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# Reductive Group Actions on Stein Spaces

Dennis M. Snow
Department of Mathematics, The Johns Hopkins University, Baltimore, MD 21218, USA

## 1. Introduction

The purpose of this paper is to extend some of the basic results about reductive group actions on affine algebraic varieties to the more general class of complex Stein spaces. These results concern both global and local aspects of such actions – the existence of a complex quotient space which approximates the orbit space and the existence of local complex subspaces transversal to a closed orbit in which the action of the group in an entire neighborhood of that orbit is determined.

To be more precise, let $X$ be a complex Stein space and $G$ a reductive complex Lie group acting holomorphically on $X$. Instead of looking at the usual orbit space of $X$ which may not even be Hausdorff, we consider the quotient of $X$ by the analytic equivalence relation defined by the *invariant holomorphic functions*, $\mathcal{O}(X)^G$. The resulting space $X/G$, the so-called *categorical quotient* of $X$ by $G$, is easily seen to be a Hausdorff $\mathbb{C}$-ringed space satisfying a certain universal property with respect to the action of $G$. For arbitrary complex spaces, such quotients need not be locally compact and hence are not *a priori* isomorphic to complex spaces. We prove the following:

**Theorem.** *The categorical quotient $X/G$ is isomorphic to a complex Stein space such that $\pi: X \to X/G$ is holomorphic.*

In particular, $\mathcal{O}(X)^G = \pi^*\mathcal{O}(X/G)$ is a Stein algebra. The corresponding theorem in the category of affine algebraic varieties is one of the central results of classical invariant theory where it is reduced via equivariant imbeddings to Hilbert's famous theorem: The algebra of invariant polynomials on a rational representation space of a reductive complex Lie group is *finitely generated*, cf. [11, 12, 24, 22, 21].

As in the algebraic case, the map $\pi: X \to X/G$ has certain geometric properties which make it a reasonable substitute for the orbit map. For example, each fiber of $\pi$ contains precisely one closed orbit so that $X/G$ is a natural parameter space for the closed orbits of $G$ in $X$. Note that $X/G$ then coincides with the usual orbit space

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of $X$ exactly when all orbits are closed. In this case, we say that $X/G$ is the *geometric quotient* of $X$ by $G$ to distinguish it from the more general situation.

Our local description of the action of $G$ on a Stein space $X$ is a generalization of Luna's slice theorem [18] for reductive group actions on affine algebraic varieties. This theorem parallels the standard local description of compact group actions, but of course is more general, since it also handles the behaviour of non-closed orbits. Let $G \cdot x$ be a closed orbit for some $x \in X$. A *slice* for the action of $G$ at $x$ is a locally closed $G_x$-stable Stein subspace $B$ of $X$ containing $x$ such that the natural map of the twisted product $G \times_{G_x} B \to X$ is biholomorphic onto a $\pi$-saturated open Stein subset of $X$. The twisted product $G \times_{G_x} B$ is defined to be the geometric quotient of $G \times B$ by the action of $G_x : h \cdot (g, b) := (gh^{-1}, hb)$; $h \in G_x$, $g \in G$, $b \in B$. The action of $G$ on $G \times_{G_x} B$ is completely determined by the action of the reductive group $G_x$ on $B$, thus yielding a natural isomorphism $(G \times_{G_x} B)/G \cong B/G_x$.

**Theorem.** *Slices exist for the action of a reductive complex Lie group on a Stein space.*

A slice at $x$ can be realized as a $G_x$-stable complex subspace of a rational representation space of $G_x$, and hence the action of $G$ on $X$ is actually modelled locally on *linear algebraic* actions. Using this fact, we show that the fibers of the categorical quotient $\pi : X \to X/G$ are equivariantly biholomorphic to affine algebraic varieties on which $G$ acts algebraically. Moreover, if a fiber is non-singular, then it is a homogeneous *vector bundle* over its unique closed orbit.

The main difficulty which must be overcome in proving the above assertions for Stein spaces is the absence of linear equivariant imbeddings – a key tool in the affine algebraic theory. Such imbeddings, if they existed, would essentially reduce the problem to the algebraic case, since every holomorphic representation of a reductive complex Lie group is *rational* with respect to its canonical structure as a linear algebraic group, cf. [13]. Nevertheless, due to the well-known theorem of Harish-Chandra [10] concerning continuous representations of compact groups on Fréchet spaces, we do have local equivariant immersions of Stein spaces at our disposal. By inspecting the local behaviour of a reductive complex Lie group in a rational representation space, as described for example in a local version of the Hilbert-Mumford-Birkes lemma [2], we maintain just enough control over certain of these local equivariant immersion to obtain equivariant imbeddings of $\pi$-saturated open subsets. It is then an easy matter to lift slices back to the Stein space $X$ and to guarantee that the categorical quotient $X/G$ is isomorphic to a complex space.

The paper is organized as follows. In Sect. 2 we recall some of the regularity properties of reductive complex Lie group actions and prove that there always exist local linear equivariant immersions of a finite dimensional Stein space which imbed any given closed orbit.

In Sect. 3 we discuss the formal properties of categorical quotients and the special geometric characteristics of the map $\pi : X \to X/G$ for Stein spaces. These properties are consequences of Theorem B and the existence of a *Reynolds operator* on $\mathcal{O}(X)$. With them, one can easily show that if $X/G$ is isomorphic to a complex space, then it is Stein. In particular, we obtain the above mentioned description of geometric quotients, since Hausdorff orbit spaces carry natural complex structures, cf. [6, 15].

This description enables us to define in Sect. 4 the notion of a *slice* for the action of a reductive group on a Stein space. Slices are not usually local objects, and since we are forced to work with local equivariant immersions, it becomes important to know whether there are arbitrarily small subsets of a slice which still retain all of the pertinent information of the slice. We call such subsets *local slices* and prove that they exist in rational representation spaces of reductive complex Lie groups. It is through these local slices that the potentially bad behaviour of local equivariant immersions is avoided.

In Sect. 5 we apply all of the foregoing results to prove the theorems quoted above as well as some direct corollaries of the existence of a slice.

Finally, we conclude the paper in Sect. 6 with a description of two stratifications of the categorical quotient $\pi: X \to X/G$ and some of their applications.

## 2. Local Equivariant Immersions

*Throughout this paper, all complex spaces are assumed to be reduced and have countable topological bases.*

A complex Lie group $G$ with finitely many connected components is said to be *reductive* if it is isomorphic to the complexification of one of its maximal compact subgroups. A reductive complex Lie group always carries a compatible structure of a linear algebraic group such that every holomorphic representation of $G$ on a finite dimensional complex vector space is in fact rational, cf. [13].

Assume the reductive Lie group $G$ acts holomorphically on a complex space $X$. This action induces a continuous representation of $G$ on the Fréchet space $\mathcal{O}(X)$. We say that a function $f \in \mathcal{O}(X)$ is *G-finite* if the vector subspace of $\mathcal{O}(X)$ spanned by the elements $f \circ g$, $g \in G$, is finite dimensional. Due to the basic result of Harish-Chandra concerning continuous representations of compact groups on Fréchet spaces, cf. [10, Lemma 5] and [4, Sect. 3], we have the following (cf. also [1, 23]):

**2.1.** *The G-finite functions are dense in* $\mathcal{O}(X)$.

As is well-known, this fact implies that reductive group actions on complex spaces are very well behaved, at least when $\mathcal{O}(X)$ is fairly large. To make this more precise, we first mention a simple but useful

**2.2. Lemma.** *Let* $h: X \to \mathbb{C}^n$ *be a linear G-equivariant holomorphic map. Then, for all* $x \in X$, $G_{h(x)}$ *is an algebraic subgroup of G. Moreover, if x is isolated in* $h^{-1}h(x)$, *then*

$$G^0_{h(x)} \subset G_x \subset G_{h(x)},$$

*and hence* $G_x$ *is also an algebraic subgroup of G.*

*Proof.* It is clear that $G_{h(x)}$ is algebraic, since the holomorphic representation of $G$ on $\mathbb{C}^n$ is rational. Furthermore, by equivariance, $G_x$ fixes $h(x)$, and $G^0_{h(x)}$ must fix $x$ because $x$ is isolated in $h^{-1}h(x)$. □

**2.3. Proposition.** *If $X$ is holomorphically spreadable, i.e. if each point $x \in X$ is isolated in the fiber of some holomorphic map $f: X \to \mathbb{C}^n$, then:*

(1) *$G_x$ is an algebraic subgroup of $G$ for all $x \in X$.*

(2) *If $x, y \in X$ with $x \in \mathrm{cl}(G \cdot y) \setminus G \cdot y$, then $\dim G \cdot x < \dim G \cdot y$.*

(3) *Each orbit of $G$ is locally closed in $X$.*

*Proof* (cf. [23, Proposition 2.4]). Let $x$ be isolated in a fiber of $f: X \to \mathbb{C}^n$. By (2.1), we may approximate each coordinate function of $f$ by a $G$-finite holomorphic function to obtain a linear equivariant holomorphic map $h: X \to \mathbb{C}^N$, $N \geqq n$, such that $x$ is still isolated in a fiber of $h$. Then (1) follows from Lemma 2.2.

Now, suppose $x \in \mathrm{cl}(G \cdot y) \setminus G \cdot y$. We claim that $h(x) \in \mathrm{cl}(G \cdot h(y)) \setminus G \cdot h(y)$ which implies $\dim G \cdot h(x) < \dim G \cdot h(y)$ by a standard fact of algebraic transformation groups, cf. [3, p. 98]; and this in turn implies $\dim G \cdot x < \dim G \cdot y$ by Lemma 2.2. To prove the claim, we note that $G \cdot h(x)$ is a locally closed submanifold of $\mathbb{C}^n$. Thus, there exists a connected neighborhood $W$ of $h(x)$ in $\mathbb{C}^N$ such that $Z := G \cdot h(x) \cap W$ is a connected closed submanifold of $W$. Since $x$ is isolated in $h^{-1}h(x)$, there exists a connected neighborhood $U$ of $x$ such that, after shrinking $W$ if necessary, the restricted map $h_{U,W}: U \to W$ is finite, cf. [9, p. 54]. Therefore, $h_{U,W}^{-1}(Z)$ has only finitely many connected components. By the equivariance of $h$, it is clear that we may choose $U$ and $W$ even smaller so that $h_{U,W}^{-1}(Z) = G \cdot x \cap U$. It follows immediately that if $h(x)$ were contained in $G \cdot h(y)$, then $x$ would be contained in $G \cdot y$, since $G \cdot h(y) \cap W = Z$ and $x \in \mathrm{cl}(G \cdot y)$. This shows that $h(x) \in \mathrm{cl}(G \cdot h(y)) \setminus G \cdot h(y)$ as claimed, finishing the proof of (2).

Finally, (3) follows directly from (2) and the fact that the set of orbits of dimension less than a fixed integer is a closed complex subspace of $X$. □

**2.4.** *Remark.* An important consequence of this proposition is that the orbits of *minimal dimension* in $X$ are always *closed.*

For convenience, we shall call a holomorphic map $f: X \to Y$ a *local immersion at* $x \in X$ if there is a neighborhood $U$ of $x$ such that $f: U \to f(U)$ is a homeomorphism onto $f(U)$ with the induced topology from $Y$. Using $G$-finite functions as above and employing a well-known technique which goes back to Cartan [5], we now prove the following:

**2.5. Proposition.** *Let $G$ be a reductive complex Lie group acting holomorphically on a finite dimensional Stein space $X$, and let $G \cdot x$ be a closed orbit for some $x \in X$. Then there exists a linear equivariant holomorphic map $h: X \to \mathbb{C}^n$ which imbeds $G \cdot x$ and is a local immersion at $x$.*

*Proof.* Since $X$ is finite dimensional, there exists a proper injective holomorphic map $f: X \to \mathbb{C}^n$, cf. [9, p. 126]. As in the proof of Proposition 2.4, we may approximate the coordinate functions of $f$ by $G$-finite functions to obtain an equivariant holomorphic map $h_0: X \to \mathbb{C}^N$, $N \geqq n$, which is a local immersion at $x$. We shall now modify $h_0$ so that it imbeds $G \cdot x$.

Since $G \cdot x$ is Stein, $G_x$ is algebraic and hence a reductive subgroup of $G$ by a theorem of Matsushima [19]. It is then well-known, cf. e.g. [14], that $G \cdot x \cong G/G_x$ admits a linear $G$-equivariant holomorphic imbedding $j: G \cdot x \to \mathbb{C}^k$, realizing $G \cdot x$ as an affine algebraic subvariety of some $\mathbb{C}^k$. By Theorem B, such a map can be extended to all of $X$, $j: X \to \mathbb{C}^k$. The idea of Cartan is that by averaging the map

$m^{-1}\cdot j\circ m$ over $m\in M$, a maximal compact subgroup of $G$, we obtain a linear $G$-equivariant holomorphic map $\hat{j}:X\to\mathbb{C}^k$. Since this map still imbeds $G\cdot x$, the holomorphic map $h:=(h_0,\hat{j}):X\to\mathbb{C}^{N+k}$ has the desired properties. □

## 3. Categorical Quotients

Let $G$ be a complex Lie group acting holomorphically on a complex space $X$. The induced representation of $G$ on the Fréchet space $\mathcal{O}(X)$ is continuous and hence the subspace of fixed points $\mathcal{O}(X)^G$ is a closed subalgebra of $\mathcal{O}(X)$, called the algebra of *invariants* of $G$ in $\mathcal{O}(X)$. This algebra defines a natural analytic equivalence relation on $X$:

$$R:=\{(x_1,x_2)\in X\times X\,|\,f(x_1)=f(x_2) \quad \text{for all} \quad f\in\mathcal{O}(X)^G\}.$$

Let $\varrho_1,\varrho_2:R\to X$ be the canonical projections and $\pi:X\to X/G$ the continuous map of $X$ onto the topological quotient space $X/G$ of $X$ by the equivalence relation $R$ such that $\pi\circ\varrho_1=\pi\circ\varrho_2$. Note that $X/G$ is Hausdorff and $\pi$ is naturally equivariant with $\pi\circ g=\pi$ for all $g\in G$. We define the *categorical quotient* of $X$ by $G$ to be the topological space $X/G$ endowed with the following sheaf $\mathcal{O}_{X/G}$ of local $\mathbb{C}$-algebras: For $U$ open in $X/G$, $\mathcal{O}_{X/G}(U)$ consists of those continuous complex valued functions on $U$ which lift via $\pi$ to holomorphic functions on $\pi^{-1}(U)\subset X$, i.e. $\pi^*\mathcal{O}_{X/G}(U)=\mathcal{O}_X(\pi^{-1}(U))^G$. In this way, $(X/G,\mathcal{O}_{X/G})$ becomes a Hausdorff $\mathbb{C}$-ringed space such that $\pi:X\to X/G$ is a surjective morphism satisfying the universal property:

**3.1.** *If $\sigma:X\to Z$ is any morphism of $\mathbb{C}$-ringed spaces such that $\sigma\circ\varrho_1=\sigma\circ\varrho_2$, then there exists a uniquely determined morphism $\sigma':X/G\to Z$ such that $\sigma=\sigma'\circ\pi$.*

In general, the categorical quotient $X/G$ is a much smaller object than the usual orbit space of $G$ acting on $X$, since invariant holomorphic functions cannot separate orbits whose closures have non-trivial intersection. Whenever the categorical quotient and the orbit space actually do coincide, i.e. when the orbits of $G$ equal the fibers of the map $\pi$, we say that $X/G$ is the *geometric quotient* of $X$ by $G$.

Such quotients of $X$ need not be isomorphic to complex spaces, cf. [8]. We would now like to mention an almost obvious criterion for when they are. First, we recall that a complex space $(X,\mathcal{O}_X)$ can be given an enriched (but still reduced) structure sheaf $\hat{\mathcal{O}}_X$ such that the identity map $(X,\hat{\mathcal{O}}_X)\to(X,\mathcal{O}_X)$ is holomorphic and the weak Riemann extension theorem holds on $(X,\hat{\mathcal{O}}_X)$, i.e. any continuous function on $X$ which is holomorphic outside of a nowhere dense analytic subset of $X$ is in fact holomorphic on all of $X$. In addition, any holomorphic map $(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ of complex spaces lifts to a holomorphic map $(X,\hat{\mathcal{O}}_X)\to(Y,\hat{\mathcal{O}}_Y)$, cf. e.g. [7, Sect. 2.29]. We shall say that $X$ has *maximal* complex structure if $\mathcal{O}_X=\hat{\mathcal{O}}_X$.

**3.2. Lemma.** *Assume $(X,\mathcal{O}_X)$ has maximal complex structure and that the categorical quotient $(X/G,\mathcal{O}_{X/G})$ can be covered by open sets $U$ with the property:*

*There exists a complex space $(Y,\mathcal{O}_Y)$ and a morphism of $\mathbb{C}$-ringed spaces $h:(U,\mathcal{O}_{X/G}|U)\to(Y,\mathcal{O}_Y)$ which is a homeomorphism of the underlying topological spaces.*

*Then, $(X/G, \mathcal{O}_{X/G})$ is isomorphic to a complex space with maximal complex structure such that $\pi:(X, \mathcal{O}_X) \to (X/G, \mathcal{O}_{X/G})$ is holomorphic.*

*Proof.* Let $U' := \pi^{-1}(U)$. The morphism $h:(U, \mathcal{O}_{X/G}|U) \to (Y, \mathcal{O}_Y)$ induces a holomorphic map $h':(U', \mathcal{O}_X|U') \to (Y, \hat{\mathcal{O}}_Y)$, since $\mathcal{O}_X|U'$ is maximal. By the universal property (3.1), we then obtain a morphism of $\mathbb{C}$-ringed spaces $(U, \mathcal{O}_{X/G}|U) \to (Y, \hat{\mathcal{O}}_Y)$. We claim that the inverse homeomorphism $h^{-1}: Y \to U$ is also an inverse morphism of these $\mathbb{C}$-ringed spaces, showing that $(U, \mathcal{O}_{X/G}|U)$ is isomorphic to a complex space. To see this, let $f \in \mathcal{O}_{X/G}(W)$ for some open set $W \subset U$. Then $f$ defines a continuous function $f \circ h^{-1}$ on $h(W)$. By lifting to $W' := \pi^{-1}(W)$ via $h'$, it is easy to check that $f \circ h^{-1}$ is holomorphic outside of the nowhere dense analytic set $S(W) \cup A$ of $h(W)$ where $S(W)$ is the set of singular points of $W$ and $A$ is the set over which the holomorphic map $h'|W'$ is degenerate. Therefore, $f \circ h^{-1} \in \hat{\mathcal{O}}_Y(h(W))$. This shows that $h^{-1}$ indeed defines an inverse morphism of $\mathbb{C}$-ringed spaces, as claimed.

The above implies that $(X/G, \mathcal{O}_{X/G})$ is isomorphic to a complex space. Since $\pi$ then induces a holomorphic map $\pi:(X, \mathcal{O}_X) \to (X/G, \hat{\mathcal{O}}_{X/G})$, the universal property (3.1) shows that $(X/G, \mathcal{O}_{X/G})$ is isomorphic to $(X/G, \hat{\mathcal{O}}_{X/G})$. □

Whenever $(X/G, \mathcal{O}_{X/G})$ is isomorphic to a complex space, it inherets not only a unique maximal complex structure from $X$, but also *connectedness, irreducibility, local irreducibility*, or *normality* from $X$. Again, this follows from the definition of $\mathcal{O}_{X/G}$ and the universal property (3.1).

We now turn our attention to the case where $G$ is a reductive complex Lie group as in Sect. 2. Let $M$ be a maximal compact subgroup of $G$ and let $f \in \mathcal{O}(X)$. We define, for $x \in X$,

$$\hat{f}(x) := \int_{m \in M} f(m \cdot x)\, d\mu,$$

where $d\mu$ is a left-invariant normalized Haar measure on $M$. Since $G$ is the complexification of $M$, it is clear that $\hat{f} \in \mathcal{O}(X)^G$. We thus obtain a surjective continuous linear projection $\mathcal{O}(X) \to \mathcal{O}(X)^G$, the so-called *Reynolds operator*. Although this map is not an algebra homomorphism, we do have the *Reynolds identity:*

**3.3.** *If $I \subset \mathcal{O}(X)^G$ is an ideal, then $(I\mathcal{O}(X))^G = I$.*

Due to the well-behaved nature of reductive group actions, as described in Sect. 2, and Theorem B for Stein spaces, we can now easily prove the following proposition and its ensuing corollaries.

**3.4. Proposition.** *Let $\pi: X \to X/G$ be the categorical quotient of a Stein space $X$ by a reductive complex Lie group $G$, and let $F$ be an arbitrary fiber of $\pi$. Then there exists a unique orbit $G \cdot z$ in $F$ satisfying any one of the following equivalent conditions:*

(1) *$G \cdot z$ is an orbit of minimal dimension in $F$.*

(2) *$G \cdot z$ is closed.*

(3) *$G \cdot z$ is contained in every non-empty closed $G$-stable subset of $F$.*

*Proof.* Since $F$ is a closed Stein subspace of $X$, any orbit of minimal dimension in $F$ is closed by (2.4). Now, suppose, $G \cdot x_0$ and $G \cdot x_1$ are two distinct closed orbits in $F$. Since $X$ is Stein, Theorem B implies there is an $f \in \mathcal{O}(X)$ such that $f|G \cdot x_0 = 0$ and

$f|G\cdot x_1=1$. But then $\hat{f}\in\mathcal{O}(X)^G$ also takes the value 0 on $G\cdot x_0$ and 1 on $G\cdot x_1$ which is impossible, since $\hat{f}$ must be constant on $F$. Therefore, $F$ has a unique closed orbit which is also the unique orbit of minimal dimension in $F$. This proves (1) and (2).

Now, let $Y$ be a non-empty closed $G$-stable subset of $F$ and let $y\in Y$. By Proposition 2.3, $\mathrm{cl}(G\cdot y)$ is a closed $G$-stable Stein subspace of $F$. By the same reasoning as above, $\mathrm{cl}(G\cdot y)$ contains a closed orbit which must also be the unique closed orbit in $F$. This proves (3). □

Retaining the above assumptions and notation, we now list a few corollaries of this proposition.

**3.5. Corollary.** *The number of connected components of any closed $G$-stable subset of $F$ does not exceed the finite number of connected components of $G$.*

Note that this means $F$ itself has at most finitely many irreducible components. We shall see in Sect. 5 that $F$ is in fact equivariantly biholomorphic to an affine algebraic variety.

*Proof.* Let $Y$ be a closed $G$-stable subset of $F$ and let $Y'$ be any connected component of $Y$. Since $Y$ is closed and $G$-stable, $G^0\cdot Y'=Y'$, and therefore $G\cdot Y'$ is the disjoint union of closed connected subsets $Y'\cup g_1Y'\cup\ldots\cup g_kY'$ for some $g_1,\ldots,g_k\in G\backslash G^0$ with $k\leqq$ number of connected components of $G$. It is easy to see that each set $g_1Y',\ldots,g_kY'$ is a connected component of $Y$ so that both $G\cdot Y'$ and $Y\backslash G\cdot Y'$ are closed $G$-stable subsets of $F$. Now, $Y\backslash G\cdot Y'$ must be empty, for otherwise both it and $Y'$ would contain the unique closed orbit in $F$ by Proposition 3.4. Thus, $Y=Y'\cup g_1Y'\cup\ldots\cup g_kY'$. □

**3.6. Corollary.** *If $Y_1$ and $Y_2$ are disjoint closed $G$-stable subsets of $X$, then $\pi(Y_1)$ and $\pi(Y_2)$ are closed and disjoint in $X/G$.*

*Proof.* Let $Y=Y_1$ or $Y_2$. We first show that $\pi(Y)$ is closed. Let $\{\pi(y_n)|n\in\mathbb{N}, y_n\in Y\}$ be a sequence of distinct points in $\pi(Y)$ which converge to $p\in X/G$. Choose $z_n\in\pi^{-1}\pi(y_n)$ such that $G\cdot z_n$ is closed and note that $G\cdot z_n\subset Y$. If $\bigcup G\cdot z_n$ $(n\in\mathbb{N})$ is a closed subspace of $X$, then by Theorem B there exists an $f\in\mathcal{O}(X)$ such that $f|G\cdot z_n=n$. But then $\hat{f}\in\mathcal{O}(X)^G$ defines a continuous function on $X/G$ such that $\hat{f}(\pi(y_n))=n$, contradicting the assumption that $\{\pi(y_n)\}$ converges. Therefore, there exists a sequence $\{z_n'|z_n'\in G\cdot z_n\}$ in $Y$ which converges say to $y\in Y$ with $\pi(y)=p$, and hence $\pi(Y)$ is closed.

Now, let $y_1\in Y_1$ and $y_2\in Y_2$. Since the unique closed orbits $G\cdot z_1\subset\pi^{-1}\pi(y_1)$ and $G\cdot z_2\subset\pi^{-1}\pi(y_2)$ are contained in $Y_1$ and $Y_2$, respectively, we have $\pi(y_1)=\pi(z_1)\neq\pi(z_2)=\pi(y_2)$. Therefore, $\pi(Y_1)$ and $\pi(Y_2)$ are disjoint. □

**3.7. Corollary.** *If $(X/G,\mathcal{O}_{X/G})$ is isomorphic to a complex space, then it is Stein and $\pi:(X,\mathcal{O}_X)\to(X/G,\mathcal{O}_{X/G})$ is holomorphic.*

*Proof.* To see that $(X/G,\mathcal{O}_{X/G})$ is Stein, we need only check that it is holomorphically convex, since it is already holomorphically separable by definition. Let $\{p_n|n\in\mathbb{N}\}$ be an infinite discrete sequence of distinct points in $X/G$. Then, there exists an infinite sequence of distinct closed orbits $G\cdot z_n\subset\pi^{-1}(p_n)$ with $\bigcup G\cdot z_n$ $(n\in\mathbb{N})$ closed in $X$. Again, by Theorem B, there is an $f\in\mathcal{O}(X)$ such that $f|G\cdot z_n=n$.

Then $\hat{f} \in \mathcal{O}(X)^G$ defines a holomorphic function on $X/G$ such that $\hat{f}(p_n) = n$. This proves $X/G$ is holomorphically convex, cf. [9, p. 110]. □

We close this section with a criterion for when a categorical quotient $X/G$ is a geometric quotient, i.e. when $X/G$ is homeomorphic to the orbit space of the action of $G$ on $X$. Necessary conditions for $X/G$ to be a geometric quotient are that all orbits be closed or that they all have the same dimension, cf. [15, Satz 7]. We now show that in our special case either of these conditions is also sufficient.

**3.8. Theorem.** *Let $\pi: X \to X/G$ be the categorical quotient of a Stein space $X$ by a reductive complex Lie group $G$. The following are equivalent:*

(1) *$X/G$ is a geometric quotient of $X$ by $G$.*

(2) *The orbits of $G$ are closed in $X$.*

(3) *The orbits of $G$ are equidimensional in $X$.*

*In all cases, $(X/G, \mathcal{O}_{X/G})$ is isomorphic to a Stein space such that $\pi: (X, \mathcal{O}_X) \to (X/G, \mathcal{O}_{X/G})$ is holomorphic.*

*Proof.* We have already remarked that (1) implies (2) and (3), so we shall prove (3)⇒(2)⇒(1). If all orbits have the same dimension, they must all be closed by (2.4); and if all orbits are closed, then each fiber of $\pi: X \to X/G$ consists of exactly one closed orbit by Proposition 3.4. Therefore, the orbit space is homeomorphic to the Hausdorff space $X/G$. Hence, we may apply [15, Satz 17] directly to conclude that $(X/G, \mathcal{O}_{X/G})$ is isomorphic to a complex space (cf. also [6]). By Corollary 3.7, this space is Stein and $\pi$ is holomorphic. □

## 4. Local Slices

Let $H$ be a reductive complex subgroup of a Stein Lie group $G$, and let $Y$ be a Stein space on which $H$ acts holomorphically. We define a holomorphic action of $H$ on $G \times Y$ by $h(g, y) := (gh^{-1}, hy)$; $h \in H$, $g \in G$, $y \in Y$. The orbits of $H$ in $G \times Y$ are all isomorphic to $H$. By Theorem 3.8, a geometric quotient of $G \times Y$ under this action, which we denote by $G \times_H Y$, exists and has the structure of a complex Stein space such that $G \times Y \to G \times_H Y$ is holomorphic. We write $[g, y]$ for the image of $(g, y) \in G \times Y$ in $G \times_H Y$. Note that $G$ acts holomorphically on $G \times_H Y$ and the map $G \times_H Y \to G/H$, defined by $[g, y] \to gH$, is easily seen to be a $G$-equivariant, locally trivial, holomorphic fiber bundle with Stein fiber $Y$ and Stein base $G/H$. The action of $G$ on $Z := G \times_H Y$ is entirely determined by the action of $H$ on $Y$. For example, the isomorphisms

$$\mathcal{O}(Y)^H \cong (\mathcal{O}(G \times Y)^G)^H \cong \mathcal{O}(G \times_H Y)^G$$

immediately induce an isomorphism of categorical quotients: $Z/G \cong Y/H$. If we identify $Y$ with the closed complex subspace $Y_0 := \{[1, y] \mid y \in Y\} \subset Z$, these isomorphisms can be seen geometrically by the fact that:

**4.1.** *$Z = G \cdot Y_0$ and, for all $y \in Y_0$, $G \cdot y \cap Y_0 = H \cdot y$.*

This also implies:

**4.2.** *If $Z'$ is a closed $G$-stable complex subspace of $Z$, then $Z'$ is equivariantly biholomorphic to $G \times_H Y'$ where $Y' := Z' \cap Y_0$.*

Now let $G$ be a reductive complex Lie group (necessarily Stein) acting holomorphically on a Stein space $X$ and let $G\cdot x$ be a closed orbit for some $x\in X$ so that $G_x$ is a reductive subgroup of $G$, cf. [19]. We say that the action of $G$ has a *slice* at the point $x$ if there exists a locally closed $G_x$-stable complex Stein subspace $B\subset X$ containing $x$ such that the natural $G$-equivariant holomorphic map

$$G\times_{G_x}B\to U:=G\cdot B,[g,b]\to g\cdot b,$$

is biholomorphic onto a $G$-saturated open subset $U$ of $X$. By $G$-saturated we shall always mean saturated with respect to the map $\pi:X\to X/G$. Note that $U$ is then necessarily Stein. We remark that the holomorphic map $G\times_{G_x}B\to U$ will be biholomorphic whenever it is a homeomorphism and $B$ is finite dimensional. For, in this case, (4.1) implies

$$\mathcal{O}(U)\cong\mathcal{O}(G\times U)^G\cong\mathcal{O}(G\times B)^{G_x}\cong\mathcal{O}(G\times_{G_x}B)$$

which is enough to conclude that the two finite dimensional Stein spaces are isomorphic, cf. [9, p. 184]. We say that slices exist for the action of $G$ on $X$ if they exist at every point $x\in X$ for which $G\cdot x$ is closed.

We define a *local slice* for the action of $G$ at $x\in X$ to be a relatively compact, locally closed complex subspace $C$ of $X$ containing $x$ such that $B:=G_x\cdot C$ is a slice for the action of $G$ at $x$ and $C$ is open in $B$. We say that local slices exist for the action of $G$ on $X$ if, for any point $x\in X$ with $G\cdot x$ closed and any neighborhood of $x$, there exists a local slice for the action of $G$ at $x$ contained in that neighborhood.

The existence of a slice immediately implies:

**4.3.** *$G_b\subset G_x$ for all $b\in B$ and hence $G_u$ is conjugate to a subgroup of $G_x$ for all $u\in U$.*

We also obtain the isomorphisms of categorical quotients:

**4.4.** $U/G\cong(G\times_{G_x}B)/G\cong B/G_x$.

For a local slice $C$ at $x$ we have:

**4.5.** *$U=G\cdot B=G\cdot C$ is a $G$-saturated open Stein subset of $X$.*

**4.6.** *For any $c\in C$, there exists a point $z\in\mathrm{cl}(G\cdot c)\cap C$ such that $G\cdot z$ and $G_x\cdot z$ are closed and contained in $\mathrm{cl}(G\cdot c)$ and $\mathrm{cl}(G_x\cdot c)$, respectively (cf. Proposition 3.4).*

We now restrict our attention to a *rational representation space $V$ of a reductive complex Lie group $G$.* A fundamental result of Luna [18, p. 98,3°] is the following

**4.7. Slice Lemma.** *Slices exist for the action of $G$ on $V$.*

Of course, this result also applies to reductive groups acting morphically on affine algebraic varieties via equivariant imbeddings and facts like (4.2), cf. [18, III].

The remainder of this section is devoted to proving that *local* slices exist for the action of $G$ on $V$. We begin by presenting a local version of the Hilbert-Mumford-Birkes lemma, cf. [2], and a local property of reductive group orbits in $V$. We use the notation $B(0,\varepsilon)$ to denote the open euclidean ball with center $0\in V$ and radius $\varepsilon$.

**4.8. Lemma.** *Let $M$ be a maximal compact subgroup of $G$. Then, there exist coordinates for $V$ such that the following hold:*

(1) *For any $x \in B(0,\varepsilon)$, $\varepsilon > 0$, and any closed $G$-stable subset $Y$ contained in the closure of $G \cdot x$, there exists a one-parameter subgroup $\lambda: \mathbb{C}^* \to G$ such that*

$$\lambda(t) \cdot x \in B(0,\varepsilon), \quad 0 < |t| < 1 + \delta \quad \textit{for some} \quad \delta > 0$$

*and*

$$\lim_{t \to 0} \lambda(t) \cdot x \in Y \cap B(0,\varepsilon).$$

(2) *If $x, y \in B(0,\varepsilon)$, $\varepsilon > 0$, and $y \in G \cdot x$, then there is an $m \in M$ and a smooth path $\varphi: [0,1] \to G^0$ with $\varphi(0) = 1_G$ such that $m\varphi(r) \cdot x \in B(0,\varepsilon)$ for all $r \in [0,1]$, and $m\varphi(1) \cdot x = y$.*

*Proof.* We may clearly assume $G$ is an algebraic subgroup of $\mathrm{GL}(V)$. Fix coordinates in $V$ such that $M$ is unitary and such that a maximal compact torus of $M$ is diagonal. Let $T \cong (\mathbb{C}^*)^k$ be the complexification of this compact torus to a maximal torus in $G$. Then $T$ is also diagonal.

The proof of (1) is now an easy adaptation of the proof given in [2, p. 464]. We claim:

(∗) *For some $m \in M$,* $\mathrm{cl}(Tm \cdot x) \cap Y \neq \emptyset$.

If (∗) is true, then by [2] there exists a one-parameter subgroup $\mu: \mathbb{C}^* \to T$ such that $\lim_{t \to 0} \mu(t) m \cdot x \in Y$. Now, since $\mu(t) m \cdot x = (t^{p_1} z_1, \ldots, t^{p_n} z_n)$ where $(z_1, \ldots, z_n)$ are coordinates for $m \cdot x \in B(0,\varepsilon)$ and $p_1, \ldots, p_n \geqq 0$, it follows immediately that

$$\mu(t) m \cdot x \in B(0,\varepsilon), \quad 0 < |t| < 1 + \delta \quad \text{for some} \quad \delta > 0,$$

and

$$\lim_{t \to 0} \mu(t) m \cdot x \in Y \cap B(0,\varepsilon).$$

Since $B(0,\varepsilon)$ and $Y$ are stable under $M$, the one-parameter subgroup $\lambda := m^{-1} \mu m : \mathbb{C}^* \to G$ satisfies the assertions of (1).

We now sketch the proof of (∗). If (∗) is not true, then using $T$-invariant functions one easily shows that $\mathrm{cl}(TM \cdot x) \cap Y = \emptyset$ and hence $M \cdot \mathrm{cl}(TM \cdot x) \cap Y = \emptyset$. But then, since $G = MTM$, $G \cdot x \subset M \cdot \mathrm{cl}(TM \cdot x) \subset \mathrm{cl}(G \cdot x)$; and since $M \cdot \mathrm{cl}(TM \cdot x)$ is closed, it equals $\mathrm{cl}(G \cdot x)$, contradicting $Y \subset \mathrm{cl}(G \cdot x)$.

To prove (2), let $y = g \cdot x \in G \cdot x$. Since $G = MTM$, there are elements $m, m_1 \in M$ and $g' \in T$ such that $g = m(m_1 g' m_1^{-1})$. Now, $g' = \exp(z)$ for some $z$ in the Lie algebra of $T$. Define $\psi(r) := \exp(rz) \in G$ for $r \in [0,1]$. Since $\psi(r)$ is a diagonal matrix whose diagonal entries are $\exp(ra_1), \ldots, \exp(ra_n)$ for some $(a_1, \ldots, a_n) \in \mathbb{C}^n \cong V$, we see that $|\psi(r) \cdot v|^2$ is a strictly concave function of $r$ for any $v \in V$, $v \neq 0$. Define $\varphi(r) := m_1 \psi(r) m_1^{-1}$, $r \in [0,1]$. Then $\varphi: [0,1] \to G^0$ is a smooth path with $\varphi(0) = 1_G$ and $m\varphi(1) \cdot x = y$. Furthermore, if $|x|, |y| < \varepsilon$, then $|\psi(0) m_1^{-1} \cdot x| < \varepsilon$ and $|\psi(1) m_1^{-1} \cdot x| < \varepsilon$. By the above remarks we have $|\psi(r) m_1^{-1} \cdot x| < \varepsilon$ for all $r \in [0,1]$ and hence $|\varphi(r) \cdot x| < \varepsilon$ for all $r \in [0,1]$, as claimed. □

**4.9. Corollary.** *For all $\varepsilon>0$, $G\cdot B(0,\varepsilon)$ is a $G$-saturated open neighborhood of $0\in V$.*

*Proof.* Let $z\in B(0,\varepsilon)$ and let $y\in\pi^{-1}\pi(z)$ where $\pi: V\to V/G$. By Proposition 3.4, there exists a unique closed orbit $Y\subset \mathrm{cl}(G\cdot z)\cap \mathrm{cl}(G\cdot y)$. By Lemma 4.8, $Y\cap B(0,\varepsilon)\neq\emptyset$ and therefore $G\cdot y\cap B(0,\varepsilon)\neq\emptyset$. This implies $\pi^{-1}\pi B(0,\varepsilon)=G\cdot B(0,\varepsilon)$. □

**4.10. Local Slice Lemma.** *Local slices exist for the action of $G$ on $V$.*

*Proof.* Let $G\cdot x$ be closed for some $x\in V$ so that $G_x$ is reductive. A slice $B$ for the action of $G$ at $x$ is obtained by taking a $G_x$-splitting of $V\cong T_x(V)=T_x(G\cdot x)\oplus W$ where $W$ is a $G_x$-stable complex subspace of the tangent space $T_x(V)$, choosing a $G_x$-saturated neighborhood $U$ of 0 in $W$, and defining $B:=x+U$, cf. [18, III, Sect. 1]. For appropriate coordinates in $W$ and $\varepsilon>0$, we have $G_x\cdot B(0,\varepsilon)\subset U$ is a $G_x$-saturated open neighborhood of 0 in $W$ by Corollary 4.9. Hence, $\pi_x B(0,\varepsilon)$ is an open neighborhood of $\pi_x(0)$ in $W/G_x$ where $\pi_x: W\to W/G_x$. Since $W/G_x$ is an affine algebraic variety, cf. [21, p. 27] we can choose a Stein neighborhood $S$ of $\pi_x(0)$ in $W/G_x$ which is contained in $\pi_x B(0,\varepsilon)$. Define $B'_\varepsilon$ to be the open neighborhood $B(0,\varepsilon)\cap\pi_x^{-1}(S)$ of 0 in $W$. It is not hard to show that the open $G_x$-saturated subset $G_x\cdot B'_\varepsilon$ of $W$ is holomorphically convex and therefore Stein. Defining $B_\varepsilon:=x+B'_\varepsilon$, we see that the natural map $G\times_{G_x}G_x\cdot B_\varepsilon\to G\cdot B_\varepsilon\subset G\cdot U$ is biholomorphic onto an open $G$-saturated Stein subset of $V$. Therefore, $B_\varepsilon$ is a local slice at $x$. □

We note for future reference that the local slices $B_\varepsilon$ at $x\in V$ just constructed clearly possess the same two properties as $B(0,\varepsilon)$ in Lemma 4.8, namely:

**4.11.** *For any $z\in B_\varepsilon$ and any closed $G_x$-stable subset $Y$ contained in the closure of $G_x\cdot z$ in $V$, there exists a one-parameter subgroup $\lambda:\mathbb{C}^*\to G_x$ such that*

$$\lambda(t)\cdot z\in B_\varepsilon,\quad 0<|t|<1+\delta \quad \textit{for some}\quad \delta>0,$$

*and*

$$\lim_{t\to 0}\lambda(t)\cdot z\in Y\cap B_\varepsilon.$$

**4.12.** *If $y,z\in B_\varepsilon$ and $y\in G_x\cdot z$, then there is an $m\in M_x$ and a smooth path $\varphi:[0,1]\to G_x^0$ with $\varphi(0)=1_{G_x}$ such that $m\varphi(r)\cdot y\in B_\varepsilon$ for all $r\in[0,1]$, and $m\varphi(1)\cdot y=z$.*

The construction of $B_\varepsilon$ also shows it to be stable under a maximal compact subgroup of $G_x$ and:

**4.13.** *The categorical quotient $G\cdot B_\varepsilon/G\cong G_x\cdot B_\varepsilon/G_x$ is a relatively compact Stein open subset of the affine algebraic variety $V/G$.*

Using the existence of local slices with the above properties, we can now prove the following regularity lemmas.

**4.14. Lemma.** *Let $A$ be a locally closed complex subspace of $V$ and let $G\cdot a$ be closed for some $a\in A$. Assume there is a neighborhood $N$ of $1_G$ in $G$ such that $A$ is open in $N\cdot A$. Then there exists a local slice $B_\varepsilon$ at $a$ and a neighborhood $A'$ of $a$ in $A$ such* *that:*

(1) $G \cdot A' = G \cdot A'_\varepsilon$ *where* $A'_\varepsilon := A' \cap B_\varepsilon$;

(2) $G \cdot A'$ *is a closed G-stable complex subspace of the G-saturated open Stein subset* $G \cdot B_\varepsilon$ *of* $V$.

*Proof.* (1) Since $A$ is locally closed and open in $N \cdot A$, we may choose a suitable local slice $B_\varepsilon$ at $a$ and a symmetric neighborhood $N_0 \subset N$ of $1_G$ in $G$ such that $A' := A \cap N_0 \cdot B_\varepsilon$ is a closed complex subspace of the open neighborhood $N_0 \cdot B_\varepsilon$ of $a$ in $V$ and such that $N_0 \cdot A' \cap N_0 \cdot B_\varepsilon = A'$. Then, for all $a' \in A'$ we have $a' = h \cdot b$ for some $h \in N_0$, $b \in B_\varepsilon$; and since $h^{-1} \in N_0$, $b \in N_0 \cdot A' \cap B_\varepsilon = A' \cap B_\varepsilon = A'_\varepsilon$. Therefore, $A' = N_0 \cdot A'_\varepsilon$ and $G \cdot A' = G \cdot A'_\varepsilon$.

(2) Let $\{g_n \cdot a'_n | n \in \mathbb{N}, g_n \in G, a_n \in A'_\varepsilon\}$ be a sequence of points in $G \cdot A'_\varepsilon$ converging to $g \cdot b \in G \cdot B_\varepsilon$. Since $\{g^{-1} g_n \cdot a'_n\}$ converges to $b \in B_\varepsilon$, we have $g^{-1} g_n \cdot a'_n \in N_0 \cdot B_\varepsilon$ for large enough $n$. Then, there are $h_n \in N_0$, $b_n \in B_\varepsilon$ such that $g^{-1} g_n \cdot a'_n = h_n \cdot b_n$. Now, $b_n \in G \cdot a'_n \cap B_\varepsilon = G_x \cdot a'_n \cap B_\varepsilon$ by (4.1). Applying (4.12) above, we see that there is an $m \in M_x$ and a path $\varphi : [0,1] \to G_x^0$ with $\varphi(0) = 1_{G_x}$ such that $m\varphi(1) \cdot a'_n = b_n$ and $m\varphi(r) \cdot a'_n \in B_\varepsilon$ for all $r \in [0,1]$. Again, since $A'$ is closed in $N_0 \cdot B_\varepsilon$ and open in $N_0 \cdot A'$, it follows that $\varphi(r) \cdot a'_n \in N_0 \cdot A' \cap B_\varepsilon = A'_\varepsilon$ for all $r \in [0,1]$, and hence $b_n \in M_x \cdot A'_\varepsilon$. We conclude that $b = \lim b_n \in M_x \cdot A'_\varepsilon$ and that $g \cdot b \in G \cdot A'_\varepsilon$. □

**4.15. Lemma.** *Let $A$ be a closed G-stable complex subspace of a G-saturated open subset $U \subset V$, and let $\pi : U \to \pi(U)$ be the restriction of the categorical quotient $V \to V/G$. Then $\pi|A : A \to \pi(U)$ is a semi-proper holomorphic map and hence $\pi(A)$ is a closed complex subspace of the open set $\pi(U)$ in the affine algebraic variety $V/G$.*

*Proof.* Let $K$ be a compact subset of $\pi(U)$. For each $p \in K$ let $C$ be a local slice at $u$ in $U$ where $G \cdot u$ is the closed orbit in $\pi^{-1}(p)$. There exists a finite number of these slices, call them $C_1, \ldots, C_k$, such that $\pi(C_1) \cup \ldots \cup \pi(C_k) \supset K$. Define $K' := A \cap \pi^{-1}(K) \cap \mathrm{cl}(C_1 \cup \ldots \cup C_k)$. Then, $K'$ is compact in $A$. Note that if $p \in \pi(A) \cap K$, then there is a point $c \in \pi^{-1}(p) \cap C_j$ for some $j$ such that $G \cdot c$ is closed, by (4.6). Since $A$ is closed and $G$-stable, it follows from Proposition 3.4 that $c \in A \cap C_j$. Therefore, $\pi(K') = \pi(A) \cap K$, showing that $\pi|A$ is semi-proper. It is then well-known that $\pi(A)$ is a closed complex subspace of $\pi(U)$ as claimed, cf. e.g. [7, Sect. 1.19]. □

## 5. Main Theorems

Throughout this section $X$ will denote a Stein space and $G$ a reductive complex Lie group acting holomorphically on $X$.

**5.1. Proposition.** *Let $G \cdot x$ be a closed orbit for some $x \in X$ and let $h : X \to \mathbb{C}^n$ be a linear equivariant holomorphic map which imbeds $G \cdot x$ and is a local immersion at $x$. Then, there exists a local slice $C$ for the action of $G$ at $x$ such that $h$ is injective on the G-saturated open Stein subset $U := G \cdot C$ and $h(U)$ is a closed G-stable complex subspace of a G-saturated open Stein subset of $\mathbb{C}^n$.*

*Proof.* Let $U_0$ be a relatively compact neighborhood of $x$ in $X$ and $N$ a sufficiently small neighborhood of $1_G$ in $G$ such that $U_1 := N \cdot U_0$ is still relatively compact in $X$ and $h : U_1 \to h(U_1)$ is a homeomorphism with $h(U_1)$ relatively compact and locally closed in $\mathbb{C}^n$. Then, $h(U_0)$ is open in $h(U_1)=N\cdot h(U_0)$, so we may apply Lemma 4.14: There exists a local slice $B_\varepsilon$ for the linear action of $G$ at $h(x)$ in $\mathbb{C}^n$ and a neighborhood $h(U_0')$ of $h(x)$ in $h(U_0)$ such that $G\cdot h(U_0')=G\cdot(h(U_0')\cap B_\varepsilon)$ is a closed $G$-stable complex subspace of the $G$-saturated open Stein subset $G\cdot B_\varepsilon$ of $\mathbb{C}^n$. Define $C:=U_0'\cap h^{-1}(h(U_0')\cap B_\varepsilon)$ and $U:=G\cdot C$. Then $U$ is open and $G$-stable in $X$, and $h(U)=G\cdot h(C)=G\cdot h(U_0')$. Moreover, by (4.2) and the fact that $G_{h(x)}=G_x$, $h(U)$ is $G$-equivariantly biholomorphic to $G\times_{G_x}D$, where $D$ is the closed $G_x$-stable complex subspace $h(U)\cap G_x\cdot B_\varepsilon=G_x\cdot h(C)$ of $G_x\cdot B_\varepsilon$. Then, $B:=G_x\cdot C=U\cap h^{-1}(D)$ is a closed $G_x$-stable complex subspace of $U$ and so locally closed in $X$. It remains to prove:

(1) $U$ is $G$-saturated;

(2) $h:U\to\mathbb{C}^n$ is injective;

for these assertions immediately imply that $B$ is Stein and $G\times_{G_x}B\to U$ is a homeomorphism, i.e. that $C$ is a local slice at $x$.

(1) We must show $\pi^{-1}\pi(c)$ is contained in $U$ for all $c\in C$ where $\pi:X\to X/G$. Let $y\in\pi^{-1}\pi(c)$ and let $G\cdot z$ be the unique closed orbit in $\pi^{-1}\pi(c)$, cf. Proposition 3.4. If we show that $G\cdot z\subset U$, then $G\cdot y\cap U\neq\emptyset$ and hence $y\in U$ since $U$ is $G$-stable. Now, by (4.6), there exists a point $z'\in C$ such that $h(z')\in \mathrm{cl}(G\cdot h(c))\cap B_\varepsilon\subset h(C)$ with $G\cdot h(z')$, $G_x\cdot h(z')$ closed and contained in $\mathrm{cl}(G\cdot h(c))$, $\mathrm{cl}(G_x\cdot h(c))$, respectively. By (4.11), there exists a one-parameter subgroup $\lambda:\mathbb{C}^*\to G_x$ such that

$$\lambda(t)\cdot h(c)\in h(U)\cap B_\varepsilon=h(C),\qquad 0<|t|\leqq 1,$$

and

$$\lim_{t\to 0}\lambda(t)\cdot h(c)\in G_x\cdot h(z')\cap B_\varepsilon.$$

Since $H$ is injective on $C$, it follows that $\lambda(t)\cdot c\in C$, $0<|t|\leqq 1$, and $\lim\lambda(t)\cdot c\in G_x\cdot z'\cap C$. Therefore, $G\cdot z'\subset\mathrm{cl}(G\cdot c)$. Also, since $h:G\cdot z'\to G\cdot h(z')$ is a finite unramified homogeneous covering map by Lemma 2.2, $G\cdot z'$ is closed. By uniqueness of $G\cdot z$ we then have $G\cdot z=G\cdot z'\subset U$ as desired.

(2) Due to the equivariance of $h$ and the fact that $U=G\cdot C$ with $h$ injective on $C$, we need only show that $h$ is injective on each orbit $G\cdot c$, $c\in C$. Let $G\cdot z$ be the closed orbit in $\mathrm{cl}(G\cdot c)$. By (1) we may assume $z\in C$. Since $G_z$ is reductive, Lemma 2.2 implies that $G_{h(z)}$ is reductive also. Let $M$ be a maximal compact subgroup of $G$. By (4.3), $M_{h(z)}\subset M_{h(x)}=M_x$ and so $M_{h(z)}\cdot z\subset C$. It follows that $M_{h(z)}\cdot z\to M_{h(z)}\cdot h(z)=h(z)$ is $M$-equivariant and injective, i.e. $M_{h(z)}\cdot z=z$. But $M_z\subset M_{h(z)}$ by Lemma 2.2. Thus, $M_z=M_{h(z)}$ and $G_z=G_{h(z)}$. Hence $h$ is injective on $G\cdot z$. Let $h_z$ be the restriction of the map $h$ to $F_z:=\pi^{-1}\pi(z)$. Then, by uniqueness of $G\cdot z$ in $F_z$ and the equivariance of $h_z$ on $F_z$, we conclude that $h_z^{-1}h_z(G\cdot z)=G\cdot z$, cf. Proposition 3.4. Therefore, since $h_z$ is locally injective, there exists a neighborhood $W$ of $G\cdot z$ in $F_z$ such that $h_z$ is injective on $W$ and $h_z^{-1}h_z(W)=W$. Since $G\cdot c\cap W\neq\emptyset$, it follows that $G_c=G_{h(c)}$ and $h$ is injective on $G\cdot c$. □

Our main theorems are now easy consequences of this proposition.

**5.2. Theorem.** *Local slices exist for the action of G on X.*

*Proof.* Let $G\cdot x$ be a closed orbit and let $X_0$ be the union of the finitely many irreducible components of $X$ containing $G\cdot x$. Then $X':=G\cdot X_0$ is a Stein subspace of $X$ with finitely many irreducible components, since $G^0 \cdot X_0 = X_0$ and $G$ has finitely many connected components. By Proposition 2.5, there exists a linear equivariant holomorphic map $h: X' \to \mathbb{C}^n$ which imbeds $G \cdot x$ and is a local immersion at $x$. Proposition 5.1 then implies that a local slice $C$ exists at $x$; and since $U = G \cdot C$ is a $G$-saturated open Stein neighborhood of $G \cdot x$ in $X'$, it follows that $U$ is also $G$-saturated and open in $X$. □

**5.3. Theorem.** *If $X$ has maximal complex structure, then the categorical quotient $X/G$ is isomorphic to a complex Stein space with maximal complex structure such that $\pi: X \to X/G$ is holomorphic.*

*Proof.* Let $G \cdot x$ be a closed orbit and let $h: X' \to \mathbb{C}^n$ be a linear equivariant holomorphic map which imbeds $G \cdot x$ and is a local immersion at $x$, as in the proof of Theorem 5.2. By Proposition 5.1, there is a local slice $C$ at $x$ such that $h: U = G \cdot C \to h(U)$ is an equivariant homeomorphism and $h(U)$ is a closed $G$-stable complex subspace of a $G$-saturated open Stein subset $W$ of $\mathbb{C}^n$. Note that $W/G$ is an open subset of the affine algebraic variety $\mathbb{C}^n/G$, and we obtain an equivariant holomorphic map $\tau \circ h: U \to W/G$ where $\tau: \mathbb{C}^n \to \mathbb{C}^n/G$. By Lemma 4.15, $\tau \circ h(U)$ is a closed complex subspace of $W/G$, and since $U$ is $G$-saturated, the above map yields a morphism of $\mathbb{C}$-ringed spaces $U/G \to \tau \circ h(U)$ which is a homeomorphism of the underlying topological spaces. The theorem now follows from Lemma 3.2. □

**5.4.** *Remark.* Since a local slice $C$ at $x \in X$ is invariant under a maximal compact subgroup $M_x$ of $G_x$, we can $M_x$-equivariantly imbed it into the Zariski-tangent space $T_x(X)$ where $M_x$ acts linearly, cf. [25, p. 63]. This shows that we may also $G_x$-equivariantly identify the slice $B := G_x \cdot C$ with a locally closed $G_x$-stable complex subspace of $T_x(X)$ where $G_x$ acts rationally. Therefore, the action of $G$ on the entire open $G$-saturated subset $U = G \cdot B = G \times_{G_x} B$ is modelled on the restriction of a *linear algebraic* action with $U/G \cong B/G_x$. This description is particularly useful when $X$ is smooth at $x$, for then $B$ is *open* and $G_x$-saturated in a $G_x$-stable $\mathbb{C}$-linear subspace of $T_x(X)$.

**5.5. Corollary.** *If every isotropy subgroup $G_x$, $x \in X$, is conjugate to a fixed subgroup $H$ of $G$, then $X$ is a locally trivial $G$-equivariant holomorphic fiber bundle with Stein fiber $G/H$ and Stein base $X/G$. In particular, $X$ is a holomorphic principal $G$-bundle if and only if $G$ acts freely on $X$.*

*Proof.* Since every orbit of $G$ has the same (minimal) dimension, every orbit is closed and $X \to X/G$ is the geometric quotient of $X$ by $G$, cf. Theorem 3.8. Thus, a local slice $C$ exists at any point $x \in X$ by Theorem 5.2 and $X$ has a covering by $G$-saturated open Stein subsets $U = G \cdot C \cong G \times_{G_x} B$, $B := G_x \cdot C$. Now, for any $c \in C$, we have $G_c \subset G_x$ by (4.3); and our assumption says that $G_x = g G_c g^{-1}$ for some $g \in G$. Therefore, $G_x = G_c$ for all $c \in C$ and $G_x$ acts trivially on the slice $B = G_x \cdot C = C$. We conclude that $U \cong G/G_x \times C \cong G/H \times C$ and $U/G \cong C$ which provides a local trivialization for a holomorphic fiber bundle structure on $X$. □

**5.6. Corollary.** *Assume $G$ has exactly one closed orbit in $X$. Then there exists a reductive complex subgroup $H$ of $G$ and a closed $H$-stable complex subspace $Y$ of $X$ such that:*

(1) *$Y$ is $H$-equivariantly biholomorphic to an affine algebraic variety on which $H$ acts algebraically with exactly one closed orbit, a fixed point of $H$.*

(2) *$X$ is $G$-equivariantly biholomorphic to the affine algebraic variety $G\times_H Y$ on which $G$ acts algebraically.*

(3) *If $X$ is smooth at some point in the closed orbit of $G$, then $Y$ is $H$-equivariantly biholomorphic to a rational representation space of $H$. In particular, $X$ is then a homogeneous vector bundle over the homogeneous affine variety $G/H$.*

*Proof.* Let $G\cdot x$ be the unique closed orbit in $X$, let $B$ be a slice at $x$, and let $U=G\cdot B$. Since $U$ is a $G$-saturated, open, and contains the unique closed orbit of $G$, it follows that $U$ is all of $X$, cf. Proposition 3.4. Setting $Y=B$ and $H=G_x$, we have $X\cong G\times_H Y$ and $X/G\cong Y/H$.

To prove that $Y$ is equivariantly biholomorphic to an affine algebraic variety, we first realize $Y$ as an $H$-stable locally closed complex subspace of the rational representation space $T_x(X)$ of $H$, cf. (5.4). Since $Y=H\cdot C$ for some local slice $C$ at $x$, it follows from Lemma 4.14 that $Y$ is closed in $T_x(X)$.

Let $T$ be a maximal (connected) torus in $G$ and choose coordinates in $T_x(X)$ such that $T$ acts on $T_x(X)$ diagonally. The set of all one-parameter subgroups $\mu:\mathbb{C}^*\to T$ forms a finitely generated abelian group and is in particular countable, cf. [3]. Let $\{\mu_i \mid i\in\mathbb{N}\}$ be an indexing of these one-parameter subgroups and define for $i\in\mathbb{N}$

$$Y_i:=\{v\in Y \mid \mu_i(t)\cdot v=(t^{k_1}v_1,\dots,t^{k_n}v_n),\ k_j>0 \quad \text{if} \quad v_j\neq 0\},$$

where $k_1,\dots,k_n$ are fixed integers depending on $\mu_i$. Since $0\in T_x(X)$ is the only closed orbit of $H$ in $Y$, Lemma 4.8 implies that for each $y\in Y$ there is a one-parameter subgroup $\lambda_y:\mathbb{C}^*\to G$ with $\lim\limits_{t\to 0}\lambda_y(t)\cdot y=0$. Now, every $\lambda_y$, $y\in Y$, is conjugate in $G$ to some $\mu_i$ and therefore we must have $Y=\bigcup G\cdot Y_i$ ($i\in\mathbb{N}$). Clearly, $Y$ can have only finitely many irreducible components (cf. also Corollary 3.5), so the assertion will follow if we prove that each $Y_i$ is an algebraic variety. Let $Y'$ be an irreducible component of $Y_i$ and let $V'$ be the minimal $\mathbb{C}$-linear subspace of $T_x(X)$ containing $Y'$. Then, for all $y\in Y'$, $\mu_i(t)\cdot y=(t^{k_1}y_1,\dots,t^{k_m}y_m)$ for some *positive* integers $k_1,\dots,k_m$. Let $f=\sum c_\alpha z^\alpha$ (multi-index) be a holomorphic function defined in a neighborhood $B(0,\varepsilon)\subset V'$ for some $\varepsilon>0$ which vanishes on $Y'\cap B(0,\varepsilon)$. Let $k\cdot\alpha:=k_1\alpha_1+\dots+k_m\alpha_m$ and define $f_r:=\sum\limits_{k\cdot\alpha=r} c_\alpha z^\alpha$ so that $f=\sum\limits_{r=1}^{\infty} f_r$. Then, for all $y\in Y'\cap B(0,\varepsilon)$, $|t|\leqq 1$, we have $\mu_i(t)\cdot y\in Y'\cap B(0,\varepsilon)$ and $f(\mu_i(t)\cdot y)=\sum\limits_{r=1} t^r f_r(y)$.

Therefore, for $s>0$, $|t|\leqq 1$,

$$0=\left(\frac{d}{dt}\right)^s f(\mu_i(t)\cdot y)=\sum_{r=s}^{\infty}\frac{r!}{(r-s)!}t^{r-s}f_r(y).$$

Setting $t=0$, we see that the polynomials $f_r$, $r>0$, also vanish on $Y'\cap B(0,\varepsilon)$. This shows that $Y'$ is an algebraic subvariety of $V'$.

Finally, if $X$ is smooth at $x$, $Y$ is *open* in an $H$-stable $\mathbb{C}$-linear subspace $V$ of $T_x(X)$, cf. (5.4), and hence equal to all of $V$ by the same remarks as in the beginning of the proof. □

In a rational representation space of a reductive group $G$, there is a well-known criterion for when an orbit of $G$ is closed, cf. [16, p. 354 and 17, Corollary 1]. Since the above corollary implies that every fiber of $\pi: X \to X/G$ is an affine algebraic variety on which $G$ acts algebraically, this criterion also holds on Stein spaces:

**5.7. Corollary.** *Let $H$ be any reductive complex subgroup of $G$, let $N := N_G(H)$ be the normalizer of $H$ in $G$, and let $x \in X$ be a fixed point of $H$. Then $G \cdot x$ is closed if and only if $N \cdot x$ is closed. In particular, if an isotropy subgroup $G_x$, $x \in X$, contains a maximal torus of $G$, then $G \cdot x$ is closed.*

We conclude this section with a few remarks about holomorphically convex spaces. Recall that a complex space $X$ is *holomorphically convex* if for all compact subsets $K \subset X$ the set

$$\hat{K} := \left\{ x \in X \,\middle|\, |f(x)| \leq \max_K |f|, \quad \text{for all} \quad f \in \mathcal{O}(X) \right\}$$

is also compact. It is a well-known theorem of Remmert (cf. [7, 1.25]) that under these circumstances there exists a proper surjective holomorphic map with connected fibers $\tau: X \to Y$ where $Y$ is a complex Stein space and $\tau^* \mathcal{O}(Y) \cong \mathcal{O}(X)$.

**5.8. Corollary.** *Let $X$ be a holomorphically convex space with maximal complex structure, and let $G$ be a reductive complex Lie group acting holomorphically on $X$. Then the categorical quotient $X/G$ has the structure of a complex Stein space with maximal complex structure such that $\pi: X \to X/G$ is holomorphic.*

*Proof.* The action of $G$ on $X$ induces a natural holomorphic action of $G$ on the Stein space $Y$ since $\tau$ is proper with connected fiber. The proof of this well-known fact is as follows: For all $y \in Y$ and $g \in G$, $g \cdot \tau^{-1}(y)$ is compact and connected, so that $\tau g \cdot \tau^{-1}(y)$ is a connected compact complex subvariety of $Y$ by the Proper Mapping Theorem. Therefore, $\tau g \cdot \tau^{-1}(y)$ must be a point. The action of $G$ on $Y$ is then defined by $g \cdot y := \tau g \cdot \tau^{-1}(y)$ which is easily seen to be holomorphic. Since $Y$ must also have maximal complex structure (cf. [7, 2.29]), we may apply Theorem 5.3 to obtain the categorical quotient $\pi': Y \to Y/G$. It is trivial to verify that $X/G \cong Y/G$ and $\pi = \pi' \circ \tau$. □

Note that the categorical quotient $\pi: X \to X/G$ is this setting does not necessarily possess the same geometric properties as it does for Stein spaces. For example, local slices may not exist and the fibers of $\pi$ may not contain unique closed orbits.

## 6. Stratifications of the Quotient

We would like to conclude this paper by describing two related stratifications of the categorical quotient $X/G$ of a Stein space $X$ by a reductive complex Lie group $G$ and some of their applications. We assume $X$ has maximal complex structure so that $X/G$ is isomorphic to a Stein space with maximal complex structure and $\pi: X \to X/G$ is holomorphic, cf. Theorem 5.3.

**6.1. Proposition.** *There exists a natural stratification of $X/G$ into a finite number of mutually disjoint, locally closed complex subspaces $X/G = Y_1 \cup \ldots \cup Y_t$ such that, for $1 \leqq k \leqq t$:*

(1) $\mathrm{cl}(Y_k) \subset Y_1 \cup \ldots \cup Y_k$.

(2) *The map $\pi_k : X_k \to Y_k$, where $X_k := \pi^{-1}(Y_k)$ and $\pi_k := \pi | X_k$, is the categorical quotient of $X_k$ by $G$ and assertions* (3.5)–(3.7) *remain valid for $\pi_k$.*

(3) *The closed orbits of $G$ in $X_k$ are closed in $X$ and they all have the same dimension.*

(4) *If $G \cdot x$ and $G \cdot y$ are closed orbits in $X_k$ such that $\pi_k(x)$, $\pi_k(y)$ lie in the same connected component of $Y_k$, then $G_x^0$ is conjugate to $G_y^0$.*

*Proof.* Let $Z_1$ be the closed $G$-stable complex subspace of $X$ consisting of the orbits of $G$ of minimal dimension. By (2.4), these orbits are closed. Define $Y_1 := \pi(Z_1)$ and $X_1 := \pi^{-1}(Y_1)$. For $k > 1$, we define $Z_k$ recursively to be the closed subspace of $X'_{k-1} := X \backslash \mathrm{cl}(X_{k-1})$ consisting of the orbits of minimal dimension in $X'_{k-1}$. Again by (2.4), these orbits are closed in $X'_{k-1}$. By Propositions 2.3 and 3.4, it is easy to see that the orbits in $Z_k$ are also closed in $X$, that $Y_k := \pi(Z_k)$ is locally closed in $X/G$ with $\mathrm{cl}(Y_k) \subset Y_1 \cup \ldots \cup Y_k$, and that $\pi_k : X_k := \pi^{-1}(Y_k) \to Y_k$ is the categorical quotient as described in (3). Since the dimension of the minimal orbits in $X'_{k-1}$ strictly increases with $k$, this process stops after a finite number of steps. It remains to prove (4). Since $\pi_k(x)$ and $\pi_k(y)$ lie in the same connected component of $Y_k$, there exists a path $\alpha : [0, 1] \to X_k$ and $g \in G$ with $\alpha(0) = x$, $\alpha(1) = g \cdot y$, and $G \cdot \alpha(r)$ a closed orbit in $X_k$ of fixed dimension, $0 \leqq r \leqq 1$. It follows from Theorem 5.3 and (4.3) that $G^0_{\alpha(r)}$ is conjugate to $G^0_{\alpha(s)}$, $0 \leqq r, s \leqq 1$. □

We now cite an example where the above stratification is trivial. We say that a fixed point $x \in X$ of $G$ is *attractive* if there exists a neighborhood $U$ of $x$ such that for all $y \in U$, $\mathrm{cl}(G \cdot y)$ contains a fixed point of $G$. We remark that in the important special case of $\mathbb{C}^*$-actions, attractive fixed points can be identified by a simple differential criterion, cf. [20].

**6.2. Theorem.** *Let $X$ be an irreducible Stein space with maximal complex structure, and let $G$ be a reductive complex Lie group acting holomorphically on $X$. Assume $G$ has an attractive fixed point in $X$. Then:*

(1) *Every fixed point of $G$ is attractive.*

(2) *The set of all fixed points $F$ is a closed irreducible Stein subspace of $X$.*

(3) *There exists a natural holomorphic retraction $X \to F$ inducing the identity on $F$.*

(4) *If $X$ is non-singular, then $X \to F$ is a holomorphic vector bundle.*

*Proof.* The assumption that $G$ has an attractive fixed point implies that $X_1$ as defined above has an interior point and hence $X = X_1$ since $X$ is irreducible. Thus, $\mathrm{cl}(G \cdot y)$ contains a fixed point of $G$ for all $y \in X$, and the only closed orbits of $G$ are its fixed points, cf. Proposition 3.4. It follows that $\tau := \pi | F : F \to X/G$ is a homeomorphism and $F$ is irreducible, cf. Sect. 3. If we endow $F$ with a maximal complex structure, then $\tau$ is biholomorphic, cf. e.g. [7, Sect. 2.20], and $\tau^{-1} \circ \pi$ defines a holomorphic retraction $X \to F$. Assertion (4) follows from Corollary 5.6 and Propositions 6.3 and 6.4. □

For the remainder of this section, we shall assume that $X$ is non-singular. As noted in (5.4), the action of $G$ on $X$ can then be locally modelled on linear algebraic actions $G \times_H B$ where $B$ is open and $H$-saturated in a rational representation space $V$ of $H$. This fact leads to a finer (possibly infinite) stratification of $X/G$ than that above and allows us to draw the same conclusions for Stein manifolds as presented in [18, III, Sects. 2–4] for smooth affine algebraic varieties.

Let $\mathcal{M}$ denote the set of isomorphism classes of rational representation spaces of the reductive complex subgroups of $G$. Define a map $\mu: X/G \to \mathcal{M}$ as follows: Let $y \in X/G$ and let $G \cdot x$ be the unique closed orbit in $\pi^{-1}\pi(y)$ where $\pi: X \to X/G$. We define $\mu(y) \in \mathcal{M}$ to be the isomorphism class of the rational representation of $G_x$ on the normal tangent directions to $G \cdot x$ at $x$ in $X$. For $\lambda \in \mathcal{M}$ we define $(X/G)_\lambda := \mu^{-1}(\lambda)$ and $X_\lambda := \pi^{-1}(X/G)_\lambda$. Contrary to the affine algebraic case, the image of $\mu$ need not be finite. However, we do have:

**6.3. Proposition.** *For all $\lambda \in \mathcal{M}$, $(X/G)_\lambda$ is a locally closed non-singular complex subspace of $X/G$ and the map $\pi: X_\lambda \to (X/G)_\lambda$ is a locally trivial, G-equivariant, holomorphic fiber bundle. In particular, the fibers of $\pi$ in $X_\lambda$ are all G-equivariantly biholomorphic.*

For the proof we refer to [18, Corollaries 4 and 5]. The point is that for a local model $G \times_H B$, $B \subset V$, we have $(G \times_H B)_\lambda = F \cap B/H = F \cap B$ where the representation of $H$ on $V$ belongs to the isomorphism class $\lambda$ and $F$ is the linear subspace of fixed points of $H$ in $V$. The construction of the fiber bundle structure uses Corollary 5.6 and is similar to that in the proof of Corollary 5.5.

Since $X = \bigcup X_\lambda (\lambda \in \mathcal{M})$ and the number of isomorphism classes $\lambda \in \mathcal{M}$ such that $X_\lambda \neq \emptyset$ is at most countable, it follows that $X_\lambda$ has non-empty interior for some $\lambda \in \mathcal{M}$. If $X/G$ is connected, it is clear that only one such $\lambda$ is possible, which we call the *principal model* of $X$.

**6.4. Proposition.** *Assume $X/G$ is connected and let $\lambda \in \mathcal{M}$ be such that $X_\lambda \neq \emptyset$. The following are equivalent:*

(1) *$\lambda$ is a principal model of $X$.*

(2) *$(X/G)_\lambda$ is open in $X/G$.*

(3) *If $G \cdot x$ is a closed orbit, $x \in X_\lambda$, then $G \cdot x$ fibers equivariantly over every closed orbit in $X$.*

(4) *If $G \cdot x$ is a closed orbit, $x \in X_\lambda$, then $\pi: X \to X/G$ is non-degenerate at $x$.*

For the proof see [18, Corollary 6].

With the existence of a principal model, one can apply the arguments of Luna to obtain the following two theorems, the second of which was originally proved by Richardson [23] using different methods. We omit the proofs, since they are the same as in the affine algebraic case, cf. [18, Corollaries 7 and 8].

**6.5. Theorem.** *Let G be a reductive complex Lie group acting holomorphically on a Stein manifold X. Suppose the tangent space at each point possesses a non-degenerate symmetric bilinear form which is invariant under the isotropy subgroup of that point. Then, there exists a dense open subset of X consisting of orbits which are closed in X.*

**6.6. Theorem** (Richardson). *Let G be a reductive complex Lie group acting holomorphically on a Stein manifold X. Then, there exists a subgroup H of G, not* *necessarily reductive, such that the set of points in $X$ whose isotropy subgroups are conjugate to $H$ is dense and has non-empty interior.*

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