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Complexifications of Transversely Holomorphic Foliations

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1. Introduction

Let $\mathcal{F}$ be a transversely holomorphic foliation on a paracompact manifold X, of dimension p and complex codimension n. This means that $\mathcal{F}$ is given by an open covering $\{U_i\}_{i\in I}$ and local submersions $f_i: U_i \to \mathbb{C}^n$ with fibers of dimension p such that, for $i, j \in I$, there is a holomorphic isomorphism g_{ji} of open sets of C^n such that $f_j = g_{ji} \cdot f_i$ on $U_i \cap U_j$.

A *complexification* of $\mathcal{F}$ is a complex analytic manifold $\hat{X}$ of complex dimension $n+p$ with a holomorphic foliation $\hat{\mathcal{F}}$ of codimension n, and an embedding $j: X \to \hat{X}$ such that $\mathcal{F} = j^{-1}(\hat{\mathcal{F}})$ as transversely holomorphic foliations, and such that the images of the leaves of $\mathcal{F}$ by j are totally real in the leaves of $\hat{\mathcal{F}}$.

So locally the leaves of $\hat{\mathcal{F}}$ are complexifications of the leaves of $\mathcal{F}$. In fact, we are only interested in the germ of $\hat{\mathcal{F}}$ along $j(X)$.

In this note, we show that a complexification of $\mathcal{F}$ always exists when the codimension is one, but that in general complexifications do not exist when the codimension is greater than one.

2. Complexification of the Leaves

In this paragraph we show that we can construct in an essentially unique way a complexification in the following weak sense.

2.1. Proposition. *Let $\mathcal{F}$ be a transversely holomorphic foliation on X. One can construct a real analytic manifold $\hat{X}$ with a real analytic transversely holomorphic foliation $\hat{\mathcal{F}}$ with leaves complex manifolds and an embedding $j: X \to \hat{X}$ such that $\mathcal{F} = j^{-1}(\hat{\mathcal{F}})$ as transversely holomorphic foliations, the leaves of $\hat{\mathcal{F}}$ being complexifications of the leaves of $\mathcal{F}$. More precisely, on $\hat{X}$ we have an open covering $\{\hat{U}_i\}_{i\in I}$ with real analytic charts $\hat{\varphi}_i: \hat{U}_i \to \mathbb{C}^n \times \mathbb{C}^p$ such that the change of charts $\hat{\varphi}_j \hat{\varphi}_i^{-1}$ is of the form*

$$(z, w) \to (g_{ji}(z), h_{ji}(z, \bar{z}, w))$$

with g_{ji} holomorphic in z, g_{ji} holomorphic in w and real analytic in z. The foliation $\hat{\mathscr{F}}$ is defined by the local projections obtained by composing the $\hat{\varphi}_i$'s with the linear projection on C^n. Moreover, the charts $\varphi_i = \hat{\varphi}_i \circ j$ map $U_i = j^{-1}(\hat{U}_i)$ in $C^n \times \mathbb{R}^n$.

The germ of $\hat{\mathscr{F}}$ along $j(X)$ is unique up to isomorphism.

For complexifications in the sense of the introduction, in general, we don't have existence nor uniqueness.

Proof. First of all, we can introduce on X a real analytic structure (inducing the given smooth C^∞-structure on X) such that the given foliation $\mathscr{F}$ is real analytic, and this can be done in an essentially unique way. Indeed, consider provisionally on X a real analytic structure X' inducing the given smooth structure. Then there is a unique germ of real analytic manifold Y with a transversely holomorphic foliation $\mathscr{G}$ (sometimes called the graph of $\mathscr{F}$) and a smooth embedding $i: X \to Y$ such that $i^{-1}(\mathscr{G}) = \mathscr{F}$ as transversely holomorphic foliations. Using Grauert [1] and Royden [5], we can construct a smooth isotopy i_t of $i = i_0$ such that i_1 is analytic and i_t is transversal to $\mathscr{G}$ for all t. So there is a smooth isotopy h_t of X such that $h_t(\mathscr{F}) = i_t^{-1}(\mathscr{G})$; when X is not compact, we have to construct i_t approaching the identity at the infinity of X. Then $\mathscr{F}$ will be real analytic for the real analytic structure $h_1^{-1}(X')$.

The existence and uniqueness of the germ of the foliation $\hat{\mathscr{F}}$ and of the real analytic embedding $j: X \to \hat{X}$ as in the proposition follows from Haefliger [2, p. 296, Theorem 1]. Indeed, as $\mathscr{F}$ is real analytic, we have a real analytic atlas $\varphi_i: U_i \to \mathbb{C}^n \times \mathbb{R}^p$ on X defining $\mathscr{F}$, the change of charts being of the form $(z, w) \mapsto (g_{ji}(z), h_{ji}(z, \bar{z}, w))$, where g_{ji} is holomorphic in z and h_{ji} real analytic in z and w. Those charts can be extended to charts of $\hat{X}$ and the change of charts is given by the unique extension of h_{ji} as a holomorphic function in w.

3. The Integrability Condition

We are looking now for a complex structure on $\hat{X}$ inducing the given complex structure on the leaves of $\hat{\mathscr{F}}$ and such that the projections obtained by composing $\hat{\varphi}_i$ with the projection on $\mathbb{C}^n$ are holomorphic.

The restriction to $X = j(X)$ of a *real analytic almost complex structure* on a neighbourhood of $j(X)$ in $\hat{X}$, inducing the given structure along the leaves of $\hat{\mathscr{F}}$ and the given holomorphic transverse structure, is given by a vector valued 1-form ω on X defined on the complexified tangent bundle $T^c(X)$ of X with value in the complexified tangent bundle $T^c(\mathscr{F})$ to the leaves of $\mathscr{F}$ whose restriction to $T^c(X)$ is the identity. The kernel of ω will be by definition the complex subspace for the given almost complex structure complementary to the tangent spaces to the leaves.

In the local coordinates $(z_1, \ldots, z_n, w_1, \ldots, w_p)$ given by the chart $\varphi_i: U_i \to \mathbb{C}^n \times \mathbb{R}^p$, ω is of the form

$$\omega = \sum_k \omega_k \otimes \partial/\partial w_k,$$

where

$$\omega_k = dw_k + \sum_j a_{kj}(z, \bar{z}, w) dz_j + \sum_j b_{kj}(z, \bar{z}, w) dz_j$$

the functions a_{kj} and b_{kj} being real analytic in z, $\bar{z}$, and w.

There is a canonical way of extending ω to a neighbourhood of $j(X)$ in X, namely for complex values of $w_1, \ldots, w_p$, the functions a_{kj} and b_{kj} extend uniquely as holomorphic functions of the w_j's.

If the almost complex structure on $\hat{X}$ derives from a complex analytic structure on $\hat{X}$ satisfying the prescribed compatibility conditions, then it can be defined by such a 1-form ω, with a_{kj} and b_{kj} holomorphic in w.

Indeed, the integrability condition can be expressed by the condition that the component of type (0, 2) of $d\omega_k$, $k=1,\ldots,p$, with respect to the almost complex structure defined by the coframe $dz_1, \ldots, dz_n, \omega_1, \ldots, \omega_p$ should vanish.

A straightforward computation gives

$$(d\omega_k)^{0,2} = \sum_{i,j}\left(\partial b_{kj}/\partial \bar{z}_i - \sum_l \partial b_{kj}/\partial w_l\, b_{li} - \sum_l \partial b_{kj}/\partial \bar{w}_l\, \bar{a}_{li}\right) d\bar{z}_i \wedge d\bar{z}_j$$
$$+ \sum_{l,j} \partial b_{kj/\partial \bar{w}_l} d\bar{w}_l \wedge d\bar{z}_j .$$

So the integrability conditions are

3.0. $\partial b_{kj}/\partial \bar{w}_l = 0$, for $k, l = 1, \ldots, p$ and $j=1, \ldots, n$.

3.1. $\sum_{i,j}\left(\partial b_{kj}/\partial \bar{z}_i - \sum_l \partial b_{kj}/\partial \bar{w}_l\, b_{li}\right) d\bar{z}_i \wedge d\bar{z}_j = 0$ for $k=1, \ldots, p$.

3.0 can always be achieved by taking the holomorphic extensions of the real analytic functions b_{kj}. Another choice for the coefficients a_{kj} does not change the almost complex structure defined by ω, so that we can also choose the coefficients a_{kj} holomorphic in w.

The integrability condition can also be expressed in a more intrinsic way in terms of the Nijenhuis bracket, namely the component of type (0, 2) of $[\omega, \omega]$ should vanish (cf. Malgrange [4]).

If the integrability conditions are satisfied, then the almost complex structure defined by ω derives from a complex structure (cf. for instance Weil [6] for an elementary proof in the real analytic case).

If the codimension n is one, then the integrability conditions are always satisfied. Hence we get the following result.

3.2. Theorem. *Any transversely holomorphic foliation of codimension one admits a complexification.*

4. Principal Circle Bundles over Complex Manifolds

We consider a principal circle bundle X over a complex manifold M with projection π. Such a bundle carries a natural transversely holomorphic foliation $\mathscr{F}$ whose leaves are the fibers of π, namely the inverse image by π of the holomorphic foliation of M by points.

4.1. Theorem. *The natural transversely holomorphic foliation on X admits a complexification if and only if the associated complex line bundle over M comes from a holomorphic line bundle.*

Proof. It is clear that if the associated line bundle L comes from a holomorphic line bundle, then $\mathscr{F}$ admits a complexification. The embedding j of X in $\hat{X} = L$ is obtained by introducing a hermitian metric on L and by identifying X with the set of vectors of length one in L.

Conversely assume that $\mathscr{F}$ admits a complexification. The germ along $j(X)$ of the corresponding almost complex structure will be determined by a vector valued 1-form ω satisfying the integrability conditions.

For a diffeomorphism h of X inducing an automorphism of $\mathscr{F}$ (in this special case this means that h maps fibers in fibers and projects on a complex analytic automorphism of the base space M), we define $h^*\omega$ as the vector valued 1-form

$$\langle h^*\omega, \xi\rangle = dh^{-1}\langle \omega, dh(\xi)\rangle$$

for any $\xi \in T^c X$.

If ω satisfies the integrability conditions, then so does $h^*\omega$. Replacing ω by its average under the right action of the circle $SO(2)$ on X, we get a 1-form still denoted by ω which is $SO(2)$-invariant and satisfies the integrability conditions.

Let P be the principal $Gl(1, \mathbb{C})$ bundle associated to the $SO(2)$-bundle X; we identify $SO(2)$ with the subgroup $U(1)$ of $Gl(1, \mathbb{C})$ so that X is identified with a subspace of P and the complexified tangent bundle to the fibers of $\pi : X \to M$ is the restriction to X of the tangent bundle to the fibers of P.

The 1-form ω being $U(1)$-invariant extends uniquely as a connection form $\hat{\omega}$ on P [a connection form for P is nothing else than a vector valued 1-form on P with values in the tangent space to the fibers of P, whose restriction to the tangent spaces to the fibers is the identity and which is $Gl(1, \mathbb{C})$-invariant]. The bracket $[\hat{\omega}, \hat{\omega}]$ is the curvature form of the connection and the integrability condition means that the component of type $(0, 2)$ of $[\hat{\omega}, \hat{\omega}]$ vanishes. This is precisely the condition that the principal bundle P (or equivalently the associated line bundle L) can be given a holomorphic structure (cf. Koszul and Malgrange [3]).

4.2. Example. A principal circle bundle $X \to M$ is characterized by its chern class $c_1 \in H^2(M, Z)$. Its associated line bundle comes from a holomorphic line bundle if and only if the image of c_1 in $H^2(M, \theta)$ by the homomorphism i induced by the inclusion of $\mathbb{Z}$ in the sheaf θ of germs of holomorphic functions in M vanishes.

If M is a compact Kaehler manifold, $i(H^2(M, Z))$ generates $H^2(M, \theta)$ (this follows from Hodge theory). So if $H^2(M, \theta) \neq 0$, for instance if M is a complex torus of dimension bigger than one, then there are circle bundles for which the underlying transversely holomorphic foliation does not admit a complexification.

4.3. Remark. The same considerations can be applied to the more general case of a principal K-bundle X with base space a complex manifold, where K is a compact Lie group. The underlying transversely holomorphic foliation $\mathscr{F}$ on X, whose leaves are the fibers admits a complexification if and only if the associated principal bundle with group K^c (the complexification of K) admits a holomorphic structure.

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