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Singularities of Transmission Problems

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1. Introduction

Let M be a C^∞ manifold. Let $S \subset M$ be a closed submanifold of codimension 1. Assume that the conormal bundle $N^*S \subset T^*M$ is oriented. Let P be a second order linear differential operator in $M-S$ with coefficients in $\bar{C}^\infty(M-S)$, the space of C^∞ functions in $M-S$ having locally C^∞ extensions across S from either side. Assume S to be non-characteristic for P from both sides. Under principal type conditions on P we can discuss the C^∞ singularities of a distribution $u \in \bar{\mathscr{D}}'(M-S)$, where $\bar{\mathscr{D}}'(M-S)$ is the space of distributions in $M-S$ which are locally extendible across S from either side, solving the transmission problem

$$(T) \qquad \begin{aligned} Pu &\in \bar{C}^\infty(M-S)\,, \\ u|S_+ - u|S_- &\in C^\infty(S)\,, \\ D_+u|S_+ + D_-u|S_- &\in C^\infty(S)\,. \end{aligned}$$

Here $u|S_+$ and $u|S_-$ denote the boundary values of u on S when approaching S from its positive and its negative side, respectively. (The sides of S are defined through the orientation of N^*S.) These boundary values, as well as those of higher derivatives of u, exist by Peetre's theorem. D_+ and D_- are the normal vectorfields, canonically associated with P, for the positive and the negative side of S, respectively. To define them let $x \in C^\infty(M)$ be a local defining function for S and for the orientation of N^*S, i.e. $S \cap U = x^{-1}(0) \cap U$ and $dx|U \cap S$ is positively oriented in an open subset $U \subset M$. $D_\pm$ equals, in U, the principal part of the commutator

$$\frac{i}{2}|g|^{-1/2}[P, \pm x]\,.$$

The function $g = -\frac{1}{2}[[P,x],x]$ has nonvanishing restrictions $g|S_\pm$. Although $D_\pm$ may depend on the choice of x the restriction $D_\pm u|S_\pm$ does not. Let $\Omega_\pm$ denote the

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function on S given by $\Omega_\pm = \mathrm{sign}(g|S_\pm)$. Assume that

$$\Omega_+ \equiv \Omega_- \quad \text{on } S.$$

This completes the setup of (T).

Note that no further generality is gained by relaxing the last condition in (T) to $D_+u|S_+ + a \cdot D_-u|S_- \in C^\infty(S)$ with some positive function $a \in C^\infty(S)$. In fact, if $b \in \bar{C}^\infty(M-S)$ is positive up to S then the normal vectorfields of P and $P' = bP$ are related through $D'_\pm = \sqrt{b|S_\pm} \cdot D_\pm$.

We shall introduce, following Melrose's theory [5], the singular spectrum $ss_T(u)$ of a solution u of (T) and the characteristic variety Σ_T associated with (T) as closed conic subsets of $\tilde{T}^*(M,S)$, the cotangent bundle of M compressed along S. The Hamilton vectorfield H_p is defined on Σ^0, the part of Σ_T not lying over S. The gliding fields H_+ and H_- are defined on $\Sigma_+^{(2)}$ and $\Sigma_-^{(2)}$, the glancing sets for the positive and the negative side of S, respectively.

We shall assume that P is of real principal type with respect to S. By this we mean that the principal symbol p of P is a realvalued function on $\tilde{T}^*(M,S)$, singular over S, such that

(1.1) the radial direction in $\tilde{T}^*(M,S)$ is linearly independent of H_p at Σ^0,

(1.2) at $\Sigma_+^{(2)} \cup \Sigma_-^{(2)}$ the radial direction in $\tilde{T}^*(M,S)$ is linearly independent of H_+, H_- and of any convex combination of H_+, H_- whenever these are defined.

Using only the Hamilton vectorfield and the gliding fields we shall define rays for P. Our result can then be stated as follows.

(1.3) **Theorem.** *Let $u \in \mathscr{D}'(M-S)$ be a solution of (T). Suppose that P is of real principal type with respect to S. Then $ss_T(u) \subset \Sigma_T$ and $ss_T(u)$ is a union of maximally extended rays.*

Away from glancing points rays just consist of pieces of H_p-bicharacteristics reflected and refracted at S in the natural way. The propagation result given in the theorem is wellknown for this case (see Hörmander [1], Nosmas [8], Taylor [9]). That essentially the same propagation result holds true when the only glancing points involved are nondegenerate diffractive points was shown by Taylor [10, 11]. Near a point over S which is glancing for just one side of S the theorem can readily be deduced from the results of Melrose and Sjöstrand [7]. The main novelty in the theorem above is a propagation result near points which are glancing for both sides of S. In proving this result we shall follow the ideas of [7] very closely. We should mention that the result stated in Theorem 1.3 may not be optimal at points which are gliding for one side of S, diffractive for the other, and where the glancing sets, $\Sigma_+^{(2)}$ and $\Sigma_-^{(2)}$, do not intersect symplectically. Also we wish to point out that we do not give results on uniform approximation of rays by broken bicharacteristics.

The plan of this paper is as follows. In Sect. 2 we recall some of those notions and facts of microlocal theory which we shall use. The basic energy estimates on $ss_T(u)$, still crude geometrically, are given in Sect. 3. Rays are defined in Sect. 4 were also some of their geometric properties are exhibited. Finally, in Sect. 5, we complete the proof of Theorem 1.3 by iterating the estimates on $ss_T(u)$ obtained in Sect. 3 to construct rays which are completely contained in $ss_T(u)$.

2. Singular Spectrum and Characteristic Variety

We recall some concepts and results of Melrose [5], referring to this paper for details, on the microlocal analysis of boundary problems on a C^∞ manifold M_0 with boundary ∂M_0. The boundary singular spectrum $ss_b(u)$ [or boundary wavefront set $WF_b(u)$] of a distribution u supported on M_0 is a closed conic subset of the compressed cotangent bundle $\tilde{T}^*M_0$. It is defined using the algebra of totally characteristic operators on M_0, $L_b(M_0)$. The principal symbols of these operators are sections in a line bundle over $\tilde{T}^*M_0$ and, therefore, their characteristic varieties are subsets of $\tilde{T}^*M_0$. $ss_b(u)$ is the intersection of the characteristic varieties of all those zeroth order totally characteristic operators mapping u into the space $\dot{A}(M_0)$ of Lagrangian distributions, supported in M_0, which are associated with the conormal bundle of ∂M_0. Boundary Fourier integral operators transform $ss_b(u)$ naturally.

There is a natural map

$$\pi: T^*M_0 \to \tilde{T}^*M_0$$

given by $(x, y, \chi, \eta) \mapsto (x, y, \lambda, \eta)$, $\lambda = x\chi$, in local canonical coordinates, $x \geqq 0$ in M_0. Its range is

$$T^*\partial M_0 \cup T^*\mathring{M}_0 \tag{2.1}$$

which can be regarded as a subset of $\tilde{T}^*M_0$. Under this identification ss_b agrees, in $T^*\mathring{M}_0$, with the usual notion of singular spectrum in manifolds without boundary.

Solutions to noncharacteristic boundary problems belong to the space of normally regular distributions, $\mathcal{N}(M_0)$. A normally regular distribution has its boundary singular spectrum contained in (2.1). Furthermore, choosing any local coordinates (x, y) with $x \geqq 0$ in M_0, and with coordinate patch U, we have

$$\mathcal{N}(M_0) \subset C^\infty(\overline{\mathbb{R}}_x^+; \mathcal{D}'(\mathbb{R}_y^{n-1}))$$

in $U \cap \{0 \leqq x < \varepsilon\}$ for some $\varepsilon > 0$. For $u \in \mathcal{N}(M_0)$ one can determine $T^*\partial M_0 \cap ss_b(u)$, over U, with tangential pseudodifferential operators $Q(x, y, D_y)$ using the WF_b-definition given in [6]. In fact, choosing $J \in L_b^0(M_0)$ such that $Id - J$ is smoothing on $\mathcal{N}(M_0)$ and such that the symbol of J vanishes in $|\lambda| \geqq c|\eta|$, TJ becomes a boundary Fourier integral operator, with the same ellipticity properties as T at $T^*\partial M_0$, if T is a tangential Fourier integral operator.

Let p denote the principal symbol of a second order differential operator P_0 on M_0, noncharacteristic with respect to ∂M_0. The characteristic variety Σ_b of P_0 is defined as the image under π in $\tilde{T}^*M_0$ of the set $p^{-1}(0) \subset T^*M_0$. We have, by (2.1), the decomposition $\Sigma_b = \Sigma_b^0 \cup \Sigma_b^\partial$, $\Sigma_b^0 = \Sigma_b \cap T^*\mathring{M}_0$, $\Sigma_b^\partial = \Sigma_b \cap T^*\partial M_0$. $\mathscr{E} = T^*\partial M_0 - \Sigma_b^\partial$ is the set of elliptic boundary points. On Σ_b^0 the Hamilton vectorfield H_p is defined,

$$H_p = x\frac{\partial p}{\partial \lambda}\partial_x - x\frac{\partial p}{\partial x}\partial_\lambda + \sum_j \left(\frac{\partial p}{\partial \eta_j}\partial_{y_j} - \frac{\partial p}{\partial y_j}\partial_{\eta_j}\right). \tag{2.2}$$

Here we have chosen local coordinates (x, y), $x \geqq 0$ in M_0, giving rise to canonical coordinates (x, y, ξ, η) in T^*M_0 and compressed canonical coordinates (x, y, λ, η), $\lambda = x\xi$, in $\tilde{T}^*M_0$. Consider

$$r = p - \tfrac{1}{2}\{p, x\}^2/\{\{p, x\}, x\}, \tag{2.3}$$

$\{,\}$ the Poissonbracket, and the (noninvariant) vectorfield

$$H_r = -x\frac{\partial r}{\partial x}\partial_\lambda + \sum_j \left(\frac{\partial r}{\partial \eta_j}\partial_{y_j} - \frac{\partial r}{\partial y_j}\partial_{\eta_j}\right). \tag{2.4}$$

Note that r does not depend on the variable ξ, conormal to $x=0$. The restrictions $r_0 = r|G$, G being the symplectic manifold $x = \{r, x\} = 0$, and $H^g = H_r|\Sigma_b^{(2)}$, $\Sigma_b^{(2)}$ being the glancing set, are invariantly defined. By definition, $\Sigma_b^{(2)} = \pi(G \cap \{r_0 = 0\})$. $\Sigma_b^1 = \Sigma_b^\partial - \Sigma_b^{(2)}$ is the set of hyperbolic boundary points [6]. H^g is the gliding field. It is tangent to $T^*\partial M_0 = \{x = \lambda = 0\}$, and it approximates the Hamilton vectorfield,

$$|H_p f - H_r f| \leqq C_f |r|^{1/2} \quad \text{on} \quad K \cap \Sigma^0, \tag{2.5}$$

$K \Subset \tilde{T}^*M_0$, for every $f \in C^\infty(\tilde{T}^*M_0)$. This is easily checked using (2.2), (2.3), and (2.4). The glancing set can be further decomposed into the set of nondegenerate diffractive points, $\Sigma_b^{2,-} = \Sigma_b^{(2)} \cap \pi(\{p, \{p, x\}\} > 0)$, and $\Sigma_b^g = \Sigma_b^{(2)} - \Sigma_b^{2,-}$, the set of all nondegenerate gliding points and all points of higher order bicharacteristic tangency. The characteristic variety Σ_b, the decomposition of Σ_b, the Hamilton vectorfield and the gliding field transform naturally under boundary and tangential canonical transformations.

We can now associate a characteristic variety Σ_T with (T) and define the singular spectrum $ss_T(u)$ for solutions u of (T). These are closed conic subsets of the cotangent bundle compressed along S, $\tilde{T}^*(M, S)$, which is the dual bundle to the compressed tangent bundle which has the vectorfields tangent to S as its sections. Let M_+ and M_- be the manifolds, with boundary S, forming the positive and the negative halfspace in M, respectively. Then we have canonical embeddings

$$\tilde{T}^*M_\pm \mapsto \tilde{T}^*(M, S) \tag{2.6}$$

agreeing on T^*S. The images, $\Sigma_\pm$, of the characteristic varieties of $P_\pm = P|M_\pm$ under the mappings (2.6) now give the characteristic variety $\Sigma_T = \Sigma_+ \cup \Sigma_-$. Σ_T inherits a natural decomposition, $\Sigma_T = \Sigma^0 \cup \Sigma^s$, $\Sigma^0 = \Sigma_+^0 \cup \Sigma_-^0 \subset T^*(M - S)$, $\Sigma^S = \Sigma_+^S \cup \Sigma_-^S \subset T^*S$. Furthermore, $\Sigma_\pm^S = \Sigma_\pm^1 \cup \Sigma_\mp^{(2)}$, $\Sigma_\mp^{(2)} = \Sigma_\mp^{2,-} \cup \Sigma_\pm^g$, and $\mathscr{E}_\pm = T^*S - \Sigma_\pm^S$, the set of elliptic points. At Σ^0 we can define the Hamilton vectorfield H_p as in (2.2). The symbol

$$r_\pm = p - \tfrac{1}{2}\{p, x\}^2/\{\{p, x\}, x\} \quad \text{in} \quad \pm x \geqq 0 \tag{2.7}$$

defines, via $H_{r_\pm}$, the gliding fields $H_\pm$, invariantly at $\Sigma_\mp^{(2)}$. As in (2.5) H_p and $H_\pm$ are related through

$$|H_p f - H_\pm f| \leqq C_f |r_\pm|^{1/2} \quad \text{on} \quad K \cap \Sigma_\pm^0, \tag{2.8}$$

$K \Subset \tilde{T}^*(M, S)$, for every $f \in C^\infty(\tilde{T}^*(M, S))$. Here and elsewhere in the paper $H_\pm$ also denotes an extension, into a neighbourhood of the glancing set, of the gliding field by $H_{r_\pm}$.

The radial direction in $\tilde{T}^*(M,S)$, referred to in (1.1) and (1.2), is in compressed canonical coordinates given by

$$\lambda\partial_\lambda + \sum_j \eta_j \partial_{\eta_j}.$$

Let u be a solution of (T). Then $ss_T(u)$ is defined as the union of the images under (2.6) of $ss_b(u_\pm)$, $u_\pm = u|M_\pm$. $u_\pm$ is normally regular, implying $ss_T(u) \subset T^*S \cup T^*(M-S)$. In local coordinates, (x,y), near $(0,y_0)$ we may write

$$P = g_\pm (D_x + V_\pm)^2 + R_\pm \quad \text{in} \quad \pm x > 0,$$

with $g_\pm \in C^\infty$, $g_+ \cdot g_- > 0$ at $x=0$, and tangential differential operators $V_\pm$ and $R_\pm$ of order 1 and 2, respectively. The normal derivatives D_+ and D_- are the vectorfield parts of $\operatorname{sign}(g_+) \cdot \sqrt{|g_+|} \cdot D_x$ and $-\operatorname{sign}(g_-) \cdot \sqrt{|g_-|} \cdot D_x$, respectively. Changing P outside a neighbourhood of $(0,y_0)$ if necessary, we can find tangential Fourier integral operators $T_\pm$ in $x \geqq 0$, elliptic near $(0,y_0)$ globally in η, restricting to the identity on $x=0$ (modulo a smoothing operator), such that $(D_x + V_\pm) T_\pm - T_\pm D_x$ is smoothing on normally regular distributions supported near $(0,y_0)$. (For a proof see the proof of Theorem 5.10 in [7].) When studying the singularities of u near $(0,y_0)$ we may thus assume, at the expense of introducing a term $fu_\pm$, $f \in C^\infty(S)$, into the last boundary condition of (T), that P can be written $P = g_\pm D_x^2 + R_\pm$ in $\pm x > 0$.

Remark. We use the orientation of N^*S only as a means to label the sides of S globally. Changing the orientation does not change (T). Suitably reformulated (T) also makes sense in the nonorientable case. Our results extend to this situation since they are essentially local.

As in [7], the tangential pseudodifferential operators $Q = q(x,y,D_y)$ we shall work with will have variable order. The symbols q are C^∞ functions in $x \geqq 0$ with values in the symbol classes $S(m,g)$ of Hörmander [2], where g is the "variable order metric"

$$g_{y,\eta}(y',\eta') = |y'|^2 (\log\langle\eta\rangle)^2 + |\eta'|^2 (\log\langle\eta\rangle/\langle\eta\rangle)^2,$$

$\langle\eta\rangle^2 = e + |\eta|^2$, and $m(y,\eta)$ is a positive weight function satisfying the continuity and temperateness conditions of [2] (for the nonsymmetric calculus). Let $\Psi(m)$ denote the space of tangential pseudodifferential operators corresponding to the symbolspace $C^\infty(\overline{\mathbb{R}_x^+}; S(m,g))$ and equip it with its natural Frechét space topology. For any $\mu \in S_{1,0}^0$, $\langle\eta\rangle^\mu$ is a weight function and $\langle\eta\rangle^\mu \in S(\langle\eta\rangle^\mu, g)$. Note also the continuous inclusions

$$S_{1,0}^\mu \subset S(\langle\eta\rangle^\mu, g) \subset S_{1,0}^\nu, \qquad \mu < \nu \quad \text{real}.$$

3. Estimates on $ss_T(u)$

Away from T^*S the propagation of singularities of solutions u of (T) is wellknown, $ss_T(u) \cap T^*(M-S)$ is contained in Σ^0 and invariant under the Hamilton flow [1]. At T^*S but outside $\Sigma_+^g \cup \Sigma_-^g$ the analysis of the singularities of u can be reduced to that of solutions of the Dirichletproblem (for either side of S) by using the Neumann operators. We recall this reduction. Let $\sigma_0 \in T^*S$. Neumann operators $N_\pm$ at σ_0 are operators on S which relate the boundary data $v_\pm = u|S_\pm$,

$w_\pm = D_\pm u|S_\pm$, of solutions to

(3.1) $$Pu \equiv 0 \mod \bar{C}^\infty(M-S)$$

microlocally near σ_0 by

(3.2) $$\sigma_0 \notin ss(w_\pm - N_\pm v_\pm).$$

It follows from the theory of elliptic, hyperbolic, and diffractive boundary problems that there exist pseudodifferential operators $N_\pm$ satisfying (3.2) on all solutions u of (3.1) if $\sigma_0 \in \mathscr{E}_\pm$ and pseudodifferential operators $N^b_\pm$ (respectively $N^f_\pm$) satisfying (3.2) on all solutions u of (3.1) having no singularities on the forward (respectively backward) half-bicharacteristics through σ_0 if $\sigma_0 \in \Sigma^1_\pm$ or $\sigma_0 \in \Sigma^{2,-}_\pm$. These operators are classical first order pseudodifferential operators at $\mathscr{E}_\pm$ and at $\Sigma^1_\pm$ with principal symbols near σ_0

$$\sigma(N_\pm) = i\Omega_\pm \sqrt{\Omega_\pm r_\pm}, \qquad \sigma_0 \in \mathscr{E}_\pm,$$

$$\sigma(N^b_\pm) = -\sigma(N^f_\pm) = -\sqrt{-\Omega_\pm r_\pm}, \qquad \sigma_0 \in \Sigma^1_\pm.$$

[$\Omega_\pm$ and $r_\pm$ are defined in the introduction and in (2.7), respectively.] At nondegenerate diffractive points $N^b_\pm$ and $N^f_\pm$ are nonclassical pseudodifferential operators [4].

Let u be a solution of (T) which has no singularities on the forward (respectively backward) half-bicharacteristics emanating from σ_0 (if they exist). Then $v \equiv u|S_+ \equiv u|S_-$ [modulo $C^\infty(S)$] satisfies

(3.3) $$\sigma_0 \notin ss(N_+ v + N_- v),$$

where $N_\pm = N^b_\pm$ (respectively $N_\pm = N^f_\pm$) in case $\sigma_0 \in \Sigma^1_\pm \cup \Sigma^{2,-}_\pm$. Then $\sigma_0 \notin ss_T(u)$ follows from the known regularity results for the Dirichletproblem once the hypoellipticity of the operator $N_+ + N_-$ is shown. At $(\mathscr{E}_+ \cup \Sigma^1_+) \cap (\mathscr{E}_- \cup \Sigma^1_-)$ this is a classical elliptic pseudodifferential operator. (Use $\Omega_+ = \Omega_-$ at $\mathscr{E}_+ \cap \mathscr{E}_-$.) By the symbolic calculus for Airy operators [4] $N_+ + N_-$ is hypoelliptic if σ_0 is diffractive for at most one side of S. $N_+ + N_-$ is also hypoelliptic at $\Sigma^{2,-}_+ \cap \Sigma^{2,-}_-$. This was shown by Taylor [11] via estimates (nonsymbolically). So we know that $ss_T(u) \subset \Sigma_T$ and that the assertions in Theorem 1.3 on propagation of singularities hold at least locally outside $\Sigma^g_+ \cup \Sigma^g_-$ because rays will be broken bicharacteristics there (see Sect. 4).

We now consider the case, where $\sigma_0 \in T^*S$ is glancing for precisely one side of S, say $\sigma_0 \in \Sigma^{(2)}_+ - \Sigma^{(2)}_-$. Let u solve (T) and assume that u has no singularities on the forward half-bicharacteristics (provided there is one) emanating from σ_0 to the negative side of S, $x<0$. Then u satisfies, microlocally near σ_0, the boundary problem

(3.4) $$Pu \equiv 0 \quad \text{in} \quad x \geqq 0,$$
$$(D_+ + L)u \in C^\infty(S) \quad \text{at} \quad x=0,$$

with $L = N_-$ if $\sigma_0 \in \mathscr{E}_-$ and $L = N^b_-$ if $\sigma_0 \in \Sigma^1_-$. The principal symbol l of L satisfies $\operatorname{Re} l \leqq 0$ in a conic neighbourhood of σ_0. Recalling the definition of D_+ we see that the boundary problem (3.4) is, after conjugating with a tangential Fourier integral operator, of the type studied in Theorem 2.3 of [7]. When $\Omega_+ = \Omega_- = 1$ (respectively $\Omega_+ = \Omega_- = -1$) condition $(2.2)_+$ [respectively $(2.2)_-$] of [7] holds true. From this we get the following estimate on $ss_T(u)$.

(3.5) **Theorem.** *Let u be a solution of (T). Let $\sigma_0 \in \Sigma_+^{(2)} - \Sigma_-^{(2)}$. Let $\varepsilon > 0$. Suppose that the forward half-bicharacteristic emanating from σ_0 into Σ_-^0, provided it exists, is disjoint from $ss_T(u)$ near σ_0. If for some t, $0 < t < t(\sigma_0, \varepsilon)$, the set $ss_T(u)$ does not intersect*

$$\{\sigma \in \Sigma_+;\, d(\sigma, \exp(tH_+)(\sigma_0)) \leqq \varepsilon t\}$$

then $\sigma_0 \notin ss_T(u)$. The same assertion holds with forward replaced by backward and H_+ replaced by $-H_+$.

Here d is a metric on $\tilde{T}^*(M, S)$ which is induced by some Riemannian metric on $\tilde{T}^*(M, S)$. We shall keep d fixed throughout the paper.

Remark. It would be interesting to know whether Theorem 3.5 also holds at $\sigma_0 \in \Sigma_+^{(2)} \cap \Sigma_-^{2,-}$. The reduction to the boundary problem (3.4) is valid in this case, too. However, L becomes an Airy operator then.

Our main result in this section is the following estimate on $ss_T(u)$ at double glancing points.

(3.6) **Theorem.** *Let u be a solution of (T). Let $\sigma_0 \in \Sigma_+^{(2)} \cap \Sigma_-^{(2)}$. Let $\varepsilon > 0$. If, for some t, $0 < t < t(\sigma_0, \varepsilon)$, $ss_T(u)$ does not intersect the set*

$$\left\{\sigma \in \Sigma_T;\, \inf_{0 \leqq \lambda \leqq 1} d(\sigma, \exp(t(\lambda H_+ + (1-\lambda)H_-)(\sigma_0))) \leqq \varepsilon t\right\},$$

then $\sigma_0 \notin ss_T(u)$.

Proof. We fix coordinates (x, y) near the base point of $\sigma_0 = (0, y_0, 0, \eta_0)$ such that $x > 0$ precisely on the positive side of S. In a neighbourhood of $(0, y_0)$, U, (T) is equivalent, after changing to a system in $x \geqq 0$ and conjugating with tangential Fourier integral operators, to the boundary problem for normally regular distributions $u_\pm$

$$P_+ u_+ \equiv P_- u_- \equiv 0 \mod C^\infty(U \cap \{x \geqq 0\}), \tag{3.7}$$

$$(u_+ - u_-)|(x = 0) \equiv 0 \mod C^\infty(U \cap \{x = 0\}), \tag{3.8}$$

$$(D_+ u_+ + D_- u_- - f u_\pm)|(x = 0) \equiv 0 \mod C^\infty(U \cap \{x = 0\}), \tag{3.9}$$

with f a smooth function on $x = 0$. Here

$$P_\pm = g_\pm D_x^2 + R_\pm, \tag{3.10}$$

$$D_\pm = \operatorname{sign}(g_\pm) \cdot \sqrt{|g_\pm|} \cdot D_x, \tag{3.11}$$

where $g_\pm \in C^\infty$ with $g_+ \cdot g_- > 0$, and where $R_\pm$ is a classical tangential pseudodifferential operator of order 2 with real principal symbol $r_\pm$. The restriction of $r_\pm$ to $x = 0$ does not change under the reduction of (T) to (3.7)–(3.9).

For every smooth function v, compactly supported in U, we have, using partial integrations (cf. Lemma 2.2 in [6])

$$\langle v, Pu \rangle = \langle P^* v, u \rangle - i\langle \sqrt{|g|}\, v, Du \rangle_\partial - i\langle \sqrt{|g|}\, Dv, u \rangle_\partial - \langle g_x' v, u \rangle_\partial, \tag{3.12}$$

where $\langle\,,\rangle$ and $\langle\,,\rangle_\partial$ are continuous extensions of the (sesquilinear) L^2 – inner products in $x\geqq 0$ and $x=0$, respectively. To ease the notation we have omitted in (3.12) the subscript $\pm$ at u, P, D, etc., and we will continue to do so in formulas which are valid for both subscripts. Taking $v=Qu$ in (3.12) with a smoothing tangential pseudodifferential operator $Q_\pm$ and adding $\langle -QPu,u\rangle$ we get the following identities

$$(3.13)_\pm \qquad \langle (P^*Q-QP)u,u\rangle=\langle Qu,Pu\rangle-\langle QPu,u\rangle$$
$$+i\langle\sqrt{|g|}Qu,Du\rangle_\partial+i\langle\sqrt{|g|}QDu,u\rangle_\partial+\langle\tilde{M}u,u\rangle_\partial,$$

where

$$(3.14) \qquad \tilde{M}=g'_x\cdot Q+i\sqrt{|g|}[D,Q].$$

Assuming

$$(3.15) \qquad \sqrt{|g_+|}Q_+=\sqrt{|g_-|}Q_- \quad \text{at} \quad x=0$$

and using the boundary conditions (3.8) and (3.9) we get

$$(3.16) \qquad \langle\sqrt{|g_+|}Q_+u_+,D_+u_+\rangle_\partial+\langle\sqrt{|g_+|}Q_+D_+u_+,u_+\rangle_\partial+\langle\sqrt{|g_-|}Q_-u_-,D_-u_-\rangle_\partial$$
$$+\langle\sqrt{|g_-|}Q_-D_-u_-,u_-\rangle_\partial\equiv\langle\sqrt{|g_+|}Q_+u_+,fu_+\rangle_\partial+\langle\sqrt{|g_-|}Q_-fu_-,u_-\rangle_\partial$$

modulo boundary brackets containing in one entry one of the smooth functions $(u_+-u_-)|(x=0)$ or $(D_+u_++D_-u_--fu_\pm)|(x=0)$. Let $\mathscr{B}\subset\Psi(\langle\eta\rangle^{-N_0})$, where $N_0\in\mathbb{N}$ is large enough for $(3.13)_\pm$ to make sense if $Q_\pm\in\mathscr{B}$, be bounded as a subset of $\Psi(\langle\eta\rangle^\mu)$ for some $\mu\in S^0_{1,0}$. In addition, suppose that the symbols of all operators in $\mathscr{B}$ vanish outside a compact set contained in U. Then, adding the Eqs. $(3.13)_+$ and $(3.13)_-$ and using (3.7) and (3.16), we obtain the estimate

$$(3.17) \qquad |\langle(P^*_+Q_+-Q_+P_+)u_+,u_+\rangle+\langle(P^*_-Q_--Q_-P_-)u_-,u_-\rangle|$$
$$\leqq C+|\langle M_+u_+,u_+\rangle_\partial|+|\langle M_-u_-,u_-\rangle_\partial|,$$

uniformly for all $Q_\pm\in\mathscr{B}$ satisfying (3.15).

Here

$$(3.18) \qquad \begin{aligned} M_+&=\tilde{M}_++i\bar{f}\sqrt{|g_+|}Q_+,\\ M_-&=\tilde{M}_-+i\sqrt{|g_-|}Q_-f.\end{aligned}$$

If (3.7)–(3.9) hold only microlocally near σ_0 then (3.17) is also true, provided $-\mu$ is large outside a conic neighbourhood of (y_0,η_0).

We shall find such a family of operators $\mathscr{B}$ so that for $Q\in\mathscr{B}$ the dominant terms in (3.17) are those involving the modified commutators

$$(3.19) \qquad R^*Q-QR.$$

Actually, the negative imaginary part of (3.19) will have a squareroot $A_\pm$, elliptic at (y_0,η_0), modulo cutoffs supported in regions, where $u_\pm$ has no singular spectrum. This will lead to a bound on $\|A_\pm u_\pm\|$, as desired.

Suppose that the gliding fields H_+ and H_- are linearly independent at σ_0. By the principal type assumption (1.2) we can find symbols in the (y,η)-variables, $\varphi_\pm$,

$\psi_j \in S^0_{1,0}$, $j=1,\dots,2n-5$, and $\tilde{\psi} \in S^1_{1,0}$, with

$$(3.20)\qquad \begin{aligned} &H_\pm \varphi_\pm(y_0,\eta_0)=1, \quad H_\pm \varphi_\mp(y_0,\eta_0)=0,\\ &H_\pm \psi_j(y_0,\eta_0)=0 \quad \text{for all } j. \end{aligned}$$

$\varphi_\pm$ and all ψ_j vanish at (y_0,η_0) and are, in a conic neighbourhood Γ of (y_0,η_0), homogeneous of degree zero, $\tilde{\psi}$ is 1-homogeneous in Γ, $\tilde{\psi}(y_0,\eta_0)=1$, and the differentials

$$d\varphi_+, d\varphi_-, d\psi_1, \dots, d\psi_{2n-5}, d\tilde{\psi},$$

are linearly independent at (y_0,η_0). The assumption on $ss_T(u)$ implies that, near σ_0, the boundary singular spectra of u_+ and u_- do not contain points with

$$(3.21)\qquad \begin{aligned} &|\varphi_+ + \varphi_- - t| < \varepsilon t, \quad \varphi_\pm > -\varepsilon t,\\ &\sum_j |\psi_j| < \varepsilon t, \quad 0 \leqq x < \varepsilon t, \end{aligned}$$

for some small $t>0$. Here we have absorbed into ε a constant caused by the change of the metric.

Fix C^∞ functions Φ_+ and Φ_-, compactly supported in U, $\Phi_\pm(0,y_0)>0$, with

$$\sqrt{|g_+|}\,\Phi_+ = \sqrt{|g_-|}\,\Phi_- \quad \text{at} \quad x=0.$$

Next, choose a nonnegative C^∞ function b on the real line, $b(s)=1$ for $s<\frac{1}{2}$, $b(s)=0$ for $s>1$, such that $b^{1/2}$ is also C^∞. Let $N \geqq 1$. With $p \in \mathbb{N}$ still to be determined, depending on ε, and with $t>0$ to be chosen later on, small enough, we consider the family $\mathscr{B}$ of operators $Q_\pm$ with symbols

$$q_\pm(x,y,\eta) = \Phi_\pm(x,y)\cdot b(xp/t)\cdot m(y,\eta),$$

where $m=m^\lambda$, $\lambda \geqq 1$, is the family of weight functions given by

$$(3.22)\quad (\log m)/(\log\langle\eta\rangle) = -N_0 - N(\omega + \chi((\varphi_+ + \varphi_- - t)p/t) + \chi(\langle\eta\rangle/\lambda) - 1).$$

Here $\chi \in C^\infty(\mathbb{R};[0,2])$, $\chi(s)=0$ for $s \leqq \frac{1}{2}$, $\chi(s)=2$ for $s \geqq 1$, and, in a conic neighbourhood of (y_0,η_0),

$$(3.23)\qquad \omega(y,\eta) = \tfrac{1}{4}(4t)^{-2p}((\varphi_+ - 4t)^{2p} + (\varphi_- - 4t)^{2p} + \Sigma|p\psi_j|^{2p}).$$

We can extend ω, provided $t>0$ is small enough, as a zeroth order symbol of type $(1,0)$ such that formula (3.23) holds, where $\omega<2$. More specifically, we extend ω by applying to the right-hand side of (3.23) a realvalued C^∞ function f with $f(s)=s$ for $s<2$, $f(s) \geqq 2$ for $s \geqq 2$, $f(s)=3$ for $s \geqq 3$, and setting $\omega=3$ everywhere else. Note that the sets $\{\omega<2\}$ form a conic neighbourhood basis of (y_0,η_0) as t shrinks to zero.

Observe that (3.15) holds and that $Q_\pm \in \Psi(m)$. The operator $R^*Q - QR$ belongs to $\Psi(m\langle\eta\rangle\log\langle\eta\rangle)$. Moreover, its symbol $b(xp/t)\cdot\Delta_\pm$ satisfies, modulo symbols in $C^\infty(\overline{\mathbb{R}}^+_x; S(m\langle\eta\rangle, g))$,

$$(3.24)\qquad \begin{aligned} i\Delta &\equiv \{r, \Phi m\} \equiv \Phi m\{r, \log m\}\\ &\equiv -N\Phi m\cdot(\log\langle\eta\rangle)\{r, \omega + \chi(\langle\eta\rangle\lambda^{-1})\} \end{aligned}$$

in the region, where $\omega<2$ and $\varphi_+ + \varphi_- < t$, t small. If $\omega<2$ then, using (3.20) (3.23),

$$(3.25)\qquad \{r_\pm, \omega\} = \frac{p}{2}(4t)^{-2p}(\{r_\pm, \varphi_\pm\}(\varphi_\pm - 4t)^{2p-1} + O(t^{2p})),$$

uniformly as $t \to 0$, and if, in addition, $\varphi_+ + \varphi_- < 2t$, $p \geqq 3$, then

$$(\varphi - 4t)^{2p-1} \leqq -C_p t^{2p-1}. \tag{3.26}$$

Note that $\{r, \chi(\langle\eta\rangle\lambda^{-1})\}$ stays uniformly bounded in $S^1_{1,0}$ for $\lambda \geqq 1$. By (3.20) $\{r, \varphi\} \geqq C\langle\eta\rangle$, $C>0$, in $\omega<2$, $0 \leqq x < t/p$, and η large. This together with (3.24)–(3.26) implies

$$C^{-1} \leqq -\mathrm{Im}(t\Delta/m\langle\eta\rangle \log\langle\eta\rangle) \leqq C \quad \text{in} \quad \omega<2, \quad 0 \leqq x < t/p, \quad \varphi_+ + \varphi_- < t, \tag{3.27}$$

with a positive constant C independent of λ and t.

Using (3.27), $b^{1/2} \in C^\infty$, we can carry out the construction of an approximate squareroot $A_\pm \in \Psi((m\langle\eta\rangle \log\langle\eta\rangle)^{1/2})$,

$$\mathrm{Re}(iR^*Q - iQR) = A^*A + W, \tag{3.28}$$

with $W_\pm \in \Psi(\langle\eta\rangle^\mu)$. Here, and throughout the rest of the proof, all bounds, in particular those on symbolnorms, hold uniformly for $\lambda \geqq 1$. $A_\pm$ is elliptic in the region, where $\omega<2$, $0 \leqq x < t/2p$, $\varphi_+ + \varphi_- < t$. μ is a zeroth order symbol of type (1,0), independent of λ, with $\mu \leqq 1 - N_0$ holding outside the cutoff region

$$\omega < \tfrac{3}{2}, \quad 0 \leqq x < t/p, \quad \text{and} \quad |\varphi_+ + \varphi_- - t| < t/p. \tag{3.29}$$

We can now fix p, of size $1/\varepsilon$, so that the region (3.29) is contained in (3.21) for small $t>0$. Although t may still have to be decreased later on, we can assume that the region (3.29) does not meet the singular spectra of u_+ and u_-. Hence, in particular, there is a constant C such that

$$|\langle Wu, u\rangle| < C. \tag{3.30}$$

We shall encounter more general remainders W below. For bounding these as in (3.30) it will suffice to assume that $ss_b(u_\pm)$ also does not intersect the region

$$\omega<2, \quad \varphi_+ + \varphi_- < 2t, \quad t/3p < x < 2t/p. \tag{3.31}$$

A priori this assumption need not hold. However, we can insure it, without affecting (3.17) and the regularity of u at σ_0, by replacing u by $u - Hu$ with operators $H_\pm \in L^0_b$ which satisfy

$$ss_b([P, H]u) \cap \{\omega<2,\ \varphi_+ + \varphi_- < 2t,\ 0 \leqq x < 2t/p\} = \emptyset,$$

and which have total symbols vanishing, where $0 \leqq x < t/4p$ and equaling 1 in the region (3.31). Such operators exist. Following the standard method of "exact commutators" their symbols can be constructed by integrating along those bicharacteristics of P which are contained in $\{\omega<2,\ \varphi_+ + \varphi_- < 2t,\ 0 \leqq x < 2t/p\}$. Note that these bicharacteristics can meet at most one of the regions $0 \leqq x < t/4p$, (3.31), if t is small enough. This follows, using (2.8), from (3.20), $H_\pm x = 0$ and $r(y_0, \eta_0) = 0$.

Given $\delta > 0$ we shall prove, for sufficiently small t,

$$|\langle((P^* - R^*)Q - Q(P-R))u, u\rangle| \leqq \delta \|Au\|^2 + C_\delta, \tag{3.32}$$

$$|\langle Mu, u\rangle_\partial| \leqq \delta \|Au\|^2 + C_\delta, \tag{3.33}$$

with a constant C_δ, depending also on t. Then we have, using (3.17), (3.28), (3.30), (3.32), and (3.33), a bound

$$\|A_+u_+\|^2+\|A_-u_-\|^2<C. \tag{3.34}$$

The construction of $A^\lambda_\pm$ can be done so that

$$A^\lambda \to A^\infty \quad \text{as} \quad \lambda\to\infty, \tag{3.35}$$

$A^\infty \in \Psi((m^\infty\langle\eta\rangle\log\langle\eta\rangle)^{1/2})$, with convergence in $\Psi(\sqrt{m^\infty}\langle\eta\rangle)$, say. Here m^∞ is the weight function defined as m^λ above, however, with the term in (3.22) involving λ omitted. Then (3.34) and (3.35) give

$$\|A^\infty_+u_+\|^2+\|A^\infty_-u_-\|^2<+\infty. \tag{3.36}$$

$A^\infty_\pm$ is elliptic in $\Psi((m^\infty\langle\eta\rangle\log\langle\eta\rangle)^{1/2})$ and of high order at σ_0 because $\omega(y_0,\eta_0)=\frac{1}{2}<1$. Therefore, $N\geqq 1$ being arbitrary, the assertion $\sigma_0\notin ss_T(u)$ follows from (3.36).

To prove (3.32) we note that [recall (3.10)]

$$(P^*-R^*)Q-Q(P-R)=[D_x,[D_x,gQ]]+2[D_x,gQ]D_x+[g,Q]D_x^2, \tag{3.37}$$

with $[D_x,[D_x,gQ]]$, $[D_x,gQ]\in\Psi(m)$, $[g,Q]\in\Psi(m\langle\eta\rangle^{-1}\log\langle\eta\rangle)$. Let Λ be the tangential pseudodifferential operator with symbol $\langle\eta\rangle^{-1}$. We can solve, modulo operators W satisfying (3.30),

$$[D_x,[D_x,gQ]]\equiv A^*B\Lambda A, \tag{3.38}$$

$$[D_x,gQ]\equiv A^*B\Lambda A, \tag{3.39}$$

$$[g,Q]\equiv A^*B\Lambda^2A. \tag{3.40}$$

Here and in the following B, and B', denote operators in $\Psi(1)$ which are, in general, different in different formulas. Recall that operators in $\Psi(1)$ are bounded on L^2. Therefore, (3.38) implies

$$|\langle[D_x,[D_x,gQ]]u,u\rangle|\leqq\delta\|Au\|^2+C_\delta, \qquad \delta>0. \tag{3.41}$$

Using (3.39) we get

$$|\langle 2[D_x,gQ]D_xu,u\rangle|\leqq C\|\Lambda AD_xu\|\cdot\|Au\|+C. \tag{3.42}$$

Modulo an operator which stays bounded on $u_\pm$ we have

$$[\Lambda A,D_x]\equiv B\Lambda A.$$

Thus, showing

$$\|\Lambda AD_xu\|\leqq\delta\|Au\|+C_\delta, \qquad \delta>0, \tag{3.43}$$

is equivalent to showing

$$\|D_x\Lambda Au\|\leqq\delta\|Au\|+C_\delta, \qquad \delta>0. \tag{3.44}$$

A classical interpolation inequality on the halfspace $x\geqq 0$ gives

$$\|D_x\Lambda Au\|\leqq C\|Au\|\cdot\|D_x^2\Lambda^2Au\|.$$

We can write

$$[D_x^2, \Lambda^2 A] \equiv B\Lambda^2 A + B'\Lambda^2 A D_x,$$

modulo an operator which stays bounded on $u_\pm$. Hence,

$$\|[D_x^2, \Lambda^2 A]u\| \leqq \delta\|Au\| + \delta\|\Lambda A D_x u\| + C_\delta, \qquad \delta > 0,$$

thereby reducing the proof of (3.43) and (3.44), to that of

$$\|\Lambda^2 A D_x^2 u\| \leqq \delta\|Au\| + C_\delta, \qquad \delta > 0. \tag{3.45}$$

Replacing D_x^2 by $\frac{1}{g}(P-R)$ (3.45) follows from (3.7) and

$$\left\|\Lambda^2 A \frac{1}{g} R u\right\| \leqq \delta\|Au\| + C_\delta.$$

This inequality holds for $0 < t < t_\delta$ (C_δ may also depend on t) because the principal symbol of R is small, where $\omega < 2$, $0 \leqq x < t/p$, when t is small. Using (3.40) we obtain

$$|\langle [g, Q] D_x^2 u, u\rangle| \leqq C\|\Lambda^2 A D_x^2 u\| \cdot \|Au\|. \tag{3.46}$$

The estimates (3.41)–(3.46) now give (3.32).

Finally, we show (3.33). By (3.14) and (3.18), we can write

$$M \equiv A^* \Lambda^{1/2} B \Lambda^{1/2} A + A^* \Lambda^{1/2} B' \Lambda^{3/2} A D_x,$$

modulo an operator $W + W'D_x$ bounded on $u_\pm$, i.e.

$$|\langle (W + W'D_x)u, u\rangle_\partial| \leqq C.$$

Hence,

$$|\langle Mu, u\rangle_\partial| \leqq C\|\Lambda^{1/2} Au\|_\partial^2 + C\|\Lambda^{3/2} A D_x u\|_\partial \|\Lambda^{1/2} Au\|_\partial + C.$$

The classical trace inequality

$$\|\Lambda^{1/2} v\|_\partial^2 \leqq C\|v\| \cdot \|\Lambda D_x v\|$$

then implies

$$|\langle Mu, u\rangle_\partial| \leqq C\|Au\| \cdot \|\Lambda D_x A u\| + C\|\Lambda A D_x u\| \cdot \|\Lambda D_x \Lambda A D_x u\| + C. \tag{3.47}$$

(3.43)–(3.45), and analogous estimates now give (3.33).

This completes the proof of the theorem in the case, where H_+ and H_- are linearly independent at σ_0. In case the gliding fields are linearly dependent at σ_0 we can find by (1.2) symbols in the (y, η)-variables, $\varphi_\pm, \psi_j \in S^0_{1,0}, j = 1, \ldots, 2n-4$, and $\tilde{\psi} \in S^1_{1,0}$, 0- and 1-homogeneous near (y_0, η_0), respectively, $\varphi_\pm(y_0, \eta_0) = \psi_j(y_0, \eta_0) = 0$, with

$$H_\pm \varphi_\pm(y_0, \eta_0) = 1, \qquad H_\pm \psi_j(y_0, \eta_0) = 0 \quad \text{for all } j,$$

$\varphi_+ = a \cdot \varphi_-$ with a positive constant a, and such that the differentials $d\varphi_+$, $d\psi_1, \ldots, d\psi_{2n-4}$, $d\tilde{\psi}$ are linearly independent at (y_0, η_0). Now, the same proof as above goes through with only obvious modifications. We may leave these to the reader.

4. Rays

A curve $\gamma: I \to N$, $I \subset \mathbb{R}$ an interval, N a C^∞ manifold, is called locally Lipschitz if it is locally Lipschitz with respect to every chart of N. For such a curve γ it makes sense to define its derivative $\gamma'(t)$, $t \in I$, as the subset of $T_{\gamma(t)}N$ consisting of all limit points of the difference quotients

$$C^\infty(N) \ni \varphi \to \frac{\varphi(\gamma(s)) - \varphi(\gamma(t))}{s-t}$$

as $s \to t$, $s \neq t$. The restrictions $s > t$ and $s < t$ define the forward, $\gamma'_+(t)$, and the backward derivative, $\gamma'_-(t)$, respectively. γ is differentiable at t precisely when $\gamma'(t)$ consists of only one tangent vector in which case $\gamma'(t)$ will also be regarded as an element of $T_{\gamma(t)}N$. It is clearly meaningful to call γ tangent to a submanifold $N' \subset N$ at $\gamma(t) \in N'$ iff $\gamma'(t) \subset T_{\gamma(t)}N'$.

(4.1) *Definition.* A ray (for P) is a locally Lipschitz curve $\gamma: I \to \tilde{T}^*(M, S)$ with $\gamma(t) \in \Sigma_T$ for all $t \in I$, $I \subset \mathbb{R}$ an interval, satisfying the following conditions

(4.2) If $\gamma(t) \in \Sigma^0$, $t \in I$, then $\gamma'(t) = H_p(\gamma(t))$.

(4.3) If $\gamma(t) \notin \Sigma^g_+ \cup \Sigma^g_-$, $t \in I$, then $\gamma(s) \in \Sigma^0$ for $|s-t| > 0$ small.

(4.4) If $\gamma(t) \in \Sigma^g_\pm - \Sigma^{(2)}_\mp$, $t \in I$, then

$$\gamma(s) \in \Sigma^0_\mp, \qquad s - t > 0 \quad \text{small, or else} \quad \gamma'_+(t) = H_\pm(\gamma(t)),$$

and

$$\gamma(s) \in \Sigma^0_\mp, \qquad t - s > 0 \quad \text{small, or else} \quad \gamma'_-(t) = H_\pm(\gamma(t)).$$

(4.5) If $\gamma(t) \in \Sigma^g_+ \cap \Sigma^{(2)}_-$ or $\gamma(t) \in \Sigma^g_- \cap \Sigma^{(2)}_+$, $t \in I$, then

$$\gamma'(t) \subset \{(\lambda H_+ + (1-\lambda) H_-)(\gamma(t));\ 0 \leqq \lambda \leqq 1\}.$$

(4.6) *Remark.* Let $\gamma: I \to \Sigma_T$ be a ray. For any $t \in I$ the derivative $\gamma'(t)$ can only consist of convex combinations of the gliding fields $H_\pm$ and of (limits of) the Hamilton field H_p at $\gamma(t)$. In fact, $\gamma'(t) = \gamma'_+(t) \cup \gamma'_-(t)$, and if $\gamma(s) \in \Sigma^0_\pm$ for $s - t > 0$ small, $s, t \in I$, then

$$\gamma'_+(t) = \lim_{s \searrow t} H_p(\gamma(s)) \in T_{\gamma(t)}(\tilde{T}^*(M, S)).$$

By (2.8) the limit equals $H_\pm(\gamma(t))$ if $\gamma(t) \in \Sigma^{(2)}_\pm$. The corresponding assertion for γ'_- also holds. This allows one to estimate the variation, along γ, of smooth functions f defined in a neighbourhood of $\gamma(I)$ in terms of $H_p f$ and $H_\pm f$ because

$$f(\gamma(t)) - f(\gamma(s)) = \alpha(t-s), \qquad t > s, \qquad s, t \in I,$$

for some $\alpha = \alpha(t, s)$ contained in the closed convex hull of

$$\bigcup_{\tau \in (s,t)} \langle df(\gamma(\tau)), \gamma'(\tau) \rangle.$$

(Compare Lemma 5.10 below.)

We now discuss the geometry of rays. Locally outside $\Sigma^g_+ \cup \Sigma^g_-$ a ray consists of pieces of bicharacteristics reflected, refracted and diffracted according to the usual laws of geometric optics. This follows from the known behaviour of bicharacteristics away from boundary points of higher order bicharacteristic tangency [3]. Note that a compressed conormal variable $\lambda = x\xi$ is Lipschitz continuous on broken bicharacteristics whereas the usual conormal variable ξ has jumps at hyperbolic boundary points.

A ray passing through a point which is gliding and of finite order bicharacteristic tangency with respect to one side of S and nonglancing with respect to the other side consists of a gliding ray segment (possibly degenerating to a point) in the boundary exiting or entering as a bicharacteristic on the nonglancing side or running into boundary points of higher order degeneracy. This essentially follows from the fact (see Melrose and Sjöstrand [6, Sect. 3]) that the gliding field is transversal to each Σ_b^k, $k=3,4,\ldots$, the set of glancing points of bicharacteristic tangency precisely $k-1$. At gliding points of infinite order bicharacteristic tangency a ray may contain an infinitely reflected ray instead of a gliding ray.

To justify our definition of rays near double gliding points we show that condition (4.5) implies that rays behave very much in the way one expects uniform limits of bicharacteristics and gliding rays to behave. We consider the special cases of symplectic and of involutive intersection of the two glancing hypersurfaces.

(4.7) **Proposition.** *Let $\gamma: I \to \Sigma_T$ be a ray. Suppose $\gamma(t_0) \in \Sigma_+^{(2)} \cap \Sigma_-^{(2)}$ and $\{r_+, r_-\}(\gamma(t_0)) \neq 0$ for some $t_0 \in I$. Then $\gamma(t) \notin \Sigma_+^{(2)} \cap \Sigma_-^{(2)}$ for $|t-t_0|>0$ small and*

$$\gamma'_+(t_0) = H_+(\gamma(t_0)) \quad \text{or} \quad \gamma'_+(t_0) = H_-(\gamma(t_0)),$$

$$\gamma'_-(t_0) = H_+(\gamma(t_0)) \quad \text{or} \quad \gamma'_-(t_0) = H_-(\gamma(t_0)).$$

Proof. We may assume $\{r_+, r_-\}(\sigma_0) > 0$, $\sigma_0 = \gamma(t_0)$. So, in a neighbourhood U of σ_0,

$$H_+ r_- > 0, \quad H_- r_+ < 0, \quad \text{and} \quad H_\pm r_\pm = 0. \tag{4.8}$$

Using (2.8) and (4.8), we can find for every $\delta > 0$ a neighbourhood $U_\delta \subset U$ of σ_0 such that

$$\begin{aligned} \delta H_p r_- &> |H_p r_+| \quad \text{in} \quad U_\delta \cap \Sigma_+^0, \\ -\delta H_p r_+ &> |H_p r_-| \quad \text{in} \quad U_\delta \cap \Sigma_-^0. \end{aligned} \tag{4.9}$$

In view of Remark 4.6 the estimates (4.8) and (4.9) imply that the function $(r_- - \delta r_+)(\gamma(t))$, $\delta > 0$, is strictly increasing in an open interval containing t_0. In particular, the first assertion of the proposition follows. Furthermore, (4.8) and (4.9) imply the following alternative. Either

(4.10) for all $\delta > 0$: $|r_-(\gamma(t))| < \delta(-r_+(\gamma(t)))$, $t - t_0 > 0$ small,

or else

(4.11) there is a sequence $t_k \searrow t_0$ with $r_-(\gamma(t_k)) > 0$ and $x(\gamma(t_k)) \geqq 0$.

In fact, $|r_-| + \delta r_+ = 0$ can hold at $\gamma(t) \in U_\delta$, $t > t_0$, only if $r_-(\gamma(t)) > 0$ and since $H_p(\delta r_+ + r_-) < 0$ in $U_\delta \cap \Sigma_-^0$ we then also have $x(\gamma(t)) \geqq 0$. (4.10) clearly implies $\lambda = 1$ when $(\lambda H_- + (1-\lambda) H_+)(\sigma_0) \in \gamma'_+(t_0)$. We now show that (4.11) implies $\gamma'_+(t_0)$

$=H_+(\sigma_0)$. Starting at a point $\gamma(t)\in U_1$, $t>t_0$, with $x(\gamma(t))\geqq 0$, the function $r_-\circ\gamma$ will increase until γ either leaves U_1 or enters $x<0$, because $H_+r_->0$, $H_-r_-=0$, in U_1 and $H_pr_->0$ in $U_1\cap\Sigma^0_+$. In particular, r_- will stay positive if $r_-(\gamma(t))>0$. Therefore, assuming (4.11), we have $\gamma'(t)=H_+(\gamma(t))$ or $x(\gamma(t))>0$ for $t-t_0>0$ small. Hence, by Remark 4.6, $\gamma'_+(t_0)=H_+(\sigma_0)$. Since the proof of the assertion on $\gamma'_-(t_0)$ is the same except for obvious sign changes the proof of the proposition is complete.

A curve $\gamma: I\to\Sigma^g_+\cap\Sigma^g_-$ which is piecewise a $H_\pm$ gliding ray is, of course, a ray in the sense of Definition 4.1. We call such rays gliding ray polygons.

(4.12) **Proposition.** *Suppose that r_+ and r_- can be extended to an open subset $V\subset T^*S$ with $\{r_+,r_-\}=0$ in V. Furthermore, suppose $\{r_+=r_-=0\}\cap V\subset\Sigma^g_+\cap\Sigma^g_-$ and suppose that H_+ and H_- are linearly independent on $\{r_+=r_-=0\}\cap V$. Let $\gamma: I\to V\cap\Sigma^g_+\cap\Sigma^g_-$ be a ray. Then γ is locally a uniform limit of gliding ray polygons.*

Proof. Fix $t_0\in I$. Using Darboux's theorem we find coordinates r_+, r_-, $g_3,\ldots$, g_{n-1}, f_+, f_-, $f_3,\ldots,f_{n-1}$ in a neighbourhood U of $\gamma(t_0)$ with $\{r_\pm,f_\pm\}=\{g_j,f_j\}=1$ in U and all other Poissonbrackets vanishing in U. The submanifold $\{r_+=r_-=0\}\cap U$ is foliated by the two-dimensional leaves on which g_j,f_j are constant. Using that γ is a ray, we may assume, after passing to a subinterval of I if necessary, that $\gamma(I)$ is contained in some leaf L. On $L\cap U$ the functions f_+ and f_- are coordinates. The gliding flows commute on L. We have

$$(\lambda H_++(1-\lambda)H_-)(f_++f_-)(\sigma)=1\,,$$

$$0\leqq(\lambda H_++(1-\lambda)H_-)f_\pm(\sigma)\leqq 1\,,$$

for all $0\leqq\lambda\leqq 1$, $\sigma\in L\cap U$. In view of Remark 4.6 we thus get

$$(f_++f_-)(\gamma(t))-(f_++f_-)(\gamma(s))=t-s\,,\qquad t,s\in I\,,$$

$$f_\pm(\gamma(t))-f_\pm(\gamma(s))=\lambda_\pm(t-s)\,,\qquad t,s\in I\,,\qquad \lambda_\pm\geqq 0\,,\qquad \lambda_++\lambda_-=1\,.$$

It is now easy to construct gliding ray polygons approximating γ. We leave the details to the reader.

Remark. A ray passing through a point in $\Sigma^{2,-}_-\cap\Sigma^g_+$ may continue as an integral curve of H_-, i.e. as a gliding ray in the diffractive set $\Sigma^{2,-}_-$. In boundary problems C^∞-singularities do not propagate along such rays. Therefore, one may expect that singularities of solutions to (T) propagate along rays satisfying the stronger condition: The assumptions in (4.4) hold with $\gamma(t)\in\Sigma^g_\pm-\Sigma^{(2)}_\mp$ replaced by $\gamma(t)\in\Sigma^g_\pm-\Sigma^g_\mp$. Singularities would propagate in this way if Theorem 3.5 were also true at $\sigma_0\in\Sigma^{(2)}_+\cap\Sigma^{2,-}_-$.

5. Propagation of Singularities

In this section we shall, for any $\sigma_0\in ss_T(u)$, construct a nontrivial ray, starting at σ_0, which is contained in $ss_T(u)$. This will complete the proof of Theorem 1.3. Indeed, just note that a ray with relatively compact image in Σ_T is globally Lipschitz continuous (see Remark 4.6) and that $\check{\gamma}$, $\check{\gamma}(t)=\gamma(-t)$, is a ray for $-P$ if γ is a ray for P.

(5.1) *Definition.* Let $I \subset \mathbb{R}$ be compact and let $\gamma: I \to \Sigma_T$ be continuous. (I, γ) is called an approximate ray of mesh $\leqq \varepsilon$, $0 < \varepsilon \leqq 1$, if and only if

$$\sup_{t \in I} \inf_{t \neq t' \in I} |t' - t| \leqq \varepsilon$$

and for every $t_0 \in I$, $t_0 < \sup I$,

(5.2) If $\gamma(t_0) \in \Sigma^0$ then $t \in I$ for $t - t_0 > 0$ small and the forward derivative $\gamma'_+(t_0)$ exists and equals $H_p(\gamma(t_0))$.

(5.3) If $\gamma(t_0) \notin \Sigma^g_+ \cup \Sigma^g_-$ then $t \in I$ and $\gamma(t) \in \Sigma^0$ for $t - t_0 > 0$ small.

(5.4) If $\gamma(t_0) \in \Sigma^g_- - \Sigma^{(2)}_+$ [respectively $\gamma(t_0) \in \Sigma^g_+ - \Sigma^{(2)}_-$] then either $t \in I$ and $\gamma(t) \in \Sigma^0_+$ [respectively $\gamma(t) \in \Sigma^0_-$] for $t - t_0 > 0$ small or else there exists $t \in I$, $t > t_0$, $(t_0, t) \cap I = \emptyset$, such that $\gamma(t) \notin \Sigma^0_+$ [respectively $\gamma(t) \notin \Sigma^0_-$] and

$$d(\gamma(t), \exp(t - t_0) H_-(\gamma(t_0))) \leqq \varepsilon |t - t_0|$$

[respectively $d(\gamma(t), \exp(t - t_0) H_+(\gamma(t_0))) \leqq \varepsilon |t - t_0|$].

(5.5) If $\gamma(t_0) \in \Sigma^g_+ \cap \Sigma^{(2)}_-$ or $\gamma(t_0) \in \Sigma^{(2)}_+ \cap \Sigma^g_-$ then there exists $t \in I$, $t_0 < t$, $(t_0, t) \cap I = \emptyset$, such that for some $0 \leqq \lambda \leqq 1$

$$d(\gamma(t), \exp((t - t_0)(\lambda H_+ + (1 - \lambda) H_-)(\gamma(t_0)))) \leqq \varepsilon |t - t_0|.$$

We shall call the number $\sup I - \inf I$ the length of the approximate ray (I, γ).
It will be important to estimate the variation of functions along approximate rays. To measure this we associate a "field of tangents" with every approximate ray.

(5.6) *Definition.* Let (I, γ) be an approximate ray. $V(I, \gamma)$ denotes the closure in $T(\tilde{T}^*(M, S))$ of the set of all $(\gamma(t), \zeta) \in T_{\gamma(t)}(\tilde{T}^*(M, S))$, $t \in I$, satisfying one of the following conditions

(5.7) $\gamma(t) \in \Sigma^0$ and $\zeta = H_p(\gamma(t))$,

(5.8) $\gamma(t) \in \Sigma^{(2)}_\pm$, $\zeta = H_\pm(\gamma(t))$, provided γ does not leave into $\Sigma^0_\mp$ for times greater than t.

(5.9) $\gamma(t) \in \Sigma^g_+ \cap \Sigma^{(2)}_-$ or $\gamma(t) \in \Sigma^g_- \cap \Sigma^{(2)}_+$ and, for some $0 \leqq \lambda \leqq 1$, $\zeta = (\lambda H_+ + (1 - \lambda) H_-)(\gamma(t))$.

(5.10) **Lemma.** *Let $K \Subset \Sigma_T$. Let f be a C^∞ function defined in an open neighbourhood of K in $\tilde{T}^*(M, S)$. Then there exists a constant $C > 0$ such that every approximate ray (I, γ) with mesh $\leqq \varepsilon$ and $\gamma(I) \subset K$ satisfies*

(5.11) $$|f(\gamma(t)) - f(\gamma(s)) - \alpha(t - s)| \leqq C\varepsilon |t - s|, \qquad t, s \in I,$$

for some α (depending on s and t) which is an element of the convex hull of the set

$$\{\langle \zeta, f \rangle(\gamma(t));\ (\gamma(t), \zeta) \in V(I, \gamma)\}.$$

Proof. Let t_0 be the supremum of the set of all $T \in [\inf I, \sup I]$ such that (5.11) holds under the additional assumption $s, t \leqq T$. t_0 exists since $T = \inf I$ belongs to this set.

We have $t_0 \in I$ and by the continuity of γ (5.11) holds for $s, t \in I \cap (-\infty, t_0]$. To prove the lemma we have to show $t_0 = \sup I$. Assume that $t_0 < \sup I$. If there exists $t_1 \in I$, $t_0 < t_1$, such that (5.11) holds for $s, t \in I \cap [t_0, t_1]$ then we have for $s, t \in I$, $s < t_0 \leqq t \leqq t_1$,

$$|f(\gamma(t_0)) - f(\gamma(s)) - \alpha'(t_0 - s)| \leqq C\varepsilon(t_0 - s),$$
$$|f(\gamma(t)) - f(\gamma(t_0)) - \alpha''(t - t_0)| \leqq C\varepsilon(t - t_0).$$

This implies

$$|f(\gamma(t)) - f(\gamma(s)) - \alpha(t - s)| \leqq C\varepsilon(t - s)$$

with $\alpha = ((t_0 - s)\alpha' + (t - t_0)\alpha'')/(t - s)$. Then (5.11) holds for $s, t \in I \cap (-\infty, t_1]$ contradicting the maximality of t_0. Using Taylorexpansion it follows from the definition of approximate rays and from the definition of $V(I, \gamma)$ that such $t_1 \in I$, $t_1 > t_0$, exists except (possibly) when $\gamma(t_0) \notin \Sigma^0 \cup \Sigma^g_\pm$ and $\gamma(t) \in \Sigma^0_\pm$ for $t - t_0 > 0$ small. In the latter case we choose $t_1 > t_0$ such that $\gamma(t) \in \Sigma^0_\pm$ for $t_0 < t \leqq t_1$ and conclude that (5.11) holds for the approximate rays $(I \cap [t, t_1], \gamma | I \cap [t, t_1])$, $t_0 < t < t_1$. Letting $t \to t_0$ we see that (5.11) holds on $I \cap [t_0, t_1]$ in any case. This proves the lemma.

Since the metric d is locally equivalent to the euclidean metric in any local coordinate system we get as a corollary to Lemma 5.10 the Lipschitz continuity of approximate rays.

(5.12) **Corollary.** *Let $K \Subset \Sigma_T$. There exists $C > 0$ such that every approximate ray (I, γ) with $\gamma(I) \subset K$ satisfies*

$$d(\gamma(t), \gamma(s)) \leqq C|t - s|, \qquad s, t \in I.$$

Remark. With Definition 5.6 suitably modified Lemma 5.10 and Corollary 5.12 hold also for rays.

We can now prove the local existence of approximate rays contained in $ss_T(u)$.

(5.13) **Proposition.** *Let u be a solution of (T). Let $K_0 \Subset ss_T(u)$. Then there exists $T_0 > 0$ such that one can find for every $\sigma \in K_0$ and every $\varepsilon > 0$ an approximate ray (I, γ) of mesh $\leqq \varepsilon$ and length T_0 with $\gamma(\inf I) = \sigma$ and $\gamma(I) \subset ss_T(u)$.*

Proof. We choose $T_0 > 0$ with $CT_0 < d(K_0, \partial K)$, where $K \Subset \Sigma_T$, $K_0 \subset K$, $d(K_0, \partial K) > 0$, and where $C > 0$ is the Lipschitz constant on K given in Corollary 5.12. Let $\varepsilon > 0$ and $\sigma \in K_0$ be given. Consider the set $\mathscr{R}$ of all approximate rays (I, γ) of mesh $\leqq \varepsilon$, length $\leqq T_0$, with $\gamma(\inf I) = \sigma$ and $\gamma(I) \subset ss_T(u)$. $\mathscr{R}$ is not empty because $(\gamma_0, \{0\}) \in \mathscr{R}$, $\gamma_0(0) = \sigma$. We define an ordering $\leqq$ on $\mathscr{R}$ by saying that $(I, \gamma) \leqq (I', \gamma')$ if and only if there exists some $T \in \mathbb{R}$ such that $I = I' \cap (-\infty, T]$, $\gamma = \gamma' | I$. The ordered set $(\mathscr{R}, \leqq)$ satisfies the hypotheses of Zorn's lemma. In fact, if $\mathscr{R}_1 \subset \mathscr{R}$ is a totally ordered subset then (I_0, γ_0),

$$I_0 = \overline{\bigcup_{(I, \gamma) \in \mathscr{R}_1} I}, \qquad \gamma_0 | I = \gamma \quad \text{for all} \quad (I, \gamma) \in \mathscr{R}_1,$$

is the supremum of $\mathscr{R}_1$ and $(I_0, \gamma_0) \in \mathscr{R}$. Note that

$$I_0 - \bigcup_{(I, \gamma) \in \mathscr{R}_1} I \subset \{\sup I_0\}$$

and that the extension of γ_0 to I_0 exists and is unique by Corollary 5.12. $\gamma_0(\sup I_0) \in ss_T(u)$ since $ss_T(u)$ is closed. So there exists a maximal element (I, γ) of $(\mathscr{R}, \leqq)$. We have to show that its length T equals T_0. Suppose $T < T_0$. By our choice of T_0 we have $\gamma(\sup I) \subset \mathring{K} \cap ss_T(u)$. It follows from the $ss_T(u)$-estimates given in Sect. 3 that there exists an approximate ray (I', γ') of positive length, of mesh $\leqq \varepsilon$, with $\sup I = \inf I'$, $\gamma'(\inf I') = \gamma(\sup I)$, and $\gamma'(I') \subset ss_T(u)$. Then $(I \cup I', \gamma_0) \in \mathscr{R}$, where $\gamma_0 | I = \gamma$ and $\gamma_0 | I' = \gamma'$, is strictly greater than (I, γ). This contradiction to the maximality of (I, γ) completes the proof of the proposition.

Recalling that $ss_T(u)$ is a closed set Theorem 1.3 now follows from Proposition 5.13 and the following result.

(5.14) **Proposition.** *Let $K \Subset \Sigma_T$. Let (I_j, γ_j), $j \in \mathbb{N}$, be a sequence of approximate rays with meshes tending to zero as $j \to \infty$ and with a positive lower bound T_0 on their lengths. Assume that* $\inf I_j \to 0$ *as $j \to \infty$ and that $\gamma_j(I_j) \subset K$ for all j. Then there exists a curve $\gamma : [0, T_0] \to K$ with*

$$\liminf_{j \to \infty} \sup_{t \in I_j \cap [0, T_0]} d(\gamma(t), \gamma_j(t)) = 0 . \tag{5.15}$$

Moreover, every such curve γ is a ray.

Proof. By Corollary 5.12 the γ_j's are uniformly Lipschitz continuous. It follows from the Arzela-Ascoli theorem – or rather its proof – that a continuous γ satisfying (5.15) exists. We fix such a limit curve γ and assume without loss of generality that

$$\lim_{j \to \infty} \sup_{t \in I_j \cap [0, T_0]} d(\gamma_j(t), \gamma(t)) = 0 .$$

We have to show that γ is a ray. It is easy to see that γ is Lipschitz continuous. It follows from Lemma 5.10 that approximate rays in Σ^0 are actually bicharacteristics. This leaves us with the task of determining $\gamma'(t_0)$, say $\gamma'_+(t_0)$, at points $\gamma(t_0) \in \Sigma^g_+ \cup \Sigma^g_-$, $t_0 \in [0, T_0)$, where $\gamma(t) \in \Sigma^0_\pm$ for $t - t_0 > 0$ small does not hold if $\gamma(t_0) \notin \Sigma^g_\pm$. We consider such a point $\sigma_0 = \gamma(t_0)$ and fix coordinates (x, y) near its basepoint such that $x > 0$ (respectively $x < 0$) on the positive (respectively negative) side of S. As before, we have the canonical and the compressed canonical coordinates, (x, y, ξ, η) and (x, y, λ, η), respectively. By the choice of σ_0 we cannot have $x(\gamma_j(T_j)) > 0$ and $H_p x(\gamma_j(T_j)) > 0$ for some sequence $T_j \to t_0$, $T_j \in I_j$, if $\sigma_0 \notin \Sigma^g_+$. We now study the case $\sigma_0 \notin \Sigma^g_+$ more closely. Applying the following lemma to the approximate rays γ_j, restricted to a suitable neighbourhood of t_0 if necessary, we get a sequence $t_j \to t_0$ such that for small $\delta > 0$

$$x(\gamma_j(t)) \leqq 0 \quad \text{for all} \quad t \in I_j \cap [t_j, t_0 + \delta] . \tag{5.16}$$

(5.17) **Lemma.** *Let $([0, \tilde{T}], \tilde{\gamma})$ be an approximate ray with $x(\tilde{\gamma}(0)) < \delta \leqq 1$, $\delta > 0$, satisfying either $\inf |H_p x| > 0$ or $\inf H_p^{(2)} x > 0$, the infima being taken over $\Sigma^0_+ \cap \tilde{\gamma}([0, \tilde{T}])$. Assume that there is no $t \in [0, \tilde{T}]$ with $x(\tilde{\gamma}(t)) > 0$ and $H_p x(\tilde{\gamma}(t)) > 0$. Then,*

$$x(\tilde{\gamma}(t)) \leqq 0 \quad \textit{for all} \quad t \in [0, \tilde{T}] , \qquad t \geqq T ,$$

where $T = C \cdot \delta^{1/2}$. The constant $C > 0$ only depends on bounds on $H_p x$ and $H_p^{(2)}$ over the set $\Sigma^0_+ \cap \tilde{\gamma}([0, \tilde{T}])$.

Proof. Note that $x(\tilde{\gamma}(t)) \leqq 0$ holds if $x(\tilde{\gamma}(t')) \leqq 0$, $0 \leqq t' < t \leqq \tilde{T}$. In fact, $\tilde{\gamma}$ can only leave as a bicharacteristic into $x > 0$ with either $H_p x > 0$ or $H_p x \geqq 0$ and $H_p^{(2)} x > 0$. It suffices to assume $T \leqq \tilde{T}$ and to show that the function $\alpha(t) = x(\tilde{\gamma}(t))$, $t \in [0, T]$, cannot satisfy simultaneously $\alpha > 0$ and $\alpha' \leqq 0$ on $[0, T]$. Assume to the contrary that α satisfies these inequalities on $[0, T]$. If $|H_p x|$ has a positive lower bound C' on $\Sigma_+^0 \cap \tilde{\gamma}([0, \tilde{T}])$ then $\alpha' \leqq -C'$ on $[0, T]$. This contradicts $\alpha > 0$ when $T > \delta/C'$ since $\alpha(0) < \delta$. In case $H_p^{(2)} x$ has a positive lower bound on $\Sigma_+^0 \cap \tilde{\gamma}([0, \tilde{T}])$ we have with some positive constant C_0

$$C_0^{-1} \leqq \alpha'' \leqq C_0 \quad \text{on } [0, T].$$

It follows from elementary calculus that this cannot hold together with $\alpha > 0$ and $\alpha' \leqq 0$ on $[0, T]$ and $T \geqq C\delta^{1/2}$, $\alpha(0) < \delta$, if $C > 0$ is large enough. This proves the lemma.

Continuing with the proof of Proposition 5.14 we get as an immediate consequence of (5.16), after letting $j \to \infty$,

(5.18) $$x(\gamma(t)) \leqq 0 \quad \text{for} \quad t - t_0 > 0 \quad \text{small}.$$

[Of course, we also obtain $x(\gamma(t)) \geqq 0$ for $t - t_0 > 0$ small if $\sigma_0 \notin \Sigma_-^g$ just by changing signs.] To show that $\gamma'_+(t_0)$ is tangent to $x = 0$ it now suffices, in view of (5.18), to consider the case $\sigma_0 \in \Sigma_\pm^{(2)}$ and to show that for every $\delta > 0$

(5.19) $$\pm x(\gamma(t)) \leqq \delta |t - t_0| \quad \text{if } |t - t_0| \text{ is small}.$$

Applying (2.8) with $f = x$ and noting that $H_\pm x = 0$ we obtain

$$H_p x(\gamma(t)) \to 0 \quad \text{as} \quad t \to t_0 \quad \text{and} \quad \gamma(t) \in \Sigma_\pm^0.$$

This implies (5.19) since a violation of (5.19) leads to a sequence $t_j \to t_0$ with $|H_p x(\gamma(t_j))| > \delta$. Since $\xi = \lambda/x$ stays bounded on $\Sigma^0 \cap K$ we have actually shown that $\gamma'_+(t_0)$ is tangent to $x = \lambda = 0$, i.e. $\gamma'_+(t_0) \subset T_{\sigma_0}(T^*S)$. Consider the case $\sigma_0 \in \Sigma_-^g - \Sigma_+^{(2)}$. We have to show

(5.20) $$\gamma'_+(t_0) = H_-(\sigma_0).$$

With t_j as in (5.16) introduce the approximate rays $(I_{j\delta}, \gamma_{j\delta})$, $\delta > 0$,

$$I_{j\delta} = I_j \cap [t_j, t_0 + \delta], \qquad \gamma_{j\delta} = \gamma_j | I_{j\delta}.$$

Using (5.16) we get for small $\delta > 0$

(5.21) $$\bigcup_j V(I_{j\delta}, \gamma_{j\delta}) \subset \bar{V}_\delta,$$

where

(5.22) $$\begin{aligned} V_\delta = &\{(\sigma, H_p(\sigma));\ \sigma \in K \cap \Sigma_-^0,\ d(\sigma, \sigma_0) \leqq C\delta\} \\ &\cup \{(\sigma, H_-(\sigma));\ \sigma \in K \cap \Sigma_-^g,\ d(\sigma, \sigma_0) \leqq C\delta\}. \end{aligned}$$

We may choose functions $\varphi_1, \ldots, \varphi_{2n-2}$ defined in a neighbourhood of σ_0 forming together with x and λ a coordinate system near σ_0 such that

$$H_- \varphi_1(\sigma_0) = 1, \qquad H_- \varphi_k(\sigma_0) = 0 \quad \text{for} \quad k > 1.$$

Using (2.8) we get a constant $C>0$ such that

$$\begin{aligned} &|\langle\zeta,\varphi_1\rangle(\sigma)-1|\leqq C\delta\,, \\ &\quad|\langle\zeta,\varphi_k\rangle(\sigma)|\leqq C\delta \quad \text{for} \quad k>1\,, \end{aligned} \tag{5.23}$$

for all $(\sigma,\zeta)\in\bar{V}_\delta$, $0<\delta$ small. We apply Lemma 5.10 with $f=\varphi_k$, $(I,\gamma)=(I_{j\delta},\gamma_{j\delta})$ and obtain using (5.21) and (5.23) and letting $j\to\infty$

$$\begin{aligned} &|\varphi_1(\gamma(t))-\varphi_1(\sigma_0)-(t-t_0)|\leqq C\delta|t-t_0|\,, \\ &\quad|\varphi_k(\gamma(t))-\varphi_k(\sigma_0)|\leqq C\delta|t-t_0| \quad \text{for} \quad k>1\,, \end{aligned}$$

for all $t\in[t_0,t_0+\delta]$. The preceding estimates also hold with $\gamma(t)$ replaced by $\exp((t-t_0)H_-(\sigma_0))$. Comparing these estimates we conclude (5.20) after letting $\delta\to 0$.

Finally, we consider the case $\sigma_0\in\Sigma_-^g\cap\Sigma_+^{(2)}$. $\gamma'(t_0)$ is tangent to $x=\lambda=0$ by (5.19). We still have to show

$$\gamma'(t_0)\subset\{\lambda H_+(\sigma_0)+(1-\lambda)H_-(\sigma_0);\ 0\leqq\lambda\leqq 1\}\,. \tag{5.24}$$

First, assume that $H_+(\sigma_0)$ and $H_-(\sigma_0)$ are linearly independent. We can find smooth functions $\varphi_\pm$ defined near σ_0 with

$$H_\pm\varphi_\pm(\sigma_0)=1\,,\qquad H_+\varphi_-(\sigma_0)=H_-\varphi_+(\sigma_0)=0\,.$$

Thus,

$$(\lambda H_+ +(1-\lambda)H_-)\varphi_\pm(\sigma_0)\geqq 0\,,\qquad 0\leqq\lambda\leqq 1\,, \tag{5.25}$$

and, setting $\varphi=\varphi_+ +\varphi_-$,

$$(\lambda H_+ +(1-\lambda)H_-)\varphi(\sigma_0)=1\,,\qquad 0\leqq\lambda\leqq 1\,. \tag{5.26}$$

Using (2.8), (5.25), and (5.26) we get a constant $C>0$ such that

$$\langle\zeta,\varphi_\pm\rangle(\sigma)\geqq -C\delta\,,\qquad \delta>0\,, \tag{5.27}$$

$$|\langle\zeta,\varphi\rangle(\sigma)-1|\leqq C\delta\,,\qquad \delta>0\,, \tag{5.28}$$

for all $(\sigma,\zeta)\in W$, $d(\sigma,\sigma_0)\leqq\delta$, where

$$\begin{aligned} W=&\{(\sigma,H_p(\sigma));\ \sigma\in K\cap\Sigma^0\} \\ &\cup\{(\sigma,(\lambda_+H_+ +\lambda_-H_-)(\sigma); \\ &\sigma\in K-\Sigma^0,\ 0\leqq\lambda_\pm,\ \lambda_+ +\lambda_- =1,\ \lambda_\pm>0 \text{ only if } \sigma\in\Sigma_\pm^{(2)}\}\,. \end{aligned}$$

We define the approximate rays $(I_{j\delta},\gamma_{j\delta})$, $j\in\mathbb{N}$, $\delta>0$,

$$I_{j\delta}=I_j\cap[t_0-\delta/C_0,t_0+\delta/C_0]\,,\qquad \gamma_{j\delta}=\gamma_j|I_{j\delta}\,.$$

Here $C_0>0$ is chosen strictly greater than the uniform Lipschitz constant for the (I_j,γ_j)'s. Then, for small $\delta>0$,

$$V(I_{j\delta},\gamma_{j\delta})\subset\bar{W}\cap\{(\sigma,\zeta);\ d(\sigma,\sigma_0)\leqq\delta\} \quad \text{if } j \text{ is large}\,. \tag{5.29}$$

Using Lemma 5.10 we get from (5.27)–(5.29) after letting $j\to\infty$

$$\varphi_\pm(\gamma(t))-\varphi_\pm(\sigma_0)\geqq -C\delta|t-t_0|\,, \tag{5.30}$$

$$(5.31) \qquad |\varphi(\gamma(t))-\varphi(\sigma_0)-(t-t_0)| \leqq C\delta|t-t_0|,$$

for $|t-t_0| \leqq \delta/C_0$. Similarly, we get for any C^∞ function ψ near σ_0 with

$$(5.32) \qquad H_\pm \psi(\sigma_0)=0$$

an estimate

$$(5.33) \qquad |\psi(\gamma(t))-\psi(\sigma_0)| \leqq C\delta|t-t_0|, \qquad |t-t_0| \leqq \delta/C_0.$$

From (5.25), (5.26), and (5.30)–(5.33) we conclude that there exists $\lambda \in [0,1]$, depending on t and δ, such that

$$|\varphi_\pm(\gamma(t))-\varphi_\pm(\exp(t-t_0)(\lambda H_+ + (1-\lambda)H_-)(\sigma_0))| \leqq C\delta|t-t_0|,$$
$$|\psi(\gamma(t))-\psi(\exp(t-t_0)(\lambda H_+ + (1-\lambda)H_-)(\sigma_0))| \leqq C\delta|t-t_0|.$$

Choosing finitely many ψ's such that the differentials of the ψ's together with the differentials of x, λ, φ_+, and φ_- span the cotangent space to $\tilde{T}^*(M,S)$ at σ_0 we obtain the inclusion (5.24), after letting $\delta \to 0$.

If $H_+(\sigma_0)$ and $H_-(\sigma_0)$ are linearly dependent we may find, by (1.2), a smooth function φ near σ_0 with $H_+\varphi(\sigma_0)=1$ and $H_-\varphi(\sigma_0)>0$. Choosing functions ψ which together with x, λ and φ form a coordinate system near σ_0 and which satisfy $H_\pm\psi(\sigma_0)=0$ we obtain the inclusion (5.24) also in this case by reasoning as in the preceding case.

Since we have analysed – up to trivial sign changes – all possible cases the proof of the proposition, and thus, also of Theorem 1.3, is now complete.

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