega \in \Omega(\hat{\mathbf{M}}, \hat{\mathbf{T}})$ (cf. Proposition 2.3.1). Then $\mathfrak{s}(\bar{x})^g$ is equivalent to $\mathfrak{s}_H^{\hat{M}}$, and the proof is completed.

We choose $\mathfrak{s}_H$ and g as in the statement of the proposition. Let $\hat{M}_H$ denote the endoscopic group for $\hat{M}$ determined by $\mathfrak{s}_H^M$ [cf. (3.4)]. If $\hat{f} \in \mathscr{C}(\hat{M})$ then $\hat{f}_H$ will denote a function in $\mathscr{C}(\hat{M}_H)$ corresponding to $\hat{f}$ under the $(\hat{M}, \hat{M}_H)$-orbital-integral transfer determined by the fixed data for H [cf. (3.4)].

Let $\hat{\varphi} = \mathrm{ad}\, g \circ \bar{\varphi}: W \to {}^L\hat{M}$. Then $\hat{\varphi}$ has parameters $(g\bar{\mu}, g\bar{\lambda})$. The corresponding L-packet $\hat{\Pi}$ of representations of $\hat{M}$ is $\{\hat{\pi} = \mathrm{ad}\, g \circ \bar{\pi}, \bar{\pi} \in \bar{\Pi}\}$. If $\bar{\pi} = \Theta(\bar{\omega}\bar{\mu}, \bar{\lambda}, \bar{\omega}\bar{\Psi})$ then $\hat{\pi} = \Theta(g\bar{\omega}g^{-1}(g\bar{\mu}), g\bar{\lambda}, g\bar{\omega}g^{-1}(g\bar{\Psi}))$. Suppose that $\mathrm{ad}\,\hat{m}$: $\mathfrak{s}(\bar{x})^g \to \mathfrak{s}_H^{\hat{M}}$, $\hat{m} \in {}^L\hat{M}^0$. Then $\mathrm{ad}\,\hat{m} \circ \hat{\varphi}$ is the lift to ${}^L\hat{M}$ of some $\hat{\varphi}'$: $W \to {}^L M_H$ with parameters $(\hat{m}\hat{\mu} - \mu^*, \hat{\lambda} - \lambda^*) = (\hat{m}g\bar{\mu} - \mu^*, g\bar{\lambda} - \lambda^*)$. From the arguments for Lemma 5.4.22 we conclude that

$$\sum_{\hat{\pi} \in \hat{\Pi}} \langle \mathfrak{x}, \hat{\pi} \rangle \chi_{\hat{\pi}}(\hat{f}) = c\chi_{\hat{\varphi}'}(\hat{f}_H), \hat{f} \in \mathscr{C}(\hat{M}), \tag{5.4.25}$$

for some $c = \pm 1$. Here we have used $\langle \mathfrak{x}, \Theta^G(\bar{\omega}\bar{\mu}, \bar{\lambda}, \bar{\omega}\bar{\Psi}) \rangle = \mathfrak{x}(\bar{\omega}_\sigma) = \mathfrak{x}^g((g\bar{\omega}g^{-1})_\sigma) = \hat{\kappa}^{\hat{m}^{-1}}(g\bar{\omega}g^{-1})$. Now regard $\hat{\varphi}'$ as a parameter for H, and write $\chi_{\hat{\varphi}'}^H$ for the attached character. Let $f \in \mathscr{C}(G)$. Then f_H corresponds to f under (G, H)-transfer and $(f_H)_{\hat{M}_H}$ corresponds to $f_{\hat{M}}$ under $(\hat{M}, \hat{M}_H)$-transfer. Thus we set

$$\hat{\chi}_{(\{\varphi\}, \mathfrak{x})}(f) = c\chi_{\hat{\varphi}'}^H(f_H) = c\chi_{\hat{\varphi}'}((f_H)_{\hat{M}_H}),$$

with c as in (5.4.25), to obtain the following result.

Theorem 5.4.26. *Let $\varphi \in \Phi_0(G)$. Then*

$$\sum_{\pi \in \bar{\Pi}_\varphi} \langle \mathfrak{x}, \pi \rangle \chi_\pi(f) = \hat{\chi}_{(\varphi, \mathfrak{x})}(f), \mathfrak{x} \in \mathbb{S}_\varphi,$$

$f \in \mathscr{C}(G)$.

The inversion is immediate.

Theorem 5.4.27. *Let $\varphi \in \Phi_0(G)$. Then*

$$\chi_\pi(f) = \frac{1}{[\mathbb{S}_\varphi]} \sum_{\mathfrak{x} \in \mathbb{S}_\varphi} \langle \mathfrak{x}, \pi \rangle \hat{\chi}_{(\varphi, \mathfrak{x})}(f), \pi \in \bar{\Pi}_\varphi,$$

$f \in \mathscr{C}(G)$.

Corollary 5.4.28. *If* $\pi \in \bar{\Pi}_\varphi - \Pi_\varphi$ *then*

$$\sum_{x \in \mathbf{S}_\varphi} \langle x, \pi \rangle \hat{\chi}_{(\varphi, x)}(f) = 0, f \in \mathscr{C}(G).$$

Some further analysis of these identities will be carried out in another paper.

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