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A Grothendieck-Riemann-Roch Formula for Maps of Complex Manifolds

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Contents

Introduction

This is the detailed proof of the formula announced in [11]. This is a generalization of the formula first discovered by Grothendieck [2] for a proper map $f: X \to Y$ of nonsingular projective varieties and a coherent sheaf $\mathscr{F}$ on X. In this case

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Grothendieck proves that if $R^if_*\mathscr{F}$ is the i^{th} direct image sheaf of $\mathscr{F}$ then for a suitable theory of characteristic classes for vector bundles over projective varieties the Chern character of a coherent sheaf can be defined and the formula

$$\sum_i (-1)^i \operatorname{ch}(R^if_*\mathscr{F})\operatorname{Todd}(Y) = f_*(\operatorname{ch}(\mathscr{F})\operatorname{Todd}(X))$$

holds in the characteristic ring of Y. In [9] and [10] the Chern character of a coherent sheaf on an arbitrary complex manifold X is defined as an element of the Hodge cohomology ring $\sum_k H^k(X, \Omega_X^k)$ and the formula above shown to be valid in case Y is a point. This is the formula first proved by Hirzebruch in case X is projective and by Atiyah and Singer, as a special case of the index theorem, in case $\mathscr{F}$ is locally free.

Here the formula is proved for any holomorphic map f of complex manifolds and any coherent sheaf $\mathscr{F}$ on X with the property that f is proper on the support of $\mathscr{F}$. Moreover, our results are derived as relations between local geometric expressions depending on local coordinate systems for the manifolds and twisting cochains for the sheaves. The Chern character of $\mathscr{F}$ and the $R^if_*\mathscr{F}$ are defined as in the earlier paper [9]. This approach extends Atiyah's definition of the characteristic ring of a holomorphic vector bundle via Čech theory, so our formula is to be interpreted as an identity in the Hodge cohomology of Y. The proof is carried out entirely within the framework of twisting cochains and their associated cochain complexes. It appears that some of these techniques can be regarded as a translation of derived category methods into more concrete geometrical terms. Those familiar with the derived category will recognise twisting cochain versions of several of the constructions used in [6], for example.

As indicated in the announcement, the proof of the formula begins, as in Grothendieck's original proof, by factoring f as an embedding followed by projection and proving the formula for each type of map separately. Functoriality shows that the formula is then valid for the composition. In our case the embedding is taken as the graph $\Gamma: X \to Y \times X$, with $\Gamma(x) = (f(x), x)$, so that composition with the projection onto Y gives f.

Multiplicativity of the Todd genus over short exact sequences of bundles shows that if $i: X \to Z$ is a holomorphic embedding the formula reduces to

$$\operatorname{ch}(i_*\mathscr{F}) = i_*(\operatorname{Todd}(N)^{-1}\operatorname{ch}(\mathscr{F})), \tag{0.1}$$

where N is the normal bundle of X in Z. We do not attempt to give a direct proof of the result in this generality. Instead we make the simplifying assumption that there exists a holomorphic map $\varrho: Z \to X$ such that $\varrho \circ i$ is the identity map. This leads to considerable simplifications in the local formulae appearing in the proof and is clearly valid for the case of the graph embedding used in the derivation of the general formula. The formula (0.1) is then shown to hold in the local cohomology space $\sum_k H_S^k(Z, \Omega_Z^k)$, where S is the support of $i_*\mathscr{F}$.

Similarly, for the projection $\pi: Y \times X \to Y$ the formula is equivalent to

$$\sum_i (-1)^i \operatorname{ch}(R^i\pi_*\mathscr{F}) = \int_X \operatorname{Todd}(X)\operatorname{ch}(\mathscr{F}) \tag{0.2}$$

for any coherent sheaf $\mathscr{F}$ on the product with support S proper over Y. The Chern character is interpreted as an element of $\sum_k H_S^k(Y \times X, \Omega_{Y \times X}^k)$.

The proof of (0.2) is the more difficult and occupies the first four chapters of the paper. In case Y is a polydisc, for example, the proof is quite similar to the proof of the Hirzebruch-Riemann-Roch formula given in [10]. The various analytic results used in [10] are replaced by their relative versions. The theory of nuclear Fréchet modules provides suitable techniques and most of the necessary results are already available: Grauert's coherence theorem and the duality theory of [12] substitute for the corresponding finiteness and duality theorems used in [10].

The extension to a general Y again depends on twisting cochain methods, this time relative to an open cover of Y. In outline this still follows the original model of [14] with an explicit construction of a cocycle in $Y \times X \times X$ which is the Gysin image, in the sense of Grothendieck duality, of a certain cocycle on $Y \times X$ under the diagonal inclusion. In more detail, the contents of the various sections are as follows.

The first chapter shows how to associate a twisting cochain and Chern character to a complex of sheaves which is "perfect" in the sense of [7]. This is applied in the case where $\mathscr{F}$ is a coherent sheaf on $Y \times X$ with the projection onto Y proper on the support of $\mathscr{F}$. If $\mathscr{U}$ is a Stein open cover of X then the complex of $\mathcal{O}_Y$-modules which associates the Čech complex $C^{\cdot}(V \times \mathscr{U}, \mathscr{F})$ to an open set V of Y is perfect, and its Chern character is shown to coincide with the left side of (0.2).

The second chapter introduces some familiar simplicial techniques into the context of twisting cochains. These are based on the "shuffle map" of Eilenberg and Maclane [4]. For a product cover $\mathscr{V} \times \mathscr{U}$ of $Y \times X$ this map is used to transform cochains on $\mathscr{V} \times \mathscr{U}$ into cochains on $\mathscr{V}$ with coefficients which are cochains on $\mathscr{U}$. The map is compatible with the usual operations on cochains and for our purposes has the useful property that twisting cochains on $\mathscr{V} \times \mathscr{U}$ can be transformed into twisting cochains on $\mathscr{V}$ in an appropriate sense. The local complexes of the transformed cochain are themselves cochain complexes with differentials coming from the original twisting cochain.

These two chapters thus describe two ways to obtain a twisting cochain for the perfect complex introduced above. In the first case the local complexes are the familiar complexes of free, finitely-generated $\mathcal{O}_Y$-modules; in the second they are themselves cochain complexes. The next chapter relates these two constructions and goes on to show how they lead to an analogue of the "Lefschetz class" on $Y \times X \times X$.

In the absolute case this is quite straightforward and is described in [10, Sect. 1]. The extension to the general case is presented in terms of the constructions of the previous two chapters. The familiar Serre-Grothendieck duality and the Künneth formula for sheaves must be replaced by their relative versions. As before the class can be characterized abstractly as the Gysin image of $Y \times X$ under the diagonal embedding and with respect to integration along the fibres $X \times X$ of the product $Y \times X \times X$. This chapter also includes a twisting cochain version of the proof of the relative duality theorem of [12].

The explicit construction of the dual class of $Y \times X$ in $Y \times X \times X$ through the cochain-level Gysin map is described in Chap. 4. As usual, this is carried out in terms of Koszul resolutions for the diagonal submanifold of $X \times X$, a twisting cochain for $\mathscr{F}$ on $Y \times X$ and explicit chain homotopies for the Koszul complexes. This gives a local formula for the dual cocycle in terms of geometric data for X and

$\mathscr{F}$. It remains to identify this expression with the appropriate characteristic classes and to show that the class constructed is the same as the Lefschetz class of the previous chapter. This latter problem is dealt with by expressing the duality properties of the explicit cocycle in terms of the chain homotopy conditions which characterize the Lefschetz class. The final section of this chapter relates the geometric formula to the characteristic classes of X and $\mathscr{F}$. This is very similar to the argument in the absolute case, and combines a stronger form of the original invariant theory argument of [14] with explicit calculation based on the product operation for twisting cochains. This concludes the proof of (0.2).

The final chapter proves the formula for an embedding $i: X \to Z$ with the retraction ϱ onto X. The proof depends heavily on the exterior algebra structure in the local Koszul resolutions for the submanifold X. The argument is based on the simple fact that for endomorphisms of an exterior algebra the trace operation is a special case of the interior product. Globally this implies that the trace map for twisting cochains of sheaves of the form $i_*\mathscr{F}$ factors through the cochain complex used in the explicit construction of the Gysin map. This complex has the property that the image of any cocycle in it has a residue on X, and the full trace of the cocycle coincides with the Gysin image of this residue in Z. Applied to the powers of the Atiyah class of $i_*\mathscr{F}$ this shows that the Chern character of $i_*\mathscr{F}$ is the Gysin image of some cocycle on X. It remains to identify this cocycle as a representative of the appropriate characteristic class. This depends on a series of reductions based on degree arguments in the exterior algebras of the Koszul complexes. The first reduction shows that Z can be replaced by the normal bundle of X, with X included as the zero section. The same argument also shows that it is only necessary to consider the case $\mathscr{F} = \mathcal{O}_X$. Finally an inductive argument on the rank of N shows that it is sufficient to prove the result for codimension one, where it follows by direct calculation.

Some attempt has been made to keep the proof reasonably self-contained. The main exceptions are the construction of the cochain level Gysin map associated to a submanifold [15], some elementary properties of twisting cochains described in [9] and various facts concerning topological tensor products and their application to coherence and duality questions on analytic spaces, for which [3] and [12] are suitable references.

Notation

For complexes $E^\cdot$, $F^\cdot$ of vector bundles or $\mathcal{O}_X$-modules over a complex manifold X the tensor product $E^\cdot \otimes F^\cdot$, homomorphism complex $\mathrm{Hom}^\cdot(E, F)$ and dual $\check{E}^\cdot$ are defined over $\mathcal{O}_X$ according to the usual conventions. Differentials and product operations for associated spaces of cochains are defined according to the sign conventions used in [9]. For M and N locally convex topological vector spaces with M nuclear, the operation $\hat{\otimes}$ is the usual completed tensor product over $\mathbb{C}$, unless otherwise indicated. The sheaf Ω_X^k of holomorphic k-differentials on X is regarded as a complex, zero except in degree $-k$. For the sign conventions in use here, this gives compatibility of products with the usual wedge product in $\bar{\partial}$-cohomology under the Dolbeault isomorphism. For open sets $V_{\beta_0}, \ldots, V_{\beta_p}$ their intersection is denoted by $V_{\beta_0 \ldots \beta_p}$.

1. Twisted Resolutions of Perfect Complexes

1.1. *Mapping Cones for Twisting Cochains*

We recall some of the terminology of twisting cochains used in [9] and extend the techniques to more general situations which arise here. Throughout Y is a complex manifold, with structure sheaf $\mathcal{O}_Y$. For coherent sheaves or holomorphic bundles $\mathcal{F}, \mathcal{G}$ the notation $\mathrm{Hom}(W; \mathcal{F}, \mathcal{G})$ means the usual space of homomorphisms over $W \subset Y$. It will also be necessary to consider the case where $\mathcal{F}$ or $\mathcal{G}$ may not be coherent; for example they may be vector bundles modelled on infinite dimensional topological vector spaces. In that case the appropriate space of homomorphisms will be specified as necessary.

Suppose given an open cover $\mathscr{V} = \{V_\beta\}$ of Y together with graded $\mathcal{O}_Y$-modules $F_\beta^{\cdot}$ over V_β. Form the space of cochains

$$C^p(\mathscr{V}, \mathrm{Hom}^q(F,F)) = \prod \mathrm{Hom}^q(V_{\beta_0 \ldots \beta_p}; F_{\beta_p}^{\cdot}, F_{\beta_0}^{\cdot})$$

with corresponding Čech coboundary and product operation as in [9]. A *twisting cochain* is an element b of $\sum_k C^k(\mathscr{V}, \mathrm{Hom}^{1-k}(F,F))$ satisfying $\delta b + b \cdot b = 0$, with $b_{\beta\beta}^{1,0}$ the identity map for all β. As before this data will be denoted by $(\mathscr{V}, F, b)$.

For two twisting cochains $(\mathscr{V}, E, a)$ and $(\mathscr{V}, F, b)$ the complexes $C_a^{\cdot}(\mathscr{V}, E^{\cdot})$ and $C_{a,b}^{\cdot}(\mathscr{V}, \mathrm{Hom}^{\cdot}(E,F))$, with differentials D_a and $D_{a,b}$ are defined as in [9], except that for present purposes it is useful to require that all cochains [except for (1,0)-components of twisting cochains] are *non-degenerate* in the sense that they vanish on all simplices of the cover for which two adjacent vertices coincide. The spaces of such cochains are closed under the differentials if it is also required that the twisting cochains are non-degenerate on k-simplices for $k > 1$. From Remark 1.8 of [9] this can always be assumed to be the case.

For complexes $M^{\cdot}$, $N^{\cdot}$ with differentials d_M, d_N and a degree zero chain map $u: M^{\cdot} \to N^{\cdot}$ the *mapping cone* of u is the complex $P^{\cdot}$ with $P^i = M^i \oplus N^{i-1}$ and differential

$$\begin{bmatrix} d_M & 0 \\ u & -d_N \end{bmatrix}.$$

Similarly, given a degree zero cocycle u in $C_{a,b}^{\cdot}(\mathscr{V}, \mathrm{Hom}^{\cdot}(E,F))$ the *mapping cone* of u is the twisting cochain $(\mathscr{V}, G, c)$ with $G_\alpha^i = E_\alpha^i \oplus F_\alpha^{i-1}$ and c given by

$$c_{\beta_0 \ldots \beta_k}^k = \begin{bmatrix} a_{\beta_0 \ldots \beta_k}^k & 0 \\ (-1)^k u_{\beta_0 \ldots \beta_k}^k & (-1)^{k+1} b_{\beta_0 \ldots \beta_k}^k \end{bmatrix}.$$

The condition $\delta c + c \cdot c = 0$ is equivalent to the similar equations satisfied by a, b together with the cocycle condition on u. The local complexes are the mapping cones of the chain maps u_β.

Finally, if $E^{\cdot}$ is a globally defined complex on Y, then for any cover $\mathscr{V}$ of Y there is a naturally associated twisting cochain which will be denoted by $(\mathscr{V}, E, e)$, where $E_\beta = E^{\cdot} | V_\beta$, the map $e_\beta^{0,1}$ is the restriction of the differential of $E^{\cdot}$ and $e_{\alpha\beta}^{1,0}$ is the identity on $E^{\cdot} | V_\alpha \cap V_\beta$.

1.2. Perfect Complexes

We use the following generalization of the notion of coherence for $\mathcal{O}_Y$-modules. A more detailed discussion can be found in [7]. For our purposes a chain map between complexes of $\mathcal{O}_Y$-modules is a *quasi-isomorphism* if the induced map on complexes of sections is a cohomology isomorphism over every sufficiently small Stein open neighborhood of each point. Similarly a complex of $\mathcal{O}_Y$-modules is *acyclic* if the corresponding complex of sections is acyclic over every sufficiently small Stein open neighborhood of each point. Thus a chain map is a quasi-isomorphism if and only if its mapping cone is acyclic.

Remark. Most of the $\mathcal{O}_Y$-modules introduced here are *quasi-coherent nuclear Fréchet modules* over $\mathcal{O}_Y$ in the sense of [12, Sect. 2]. If a complex of such sheaves is bounded above and the complex of sections is acyclic over a Stein open set V, then the complex of sections is acyclic over every Stein open set $W \subset V$ (see [3, Proposition 3] for example).

Definition 1.2.1. A complex $\mathcal{E}^{\cdot}$ of $\mathcal{O}_Y$-modules is *perfect* if each point of Y has a neighborhood V such that $\mathcal{E}^{\cdot}|V$ is quasi-isomorphic to a bounded complex $E^{\cdot}$ of $\mathcal{O}_V$-modules, with each E^k free and of finite rank.

Suppose that $\mathcal{E}^{\cdot}$ is a perfect complex with differential ∂. The twisting cochain $(\mathcal{V}, \mathcal{E}, e)$ is formed as above for any open cover $\mathcal{V}$ of Y.

Definition 1.2.2. A *twisted resolution* of $\mathcal{E}^{\cdot}$ over $\mathcal{V}$ consists of a twisting cochain $(\mathcal{V}, E, a)$ together with a cocycle u of degree zero in $C^{\cdot}_{a,e}(\mathcal{V}, \mathrm{Hom}^{\cdot}(E, \mathcal{E}))$ with the property that each complex $E^{\cdot}_\beta$ is bounded, the E^k_β are free and of finite rank and each local chain map $u_\beta : E^{\cdot}_\beta \to \mathcal{E}^{\cdot}|V_\beta$ is a quasi-isomorphism.

In the case where $\mathcal{E}^{\cdot}$ consists of a single coherent sheaf in degree zero and $(\mathcal{V}, E, a)$ is a twisting cochain for $\mathcal{E}$ in the sense of [9] then there exists a natural choice for u; let u_β be the quotient map from E^0_β onto $\mathcal{E}|V_\beta$ and set $u^k = 0$ for $k > 0$. In the general case the existence of u ensures that the local maps u_β extend to a global map of the cochain complexes $C^{\cdot}_a(\mathcal{V}, E^{\cdot})$ and $C^{\cdot}_e(\mathcal{V}, \mathcal{E}^{\cdot})$ given by left multiplication by u. The construction of a twisting cochain for a single coherent sheaf given in [9, Sect. 1] can now be generalized as follows.

Proposition 1.2.3. *Given a perfect complex $\mathcal{E}^{\cdot}$ and an open cover $\mathcal{V}$ of Y, there exists a twisted resolution of $\mathcal{E}^{\cdot}$ by a twisting chochain $(\mathcal{V}', E, a)$ on some refinement $\mathcal{V}'$ of $\mathcal{V}$.*

Proof. For a suitable Stein refinement of $\mathcal{V}$ it can be assumed that there exist chain maps $u_\beta : E^{\cdot}_\beta \to \mathcal{E}^{\cdot}|V_\beta$ as in the definition, such that u_β induces cohomology isomorphisms over every Stein open subset of V_β. Suppose that $E^{\cdot}_\beta$ has differential $a^{0,1}_\beta$ and let $L^{\cdot}_\beta$ be the mapping cone of u_β. We construct a cochain A in $\sum_k C^k(\mathcal{V}, \mathrm{Hom}^{1-k}(L, L))$ satisfying $\delta A + A \cdot A = 0$. This will be the mapping cone of some cocycle u in $\sum_k C^k_{a,e}(\mathcal{V}, \mathrm{Hom}^{-k}(E, \mathcal{E}))$ extending the given u_β. In particular $A^{0,1}_\beta$ will be the differential of $L^{\cdot}_\beta$ and $A^{1,0}_{\beta\beta}$ the identity map. It follows that $(\mathcal{V}, E, a)$ and u solve the problem. The construction of A depends on the following lemmas.

Lemma 1.2.4. *Let $E^{\cdot}$ be a bounded complex of modules with each E^k free and of finite rank. If the complex $F^{\cdot}$ is acyclic then the same is true for the complex* $\mathrm{Hom}^{\cdot}(E, F)$.

Proof. Either consider the spectral sequence obtained by filtering by the $E^{\cdot}$ degree or show directly that every chain map of $E^{\cdot}$ into $F^{\cdot}$ is homotopic to zero, as in [7, Lemma 4.1].

Lemma 1.2.5. *Suppose given chain maps $r: E^{\cdot} \to F^{\cdot}$ and $s: G^{\cdot} \to F^{\cdot}$ with $E^{\cdot}$ as above and s an isomorphism on cohomology. Then there exists a chain map $r': E^{\cdot} \to G^{\cdot}$ with $s \circ r'$ homotopic to r.*

Proof [7, Lemma 4.2]. Let $K^{\cdot}$ be the mapping cone of s. The composition of r with the inclusion of $F^{\cdot}$ into the acyclic complex $K^{\cdot}$ gives a chain map (of degree $+1$). By the preceding lemma this map is homotopic to zero and the components of the homotopy give r' and the homotopy between $s \circ r'$ and r.

Now apply (1.2.5) with $r = u_\beta$ and $s = u_\alpha$ to obtain $a_{\alpha\beta}$ and $u_{\alpha\beta}$ over $V_\alpha \cap V_\beta$ satisfying $a_\alpha a_{\alpha\beta} = a_{\alpha\beta} a_\beta$ and $u_\beta - u_\alpha a_{\alpha\beta} = \partial u_{\alpha\beta} + u_{\alpha\beta} a_\beta$, where ∂ is the differential of $\mathscr{E}^{\cdot}$. This gives $A^{1,0}$ satisfying $A_\alpha A_{\alpha\beta} = A_{\alpha\beta} A_\beta$.

The remaining $A^{k,1-k}$ are now constructed inductively. If D is the differential on $\mathrm{Hom}^{\cdot}(L_{\beta_k}, L_{\beta_0})$ this means finding $A^{k,1-k}_{\beta_0 \dots \beta_k}$ on $V_{\beta_0 \dots \beta_k}$ satisfying

(i) $(-1)^{k+1} D A^k_{\beta_0 \dots \beta_k} = \left[\delta A^{k-1} + \sum_{l=1}^{k-1} A^l \cdot A^{k-l} \right]_{\beta_0 \dots \beta_k}$.

(ii) $A^k_{\beta_0 \dots \beta_k}$ vanishes on the subcomplex $\mathscr{E}^{\cdot}_{\beta_k}$ of $L^{\cdot}_{\beta_k}$.

Condition (ii) is equivalent to the fact that $A^k_{\beta_0 \dots \beta_k}$ lies in the subcomplex $\mathrm{Hom}^{\cdot}(E_{\beta_k}, L_{\beta_0})$ of $\mathrm{Hom}^{\cdot}(L_{\beta_k}, L_{\beta_0})$ and it follows inductively that the right side of (i) lies in this subcomplex for $k > 1$. The usual inductive calculation shows moreover that the same expression is a cocycle. But $\mathrm{Hom}^{\cdot}(E_{\beta_k}, L_{\beta_0})$ is acyclic, so $A^k_{\beta_0 \dots \beta_k}$ satisfying (i) exists. The resulting twisted resolution will be written as $(\mathscr{V}, E, A)$.

1.3. The Trace Map and Chern Character

With suitable modifications the theory of the trace map introduced in [9] goes through for perfect complexes. Only certain aspects of the theory are needed for present purposes however, and these are described next. Let $(\mathscr{V}, E, A)$ be a twisted resolution of a perfect complex $\mathscr{E}^{\cdot}$ on Y as above. The *trace map*

$$\tau_a : C^{\cdot}_a(\mathscr{V}, \mathrm{Hom}^{\cdot}(E, E)) \to C^{\cdot}(\mathscr{V}, \mathcal{O}_Y)$$

is the chain map into the usual Čech complex given on cochains of bidegree (p, q) by

$$(\tau_a u^{p,q})_{\beta_0 \dots \beta_{p+q}} = \sum_{k=0}^{q} (-1)^{k(p+q)+q} \cdot \mathrm{tr}(a^{q+1}_{\beta_{k+p} \dots \beta_{q+p} \beta_0 \dots \beta_k} u_{\beta_k \dots \beta_{k+p}})$$

where as usual "tr" denotes the sum of traces in even degree minus the sum of traces in odd degree. The same calculation as in [9, Sect. 3] shows that τ_a is a chain map.

Similarly the *Atiyah class* of $\mathscr{E}^{\cdot}$ is represented in $C_a(\mathscr{V}, \mathrm{Hom}^{\cdot}(E, E \otimes \Omega^1_Y))$ by the 0-cocycle da, where d is the usual exterior derivative, and the *Chern character* of $\mathscr{E}^{\cdot}$ is the class in $\sum_k H^k(Y, \Omega^k_Y)$ represented by

$$\mathrm{ch}(\mathscr{E}^{\cdot}) = \tau_a\Big(\sum_k (da)^k / k!\Big).$$

Independence of choice of twisted resolution can be shown as in [9, Sect. 3]. It follows that quasi-isomorphic perfect complexes have the same Chern character. Alternatively, independence also follows from the formula, to be proved below,

$$\text{(1.3.1)} \qquad \mathrm{ch}(\mathscr{E}^{\cdot}) = \sum_k (-1)^k \mathrm{ch}(\mathscr{H}^k(\mathscr{E}^{\cdot}))$$

which relates the Chern character for perfect complexes to the definition given in [9] through the cohomology sheaves $\mathscr{H}^k(\mathscr{E}^{\cdot})$. The formula (1.3.1) is proved by reduction to the following proposition, which generalizes the additivity property of the Chern character given in [9, Proposition 4.5].

Proposition 1.3.2. *Suppose $\mathscr{E}^{\cdot}$ and $\mathscr{F}^{\cdot}$ are perfect complexes on Y and $\Theta: \mathscr{E}^{\cdot} \to \mathscr{F}^{\cdot}$ is a chain map (of degree zero). Let $\mathscr{G}^{\cdot}$ be the mapping cone of Θ. Then $\mathscr{G}^{\cdot}$ is also perfect and*

$$\mathrm{ch}(\mathscr{G}^{\cdot}) = \mathrm{ch}(\mathscr{E}^{\cdot}) - \mathrm{ch}(\mathscr{F}^{\cdot})$$

in $\sum_k H^k(Y, \Omega_Y^k)$.

From the proposition formula (1.3.1) follows by induction on the number of integers k such that $\mathscr{H}^k(\mathscr{E}^{\cdot}) \neq 0$. If there are no such k the result is trivial. In general suppose that $\mathscr{H}^k(\mathscr{E}^{\cdot}) \neq 0$ and $\mathscr{H}^l(\mathscr{E}^{\cdot}) = 0$ for $l<k$. If $\mathscr{E}^l$ is replaced by zero for all $l<k$ and $\mathscr{E}^k$ by $\mathscr{E}^k/\partial\mathscr{E}^{k-1}$ the resulting complex is quasi-isomorphic to $\mathscr{E}^{\cdot}$, so it can be assumed that $\mathscr{E}^l = 0$ for $l<k$. Let $\mathscr{L}^{\cdot}$ be the perfect complex with $\mathscr{H}^k(\mathscr{E}^{\cdot})$ in degree k and zero elsewhere. The inclusion of $\mathscr{H}^k(\mathscr{E}^{\cdot})$ into $\mathscr{E}^k$ gives a chain map of $\mathscr{L}^{\cdot}$ into $\mathscr{E}^{\cdot}$, with mapping cone $\mathscr{K}^{\cdot}$ say. The cohomology exact sequence shows that $\mathscr{H}^i(\mathscr{K}^{\cdot}) = \mathscr{H}^{k+1}(\mathscr{E}^{\cdot})$ for $i \neq k+1$, while $\mathscr{H}^{k+1}(\mathscr{K}^{\cdot}) = 0$. Then (1.3.1) for $\mathscr{E}^{\cdot}$ follows from the proposition and the corresponding formula for $\mathscr{K}^{\cdot}$.

Proof of 1.3.2. Let $(\mathscr{V}, E, A)$ and $(\mathscr{V}, F, B)$ be twisted resolutions for $\mathscr{E}^{\cdot}, \mathscr{F}^{\cdot}$ with A as above and B the twisted resolution corresponding similarly to a twisting cochain b and 0-cocycle v in $C^{\cdot}_{b,e}(\mathscr{V}, \mathrm{Hom}^{\cdot}(F, \mathscr{F}))$. From the form of the Atiyah class and the trace map the proposition will follow from the existence of a twisted resolution for $\mathscr{G}^{\cdot}$ for which the associated twisting cochain is the mapping cone of some 0-cocycle h in $C^{\cdot}_{a,b}(\mathscr{V}, \mathrm{Hom}^{\cdot}(E, F))$. Let $L^{\cdot}_\beta$, $M^{\cdot}_\beta$ be the mapping cones of u_β, v_β.

Proposition 1.3.3. *There exists a 0-cocycle U in $C^{\cdot}_{A,B}(\mathscr{V}, \mathrm{Hom}^{\cdot}(L, M))$ of the form*

$$U^k = \begin{bmatrix} h_k & 0 \\ (-1)^k s^k & (-1)^k \Theta^k \end{bmatrix}$$

with respect to the decomposition $L^i_\beta = E^i_\beta \oplus \mathscr{E}^{i-1}_\beta$ and $M^i_\beta = F^i_\beta \oplus \mathscr{F}^{i-1}_\beta$, where $\Theta^{0,0}_\beta$ is $\Theta | V_\beta$ and $\Theta^{k,-k} = 0$ for $k>0$.

Remark 1.3.4. With this choice of signs the cocycle condition on U is equivalent to the cocycle conditions on h, Θ together with the relation

$$v \cdot h - \Theta \cdot u = D_{a,e} s.$$

Proof. Let ∂', ∂'' be the differentials of $\mathscr{E}^{\cdot}, \mathscr{F}^{\cdot}$. Lemma 1.2.5 gives the existence of a chain map $h_\beta: E^{\cdot}_\beta \to F^{\cdot}_\beta$ and s_β with

$$v_\beta h_\beta - \Theta_\beta u_\beta = \partial'' s_\beta + s_\beta a_\beta$$

and hence a cochain U^0 satisfying $B_\beta U_\beta = U_\beta A_\beta$. The argument proceeds as in the proposition above by finding sections $U^k_{\beta_0\ldots\beta_k}$ of the acyclic subcomplex $\mathrm{Hom}^\cdot(E_{\beta_k}, M_{\beta_0})$ of $\mathrm{Hom}^\cdot(L_{\beta_k}, M_{\beta_0})$ with coboundary

$$(-1)^{k+1}\left[\delta U^{k-1} + \sum_{l=1}^{k-1} (B^l \cdot U^{k-l} - U^{k-l} \cdot A^l)\right]_{\beta_0\ldots\beta_k}.$$

This expression is checked inductively to be a cocycle and to lie in the acyclic subcomplex; the existence of U^k follows.

The mapping cone of U is now a twisting cochain for local complexes $L^i_\beta \oplus M^{i-1}_\beta$. Under the isomorphism of $L^i_\beta \oplus M^{i-1}_\beta$ with $N^i_\beta \oplus \mathscr{G}^{i-1}_\beta$, where $N^i_\beta = E^i_\beta \oplus F^{i-1}_\beta$, this goes over into a twisted resolution $(\mathscr{V}, N, C)$ for $\mathscr{G}^\cdot$, with

$$C^k = \begin{bmatrix} c^k & 0 \\ (-1)^k w^k & (-1)^{k+1} e^k \end{bmatrix},$$

where now

$$c^k = \begin{bmatrix} a^k & 0 \\ (-1)^k h^k & (-1)^{k+1} b^k \end{bmatrix}, \quad w^k = \begin{bmatrix} u^k & 0 \\ (-1)^{k+1} s^k & (-1)^k v^k \end{bmatrix}.$$

Since u^0_β and v^0_β are quasi-isomorphisms, so is w^0_β. This gives the required twisting cochain for $\mathscr{G}^\cdot$.

1.4. Non-Degeneracy of the Trace Pairing

If $E^\cdot, F^\cdot$ are complexes of finite-dimensional vector spaces, then the trace gives an isomorphism between $\mathrm{Hom}^\cdot(F, E)$ and the dual of $\mathrm{Hom}^\cdot(E, F)$. In order to state and prove the global analogue of this perfect pairing we use the terminology of [9] concerning simplicial maps between complexes associated to twisting cochains. For a cover $\mathscr{V}$ of Y and simplex $\beta = \beta_0\ldots\beta_p$ of the nerve of $\mathscr{V}$, write $\gamma \leqq \beta$ in case $\gamma = \gamma_0\ldots\gamma_q$ with $\gamma_k = \beta_{v(k)}$ for some strictly increasing $v: \{0, \ldots, q\} \to \{0, \ldots, p\}$. (This is a slightly more restrictive definition than that used previously.)

Now let $C^\cdot_A(\mathscr{V}, E)$ be a twisted complex consisting of complexes of (finite-dimensional) vector bundles $E^\cdot_\beta$ over $V_\beta = V_{\beta_0} \cap \ldots \cap V_{\beta_p}$ for each simplex $\beta = \beta_0\ldots\beta_p$ of $\mathscr{V}$, with differential induced by bundle maps A^γ_β for $\gamma \leqq \beta$. Recall that if $C^\cdot_B(\mathscr{V}, F)$ is a second such complex we define $\mathrm{Hom}^\cdot(\mathscr{V}; E, F)$ as the complex of maps $T: C^\cdot_A(\mathscr{V}, E) \to C^\cdot_B(\mathscr{V}, F)$ of the form

$$(Tc)_\beta = \sum_{\gamma \leqq \beta} T^\gamma_\beta c_\gamma$$

for suitable bundle maps T^γ_β. For T of total degree r the differential is given as usual by

$$T \mapsto B \circ T + (-1)^{r+1} T \circ A.$$

Suppose that $(\mathscr{V}, E, a)$, $(\mathscr{V}, F, b)$ are twisting cochains, with $E^\cdot_\beta$, $F^\cdot_\beta$ bounded, free and of finite rank. Interpret $C^\cdot_{a,b}(\mathscr{V}, \mathrm{Hom}^\cdot(E, F))$ and the Čech complex $C^\cdot(\mathscr{V}, \mathcal{O}_Y)$ as twisted complexes $C^\cdot_D(\mathscr{V}, M)$ and $C^\cdot_\delta(\mathscr{V}, \mathcal{O}_Y)$, so that $M^r_{\beta_0\ldots\beta_p} = \mathrm{Hom}^r(E_{\beta_p}, F_{\beta_0})$, and δ^γ_β is zero unless γ is a codimension one face of β, in which case it is the restriction map, up to sign. The next proposition is the global non-degeneracy result. In a derived category setting this result appears in [7, Sect. 7].

Proposition 1.4.1. *The chain map* $u \mapsto \phi_u$ *from* $C^{\cdot}(\mathscr{V}, \mathrm{Hom}^{\cdot}(F,E))$ *into* $\mathrm{Hom}^{\cdot}(\mathscr{V}; M, \mathcal{O}_Y)$, *defined by* $\phi_u(v) = \tau_a(u \cdot v)$, *is a quasi-isomorphism of cochain complexes.*

Proof. Note that $\mathrm{Hom}^{\cdot}(\mathscr{V}; M, \mathcal{O}_Y)$ can itself be regarded as a twisted complex on $\mathscr{V}$, for which the local complex over $V_{\beta_0 \ldots \beta_r}$ is $G^{\cdot}_{\beta_0 \ldots \beta_r}$ say, with

$$G^q_{\beta_0 \ldots \beta_r} = \sum_{\gamma_0 \ldots \gamma_s \leqq \beta} \mathrm{Hom}(\mathrm{Hom}^{-q}(E_{\gamma_s}, F_{\gamma_0}), \mathcal{O}_Y)$$

and differential $T \mapsto \pm T \circ D$. Under this interpretation ϕ itself becomes a simplicial map in $\mathrm{Hom}^{\cdot}(\mathscr{V}; M, G)$. The usual comparison of spectral sequences shows that in order to prove that ϕ is a quasi-isomorphism, it is only necessary to check that each of the local chain maps ϕ^{β}_{β} induces a cohomology isomorphism over V_{β}. But ϕ^{β}_{β} factorizes into a map

$$\text{(1.4.2)} \qquad \mathrm{Hom}^{\cdot}(F_{\beta_r}, E_{\beta_0}) \to \mathrm{Hom}^{\cdot}(\mathrm{Hom}(E_{\beta_r}, F_{\beta_r}), \mathcal{O}_Y)$$

followed by the natural inclusion of the second complex into $G^{\cdot}_{\beta}$, where in (1.4.2) a bundle map $v_{\beta_0 \ldots \beta_r}$ goes into the homomorphism sending u_{β_r} into $\mathrm{tr}(b^1_{\beta_r \beta_0} v_{\beta_0 \ldots \beta_r} u_{\beta_r})$. But the second complex is isomorphic to $\mathrm{Hom}^{\cdot}(F_{\beta_r}, E_{\beta_r})$ under the trace pairing, and $b^1_{\beta_r \beta_0}$ is a chain homotopy equivalence, so (1.4.2) induces quasi-isomorphisms on the complexes of sections over V_{β}.

Let $\mathscr{H}^q_{\beta}$ be the q-cohomology in this case. If the complex $G^{\cdot}_{\beta}$ is filtered by simplicial degree then the resulting E^{pq}_1 can be identified with the space of (ordered, non-degenerate) p-chains of the simplex $\beta_0 \ldots \beta_r$, with coefficients in $\mathscr{H}^q_{\beta}$ and the usual simplicial boundary (up to sign). The inclusion map into $G^{\cdot}_{\beta}$ then corresponds to inclusion of the vertex β_r, and the acyclicity of the r-simplex gives the result.

2. Direct Images of Twisting Cochains

2.1. The Shuffle Map

Suppose given complex manifolds X, Y with open covers $\mathscr{U}$, $\mathscr{V}$ and projection map $\pi: Y \times X \to Y$. A coherent sheaf $\mathscr{F}$ on $Y \times X$ gives a complex $C^{\cdot}(\mathscr{U}, \mathscr{F})$ of $\mathcal{O}_Y$-modules with

$$C^r(\mathscr{U}, \mathscr{F})(W) = C^r(W \times \mathscr{U}, \mathscr{F})$$

for $W \subset Y$, where if $\mathscr{U} = \{U_\alpha\}$ then $W \times \mathscr{U}$ is the cover $\{W \times U_\alpha\}$ of $W \times X$. This complex has the usual Čech boundary operator and if $\mathscr{U}$ is a Stein cover, the q^{th} cohomology gives the direct image sheaf $R^q \pi_*(\mathscr{F})$. There is also a bicomplex $C^{\cdot}(\mathscr{V}, C^{\cdot}(\mathscr{U}, \mathscr{F}))$, for which a cochain c of bidegree (p, q) consists of sections $c_{\beta_0 \ldots \beta_p}$ of $C^q(\mathscr{U}, \mathscr{F})$ over $V_{\beta_0 \ldots \beta_p}$, each of which in turn corresponds to sections $c_{\beta_0 \ldots \beta_p, \alpha_0 \ldots \alpha_q}$ of $\mathscr{F}$ over $V_{\beta_0 \ldots \beta_p} \times U_{\alpha_0 \ldots \alpha_q}$. The differential is $\delta_X + \delta_Y$, where

$$(\delta_X v)_{\beta_0 \ldots \beta_p, \alpha_0 \ldots \alpha_{q+1}} = \sum_{k=0}^{q+1} (-1)^{k+p} c_{\beta_0 \ldots \beta_p, \alpha_0 \ldots \hat{\alpha}_k \ldots \alpha_{q+1}}$$

$$(\delta_Y c)_{\beta_0 \ldots \beta_{p+1}, \alpha_0 \ldots \alpha_q} = \sum_{k=0}^{p+1} (-1)^k c_{\beta_0 \ldots \hat{\beta}_k \ldots \beta_{p+1}, \alpha_0 \ldots \alpha_q}.$$

This bicomplex is related to the complex $C^{\cdot}(\mathcal{V}\times\mathcal{U},\mathcal{F})$ by the *shuffle map* of Eilenberg and Maclane [4, 8] applied to the cover $\mathcal{V}\times\mathcal{U}$. This is the map

(2.1.1) $$\sigma: C^{\cdot}(\mathcal{V}\times\mathcal{U},\mathcal{F})\to C^{\cdot}(\mathcal{V},C^{\cdot}(\mathcal{U},\mathcal{F}))$$

given by

(2.1.2) $$\sigma(u)_{\beta_0\ldots\beta_p,\alpha_0\ldots\alpha_q}=\sum \pm u_{\tau_0\ldots\tau_{p+q}},$$

where the sum runs over all strictly increasing sequences $1\leqq i_1<\ldots<i_p\leqq p+q$ and $\tau_r=(\beta_{l_r},\alpha_{m_r})$ with $l_0=0=m_0$ and

$$m_{i_r}=m_{i_r-1},\quad l_{i_r}=l_{i_r-1}+1,$$
$$l_{j_s}=l_{j_s-1},\quad m_{j_s}=m_{j_s-1}+1.$$

Here the sequence $1\leqq j_1<\ldots<j_q\leqq p+q$ is the complement of the sequence (i_r) and the sign in the formula is that of the permutation $(i_1,\ldots,i_p,j_1,\ldots,j_q)$.

Both complexes in (2.1.1) have a cup-product. A pairing $\mathcal{E}\otimes\mathcal{F}\to\mathcal{G}$ induces the usual cup-product pairing $C^{\cdot}(\mathcal{U},\mathcal{E})\otimes C^{\cdot}(\mathcal{U},\mathcal{F})\to C^{\cdot}(\mathcal{U},\mathcal{G})$ and hence a further cup-product between $C^{\cdot}(\mathcal{V},C^{\cdot}(\mathcal{U},\mathcal{E}))$ and $C^{\cdot}(\mathcal{V},C^{\cdot}(\mathcal{U},\mathcal{F}))$, also denoted by $u,v\mapsto u\cdot v$. The important properties of the shuffle map are the following.

Proposition 2.1.3. *With respect to the above operations the shuffle map satisfies*
(i) $\sigma(\delta u)=(\delta_X+\delta_Y)\sigma(u)$,
(ii) $\sigma(u\cdot v)=\sigma(u)\cdot\sigma(v)$.
Moreover σ is a quasi-isomorphism.

Proof. Equation (i) is proved by induction on degree, as in [4, Theorem 5.2]. The second formula follows from the definitions; the argument is given in [5, Sect. 3]. The last statement is also proved in [4].

2.2. Fibre Integration on Čech Cochains

Let $\pi: Y\times X\to Y$ and $\mathcal{U}$, $\mathcal{V}$ be as above with X, Y of complex dimension n, m respectively. For a coherent sheaf $\mathcal{E}$ on Y we use the standard notation $\pi^!\mathcal{E}$ for the tensor product of Ω^n_X, regarded as usual as a complex zero except in degree $-n$, with $\pi^*\mathcal{E}$ over $\mathcal{O}_{X\times Y}$. The shuffle map can be used to define a *fibre integral.*

$$\int_X : C^{\cdot}_c(\mathcal{V}\times\mathcal{U},\pi^!\mathcal{E})\to C^{\cdot}(\mathcal{V},\mathcal{E}),$$

where the subscript indicates the subcomplex of cochains with support on which π is proper. This is defined as the composition of the shuffle map

$$\sigma: C^{\cdot}_c(\mathcal{V}\times\mathcal{U},\pi^!\mathcal{E})\to C^{\cdot}(\mathcal{V},C^{\cdot}_c(\mathcal{U},\Omega^n_X)\otimes\mathcal{E})$$

with the usual integral on $C^{\cdot}_c(\mathcal{U},\Omega^n_X)$ applied pointwise with respect to Y.

Recall that this integral is defined as the composition of a Dolbeault homomorphism with the usual integration of smooth forms for the orientation given locally by $dx^1dy^1\ldots dx^ndy^n$, where $z^i=x^i+\sqrt{-1}y^i$ are holomorphic coordinates on X. Conventions for the Dolbeault homomorphism are fixed as follows.

For fixed p, let $A^{p,q}_X$ be the sheaf of smooth forms of type (p,q) on X, regarded as a complex with $A^{p,q}_X$ in degree $p-q$ and differential $(2\pi\sqrt{-1})^{-1}\bar\partial$. The corresponding bicomplex $C^r(\mathscr{U}, A_X^{p,q})$ has differential $D=D_1+D_2$ where D_1 is the usual Čech differential and $D_2=(-1)^r(2\pi\sqrt{-1})^{-1}\bar\partial$. We require that the Dolbeault homomorphism

$$\Theta : C^p(\mathscr{U}, \Omega_X^n) \to H^0(X, A_X^{n,p})$$

reduces to the cohomology isomorphism obtained from the natural inclusions of both spaces into the bicomplex. Recall that Θ can be defined as an explicit chain map in terms of a locally finite partition of unity for $\mathscr{U}$, as in [14] or [1]. In fact Θ and hence the integral can be extended to the whole bicomplex. If X has boundary ∂X then Stokes' theorem takes the form

$$\int_X D\eta = (2\pi\sqrt{-1})^{-1} \int_{\partial X} \Theta\eta$$

for η in $C^{\cdot}(\mathscr{U}, A_X^{n,\cdot})$.

2.3. *Twisting Cochains and the Shuffle Map*

Suppose given a system $F^{\cdot}_{(\beta,\alpha)}$ of graded coherent $\mathcal{O}_{X\times Y}$-modules over the sets $V_\beta \times U_\alpha$ of the product cover $\mathscr{V}\times\mathscr{U}$ of $Y\times X$. Over each V_β in $\mathscr{V}$ this gives a graded $\mathcal{O}_Y$-module $\mathbb{F}^{\cdot}_\beta$ with

$$\mathbb{F}_\beta^r(W) = \sum_{p+q=r} C^p(W\times\mathscr{U}, F_\beta^q)$$

for $W\subset V_\beta$. Here $F^{\cdot}_\beta$ is regarded as a system of graded sheaves over $W\times\mathscr{U}$ with $(F^{\cdot}_\beta)_\alpha = F^{\cdot}_{(\beta,\alpha)}$. If the $F^{\cdot}_{(\beta,\alpha)}$ are free and related by a twisting cochain $(\mathscr{V}\times\mathscr{U}, F, b)$ we will show that the cochain b contains all information necessary to obtain a twisting cochain relating the $\mathbb{F}^{\cdot}_\beta$ on the cover $\mathscr{V}$ of Y. It must first be made clear what is meant by a homomorphism between the local sheaves $\mathbb{F}^{\cdot}_\beta$.

More generally, if $E^{\cdot}_{(\beta,\alpha)}$ is a second system of coherent $\mathcal{O}_{X\times Y}$-modules with corresponding $\mathcal{O}_Y$-modules $\mathbb{E}^{\cdot}_\beta$, then we take the homomorphisms from $\mathbb{E}^{\cdot}_\beta$ to $\mathbb{F}^{\cdot}_\gamma$ as the space of *simplicial operators* as described in Sect. 1.4 above. In the notation used there

$$\mathrm{Hom}^{\cdot}(W; \mathbb{E}_\beta, \mathbb{F}_\gamma) = \mathrm{Hom}^{\cdot}(W\times\mathscr{U}; E_\beta, F_\gamma)$$

for $W\subset V_\beta\cap V_\gamma$. In this case the space consists of all operators P of the form

$$(Pu)_\mu = \sum_{\nu\leqq\mu} P_\mu^\nu u_\nu ,$$

where μ is a simplex of $\mathscr{U}$ and if $\mu=\mu_0\ldots\mu_p$ and $\nu=\nu_0\ldots\nu_q$ then P_μ^ν lies in $\mathrm{Hom}^{\cdot}(W\times U_{\mu_0\ldots\mu_p}; E_{(\beta,\nu_0)}, F_{(\gamma,\mu_0)})$. In particular, cup-product multiplication by elements of $C^{\cdot}(W\times\mathscr{U}, \mathrm{Hom}(E_\beta, F_\gamma))$ belongs to this space, as does the Čech boundary operator in case $E^{\cdot}_\beta = F^{\cdot}_\gamma$.

This is not the only possibility for maps between the local complexes. For future reference note that we also have the space of *smoothing operators* from $\mathbb{E}^{\cdot}_\beta$ to $\mathbb{F}^{\cdot}_\gamma$. We distinguish these by a subscript, and if $p_i: Y\times X\times X\to Y\times X$ are the projections on the first and second factors we write

$$\mathrm{Hom}_s^{\cdot}(W; \mathbb{E}_\beta, \mathbb{F}_\gamma) = C^{\cdot}(W\times\mathscr{U}\times\mathscr{U}, \mathrm{Hom}^{\cdot}(p_2^* E_\beta, p_1^! F_\gamma)) .$$

The action of a cochain k in this space on u in $\mathbb{E}_\beta(W)$ is given by

$$k(u)=\int_{X_2} k\cdot p_2^* u$$

where the integral is evaluated along the second copy of X in the product.

If $L_\gamma^{\cdot}$ over V_γ is free and finitely generated over $\mathcal{O}_Y$ it is also necessary to consider homomorphisms of $\mathbb{F}_\beta^{\cdot}$ into $L_\gamma^{\cdot}$. These will always be taken to be of "smoothing operator" type, so for $W\subset V_\beta\cap V_\gamma$ we have

$$\mathrm{Hom}^{\cdot}(W;\mathbb{F}_\beta,L_\gamma)=C^{\cdot}(W\times\mathcal{U},\mathrm{Hom}^{\cdot}(F_\beta,\pi^! L_\gamma)) \tag{2.3.1}$$

with ψ acting on u in $\mathbb{F}_\beta(W)$ as

$$\psi(u)=\int_X \psi\cdot u \tag{2.3.2}$$

in $L_\gamma^{\cdot}(W)$. For maps in the other direction the space $\mathrm{Hom}^{\cdot}(W;L_\gamma,\mathbb{F}_\beta)$ has the obvious interpretation.

With this definition we can use the shuffle map and the twisting cochain b to define a twisting cochain $\hat{b}$ in $C^{\cdot}(\mathcal{V},\mathrm{Hom}^{\cdot}(\mathbb{F},\mathbb{F}))$. First note that the formula (2.1.2) for the shuffle map extends immediately to give a map

$$\sigma: C^{\cdot}(\mathcal{V}\times\mathcal{U},\mathrm{Hom}^{\cdot}(E,F))\to C^{\cdot}(\mathcal{V},C^{\cdot}(\mathcal{U},\mathrm{Hom}^{\cdot}(E,F)))$$

even though in the present case the local complexes may vary. A cochain ϕ on the right has components $\phi_{\beta_0\ldots\beta_r}$ over $V_{\beta_0\ldots\beta_r}$, where each $\phi_{\beta_0\ldots\beta_r}$ is in turn a cochain of $C^{\cdot}(V_{\beta_0\ldots\beta_r}\times\mathcal{U},\mathrm{Hom}^{\cdot}(E_{\beta_r},F_{\beta_0}))$. Cup-product identifies this space with a subspace of $\mathrm{Hom}^{\cdot}(V_{\beta_0\ldots\beta_r}\times\mathcal{U};\ E_{\beta_r},\ F_{\beta_0})$ and so σ can be regarded as a map into $C^{\cdot}(\mathcal{V},\mathrm{Hom}^{\cdot}(\mathbb{E},\mathbb{F}))$.

For a third local system $G_{(\beta,\alpha)}^{\cdot}$ with corresponding $\mathcal{O}_Y$-modules $\mathbb{G}_\beta^{\cdot}$ there is the usual cup product between $C^{\cdot}(\mathcal{V},\mathrm{Hom}^{\cdot}(\mathbb{F},\mathbb{G}))$ and $C^{\cdot}(\mathcal{V},\mathrm{Hom}^{\cdot}(\mathbb{E},\mathbb{F}))$. Moreover, for u, v in $C^{\cdot}(\mathcal{V}\times\mathcal{U},\mathrm{Hom}^{\cdot}(F,G))$ and $C^{\cdot}(\mathcal{V}\times\mathcal{U},\mathrm{Hom}^{\cdot}(E,F))$ formulae (i) and (ii) of Proposition 2.1.3 remain true. In this case δ, δ_X, and δ_Y are the modified Čech differentials, with first and last face operators omitted. However, this does not affect the proof of (ii).

A twisting cochain $(\mathcal{V},\mathbb{F},\hat{b})$ is now obtained by defining the action of $\hat{b}^k_{\beta_0\ldots\beta_k}$ in $\mathrm{Hom}^{1-k}(V_{\beta_0\ldots\beta_k};\mathbb{F}_{\beta_k},\mathbb{F}_{\beta_0})$ on u_{β_k} in $\mathbb{F}_{\beta_k}^{\cdot}(V_{\beta_0\ldots\beta_k})$ as follows. For $k>0$

$$\hat{b}_{\beta_0\ldots\beta_k}(u_{\beta_k})=(\sigma(b)_{\beta_0\ldots\beta_k})\cdot(u_{\beta_k})$$

while

$$\hat{b}_{\beta_0}(u_{\beta_0})=\delta_X(u_{\beta_0})+(\sigma(b)_{\beta_0})\cdot(u_{\beta_0})$$

for $k=0$. Here δ_X acts on $C^{\cdot}(V_{\beta_0}\times\mathcal{U},\mathbb{F}_{\beta_0}^{\cdot})$ in the usual way [9]. Note that $\hat{b}^0_\beta$ is then the usual differential for the space $C^{\cdot}(V_\beta\times\mathcal{U},\mathbb{F}_\beta^{\cdot})$ with respect to the twisting cochain obtained by restricting b to the cover $V_\beta\times\mathcal{U}$ of $V_\beta\times X$. It follows from the nondegeneracy assumption on b that $\sigma(b)_{\beta\beta,\alpha_0\ldots\alpha_q}=0$ unless $q=0$, and $\sigma(b)_{\beta\beta,\alpha_0}$ is the identity. The usual derivation properties for the operator δ_X hold, so that

$$\hat{b}\cdot\hat{b}=\delta_X\sigma(b)+\sigma(b)\cdot\sigma(b)$$

and the formula $\delta_Y\hat{b}+\hat{b}\cdot\hat{b}=0$ follows from Proposition 2.1.3 and the corresponding formula for b.

If $\mathscr{U}$ is Stein and $(\mathscr{V}\times\mathscr{U},F,b)$ is associated to a coherent sheaf $\mathscr{F}$ on $Y\times X$ then it will be shown that $(\mathscr{V},\mathbb{F},\hat{b})$ gives a twisted resolution for the $\mathscr{O}_Y$-module $C^{\cdot}(\mathscr{U},\mathscr{F})$ introduced in 2.1. In this sense $(\mathscr{V},\mathbb{F},\hat{b})$ can be thought of as the *direct image* of $(\mathscr{V}\times\mathscr{U},F,b)$. For future reference we make the following remark.

Remark 2.3.3. If $\mathscr{U}$, $\mathscr{V}$ are Stein and the complexes $C^{\cdot}(\mathscr{V}\times\mathscr{U},\mathrm{Hom}^{\cdot}(F,F))$ and $C^{\cdot}(\mathscr{V},\mathrm{Hom}^{\cdot}(\mathbb{F},\mathbb{F}))$ are filtered by Čech degree and total Čech degree respectively, then the corresponding E_1^{pq} terms are the cochain complexes $C^{\cdot}(\mathscr{V}\times\mathscr{U},\mathscr{H}^q)$ and $C^{\cdot}(\mathscr{V},C^{\cdot}(\mathscr{U},\mathscr{H}^q))$ respectively. Here $\mathscr{H}^q$ is the globally defined sheaf obtained by using the local chain maps $b^{1,0}$ to glue the q^{th} cohomology sheaves of the complexes $\mathrm{Hom}^{\cdot}(F_{(\beta_0,\alpha_0)},F_{(\beta_1,\alpha_1)})$. Since σ induces an isomorphism between the E_2^{pq} terms, the comparison theorem shows that σ is a quasi-isomorphism between the original complexes.

The relation between a coherent sheaf on $Y\times X$ and an associated twisted resolution is preserved by the shuffle map, as shown by the next proposition. As before $\mathscr{F}$ is a coherent $\mathscr{O}_{Y\times X}$-module and $\mathscr{U}$ a Stein open cover of X. Write $\mathbb{G}^{\cdot}$ for the complex of $\mathscr{O}_Y$-modules $C^{\cdot}(\mathscr{U},\mathscr{F})$ defined in Sect. 2.1. Suppose also that there is a twisting cochain $(\mathscr{V}\times\mathscr{U},F,b)$ which, together with the usual cocycle u in $C^0_{b,e}(\mathscr{V}\times\mathscr{U},\mathrm{Hom}^0(F,\mathscr{F}))$, gives a twisted resolution for $\mathscr{F}$. The shuffle map applied to b and u now gives a twisted resolution for $\mathbb{G}^{\cdot}$ as follows.

Proposition 2.3.4. *The twisting cochain $(\mathscr{V},\mathbb{F},\hat{b})$ together with $\sigma(u)$ in $C^{\cdot}(\mathscr{V},\mathrm{Hom}^{\cdot}(\mathbb{F},\mathbb{G}))$ give a twisted resolution of $\mathbb{G}^{\cdot}$.*

Proof. If e is the trivial twisting cochain for $\mathscr{G}$ on $\mathscr{V}\times\mathscr{U}$ then $\hat{e}$ is the trivial twisting cochain for $\mathbb{G}^{\cdot}$ on $\mathscr{V}$, so that

$$\begin{aligned}&\delta_Y\sigma(u)+\hat{e}\cdot\sigma(u)-\sigma(u)\cdot\hat{b}\\&=\delta_Y(\sigma(u))+\delta_X(\sigma(u))+\sigma(e)\cdot\sigma(u)-\sigma(u)\cdot\sigma(b)\\&=0\quad\text{by Proposition 2.1.3.}\end{aligned}$$

The maps $\sigma(u)^0_{\beta,\alpha_0\ldots\alpha_q}$ are zero for $q>0$, while $\sigma(u)^0_{\beta,\alpha}$ is the projection of $F^0_{(\beta,\alpha)}$ onto $\mathscr{F}|V_\beta\times U_\alpha$. Since $\mathscr{U}$ is Stein the map $\sigma(u)^0_\beta:\mathbb{F}^{\cdot}_\beta\to\mathbb{G}^{\cdot}_\beta$ is therefore a quasi-isomorphism and $\sigma(u)$ provides the twisted resolution as required.

3. The Relative Lefschetz Class

3.1. Resolutions by Nuclear Fréchet Modules

As above, let $(\mathscr{V}\times\mathscr{U},F,b)$ be a twisting cochain for the coherent sheaf $\mathscr{F}$ on $Y\times X$, with each $F^k_{(\beta,\alpha)}$ free and finitely generated and $\mathscr{V}$, $\mathscr{U}$ Stein open covers. The sheaves $\mathbb{G}^{\cdot}$ and $\mathbb{F}^{\cdot}_\beta$ defined in the previous section are not coherent as $\mathscr{O}_Y$-modules, but are *quasicoherent nuclear Fréchet $\mathscr{O}_Y$-modules* in the sense of [12, Sect. 2]. Recall that, in particular, this means that for each Stein open set W of Y (resp. V_β) the complex $\mathbb{G}^{\cdot}(W)$ (resp. $\mathbb{F}^{\cdot}_\beta(W)$), with its natural topological vector space structure, is a complex of nuclear Fréchet modules over the nuclear Fréchet algebra $\mathscr{O}_Y(W)$. Moreover $\mathbb{F}^{\cdot}_\beta(W)$ is a complex of *free* $\mathscr{O}_Y(W)$-modules. In this sense $\mathbb{F}^{\cdot}_\beta$ can be regarded as a "free resolution" of $\mathbb{G}^{\cdot}|V_\beta$.

As in [12], nuclear Fréchet spaces will be referred to as spaces of type **FN** and their strong duals as spaces of type **DFN**.

Suppose now that the projection $\pi: Y \times X \to Y$ is proper on the support of $\mathscr{F}$. Then the proof [3] of Grauert's theorem shows that $\mathbb{G}^{\cdot}$ is perfect, so that for a fine enough cover $\mathscr{V}$ of Y there exists a twisted resolution for which the local complexes of the associated twisting cochain $(\mathscr{V}, L, c)$ are bounded, free and finitely generated.

One step in the proof [10] of Hirzebruch-Riemann-Roch in the absolute case is the choice of cocycles representing a basis of the finite-dimensional space $H^k(X, \mathscr{F})$, together with cocycles representing the dual basis of $\mathrm{Ext}^{n-k}(X; \mathscr{F}, \Omega^n_X)$ under Serre-Grothendieck duality. The relative case is quite analogous, except that the full twisting cochain machinery is used and the choice of cocycles becomes a comparison of the finite and infinite dimensional twisting cochains $(\mathscr{V}, L, c)$ and $(\mathscr{V}, \mathbb{F}, \hat{b})$. The precise formulation is as follows.

Proposition 3.1.1. *After a suitable refinement of $\mathscr{V}$, there exists a 0-cocycle h in the complex $C^{\cdot}(\mathscr{V}, \mathrm{Hom}^{\cdot}(L, \mathbb{F}))$ with the property that the local chain maps $h^0_\beta: L^{\cdot}_\beta \to \mathbb{F}^{\cdot}_\beta$ are compatible, up to chain homotopy, with the quasi-isomorphisms of both complexes into $\mathbb{G}^{\cdot} | V_\beta$.*

Proof. This follows directly from Proposition 1.3.3 and Remark 1.3.4.

In the absolute case the existence of a dual basis for the Ext space depends on the Serre-Grothendieck duality theorem, so it is not surprising that the *relative* duality theorem is needed here. In order to state this result, define the complex $\check{\mathbb{F}}^{\cdot}_\beta$ over V_β as in (2.3.1) by setting

$$\check{\mathbb{F}}^{\cdot}_\beta(W) = C^{\cdot}(W \times \mathscr{U}, \mathrm{Hom}^{\cdot}(F_\beta, \pi^! \mathcal{O}_Y))$$

for $W \subset V_\beta$, with differential given by the twisting cochain $\check{b}_\beta$. Similarly $\check{L}_\beta$ is the complex dual to $L^{\cdot}_\beta$. The following relative duality theorem is equivalent to that proved in [12] or [13], for example.

Theorem 3.1.2. *The map $\check{h}_\beta: \check{\mathbb{F}}^{\cdot}_\beta \to \check{L}^{\cdot}_\beta$, adjoint to h_β under the cup-product and integration pairing* (2.3.2), *is a quasi-isomorphism of $\mathcal{O}_Y$-modules.*

The published proofs of this result use the terminology of derived categories and Forster-Knorr systems. However the underlying analytic ideas can be translated into the context of twisting cochains. Such a proof is given in Sect. 3.4 as a further illustration of twisting cochain techniques.

Let $M^{\cdot}_\beta$ be the mapping cone of h_β. Then the dual complex $\check{M}^{\cdot}_\beta$ is the mapping cone of $\bar{h}_\beta: \bar{\mathbb{F}}^{\cdot}_\beta \to \check{L}^{\cdot}_\beta$, where $\bar{\mathbb{F}}^{\cdot}_\beta$ is the complex $\check{\mathbb{F}}^{\cdot}_\beta$ with the sign of the differential changed, and $\bar{h}_\beta$ acting on $\bar{\mathbb{F}}^r_\beta$ as $(-1)^{r+1}\check{h}_\beta$. The relative duality theorem implies that $\check{M}^{\cdot}_\beta$ is *acyclic*.

Proposition 3.1.3. *Let h in $C^{\cdot}(\mathscr{V}, \mathrm{Hom}(L, \mathbb{F}))$ be as above. For a suitable refinement $\mathscr{V}'$ of $\mathscr{V}$ there exists a 0-cocycle g in $C^{\cdot}(\mathscr{V}', \mathrm{Hom}^{\cdot}(\mathbb{F}, L))$ such that $g \cdot h$ is cohomologous to the identity cochain in $C^0(\mathscr{V}', \mathrm{Hom}^0(L, L))$.*

Proof. Consider the complex $C^{\cdot}(\mathscr{V}, \mathrm{Hom}^{\cdot}(M, L))$ with differential given by the mapping cone of h and the twisting cochain c. For $W \subset V_\beta \cap V_\gamma$ there is an isomorphism of $\mathrm{Hom}^{\cdot}(W; M_\beta, L_\gamma)$ with $\mathrm{Hom}^{\cdot}(W; \check{L}_\gamma, \check{M}_\beta)$ given by taking adjoints. Since $\check{L}_\gamma$ is free and $\check{M}^{\cdot}_\beta$ acyclic, this implies that the complexes $\mathrm{Hom}^{\cdot}(W; M_\beta, L_\gamma)$

are locally acyclic, which in turn shows that any cocycle of $C^{\cdot}(\mathscr{V}, \mathrm{Hom}^{\cdot}(M, L))$ bounds on a suitable refinement.

In particular this applies to the 0-cocycle corresponding to the identity cochain 1 in the subcomplex $C^{\cdot}(\mathscr{V}, \mathrm{Hom}^{\cdot}(L, L))$, which is bounded on some refinement by a cochain with components (k, g) say, with respect to the decompositions of the $M_{\beta}^{\cdot}$. But this is equivalent to the equations

$$\delta k + c \cdot k + k \cdot c = 1 - g \cdot h$$

$$\delta g + c \cdot g - g \cdot \mathfrak{b} = 0$$

as required.

3.2. The Künneth Map

Recall from Sect. 2.3 that for $W \subset V_{\beta} \cap V_{\gamma}$ we defined a complex of "smoothing operators":

$$\mathrm{Hom}_s^{\cdot}(W; \mathbb{F}_{\beta}, \mathbb{F}_{\gamma}) = C^{\cdot}(W \times \mathscr{U} \times \mathscr{U}, \mathrm{Hom}^{\cdot}(p_2^* F_{\beta}, p_1^! F_{\gamma})).$$

The twisting cochain $\mathfrak{b}$ gives a differential on the space of cochains $C^{\cdot}(\mathscr{V}, \mathrm{Hom}_s^{\cdot}(\mathbb{F}, \mathbb{F}))$ sending u into

$$\delta_Y u + p_2^* \mathfrak{b} \cdot u + (-1)^{\deg u + 1} u \cdot p_1^* \mathfrak{b}$$

where $p_i^* \mathfrak{b}$ is induced from $\mathfrak{b}$ by the refinement of $p_i^{-1} \mathscr{U}$ by $\mathscr{U} \times \mathscr{U}$. The *Künneth map* is then the chain map

$$\kappa : C^{\cdot}(\mathscr{V}, \mathrm{Hom}^{\cdot}(L, L)) \to C^{\cdot}(\mathscr{V}, \mathrm{Hom}_s^{\cdot}(\mathbb{F}, \mathbb{F}))$$

defined by $\kappa(u) = p_1^* h \cdot p^* u \cdot p_2^* g$, where $p : Y \times X \times X \to Y$ is the projection.

Proposition 3.2.1. *For $\mathscr{V}$ sufficiently fine the Künneth map is a quasi-isomorphism.*

Proof. Filter with respect to the Čech degree coming from $\mathscr{V}$. Comparison of the resulting spectral sequences shows that it is enough to prove that κ induces a quasi-isomorphism of local complexes $\mathrm{Hom}^{\cdot}(W; L_{\beta}, L_{\gamma})$ and $\mathrm{Hom}_s^{\cdot}(W; \mathbb{F}_{\beta}, \mathbb{F}_{\gamma})$ for some neighborhood W of each point in $V_{\beta} \cap V_{\gamma}$. But κ factorizes:

$$\mathrm{Hom}^{\cdot}(W; L_{\beta}, L_{\gamma})$$

(A) $\qquad \to L_{\gamma}^{\cdot}(W) \otimes_{\mathcal{O}(W)} \check{L}_{\beta}^{\cdot}(W)$

(B) $\qquad \to \mathbb{F}_{\beta}^{\cdot}(W) \hat{\otimes}_{\mathcal{O}(W)} \check{\mathbb{F}}_{\beta}^{\cdot}(W)$

(C) $\qquad \to C^{\cdot}(W \times p_2^{-1} \mathscr{U}, W \times p_1^{-1} \mathscr{U}; \mathrm{Hom}^{\cdot}(p_2^* F_{\beta}, p_1^! F_{\gamma}))$

(D) $\qquad \to C^{\cdot}(W \times \mathscr{U} \times \mathscr{U}, \mathrm{Hom}^{\cdot}(p_2^* F_{\beta}, p_1^! F_{\gamma}))$.

Here (A) is the obvious identification and (B) is the tensor product of the maps $u \mapsto h_{\gamma} \circ (\pi^* u)$ and $\eta \mapsto (\pi^* \eta) \circ g_{\beta}$. Any point of $V_{\beta} \cap V_{\gamma}$ has a neighborhood on which both maps are quasi-isomorphisms. If $M^{\cdot}, N^{\cdot}, P^{\cdot}$ are complexes of nuclear Fréchet modules over a nuclear Fréchet algebra A, bounded below and with $P^{\cdot}$ free over A, and if $\theta : M^{\cdot} \to N^{\cdot}$ is a quasi-isomorphism, then so is $\theta \otimes 1 : M^{\cdot} \hat{\otimes}_A P^{\cdot} \to N^{\cdot} \hat{\otimes}_A P^{\cdot}$. In fact since $P^{\cdot}$ is free, filtration by $P^{\cdot}$ degree gives an isomorphism on E_1 terms, and the convergence of the spectral sequence gives the isomorphism on total cohomology. But (B) factorizes as two maps of this type.

The map (C) is the formal identification with cochains of the double cover via the isomorphism

$$\begin{aligned}&\operatorname{Hom}(W\times U_{\mu_0\ldots\mu_p}\times U_{\nu_0\ldots\nu_q};\, p_2^*F^r_{(\beta,\nu_q)}, p_1^!F^s_{(\gamma,\mu_0)})\\ &\quad = F^s_{(\gamma,\mu_0)}(W\times U_{\mu_0\ldots\mu_p})\hat{\otimes}_{\mathcal{O}(W)}\operatorname{Hom}(W\times U_{\nu_0\ldots\nu_q};\, F^r_{(\beta,\nu_q)}, \pi^!\mathcal{O}_Y)\,.\end{aligned}$$

For u, v of bidegree (p,q), (r,s) in $\mathbb{F}^{\cdot}_{\gamma}(W)$ and $\check{\mathbb{F}}^{\cdot}_{\beta}(W)$ the tensor product $u^{p,q}\otimes v^{r,s}$ maps to a cochain whose value on the simplex $\mu_0\ldots\mu_p\nu_0\ldots\nu_q$ of the double cover is

$$(-1)^{qr}u^{p,q}_{\mu_0\ldots\mu_p}\otimes v^{r,s}_{\nu_0\ldots\nu_q}\,.$$

Finally, (D) is the map $w\mapsto\tilde{w}$, where

$$\tilde{w}_{(\mu_0,\nu_0)\ldots(\mu_{p+q}\nu_{p+q})}=\sum_{k=0}^{p+q} w_{\mu_0\ldots\mu_k,\nu_k\ldots\nu_{p+q}}\,.$$

The Eilenberg-Zilber theorem [8] shows that this induces an isomorphism on E_1-terms obtained by filtration with respect to Čech degree, and hence on total cohomology.

The properties of the Künneth map are most easily established by factoring it through the shuffle map as follows. By Remark 2.3.3 the shuffle map induces quasi-isomorphisms

$$\sigma: C^{\cdot}(\mathscr{V}\times\mathscr{U},\operatorname{Hom}^{\cdot}(\pi^*L,F))\to C^{\cdot}(\mathscr{V},\operatorname{Hom}^{\cdot}(L,\mathbb{F}))$$

and

$$\sigma: C^{\cdot}(\mathscr{V}\times\mathscr{U},\operatorname{Hom}^{\cdot}(F,\pi^!L))\to C^{\cdot}(\mathscr{V},\operatorname{Hom}^{\cdot}(\check{\mathbb{F}},L))\,,$$

so there exist g', h' with $\sigma(g')=g$ and $\sigma(h')=h$ in cohomology. For $i,j=1,2$ define

$$\kappa'_{ij}: C^{\cdot}(\mathscr{V},\operatorname{Hom}^{\cdot}(L,L))\to C^{\cdot}(\mathscr{V}\times\mathscr{U}\times\mathscr{U},\operatorname{Hom}^{\cdot}(p_j^*F,p_i^!F))$$

by setting $\kappa'_{ij}(u)=p_i^*(h')\cdot p^*(u)\cdot p_j^*(g')$. Then, for example, $\sigma\circ\kappa'_{12}$ is chain homotopic to κ, so κ'_{12} is also a quasi-isomorphism. From the definition of the fibre integral and the choice of g it also follows that

(3.2.2) $$\int_X g'\cdot h'=1$$

in the cohomology of $C^{\cdot}(\mathscr{V},\operatorname{Hom}^{\cdot}(L,L))$.

3.3. *Characterization of the Lefschetz Class*

This section deals with the relative version of the "Lefschetz calculation" of [10], and characterizes the image of the identity class in $C^{\cdot}(\mathscr{V},\operatorname{Hom}^{\cdot}(L,L))$ under κ'_{12} as the Gysin image of a class supported on the diagonal $Y\times X$ in $Y\times X\times X$. The main result is the following proposition.

Proposition 3.3.1. *Let λ be a cocycle in $C^{\cdot}(\mathscr{V}\times\mathscr{U}\times\mathscr{U},\operatorname{Hom}^{\cdot}(p_2^*F,p_1^!(F))$ with the property that the two maps from $C^{\cdot}(\mathscr{V},\operatorname{Hom}^{\cdot}(L,L))$ into $C^{\cdot}(\mathscr{V},\mathcal{O}_Y)$ given by*

(3.3.2) $$u\mapsto\int_{X\times X}\tau_b(\kappa'_{21}(u)\cdot\lambda)$$

and

$$u \mapsto \int_X \tau_b \Delta^* \kappa'_{22}(u) \tag{3.3.3}$$

are chain homotopic via a simplicial chain homotopy. Then $\lambda = \kappa'_{12}(1)$ in cohomology.

Proof. Choose v so that $\kappa'_{12}(v) = \lambda$ in cohomology. Then the maps sending u to $\tau_c(u)$ and $\tau_c(u \cdot v)$ are chain homotopic. This is shown by the following sequence of chain maps, where the notation indicates that the formulae define simplicially chain homotopic maps from $C^{\cdot}(\mathscr{V}, \mathrm{Hom}^{\cdot}(L, L))$ into $C^{\cdot}(\mathscr{V}, \mathcal{O}_Y)$.

$$\begin{aligned}
& u \mapsto \tau_c(u) \\
\text{(i)} \cong & \int_X \tau_c(\pi^* u \cdot g' \cdot h') \\
\text{(ii)} \cong & \int_X \tau_b(h' \cdot \pi^* u \cdot g') \\
\text{(iii)} \cong & \int_X \tau_b \Delta^* \kappa'_{22}(u) \\
\text{(iv)} \cong & \int_{X \times X} \tau_b(\kappa'_{21}(u) \cdot \kappa'_{12}(v)) \\
\text{(v)} \cong & \int_{X \times X} \tau_b(p_2^* h' \cdot \pi^* u \cdot p_1^* g' \cdot p_1^* h' \cdot \pi^* v \cdot p_2^* g') \\
\text{(vi)} \cong & \int_{X \times X} \tau_c(\pi^* u \cdot p_1^* g' \cdot p_1^* h' \cdot \pi^* v \cdot p_2^* g' \cdot p_2^* h') \\
\text{(vii)} \cong & \, \tau_c(u \cdot v) .
\end{aligned}$$

The homotopies (i) and (vii) follow from the fact that $\int_X g' \cdot h'$ is cohomologous to the unit section. Relations (ii), (vi) follow from the calculation of [9, Proposition 3.8]. The choice of v and the hypothesis give (iv). The proposition then follows from the nondegeneracy of the trace pairing.

Remark 3.3.4. The proposition will be applied with λ defined on a cover $\mathscr{V} \times \mathscr{W}$, where $\mathscr{W}$ is some Stein refinement of $\mathscr{U} \times \mathscr{U}$. Then restriction to $\mathscr{V} \times \mathscr{W}$ induces an isomorphism on cohomology, so it is still possible to find v such that the restriction of $\kappa'_{12}(v)$ to $\mathscr{V} \times \mathscr{W}$ is cohomologous to λ. Integration of cochains on $\mathscr{U} \times \mathscr{U}$ can be factored through restriction to $\mathscr{W}$ and the same argument shows that v is cohomologous to the unit section.

3.4. Relative Duality

As usual $\mathscr{F}$ is a coherent sheaf on $Y \times X$, with the projection map $\pi: Y \times X \to Y$ proper on the support of $\mathscr{F}$. Let V be a Stein open set in Y with the property that there exists a twisting cochain $(V \times \mathscr{U}, F, b)$ for $\mathscr{F}$ over $V \times X$, where $V \times \mathscr{U}$ is a locally finite cover $\{V \times U_\alpha\}$ with each U_α Stein. It will be necessary to assume that the indexing set of $\mathscr{U}$ is ordered, and that associated cochains are defined only on strictly increasing simplices of the nerve. This ensures that the corresponding cochain complexes are bounded. The complexes of such cochains are quasi-isomorphic quotients of the full cochain complexes and the quotient maps are compatible with the duality pairing, so the conclusion of the theorem is not affected by this restriction.

For V sufficiently small there is a quasi-isomorphism of $\mathcal{O}(V)$-modules,

$$L^{\cdot}(V) \to C^{\cdot}(V \times \mathcal{U}, F^{\cdot}),$$

given by the coherence theorem, where $L^{\cdot}$ is a bounded complex of free, finitely generated $\mathcal{O}_V$-modules. The duality theorem is established by showing that the adjoint map

$$C^{\cdot}(V \times \mathcal{U}, \mathrm{Hom}^{\cdot}(F, \pi^! \mathcal{O}_Y)) \to \check{L}^{\cdot}(V), \tag{3.4.1}$$

defined by the duality pairing, induces a quasi-isomorphism of $\mathcal{O}_V$-modules. The proof uses some properties of nuclear spaces and topological tensor products, but is otherwise quite analogous to the proof of Serre duality in the case when Y is a point and $\mathcal{F}$ locally free.

Let $A_X^{p,q}$ and $D_X^{p,q}$ be the sheaves of germs of smooth forms and currents of type (p, q) on X. For V, K Stein open and compact sets respectively in Y the presheaves $\mathcal{O}_Y(V) \hat{\otimes} A_X^{p,q}$ and $\mathcal{O}_Y(K) \hat{\otimes} D_X^{p,q}$ on X defined by

$$U \mapsto \mathcal{O}_Y(V) \hat{\otimes} A_X^{p,q}(U)$$

and

$$U \mapsto \mathcal{O}_Y(K) \hat{\otimes} D_X^{p,q}(U)$$

are both sheaves on X. In the first case this can be proved as in [12, Proposition 4] via the identification of $\mathcal{O}_Y(V) \hat{\otimes} A_X^{p,q}(U)$ with the space of continuous linear maps from $\mathcal{O}_Y(V)'$ into $A_X^{p,q}(U)$. The second example is the sheaf of $\mathcal{O}_Y(K)$-valued currents on X (see [16, Proposition 50.5] for example). Note that both sheaves are *fine*.

Since $\mathcal{U}$ is Stein and $\mathcal{O}_Y(V)$ nuclear the inclusion of $\mathcal{O}_X(U)$ into the $\bar{\partial}$-complex $A_X^{0,\cdot}(U)$ gives a quasi-isomorphism

$$L^{\cdot}(V) \to C^{\cdot}(\mathcal{U}, F^{\cdot} \otimes (\mathcal{O}_Y(V) \hat{\otimes} A_X^{0,\cdot})). \tag{3.4.2}$$

The coefficient sheaf is now

$$U \mapsto F^{\cdot}(U \times V) \otimes (\mathcal{O}_Y(V) \hat{\otimes} A_X^{0,\cdot}(U)),$$

where the first tensor product is taken over $\mathcal{O}_Y(V) \hat{\otimes} \mathcal{O}_X(U) = \mathcal{O}_{Y \times X}(V \times U)$.

The map (3.4.2) has the property that both complexes consist of free $\mathcal{O}_Y(V)$-modules of type **FN**. Therefore the mapping cylinder $Q^{\cdot}(V)$ is a bounded, acyclic complex of free $\mathcal{O}_Y(V)$-modules of type **FN**. By a standard property [12, p. 98] of such complexes over the nuclear Fréchet algebra $\mathcal{O}_Y(V)$, the dual complex $\mathrm{Homtop}_{\mathcal{O}(V)}(Q^{\cdot}(V), \mathcal{O}_Y(K))$ is an acyclic complex of $\mathcal{O}_Y(K)$-modules of type **DFN** for any Stein compact subset K of V. Consequently the map

$$\mathrm{Homtop}_{\mathcal{O}(V)}(C^{\cdot}(\mathcal{U}, F^{\cdot} \otimes (\mathcal{O}_Y(V) \hat{\otimes} A_X^{0,\cdot})), \mathcal{O}_Y(K)) \to \mathrm{Homtop}_{\mathcal{O}(V)}(L^{\cdot}(V), \mathcal{O}_Y(K)) \tag{3.4.3}$$

adjoint to (3.4.2) is a quasi-isomorphism. By definition

$$\mathrm{Homtop}_{\mathcal{O}(V)}(\mathcal{O}_Y(V) \hat{\otimes} A_X^{0,p}(U), \mathcal{O}_Y(K))$$

is the space of distributions with compact support in U and taking values in $\mathcal{O}_Y(K)$. Thus (3.4.3) can be interpreted as a quasi-isomorphism

$$C_{\cdot}(\mathcal{U}, \mathrm{Hom}_c^{\cdot}(F, \mathcal{O}_Y(K) \hat{\otimes} D_X^{n})) \to \check{L}^{\cdot}(K), \tag{3.4.4}$$

where the coefficients of the chain complex over an open set W in the nerve of $\mathscr{U}$ are elements of

$$\mathrm{Hom}_{\mathcal{O}(V\times W)}(F^{\cdot}(V\times W), \mathcal{O}_Y(K)\hat{\otimes} D_{X,c}^{n,\cdot}(W))$$

and the subscript as usual denotes the sections with compact support. The differential is the adjoint of the usual twisting cochain differential on the cochain complex.

Now let $\mathscr{M}=\{M_\alpha\}$ be a covering of X by Stein compact sets, so that each M_α has a fundamental sequence of Stein open neighborhoods $U_\alpha^{(k)}$ with $U_\alpha^{(0)}=U_\alpha$. Let $\{\varphi_\alpha\}$ be a partition of unity for $\mathscr{U}$, with $\operatorname{supp}\varphi_\alpha \subset M_\alpha$.

Cup-product and fibre integration, via the partition of unity, induce a chain map with components

$$(3.4.5)\quad C^p(\mathscr{M}, \mathrm{Hom}^{\cdot}(F^q, \mathcal{O}_Y(K)\hat{\otimes} D_X^{n,r})) \to \sum_{l-k=p+r} C_k(\mathscr{U}, \mathrm{Hom}_c(F^q, \mathcal{O}_Y(K)\hat{\otimes} D_X^{n,l})).$$

It remains to show that this map, and also the natural inclusion of

$$C^{\cdot}(K\times\mathscr{M}, \mathrm{Hom}^{\cdot}(F, \pi^!\mathcal{O}_Y))$$

into

$$C^{\cdot}(\mathscr{M}, \mathrm{Hom}^{\cdot}(F, \mathcal{O}_Y(K)\hat{\otimes} D_X^n))$$

are quasi-isomorphisms. For then the map (3.4.1) restricts to a quasi-isomorphism

$$C^{\cdot}(K\times\mathscr{M}, \mathrm{Hom}^{\cdot}(F, \pi^!\mathcal{O}_Y)) \to \check{L}^{\cdot}(K)$$

and the duality theorem follows easily from the fact that, if W is any Stein open subset of U and $\mathscr{U}^{(k)}$ is the cover $\{U_\alpha^{(k)}\}$, the complexes $C^{\cdot}(W\times\mathscr{U}^{(k)}, \mathrm{Hom}^{\cdot}(F, \pi^!\mathcal{O}_Y))$ are isomorphic for all $k\geqq 0$.

Over U_α let $\mathscr{H}_\alpha^{k,l}$ be the k^{th} cohomology sheaf of the complex $\mathrm{Hom}(F_\alpha^{\cdot}, \mathcal{O}_Y(K)\hat{\otimes} D_X^{n,l})$. The chain maps between the local complexes $F_\alpha^{\cdot}$ glue the $\mathscr{H}_\alpha^{k,l}$ into a globally defined, compactly supported $\mathcal{O}_X$-module $\mathscr{H}^{k,l}$. This sheaf is also fine. A suitable filtration on the complexes of (3.4.5) reduces the problem to that of showing that the induced maps

$$(3.4.6)\qquad C^p(\mathscr{M}, \mathscr{H}^{q,r}) \to \sum_{l-k=p+r} C_k(\mathscr{U}, \mathscr{H}_c^{q,l})$$

give a quasi-isomorphism on the associated complexes. The second complex is now the usual complex of chains on the nerve of $\mathscr{U}$ with coefficients which are compactly supported sections of $\mathscr{H}^{q,l}$. The map preserves the filtration induced on the first complex by r and on the second complex by l. The map on the cohomology of the associated graded complexes reduces to the identity map on the space of global sections of the fine sheaf $\mathscr{H}^{q,r}$, so that (3.4.6) is indeed quasi-isomorphism.

For the final step of the proof, it is sufficient to show that for a Stein compact set M in X, the $\bar{\partial}$-complex $\mathcal{O}_Y(K)\hat{\otimes} D_X^{n,\cdot}(M)$ is a resolution of $\mathcal{O}_{Y\times X}(K\times M) = \mathcal{O}_Y(K)\hat{\otimes}\mathcal{O}_X(M)$. All the spaces are of type **DFN**, so by [12, Proposition 1] it is enough to show that $D_X^{n,\cdot}(M)$ gives a resolution of $\mathcal{O}_X(M)$. This follows, for example, by taking limits over a fundamental sequence of Stein open neighborhoods of M.

4. Construction and Properties of the Dual Class

4.1. Koszul Complexes

Let $\mathscr{W} = \mathscr{U}' \coprod \mathscr{U}''$ be a cover of $X \times X$, where $\mathscr{U}'$ consists of the sets $U_\alpha \times U_\alpha$ for U_α in $\mathscr{U}$, and $\mathscr{U}''$ is a Stein open cover of the complement of the diagonal. Assume also that $\mathscr{W}$ refines $\mathscr{U} \times \mathscr{U}$. The sheaf $\Delta_* \mathscr{O}_X$ on $X \times X$ has a twisting cochain $(\mathscr{W}, K, a)$ as in [15]. In fact the construction given in [15] can be simplified by taking $K_\alpha^{\cdot}$ to be zero on the sets of $\mathscr{U}''$, while on $\mathscr{U}'$ the $K_\alpha^{\cdot}$ are the usual Koszul resolutions formed by the functions $\zeta_\alpha^i - z_\alpha^i$ defining $\Delta_* \mathscr{O}_X$ in $U_\alpha \times U_\alpha$. As usual z_α^i and ζ_α^i are the coordinates for $U_\alpha \times U_\alpha$ defined via p_1, p_2 by fixed coordinates z_α^i on U_α. This is the approach which will be taken here. The extension of the local differentials proceeds as before, except that the $a^k_{\alpha_0 \ldots \alpha_k}$ will be zero unless both α_0 and α_k are vertices of $\mathscr{U}'$.

Now use Δ to denote the diagonal embedding of $Y \times X$ into $Y \times X \times X$, and let $(\mathscr{V} \times \mathscr{U}, F, b)$ be a twisting cochain for $\mathscr{F}$ on $Y \times X$. The cochains a, b can be pulled back to $Y \times X \times X$ and restricted to $\mathscr{V} \times \mathscr{W}$ so that the product operation gives twisting cochains $(\mathscr{V} \times \mathscr{W}, p_i^* F \otimes K, p_i^* b \tilde{\otimes} a)$ for $\Delta_* \mathscr{F}$ as in [9]. If $\gamma = (\beta, \alpha)$ is a vertex of $\mathscr{V} \times \mathscr{U}'$ we write z_γ^i, ζ_γ^i, K_γ^i for the pull-back of z_α^i, ζ_α^i, K_α^i to $V_\beta \times U_\alpha \times U_\alpha$.

The twisting cochains $p_2^* b \tilde{\otimes} a$ and $p_1^* b$ define a differential for the complex

$$C^{\cdot}(\mathscr{V} \times \mathscr{W}, \operatorname{Hom}^{\cdot}(p_2^* F \otimes K, p_1^! F)) \tag{4.1.1}$$

with cohomology $\operatorname{Ext}^k(Y \times X \times X;\, \Delta_* \mathscr{F},\, p_1^! \mathscr{F})$. This complex is trigraded with (p, q, r) component

$$C^p(\mathscr{V} \times \mathscr{W}, \operatorname{Hom}^q(p_2^* F, p_1^* F) \otimes \operatorname{Hom}^r(K, p_1^! \mathscr{O}_Y)). \tag{4.1.2}$$

Interest will center on the behavior of these cochains on the simplexes of $\mathscr{U}'$. A cochain f of (4.1.2), with $\mathscr{W}$ replaced by $\mathscr{U}'$, corresponds to a family of cochains f^I in $C^p(\mathscr{V} \times \mathscr{U}', \operatorname{Hom}^q(p_2^* F, p_1^* F))$ depending in an alternating fashion on an r-tuple $I = i_1 \ldots i_r$ of integers with $1 \leqq i_1, \ldots, i_r \leqq n$. We also write $|I| = r$ in this case. If $e_\gamma^1, \ldots, e_\gamma^n$ is the basis used to construct the Koszul complex $K_\gamma^{\cdot}$ the cochain f^I is given explicitly by

$$(f^I)_{\gamma_0 \ldots \gamma_p} d\zeta^1_{\gamma_p} \ldots d\zeta^n_{\gamma_p} = f_{\gamma_0 \ldots \gamma_p}(e^I_{\gamma_p})$$

where $e_\gamma^I = e_\gamma^{i_1} \wedge \ldots \wedge e_\gamma^{i_r}$. For $r = 0$ the corresponding cochain is denoted by $f^{\emptyset}$, and $e_\gamma^{\emptyset} = 1$ in $K_\gamma^0 = \mathscr{O}_{Y \times X \times X}$.

Similarly, cochains in the space

$$C^p(\mathscr{V} \times \mathscr{U}', \operatorname{Hom}^q(p_2^* F, p_2^* F) \otimes \operatorname{Hom}^r(K, K)) \tag{4.1.3}$$

can be identified with families of cochains u_J^I in $C^p(\mathscr{V} \times \mathscr{U}', \operatorname{Hom}^q(p_2^* F, p_2^* F))$ with $|I| - |J| = r$ and alternating in the components of $J = j_1 \ldots j_s$ and $I = i_1 \ldots i_{r+s}$. The u_J^I are given explicitly by

$$u_{\gamma_0 \ldots \gamma_p}(e^I_{\gamma_p}) = \sum_{|J| = |I| - r} (u_J^I)_{\gamma_0 \ldots \gamma_p} e^J_{\gamma_0},$$

where the summation is carried out over strictly increasing J only, so that the u_J^I are uniquely determined. Note also that if the spaces (4.1.2) and (4.1.3) are bigraded by combining the first two degrees then the usual product operations between them can be expressed in terms of the component cochains by

$$(u^{pq} \cdot v^{rs})^I_J = (-1)^{qr} \sum_L (u^L_J) \cdot (v^I_L)$$

and

$$(f^{pq}\cdot v^{rs})^I=(-1)^{qr}\sum_L(f^L)\cdot(v_L^I).$$

The cochain-level residue map of [15] corresponds in the present situation to a chain map

$$\text{Res}: C^{\cdot}(\mathscr{V}\times\mathscr{U}', \operatorname{Hom}^{\cdot}(p_2^*F, p_1^*F)\otimes\operatorname{Hom}^{\cdot}(K, p_1^!\mathcal{O}_X))\to C^{\cdot}(\mathscr{V}\times\mathscr{U}, \operatorname{Hom}^{\cdot}(F,F)),$$

of degree zero, given by

$$(\text{Res} f)_{\gamma_0\ldots\gamma_p}dz^1_{\gamma_p}\wedge\ldots\wedge dz^n_{\gamma_p}=\Delta_*(f^{1\ldots n})_{\gamma_0\ldots\gamma_p}.$$

That this defines a chain map is a consequence of the definition of $p_2^*b\tilde{\otimes}a$ and the fact that $\Delta_*a^k=0$ for $k\geqq 1$, while Δ_*a^1 provides the transition functions for Ω^n_X. For future reference, note the explicit formula [15, (3.10)]:

$$((\Delta^*a^1)^I_J)_{\alpha\beta}=\sum_\sigma \operatorname{sgn}(\sigma)\phi^{i_1}_{j_{\sigma(1)}}\phi^{i_2}_{j_{\sigma(2)}}\ldots\phi^{i_p}_{j_{\sigma(p)}} \tag{4.1.4}$$

where $I=i_1\ldots i_p$, $J=j_1\ldots j_p$ and $\phi^i_j=\partial z^i_\beta/\partial z^j_\alpha$; the sum extends over all permutations σ of $\{1,\ldots,p\}$.

4.2. Construction of the Dual Class

As in the absolute case the residue map induces quasi-isomorphisms

$$\operatorname{Hom}^{\cdot}(p_2^*F_\gamma, p_1^*F_\gamma)\otimes\operatorname{Hom}^{\cdot}(K_\gamma, p_1^!\mathcal{O}_{Y\times X})\to\Delta_*\operatorname{Hom}^{\cdot}(F_\gamma,F_\gamma)$$

on the local complexes, and the usual comparison of spectral sequences gives the global isomorphism

$$\text{Res}: \operatorname{Ext}^k(Y\times X\times X;\Delta_*\mathscr{F}, p_1^!\mathscr{F})\to\operatorname{Ext}^k(X;\mathscr{F},\mathscr{F}).$$

This shows the existence of a unique class in $\operatorname{Ext}^0(Y\times X\times X;\Delta_*\mathscr{F}, p_1^!\mathscr{F})$ which corresponds under this isomorphism to the identity section of $\operatorname{Hom}(\mathscr{F},\mathscr{F})$. We use the procedure of [14, 15, 10] to construct an explicit cocycle representing this class. For this purpose decompose the differential D for the complex (4.1.1) as $D'+D''$, where for f of total degree m,

$$D'f=\delta f+p_1^*b\cdot f+(-1)^{m+1}\sum_{k\geqq 1} f\cdot((p_2^*b)^k\tilde{\otimes}a^k), \tag{4.2.1}$$

$$D''(f)=(-1)^{m+1}f\cdot(1\otimes a^0). \tag{4.2.2}$$

With respect to the bidegree obtained by adding first and second degrees in (4.1.2), D' has components of bidegree $(k,1-k)$ for $k>0$ and D'' has bidegree $(0,1)$. For a simplex $\gamma=\gamma_0\ldots\gamma_p$ of $\mathscr{V}\times\mathscr{U}'$ and f of tridegree (p,q,r) the differential D'' is given explicitly by

$$\begin{aligned}&(-1)^{p+q+r+1}((D''f)^I)_\gamma(z_{\gamma_p},\zeta_{\gamma_p})\\&=\sum_{l=1}^{r+1}(-1)^{l+1}(\zeta^{i_l}_{\gamma_p}-z^{i_l}_{\gamma_p})(f^{I_l})_\gamma(z_{\gamma_p},\zeta_{\gamma_p}),\end{aligned}$$

where $I=i_1\dots i_{r+1}$ and $I_l=i_1\dots\hat{i}_l\dots i_{r+1}$. A corresponding homotopy operator P is defined as in [14, 15] by the formula

$$\begin{aligned}&(-1)^{p+q+r}((Pf)^I)_\gamma(z_{\gamma_p},\zeta_{\gamma_p})\\&=\sum_{k=1}^{n}\int_0^1 t^{n-r}[\partial/\partial\zeta^k_{\gamma_p}(f^{kI})_\gamma](z_{\gamma_p},z_{\gamma_p}+t(\zeta_{\gamma_p}-z_{\gamma_p}))dt,\end{aligned}$$

where $I=i_1\dots i_{r-1}$ and $kI=ki_1\dots i_{r-1}$. The f^J are interpreted as matrix-valued cochains via the fixed trivializations of the $F^{\cdot}_{\gamma_k}$. We than have the relation

$$PD''f+D''Pf=f-\varrho''f,$$

where

$$(\varrho''f)^I(z,\zeta)=\begin{cases}f^I(z,z) & \text{if } \ |I|=n\\ 0 & \text{otherwise}.\end{cases}$$

Now define $\eta^{0,0}$ of bidegree (0, 0) and tridegree (0, 0, 0) in the complex (4.1.2) by setting $(\eta^{0,0})^{1\dots n}_\gamma=1_\gamma$, where the map $1_\gamma: p_2^*F^k_\gamma\to p_1^*F^k_\gamma$ is given in terms of the fixed trivializations by the identity matrix. The method of [14] or [15, (1.11)] shows that $\eta^{0,0}$ extends to a cocycle $\eta=\sum_{k=0}^{n}\eta^{k,-k}$, which by the definition of $\eta^{0,0}$ maps to the identity section under the residue map. A calculation, similar to that which shows the residue to be a chain map, gives the condition $\varrho''D'\eta^{0,0}=0$ and η is given explicitly on $\mathscr{U}'$ by the formula

$$\eta|\mathscr{V}\times\mathscr{U}'=\sum_{i\geqq 0}(-1)^i(PD')^i\eta^{0,0}. \tag{4.2.3}$$

4.3. *Properties of the Dual Class*

The local chain maps $u^{0,0}_\gamma: p_2^*F^{\cdot}_\gamma\to p_2^*F^{\cdot}_\gamma\otimes K^{\cdot}_\gamma$, induced by the quotient map of $p^*\mathscr{F}$ onto $\Delta_*\mathscr{F}$, extend to a 0-cocycle u in $C^{\cdot}(\mathscr{V}\times\mathscr{W}, \mathrm{Hom}^{\cdot}(p_2^*F, p_2^*F\otimes K))$. Then the map

$$\pi^0:\mathrm{Ext}^k(Y\times X\times X,\Delta_*\mathscr{F},p_1^!\mathscr{F})\to\mathrm{Ext}^k(Y\times X\times X,p_2^*\mathscr{F},p_1^!\mathscr{F})$$

induced by the same quotient map is given at cochain level by setting $\pi^0(\xi)=\xi\cdot u$ for ξ in the complex (4.1.1). In this section it is shown that $\pi^0(\eta)$ in $C^{\cdot}(\mathscr{V}\times\mathscr{W}, \mathrm{Hom}^{\cdot}(p_2^*F, p_1^!F))$ represents the dual class of the diagonal in the sense that for any cocycle w in $C^{\cdot}_c(\mathscr{V}\times\mathscr{W}, \mathrm{Hom}^{\cdot}(p_1^*F, p_2^!F))$,

$$\int_{X\times X}\tau_b(w\cdot\pi^0(\eta))=\int_X\tau_b\Delta^*(w) \tag{4.3.1}$$

in the cohomology of $C^{\cdot}(\mathscr{V}, \mathscr{O}_Y)$. Here the subscript c denotes those cochains with support proper over Y. In fact we prove a stronger result which will be used later: as chain maps from $C^{\cdot}_c(\mathscr{V}\times\mathscr{W}, \mathrm{Hom}^{\cdot}(p_1^*F, p_2^!F))$ into $C^{\cdot}(\mathscr{V}, \mathscr{O}_Y)$ the two sides of (4.3.1) are *chain homotopic* via an explicit chain homotopy. This depends on a fairly straightforward generalization of the "Čech parametrix" construction of [14]. We review the argument, using the notation of the previous section.

Since all the local complexes $K^{\cdot}_\gamma$ become acyclic on restriction to $Y\times X\times X-\Delta$ the same holds for the global complex $C^{\cdot}(\mathscr{V}\times\mathscr{W}, \mathrm{Hom}^{\cdot}(p_2^*F\otimes K, p_1^!F))$, so the cocycle η bounds on the complement of Δ. The existence of the homotopy depends on the explicit construction of a boundary of η, and this in turn needs the existence of explicit chain contractions for the $K_\gamma^{\cdot}$ off Δ. Such exist only if smooth, rather than holomorphic, coefficients are introduced. This means working in the trigraded space of cochains with (p, q, r) component

$$(4.3.2) \qquad \sum_{k+l=p} C^k(\mathscr{V} \times \mathscr{W}', \mathrm{Hom}^l(p_2^* F, p_1^* F) \otimes \mathrm{Hom}^q(K, p_1^! \mathcal{O}_Y \otimes A^{0,r}_{X \times X})).$$

Here $\mathscr{W}'$ is the restriction of $\mathscr{W}$ to the complement of the diagonal and the coefficient sheaf $p_1^! \mathcal{O}_Y \otimes A^{0,r}_{X \times X}$ is the tensor product over $\mathcal{O}_{X \times X}$ of $p_1^! \mathcal{O}_Y$ with the sheaf of smooth forms of type $(0, r)$ on $X \times X$. These cochains have differential $D = D' + D'' + D'''$ where for f of degree (p, q, r) the operators D', D'' are given by (4.2.1) and (4.2.2) with $m = p + q$, and $D''' f = (-1)^{p+q} \bar{\partial} f / 2\pi \sqrt{-1}$. In the formulae (4.2.1), (4.2.2) the third, antiholomorphic degree is ignored in the formation of products. Away from the diagonal D'' has a homotopy operator Φ defined on (p, q, r) cochains by

$$(-1)^{p+q} (\Phi f)^I_{\gamma_0 \dots \gamma_p}(z, \zeta) = \sum_{k=1}^{n} g^k_{\gamma_p}(z, \zeta) f^{kI}_{\gamma_0 \dots \gamma_p}(z, \zeta),$$

where $g^k_\gamma(z, \zeta) = (\bar{\zeta}^k_\gamma - \bar{z}^k_\gamma) / |z_\gamma - \zeta_\gamma|^2$ and the formula

$$(4.3.3) \qquad D'' \Phi + \Phi D'' = 1$$

follows from the relation $\sum_k (\zeta^k_\gamma - z^k_\gamma) g^k_\gamma(z, \zeta) = 1$. The formula (4.3.3) gives the relation $\eta | \mathscr{V} \times \mathscr{W}' = D\mu$, where

$$\mu = \sum_{k \geqq 0} (-1)^k (\Phi(D' + D'''))^k \Phi(\eta | \mathscr{V} \times \mathscr{W}').$$

The terms making up μ will have singularities of various orders along the diagonal, and in case $\mathscr{F}$ is locally free and Y is a point the cochain μ is essentially the kernel of the Čech parametrix of [14], and we use the terminology of that paper, especially Sect. 2, to describe the singularities which occur. In fact μ decomposes as $\sum_{k \geqq 1} \mu^{(k)}$ where $\mu^{(k)}$ is "regular of order $-k$". In particular this means that every component $(\mu^{(k)})^I_{\gamma_0 \dots \gamma_p}$ satisfies an estimate

$$(\mu^{(k)})^I_{\gamma_0 \dots \gamma_p}(z, \zeta) \leqq C / |\zeta_{\gamma_p} - z_{\gamma_p}|^{2n-k}$$

on compact subsets of (z, ζ). The effect on the order of regularity of the various operators used in the manufacture of μ is such that $\mu^{(1)}$ can be taken to be of tridegree $(0, -n, n-1)$, given by

$$\mu^{(1)} = (-1)^{n-1} (\Phi D''')^{n-1} \Phi(\eta^{0,0} | \mathscr{V} \times \mathscr{W}').$$

A direct calculation then shows that

$$\begin{aligned} -(\mu^{(1)})^{\emptyset}_\gamma d\zeta^1_\gamma \dots d\zeta^n_\gamma &= 1_\gamma \otimes c_n \sum_{k=1}^{n} (-1)^{k+1} g^k_\gamma \bar{\partial} g^1_\gamma \dots \widehat{\bar{\partial} g^k_\gamma} \dots \bar{\partial} g^n_\gamma d\zeta^1_\gamma \dots d\zeta^n_\gamma \\ &= 1_\gamma \otimes c_n \sum_{k=1}^{n} (-1)^{k+1} |w_\gamma|^{-2n} \bar{w}^k_\gamma d\bar{w}^1_\gamma \dots \widehat{d\bar{w}^k_\gamma} \dots d\bar{w}^n_\gamma d\zeta^1_\gamma \dots d\zeta^n_\gamma \\ &= 1_\gamma \otimes k(z_\gamma, \zeta_\gamma), \end{aligned}$$

where $w_\gamma^i = \zeta_\gamma^i - z_\gamma^i$, the constant c_n is $(-1)^{n(n-1)/2}(n-1)!/(2\pi\sqrt{-1})^{n-1}$ and $k(z,\zeta)$ is the Bochner-Martinelli kernel on $\mathbb{C}^n \times \mathbb{C}^n$, as in [14] for example. The basic property of this kernel is that if ω is a smooth $(0,n)$ form with compact support on $\mathbb{C}^n \times \mathbb{C}^n$ and coefficients in $p_1^* \Omega_X^n$, then

(4.3.4) $$\lim_{\varepsilon \to 0} \int_{\partial\varepsilon} \omega \wedge k = 2\pi\sqrt{-1} \int_{\mathbb{C}^n} \Delta^* \omega ,$$

where $\partial\varepsilon$ is the boundary of the region $|\zeta - z| \leqq \varepsilon$.

For ω of degree r in $C^\cdot(\mathscr{V} \times \mathscr{W}, \operatorname{Hom}^\cdot(p_1^* F, p_1^! F))$ with boundary $D\omega$, right multiplication by u followed by the trace map gives

(4.3.5) $$D\tau_b(\omega \cdot \pi^0(\mu)) = \tau_b(D\omega \cdot \pi^0(\mu)) + (-1)^r \tau_b(\omega \cdot \pi^0(\eta))$$

in $C^\cdot(\mathscr{V} \times \mathscr{W}', p_1^! \mathcal{O}_Y \otimes A^{0,\cdot})$. Since $\pi^0(\mu)$ is locally integrable in the $X \times X$ direction and the map Φ is (locally) independent of the Y-coordinates, there is a map

$$S : C_c^\cdot(\mathscr{V} \times \mathscr{W}, \operatorname{Hom}^\cdot(p_1^* F, p_2^! F)) \to C^\cdot(\mathscr{V}, \mathcal{O}_Y)$$

defined on cochains of degree r by

(4.3.6) $$S(\omega) = (-1)^r \int_{X \times X} \tau_b(\omega \cdot \pi^0(\mu))$$

and the duality property of $\pi^0(\eta)$ appears in the following form:

Proposition 4.3.7.

$$\int_{X \times X} \tau_b(\omega \cdot \pi^0(\eta)) - \int_X \tau_b \Delta^*(\omega) = \delta_Y S(\omega) + SD(\omega) .$$

Proof. Suppose first that ω has total degree r and vanishes except on a fixed simplex $\gamma_0 \ldots \gamma_p$ of $\mathscr{V} \times \mathscr{U}'$. From Stokes' theorem, (4.3.4) and the formula for $\mu^{(1)}$ in terms of the Bochner-Martinelli kernel:

$$\int_{X \times X} D\tau_b(\omega \cdot \pi^0(\mu)) = \delta_Y \int_{X \times X} \tau_b(\omega \cdot \pi^0(\mu)) - (-1)^r (2\pi\sqrt{-1})^{-1} \lim_{\varepsilon \to 0} \int_{\partial\varepsilon} \tau_b(\omega \cdot \pi^0(\mu))$$
$$= (-1)^r \delta_Y S(\omega) + (-1)^r \int_X \tau_b \Delta^* \omega .$$

Together with (4.3.5) this proves the formula in this case. If ω is supported on a simplex of $\mathscr{V} \times \mathscr{U}''$ the proof is the same except that the limit and diagonal terms vanish identically. The general case follows by linearity.

Corollary 4.3.8. *The dual class $\pi^0(\eta)$ is cohomologous to the image of the unit section of $C^\cdot(\mathscr{V}, \operatorname{Hom}(L, L))$ under the Künneth map κ'_{12} and restriction to $\mathscr{W}$.*

Proof. From Proposition 3.3.1 and Remark 3.3.4, with $\lambda = \pi^0(\eta)$, it suffices to find a simplicial chain homotopy between the two maps (3.3.2) and (3.3.3). But from the previous proposition,

$$\int_{X \times X} \tau_b(\kappa'_{21}(u) \cdot \pi^0(\eta)) - \int_X \tau_b \Delta^* \kappa'_{22}(u) = \delta_Y S'(u) + S'D(u) ,$$

where for u of total degree r

$$S'(u) = (-1)^r \int_{X \times X} \tau_b(p_2^* h' \cdot p^* u \cdot p_1^* g' \cdot \pi^0(\eta)) .$$

If the definitions of the trace and fibre integral are followed through, it can be verified that the value of $S'(u)$ on a simplex $\beta_0 \dots \beta_r$ of $\mathscr{V}$ depends only on the values of u on simplices of the form $\beta_k \dots \beta_{k+l}$. This gives a chain homotopy of the required type.

This corollary is the case $k=0$ of the more general formula of the next proposition, which gives explicit representatives for the images of the powers of the Atiyah class under the Künneth map.

Proposition 4.3.9. *For any* $k \geqq 0$,

$$\kappa'_{21}((dc)^k) = p_1^*(d_Y b)^k \cdot \pi^0(\eta)$$

in the cohomology of $C^{\cdot}(\mathscr{V} \times \mathscr{W}, \mathrm{Hom}^{\cdot}(p_2^* F, p_1^! F \otimes \Omega_Y^k))$. *Here* $d = d_Y + d_X$ *is the decomposition of the exterior derivative on the product space* $Y \times X$.

Proof. Since g' is a cocycle, $\delta g' + b \cdot g' - g' \cdot c = 0$. Exterior differentiation in the Y-direction shows that the cocycles $g' \cdot dc$ and $d_Y b \cdot g'$ are cohomologous. This gives

$$\begin{aligned}\kappa'_{21}((dc)^k) &= p_1^* g' \cdot p^*(dc)^k \cdot p_2^* h' \\ &= p_1^*(d_Y b)^k \cdot p_1^* g' \cdot p_2^* h' \\ &= p_1^*(d_Y b)^k \cdot \pi^0(\eta)\end{aligned}$$

in cohomology.

This result shows the existence of a local geometric expression for a cocycle on $Y \times X$ which gives the trace of the k^{th} power of the Atiyah class dc when integrated over X. In fact the previous proposition combined with Lemma 3.3.4 shows that

$$\tau_c((dc)^k) = \int_X \tau_b((d_Y b)^k \cdot \Delta^* \pi^0(\eta)) \tag{4.3.10}$$

in the cohomology of $C^k(\mathscr{V}, \Omega_Y^k)$.

It remains to identify the class $\Delta^* \pi^0(\eta)$ in the complex $C^{\cdot}(\mathscr{V} \times \mathscr{W}, \mathrm{Hom}^{\cdot}(p_2^* F, p_1^! F))$. This is quite similar to the corresponding problem in the absolute case. The next section gives a suitably modified presentation of the argument of [10].

4.4. Restriction of the Dual Class to the Diagonal

Since P involves derivatives the formula for the cocycle $\Delta^* \pi^0(\eta)$ given by (4.2.3) is a priori extremely complicated. However, it is immediately simplified by use of the following properties of P. Write Q for the operator $\Delta^* \circ P$ and note that if the lower index is kept fixed then P and Q also operate on cochains in the complex (4.1.3) so that, for example,

$$((Qu)_J^I)_{\gamma_0 \dots \gamma_p} = (-1)^{\deg u} (n - |I|)^{-1} \sum_{k=1}^{n} \Delta^* (\partial / \partial \zeta_{\gamma_p}^k (u_J^{kI})_{\gamma_0 \dots \gamma_p}). \tag{4.4.1}$$

Proposition 4.4.2. *The operator* P *satisfies*

(i) $P(\eta^{0,0}) = 0$

(ii) $P^2 = 0$

(iii) $P \circ \delta + \delta \circ P = 0$

while for f *in the complex* (4.1.3):

(iv) $P(p_1^* b \cdot f) + (p_1^* b) \cdot (Pf) = 0$.

Also, the operator Q behaves as follows with respect to products. Suppose u lies in (4.1.3) *and f in either* (4.1.3) *or* (4.1.2). *If $\Delta^* u=0$ then*

(v) $Q(f\cdot u)=(-1)^{\deg f}(\Delta^* f)\cdot Qu$

while if $u=(b)^1\tilde{\otimes}a^1$ then

(vi) $Q(f\cdot u)=(Qf)\cdot\Delta^* u+(-1)^{\deg f}(\Delta^* f)\cdot Qu$.

Proof. Formulae (i), (ii), (iii), and (v) are immediate consequences of the definitions, while (iv) follows from the fact that $p_1^* b$ is independent of the ζ-coordinates. The relation (vi) is obtained by a straightforward calculation using the above formula for Q. The main ingredient is the identity

$$\sum_{k=1}^{n}\Delta^*(\partial/\partial\zeta^k_{\gamma_{p+s}}((f^J)_{\gamma_0\ldots\gamma_p})(u^{kI}_J)_{\gamma_p\ldots\gamma_{p+s}})$$
$$=\sum_{m=1}^{r}(-1)^{m+1}\Delta^*(\partial/\partial\zeta^{j_m}_{\gamma_p}((f^J)_{\gamma_0\ldots\gamma_p})(u^I_{J_m})_{\gamma_p\ldots\gamma_{p+s}})$$

which follows from the chain rule and the explicit formula (4.1.4) for $\Delta^* a^1$. Here $J=j_1\ldots j_r$, $I=i_1\ldots i_{r-1}$ and $J_m=j_1\ldots\hat{j}_m\ldots j_r$.

Now the formula (4.2.1) for D' and the above properties of P and Q reduce the expression for $\Delta^*\pi^0(\eta)$ given by (4.2.3) to a linear combination of terms, each of which is simply a product

(4.4.3) $$(\Delta^*\eta^{0,0})\cdot(Qs^{k_1})\ldots(Qs^{k_m})$$

where $s^k=(b)^k\tilde{\otimes}a^k$ and $k_1+\ldots+k_m=n$, with each $k_i\geqq 1$.

The expression (4.4.3) can be regarded from a slightly different point of view as follows. If $\gamma=(\beta,\alpha)$, identify $\Delta^* K_\gamma^{-r}$ with $\Omega^r_X=\Lambda^r T^* X$ over $V_\beta\times U_\alpha$ via the correspondence $e^I_\gamma\mapsto dz^I_\gamma=dz^{i_1}_\gamma\ldots dz^{i_r}_\gamma$. Under this identification the formula (4.1.4) shows that $(\Delta^* a^1)_{\gamma_0\gamma_1}$ becomes the identity map on Ω^r_X and Qs^k is a cochain in the direct sum of complexes

$$\sum_l C^{\cdot}_b(\mathscr{V}\times\mathscr{U},\operatorname{Hom}^{\cdot}(F\otimes\Omega^l_X,F\otimes\Omega^{l+k}_X)).$$

For this interpretation the definition of $\eta^{0,0}$ shows that (4.4.3) is precisely the cochain

(4.4.4) $$(Qs^{k_1})\cdot(Qs^{k_2})\ldots(Qs^{k_m})$$

of $C^{\cdot}(\mathscr{V}\times\mathscr{U},\operatorname{Hom}^{\cdot}(F,F\otimes\Omega^n_X))$ under the identification of $\operatorname{Hom}(\Omega^0_X,\Omega^n_X)$ with Ω^n_X. The next proposition gives a more precise description of the cochains Qs^k. Recall that the calculation of [9, Sect. 5] shows that the product operation for the twisting cochain b defines a chain map from the usual Čech complex $C^{\cdot}(\mathscr{V}\times\mathscr{U},\mathcal{O}_{Y\times X})$ into $C^{\cdot}(\mathscr{V}\times\mathscr{U},\operatorname{Hom}^{\cdot}(F,F))$ which takes a k-cochain u^k to the cochain $(b)^k\tilde{\otimes}u^k$ of total degree k.

Proposition 4.4.5. *Define the cochains Qa^k according to* (4.4.1), *applied in the case where F and b are trivial and Y is reduced to a point. Extend Qa^k to a cochain on the cover $\mathscr{V}\times\mathscr{U}$ in the obvious way. Then*

(i) $Qs^k=(b)^k\tilde{\otimes}Qa^k$ *for* $k>1$, *while, acting on elements of* $C^{\cdot}(\mathscr{V}\times\mathscr{U}, F^{\cdot}\otimes\Omega_X^p)$,

(ii) $Qs^1=(b)^1\tilde{\otimes}Qa^1-d_Xb/(n-p)$,

where db defines cochains in $C_b^{\cdot}(\mathscr{V}\times\mathscr{U}, \mathrm{Hom}^{\cdot}(F\otimes\Omega_X^p, F\otimes\Omega_X^{p+1}))$ *via the map from* Ω_X^1 *into* $\mathrm{Hom}(\Omega_X^p,\Omega_X^{p+1})$ *given by left exterior multiplication.*

Proof. The first formula follows immediately from $\Delta^*a^k=0$. For (ii) the formula for Q gives, with $I=i_1\dots i_p$ and $\phi_j^i=\partial z_{\gamma_s}^i/\partial z_{\gamma_0}^j$,

$$
\begin{aligned}
&-(n-p)(Qs^1-(b)^1\tilde{\otimes}Qa^1)_{\gamma_0\dots\gamma_s}(dz_{\gamma_s}^I)\\
&=\sum_{k=1}^n(\partial b^s_{\gamma_0\dots\gamma_s}/\partial z^k_{\gamma_s})(a^1_{\gamma_0\gamma_s}(dz^{kI}_{\gamma_s}))\\
&=\sum_{k=1}^n(\partial b^s_{\gamma_0\dots\gamma_s}/\partial z^k_{\gamma_s})\sum_J\sum_\sigma\mathrm{sgn}(\sigma)\phi^k_{j_{\sigma(1)}}\phi^{i_1}_{j_{\sigma(2)}}\cdots\phi^{i_p}_{j_{\sigma(p+1)}}\cdot dz^{j_1}_{\gamma_0}\dots dz^{j_{p+1}}_{\gamma_0}\\
&=\sum_{k=1}^n(\partial b^s_{\gamma_0\dots\gamma_s}/\partial z^k_{\gamma_s})\sum_{j_1=1}^n\cdots\sum_{j_{p+1}=1}^n\phi^k_{j_1}\phi^{i_1}_{j_2}\dots\phi^{i_p}_{j_{p+1}}\cdot dz^{j_1}_{\gamma_0}\dots dz^{j_{p+1}}_{\gamma_0}\\
&=\sum_{k=1}^n(\partial b^s_{\gamma_0\dots\gamma_s}/\partial z^k_{\gamma_s})dz^k_{\gamma_s}dz^{i_1}_{\gamma_s}\dots dz^{i_p}_{\gamma_s}
\end{aligned}
$$

as required.

The cochains Qa^k in $\sum_l C^k(\mathscr{U},\mathrm{Hom}(\Omega_X^l,\Omega_X^{l+k}))$ are related to the characteristic ring of X by the following proposition, which is a simple modification of the invariant theory argument of [14, Sect. 5].

Proposition 4.4.6. *There exist cocycles* θ_l^k *in* $C^k(\mathscr{U},\Omega_X^k)$ *such that* θ_l^k *corresponds to the appropriate component of* Qa^k *under the injective map from* Ω_X^k *into* $\mathrm{Hom}(\Omega_X^l,\Omega_X^{l+k})$ *given by left exterior multiplication. Moreover, each* θ_l^k *represents an Atiyah-Chern class of* X.

Proof. We first show that Qa^k is a cocycle. This follows by applying (i), (v), and (vi) of Proposition 4.4.2 to the equation $Q(\delta a+a\cdot a)=0$ to obtain

$$\delta(Qa^k)+(\Delta^*a^1)\cdot Qa^k-Qa^k\cdot\Delta^*a^1=0$$

as required.

Furthermore, as in [14, Sect. 11], both a^k and P are equivariant under affine transition functions, so that the same is true for Qa^k. Therefore the argument of [14, Sect. 5] can be adapted to the universal model for Qa^k over $\mathbb{C}^n$. So let $\mathscr{A}_n^+$ be the group of local automorphisms of $\mathbb{C}^n$ which fix the origin and whose linear past is the identity. Write V for the vector space $\mathbb{C}^n$. Then the universal model for Qa^k is determined by its restrictions

$$F_+:[J^N(\mathscr{A}_n^+)]^k\to\Lambda^lV\otimes\Lambda^{l+k}V^*$$

to the N-jets of its arguments, for N sufficiently large. Under conjugation by a constant $\lambda\neq0$ the right side behaves like Λ^kV^*, so that exactly the same reasoning as in [14, Sect. 5] shows that F_+ is in turn given by a $GL(V)$-invariant form

$$\otimes^k(S^2V^*\otimes V)\otimes\Lambda^lV^*\otimes\Lambda^{l+k}V\to\mathbb{C}. \tag{4.4.7}$$

The usual invariant theory for GL(V), as applied in Theorem 5.10 of [14] for example, shows that the space of such forms is spanned by the maps obtained by contracting each copy of V with some copy of V^*. Note however that the contraction of both copies of V in an S^2V^* with $\Lambda^{l+k}V$ gives zero, so that it can be assumed that each S^2V^* contracts with one V from $\otimes^k V$ and one from $\Lambda^{l+k}V$, leaving $\Lambda^k V^*$ to contract with $\Lambda^{l+k}V$.

More explicitly, suppose σ is a permutation of $\{1, \ldots, k\}$ and

$$u = x_1 \otimes y_1 \otimes \ldots \otimes x_k \otimes y_k$$

with x_i in S^2V^* and y_i in V. Define u_σ in $\Lambda^k V^*$ by contracting x_i with $y_{\sigma(i)}$ and skew-symmetrizing the resulting element of $\otimes^k V^*$. Then for v in $\Lambda^l V^*$ and w in $\Lambda^{k+l}V$ the maps Φ_σ defined, in terms of the usual pairing of $\Lambda^{k+l}V$ and $\Lambda^{k+l}V^*$, by

$$\Phi_\sigma(u \otimes v \otimes w) = \langle u_\sigma \wedge v, w \rangle$$

span the space of GL(V)-invariant forms (4.4.7).

If Φ_σ is regarded as a GL(V)-equivariant map

$$\Phi_\sigma : \otimes^k (S^2 V^* \otimes V) \to \mathrm{Hom}(\Lambda^l V^*, \Lambda^{k+l} V^*)$$

the above expression translates into the formula $(\Phi_\sigma(u))(v) = u_\sigma \wedge v$. This proves the first part of the proposition.

In order to see that the resulting cocycle θ_l^k represents an Atiyah-Chern class, note that its universal model satisfies all the hypotheses of Theorem II of [14, Sect. 4] except for skew-symmetry. Therefore its skew-symmetrization, which represents the same class in cohomology, is an Atiyah-Chern class at the cochain level.

To summarize, the cocycle $\Delta^* \pi^0(\eta)$ is a sum of terms of the form (4.4.4), and each of these terms is itself a linear combination of products $\xi_1 \wedge \ldots \wedge \xi_m$ of cocycles in $C_b^{\cdot}(\mathscr{V} \times \mathscr{U}, \mathrm{Hom}^{\cdot}(F, F \otimes \Omega_X))$. These cocycles themselves are given by either $\xi_i = d_X b$ or $\xi_i = (b)^k \tilde{\otimes} \theta_l^k$, for some $k \geqq 1$ and $0 \leqq l \leqq n-k$.

Now multiply this expression for $\Delta^* \pi^0(\eta)$ by $(d_Y b)^l$, apply the trace map and rearrange each of the terms $\tau_b(\xi_1 \wedge \ldots \wedge \xi_m \wedge (d_Y b)^l)$ according to Propositions 3.8 and 5.11 of [9] to give, in cohomology,

$$\tau_b(\xi_1 \wedge \ldots \wedge \xi_m \wedge (d_Y b)^l) = \pm \theta_{j_1}^{i_1} \wedge \ldots \wedge \theta_{j_p}^{i_p} \wedge \tau_b((d_X b)^k \wedge (d_Y b)^l)$$

where $p + k = m$ and $i_1 + \ldots + i_p + k = n$. This gives

$$\tau_b((d_Y b)^l \cdot \Delta^* \pi^0(\eta)) = \Theta_{n-k}(X) \cdot (d_X b)^k \cdot (d_Y b)^l / k! \tag{4.4.8}$$

for certain universally defined polynomials $\Theta_k(X)$ of degree k in the Atiyah-Chern classes of X, independent of X, Y, $\mathscr{F}$ and l. But in case Y is a point and $l=0$ we know that $\Theta_k(X)$ represents the k^{th} Todd class $\mathrm{Todd}_k(X)$. Multiply (4.4.8) by $1/l!$, sum over l and integrate over X. Type considerations show that each $\Theta_{n-k}(X)$ can be replaced by $\sum_j \Theta_j(X) = \mathrm{Todd}(X)$. Formula (4.3.10) therefore gives, in cohomology,

$$\begin{aligned}\sum_k \tau_c(dc)^l / l! &= \int_X \mathrm{Todd}(X) \cdot \tau_b \Big(\sum_{k,l} (d_X b)^k \cdot (d_Y b)^l / k!\, l! \Big) \\ &= \int_X \mathrm{Todd}(X) \cdot \tau_b \Big(\sum_m (d_X b + d_Y b)^m / m! \Big) \\ &= \int_X \mathrm{Todd}(X) \cdot \mathrm{ch}(\mathscr{F})\end{aligned}$$

– the Riemann-Roch formula for the projection π.

5. Riemann-Roch for Retractible Embeddings

5.1. Factorization of the Trace Map

Let $i: X \to Z$ be a holomorphic embedding of an m-dimensional complex manifold X into a complex manifold Z of dimension $n = m + r$. Let N be the normal bundle of X in Z, and assume the existence of a holomorphic map $\varrho: Z \to X$ with $\varrho \circ i$ the identity.

Let $(\mathscr{W}, K, a)$ be a twisting cochain for $i_* \mathcal{O}_X$ on Z, constructed as in [15, Sect. 3] with Koszul resolutions and coordinates adapted to the splitting $i^*TZ = TX \oplus N$ given by ϱ. The Stein cover $\mathscr{W}$ consists of a cover $\mathscr{V}$ of $Z - X$, on which the $K_\alpha^\cdot$ are taken to be zero, and a cover $\mathscr{U}$ of a neighborhood of X which restricts to a Stein cover $\mathscr{U}_X$ of X. It can also be assumed that $\mathscr{W}$ refines $\varrho^{-1}\mathscr{U}_X$.

Let $\mathscr{F}$ be a coherent sheaf on X with associated twisting cochain $(\mathscr{U}_X, F, b)$. The product operation gives a twisting cochain $(\mathscr{W}, F \otimes K, b \tilde{\otimes} a)$ for $i_* \mathscr{F}$, where F, b stand for the restrictions of the $\varrho^* F_\alpha$ and $\varrho^* b$ to $\mathscr{W}$ (see [10, Lemma 2.1] for example). As usual the quotient map from $\mathcal{O}_Z$ onto $i_* \mathcal{O}_X$ is associated with a 0-cocycle u in $C^\cdot(\mathscr{W}, K^\cdot)$ which induces $\pi^0: C^\cdot(\mathscr{W}, \mathrm{Hom}^\cdot(K, 1)) \to C^\cdot(\mathscr{W}, 1)$ with $\pi^0(w) = w \cdot u$.

Proposition 5.1.1. *The trace map*

$$T: C^\cdot(\mathscr{W}, \mathrm{Hom}^\cdot(F \otimes K, F \otimes K)) \to C^\cdot(\mathscr{W}, 1)$$

factorizes (up to simplicial homotopy) as $T = \pi^0 \circ S$ for suitable

$$S: C^\cdot(\mathscr{W}, \mathrm{Hom}^\cdot(F \otimes K)) \to C^\cdot(\mathscr{W}, \mathrm{Hom}^\cdot(K, 1)). \tag{5.1.2}$$

Proof. The construction is similar to that used for the refined trace map of [9, Sect. 5]. For a simplex $\alpha_0 \ldots \alpha_p$ of $\mathscr{W}$ set $M_\alpha^\cdot = \mathrm{Hom}^\cdot(F_{\alpha_p}, F_{\alpha_0})$, $L_\alpha^\cdot = \mathrm{Hom}^\cdot(K_{\alpha_p}, K_{\alpha_0})$ and $\check{K}_\alpha^\cdot = \mathrm{Hom}^\cdot(K_{\alpha_p}, 1)$, so that (5.1.2) will take the form of a simplicial chain map between twisted complexes $C_A^\cdot(\mathscr{W}, M^\cdot \otimes L^\cdot)$ *and* $C_B^\cdot(\mathscr{W}, \check{K}^\cdot)$. The first complex is bigraded by combining the first two degrees, and the map S will take the form $\sum_{k \geq 0} S^k$ where, according to this convention, S^k has bidegree $(k, -k)$. The map S^0 is defined in terms of the algebra structure of the Koszul complexes. For any vertex γ there is the chain map $e_\gamma: K_\gamma^\cdot \to \mathrm{Hom}^\cdot(K_\gamma, K_\gamma)$ given by $e_\gamma(w)(v) = w \wedge v$. This has adjoint $\check{e}_\gamma: \mathrm{Hom}^\cdot(K_\gamma, K_\gamma) \to \mathrm{Hom}^\cdot(K_\gamma, 1)$ with

$$\check{e}_\gamma(\phi)(w) = \mathrm{trace}\, \phi \circ (e_\gamma(w)).$$

For simplices $\alpha = \alpha_0 \ldots \alpha_p$ and $\beta = \beta_0 \ldots \beta_q$ with $\beta \leq \alpha$ this gives chain maps $E_\alpha^\beta: L_\beta^\cdot | W_\alpha \to \check{K}_\alpha^\cdot$ with $E_\alpha^\beta(\phi_\beta) = \check{e}_{\alpha_p}(a_{\alpha_p \beta_0} \phi_\beta a_{\beta_q \alpha_p})$.

Let $\tau_b: C^\cdot(\mathscr{U}_X, \mathrm{Hom}^\cdot(F, F)) \to C^\cdot(\mathscr{U}_X, 1)$ be the usual trace map for $(\mathscr{U}_X, F, b)$ and suppose that the restriction of $\varrho^* \tau_b$ to $\mathscr{W}$ is described in terms of local vector bundle maps τ_α^β for $\beta \leq \alpha$. Then S^0 is defined by setting $(S^0)_\alpha^\beta = \tau_\alpha^\beta \otimes E_\alpha^\beta$ and the fact that this extends to a global chain map S follows by a spectral sequence argument, as in the proof of Theorem 5.7 of [9]. In fact the local complexes $\mathrm{Hom}^\cdot(\mathrm{Hom}(K_{\beta_q}, K_{\beta_0}), \mathrm{Hom}(K_{\alpha_p}, 1))$ are isomorphic to $\mathrm{Hom}^\cdot(K_{\alpha_p}, \mathrm{Hom}(K_{\beta_0}, K_{\beta_q}))$ under the adjoint operation, and two applications of Lemma 1.6 of [9] show that this latter complex is acyclic in degree $q < 0$.

Then for α, β as above, define $I^\beta_\alpha : \check{K}^\cdot_\beta | W_\alpha \to \check{K}^\cdot_\alpha$ and $J^\beta_\alpha : L^\cdot_\beta | W_\alpha \to L^\cdot_\alpha$ by $I^\beta_\alpha(\varphi_\beta) = \varphi_\beta a_{\beta_q \alpha_p}$ and $J^\beta_\alpha(\psi_\beta) = a_{\alpha_0 \beta_0} \psi_\beta a_{\beta_q \alpha_p}$. The argument of [9] now applies once it is established that for $\gamma = \gamma_0 \ldots \gamma_s$ with $\gamma \leqq \beta$ the maps $I^\beta_\alpha E^\gamma_\beta$, $E^\beta_\alpha J^\gamma_\beta$, and E^γ_α are chain homotopic. For the last two maps this is straightforward. For the first two maps the problem reduces to the following lemma.

Lemma 5.1.3. *The two maps from* $\mathrm{Hom}^\cdot(K_{\beta_q}, K_{\alpha_p})$ *to* $\mathrm{Hom}^\cdot(K_{\alpha_p}, 1)$ *sending* ψ *into* $\check{e}_{\alpha_p}(\psi a_{\beta_q \alpha_p})$ *and* $\check{e}_{\beta_q}(a_{\beta_q \alpha_p} \psi) \circ a_{\beta_q \alpha_p}$ *are chain homotopic.*

Proof. Let $\tilde{a}_{\beta_q \alpha_p} = \Lambda^\cdot(a_{\beta_q \alpha_p} | K^{-1}_{\alpha_p})$. Then the chain maps $a_{\beta_q \alpha_p}$ and $\tilde{a}_{\beta_q \alpha_p}$ are chain homotopic over $W_{\beta_q} \cap W_{\alpha_p}$, since both lift the identity map on $i_* \mathcal{O}_X$. Therefore $a_{\beta_q \alpha_p}$ can be replaced by $\tilde{a}_{\beta_q \alpha_p}$ without changing the homotopy classes of the maps. But then the maps become identical, since for w_{α_p} in $K^\cdot_{\alpha_p}$ we have

$$\tilde{a}_{\beta_q \alpha_p} \circ e_{\alpha_p}(w_{\alpha_p}) = e_{\beta_q}(\tilde{a}_{\beta_q \alpha_p} w_{\alpha_p}) \circ \tilde{a}_{\beta_q \alpha_p}.$$

Since $(\pi^0)^\alpha_\alpha$ is just the projection of $K^\cdot_\alpha$ onto $K^0_\alpha = \mathcal{O}_Z | W_\alpha$, the formula for S^α_α shows that $(\pi^0)^\alpha_\alpha S^\alpha_\alpha$ is a local trace map acting on $\mathrm{Hom}^\cdot(F_\alpha \otimes K_\alpha, F_\alpha \otimes K_\alpha)$. Uniqueness of trace [9] then shows that $\pi^0 \circ S$ and T are chain homotopic and completes the proof of the proposition.

This result shows that the trace map factors through the complex used in [15] for the cochain-level description of the Gysin map. Recall the construction of the Gysin map in this context: for any vector bundle E over Z there is a chain map

$$(5.1.4) \qquad Res: C^\cdot(\mathscr{W}, \mathrm{Hom}^\cdot(K \otimes E, \Omega^n_Z)) \to C^\cdot(\mathscr{U}_X, \mathrm{Hom}(i^* E, \Omega^m_X))$$

of degree zero, induced by the isomorphism $i^* \Omega^n_Z = \Lambda^r \check{N} \otimes \Omega^m_X$ and the identification of $i^* K^{-r}_\alpha$ with $\Lambda^r \check{N}$ over $W_\alpha \cap X$. If $K^\cdot_\alpha$ is the Koszul complex on generators $e^1_\alpha, \ldots, e^r_\alpha$ defined by coordinates $\zeta^1_\alpha, \ldots, \zeta^r_\alpha$ vanishing on X, then this identification is compatible with the action of $i^* a^1_{\alpha\beta}$ and takes $e^1_\alpha \wedge \ldots \wedge e^r_\alpha$ into $d\zeta^1_\alpha \wedge \ldots \wedge d\zeta^r_\alpha$. The map (5.1.4) induces an isomorphism on cohomology; this is essentially the "fundamental local isomorphism" of Grothendieck duality theory. If $E = \Omega^{m-k}_Z$ then the natural map $\mu : \Omega^k_X \to \mathrm{Hom}^\cdot(i^* \Omega^{m-k}_Z, \Omega^m_X)$, defined by restriction and exterior product, composed with π^0 and the cohomology inverse of (5.1.4), gives the Gysin map i_* from $H^k(X, \Omega^l_X)$ into the local cohomology $H^{k+r}_X(Z, \Omega^{l+r}_Z)$, if this is defined in terms of relative cochains for the covers $\mathscr{W}$, $\mathscr{V}$.

5.2. *Reduction to the Normal Bundle*

If S is applied to powers of the Atiyah cocycle $d(b \tilde{\otimes} a)$ for $i_* \mathscr{F}$, it turns out that none of the correcting homotopies making up π^0, S and a can appear in the expression for the residue. This follows from the simple degree arguments given below and implies that the residue is unchanged when the embedding of X into Z is replaced by the embedding of X into the zero section of N. This situation is of course much simpler, since in this case $i_* \mathcal{O}_X$ is resolved by the globally defined Koszul complex on N, and in fact the splitting principle for vector bundles allows us to reduce to the case where N is a line bundle, where the required formula is an easy explicit calculation. The next proposition gives the reduction to the embedding into the normal bundle.

Proposition 5.2.1. *For $l \geqq 0$ there exist cocycles $\Psi_l(N)$ in $C^l(\mathscr{U}_X, \Omega_X^l)$, representing classes in $H^l(X, \Omega_X^l)$ which depend only on the normal bundle N of X in Z, such that for all $p \geqq 0$*

$$\text{(5.2.2)} \qquad \operatorname{Res} \circ S(d(b \tilde{\otimes} a))^{p+r}/(p+r)! = \sum_{k+l=p} \mu(\tau_b((db)^k/k!)\Psi_l(N))$$

in the cohomology of $C^p(\mathscr{U}_X, \operatorname{Hom}(i^*\Omega_Z^{m-p}, \Omega_X^m))$.

Proof. From [15] the a^k for $k > 1$ vanish on X, so $d(b \tilde{\otimes} a)$ can be replaced by $db \tilde{\otimes} a^1 + b \tilde{\otimes} da$ without changing the residue. Since the residue involves only components of Koszul degree r and all components of S have Koszul degree $\leqq 0$, only terms involving r or more da^0 factors will contribute to the first expression in (5.2.2). On the other hand, on X the coefficients of da^0 operate by exterior multiplication by normal cotangents, and since $\Lambda^q \check{N} = 0$ for $q > r$ any term contributing to the residue must involve *exactly* r copies of da^0. This in turn implies that no factors of negative Koszul degree can appear in the residue, so that in the formula (5.2.2) da can be replaced by $da^0 + da^1$ and S^q can be taken to be zero for $q > 0$. Moreover, the normal component of da^1 must also disappear in (5.2.2) since the r copies of da^0 already fill up the normal cotangent directions. From all this one concludes that the residue is unchanged when a is replaced by the twisting cochain for the standard Koszul complex of N, with respect to local trivializations given by the above identifications of the $i^*K_\alpha^{-q}$ with $\Lambda^q \check{N}$.

In order to see that the residue lies in the image of μ it is enough to check that for $l > 0$ it vanishes on the components $\Lambda^l \check{N} \otimes \Omega_X^{m-p-l}$ of the decomposition of $i^*\Omega_Z^{m-p}$. This again follows since $\Lambda^q \check{N} = 0$ for $q > r$.

The complex $C^{\cdot}(\mathscr{U}_X, \operatorname{Hom}^{\cdot}(F \otimes i^*K, F \otimes i^*K))$ decomposes into a direct sum of complexes, each of which is acted on by the trace map,

$$\tau_b : C^{\cdot}(\mathscr{U}_X, \operatorname{Hom}^{\cdot}(F \otimes \Lambda^p \check{N}, F \otimes \Lambda^q \check{N})) \to C^{\cdot}(\mathscr{U}_X, \operatorname{Hom}(\Lambda^p \check{N}, \Lambda^q \check{N})).$$

Since a^1 operates as the identity on $\Lambda^{\cdot} \check{N}$, for w in $C^p(\mathscr{U}_X, \operatorname{Hom}^{s-p}(F \otimes i^*K, F \otimes i^*K \otimes i^*\Omega_Z^s))$ the identities

$$\tau_b((db \tilde{\otimes} a^1) \cdot w) = \tau_b(w \cdot (db \tilde{\otimes} a^1))$$
$$\tau_b(w \cdot (b \tilde{\otimes} da)) = \tau_b(w) \cdot da$$

hold in cohomology, where the cochain coefficients are composed using wedge product. This is proved as in [9, Proposition 3.8, 5.11], and allows us to rearrange the residue in (5.2.2) into an expression of the type occuring on the right of the same formula, where $\Psi_l(N)$ is independent of $\mathscr{F}$.

In order to see that the class of $\Psi_l(N)$ is independent of the twisting cochains and partial trace (5.1.2) on different covers $\mathscr{W}$, $\mathscr{W}'$, the argument of [9] can be adapted and the entire construction carried out on the union $\mathscr{W} \coprod \mathscr{W}'$ of the two covers in such a way that all cochains take on their original values when all vertices lie in $\mathscr{W}$ or $\mathscr{W}'$ respectively.

Since $\pi^0 \circ S$ is a trace map for $i_*\mathscr{F}$ the theory of the cochain-level Gysin map gives

$$\operatorname{ch}(i_*\mathscr{F}) = i_*[\operatorname{ch}(\mathscr{F}) \cdot \Psi_l(N)]$$

in $\sum_k H_X^k(Z, \Omega_Z^k)$. The class represented by $\Psi_l(N)$ is related to the Todd genus of the bundle N in the next section.

5.3 Identification of the Todd Genus

The proof of the Riemann-Roch theorem is completed by showing that for any holomorphic vector bundle E over X the class $\Psi(E)=\sum_l \Psi_l(E)$ coincides with the inverse $\mathrm{Todd}(E)^{-1}$ of the total Todd class of the bundle. In fact it is sufficient to establish the following properties.

Proposition 5.3.1. (a) *For L a line bundle, $\Psi(L)=\mathrm{Todd}(L)^{-1}$.*

(b) *If $0\to E\to F\to G\to 0$ is an exact sequence of bundles, then $\Psi(F)=\Psi(E)\cdot\Psi(G)$.*

(c) *For a holomorphic map $f:Z\to X$ and bundle E over X, we have $\Psi(f^*E)=f^*\Psi(E)$.*

These properties imply that $\Psi(E)=\mathrm{Todd}(E)^{-1}$ in general: assume inductively that this is true for bundles of rank less than r. If the rank r bundle E has a line sub-bundle L, formulae (a), (b) show that

$$\Psi(E)=\Psi(L)\cdot\Psi(E/L)=\mathrm{Todd}(L)^{-1}\cdot\mathrm{Todd}(E/L)^{-1}=\mathrm{Todd}(E)^{-1}.$$

But by the splitting principle [6], for any E there exists a map $f:Z\to X$ such that f^*E has line sub-bundle and $H^k(X,\Omega_X^k)$ injects into $H^k(Z,\Omega_Z^k)$. Then (c) gives the required result.

Proof of 5.3.1. Assume that L is trivialized by local sections e_α with corresponding fibre coordinates ζ_α. These trivializations give a twisting cochain for the sheaf of the zero section, for which da_α^0 operates by interior product with $e_\alpha\otimes d\zeta_\alpha$. The component $da_{\alpha\beta}^1$ vanishes on K_β^0 and operates on $K_\beta^{-1}=\check{L}$ as multiplication by $-A_{\alpha\beta}$, where $\zeta_\alpha=\varphi_{\alpha\beta}\zeta_\beta$ and $A_{\alpha\beta}=\varphi_{\alpha\beta}^{-1}d\varphi_{\alpha\beta}$ is the Atiyah class of L. The component of Koszul degree 1 in $(da)^p$ is therefore $(da^0)\cdot(-A)^{p-1}$, with residue $\mu((-A)^{p-1})$. Now multiply by $1/p!$ and sum to obtain

$$\begin{aligned}\mathrm{Res}\Big(\sum_{p\geqq 0}(da)^p/p!\Big)&=\sum_{p\geqq 0}\mu((-A)^p/(p+1)!)\\&=\mu(\mathrm{Todd}(L)^{-1}).\end{aligned}$$

For part (b) the sequence can be locally split over a fine enough cover $\mathscr{U}$ of X so that F is described in terms of transition matrices

$$b_{\alpha\beta}=\begin{bmatrix}a_{\alpha\beta} & 0\\ h_{\alpha\beta} & c_{\alpha\beta}\end{bmatrix}$$

with respect to decompositions $E_\alpha\oplus G_\alpha$ of $F_\alpha=F|U_\alpha$. Suppose that E, G, F have fibre dimensions p, q, r respectively. The Koszul degree has two components corresponding to the isomorphisms $\Lambda^{\cdot}\check{F}_\alpha=\Lambda^{\cdot}\check{E}_\alpha\otimes\Lambda^{\cdot}\check{G}_\alpha$ and the twisting cochain for the zero section of F has components of bidegree $(1,0)$ and $(0,1)$ given by

$$b^0=a^0\otimes 1+1\otimes c^0, \tag{5.3.2}$$

while b^1 has components of Koszul bidegree $(k,-k)$ for $k\geqq 0$, the $(0,0)$ component of which does not involve the $h_{\alpha\beta}$.

The degree argument of the previous section, applied to the second Koszul degree, shows that no more than q copies of dc^0 can appear in the residue in this case, since the coefficients of dc^0_α lie in the sub-bundle $\check{G}$ of the conormal bundle $\check{F}$, and $\Lambda^{q+1}\check{G}=0$. Therefore no terms of negative second Koszul degree can appear and the residue does not involve the $h_{\alpha\beta}$. It follows that the residue is unchanged when F is replaced by EG. In this case b^0 is still given by (5.3.2), while

$$db^1_{\alpha\beta}=da^1_{\alpha\beta}\otimes c^1_{\alpha\beta}+a^1_{\alpha\beta}\otimes dc^1_{\alpha\beta}.$$

Now write db as $db'+db''$ where $db'=da^0\otimes 1+da^1\otimes c^1$ and $db''=1\otimes dc^0+a^1\otimes dc^1$. The cochains db' and db'' commute in cohomology: $db'\cdot db''-db''\cdot db'=Df$, where f is the cochain $da^1\otimes dc^1$. Therefore

$$\begin{aligned}\Psi_s(F)&=\operatorname{Res}\circ S[(db)^{r+s}/(r+s)!]\\&=\sum_{k+l=s}\operatorname{Res}\circ S[((db')^{p+k}/(p+k)!)\cdot((db'')^{q+l}/(q+l)!)]\\&=\sum_{k+l=s}\Psi_k(E)\cdot\Psi_l(G)\end{aligned}$$

as required.

Finally, property (c) is an immediate consequence of the functoriality of the construction.

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