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Einstein-Kähler V-Metrics on Open Satake V-Surfaces with Isolated Quotient Singularities

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Dedicated to the memory of the late Professor Takehiko Miyata

0. Introduction

In [A] and [Y 1], Aubin and Yau proved the existence of a Ricci-negative Einstein-Kähler metric on compact complex manifolds with ample canonical bundles. The purpose of this paper is to generalize the Aubin-Yau theorem to the category of open Satake V-surfaces with isolated quotient singularities. We note that this category includes as a special case the logarithmic canonical models of minimal surfaces of logarithmic general type which are treated in [S, K1, K2, K2 Added in Proof]. In this paper, we use the terminology "Satake V-manifolds" in the following sense. A complex space X is called a *(Satake) V-manifold* if it has at worst isolated quotient singularities. Let X be a V-manifold and ds^2 a Riemannian metric defined in the regular part of X. ds^2 is called a *V-metric* if we obtain it by pushing down a smooth metric defined in local uniformizations. Therefore a V-metric looks like a smooth metric in terms of local uniformizing coordinates. Two-dimensional V-manifolds are called V-surfaces. Our main result is:

Theorem 1. *Let X be a compact complex surface with at worst isolated quotient singularities and C a divisor which lies in the regular part of X with at worst normal crossings. Let $\bar{X}\to X$ be the minimal resolution of X and $D=\sum_i D_i$ its exceptional divisor. We can determine the nonnegative rational numbers μ_i by requiring $K_{\bar{X}}+\sum_i \mu_i D_i$ to be trivial near D. Assume the following conditions are satisfied:*

(1) *$(K_{\bar{X}}+C)\cdot C_j \geqq 0$ for every irreducible component C_j of C;*

(2) *the divisor consisting of C_j's with $(K_{\bar{X}}+C)\cdot C_j>0$ has at worst simple normal crossings;*

(3) *there is a no (-1)-curve (exceptional curve of the first kind) which meets* Supp(C) *at most one point;*

(4) *for every (-2)-curve $E\notin$* Supp(D) *which meets D, there exists a component D_i of D meeting E with $\mu_i>0$.*

If $\kappa(K_{\bar{X}}+D+C)=2$, then there exists a complete Ricci-negative Einstein-Kähler V-metric on $X-C$ with finite volume, which is unique up to multiplication by positive numbers.

Let B^2 be the open unit ball in $\mathbb{C}^2$ and Γ a discrete subgroup of $PSU(2,1)$ acting on B^2 properly discontinuously with at worst isolated fixed points. The ball-metric is Einstein-Kähler whose holomorphic sectional curvature is negative and constant, which is invariant under the action of $PSU(2,1)$. Assume the volume of $X'=\Gamma\backslash B^2$ is finite. Then we can compactify X' by adding a finite number of cusp points. Let X be the partial minimal resolution of X' over the cusp points and C the exceptional divisor. C consists of disjoint elliptic curves. (X, C) with the ball-metric yields an example of Theorem 1. Hirzebruch-Mumford's proportionality theorem tells us that the following equality

$$\left(K_{\bar{X}}+\sum_i \mu_i D_i + C\right)^2 = 3\left\{e(\bar{X})-e(C)-e(D)+\sum_p (1/|G_p|)\right\}$$

holds, where $\{p\}$ runs over the quotient singularities of X. We obtain the following uniformization theorem by integrating the pointwise deviation of the canonical Einstein-Kähler V-metric in Theorem 1 from the ball-metric (cf. the proof of Theorem 2 in [K2]). It is a converse of the proportionality theorem.

Theorem 2. *Let $X, C, \bar{X}, D$ be as in Theorem 1. If we assume $\kappa(K_{\bar{X}}+D+C)=2$ and conditions* (1) (2) (3) (4) *in Theorem 1, then we have the following inequality*

$$\left(K_{\bar{X}}+\sum_i \mu_i D_i + C\right)^2 \leqq 3\left\{e(\bar{X})-e(C)-e(D)+\sum_p (1/|G_p|)\right\},$$

where $\{p\}$ runs over the quotient singularities of X and G_p is the finite subgroup of $U(2)$ corresponding to p. The equality holds if and only if the universal covering of the regular part of $X-C$ is biholomorphic to the open unit ball B^2 in $\mathbb{C}^2$ minus a discrete set of points. In other words, the equality occurs if and only if there exists a discrete subgroup Γ of $PSU(2,1)$ which acts on B^2 properly discontinuously with at worst isolated fixed points such that $\bar{X}-C$ is biholomorphic to the minimal resolution of $\Gamma\backslash B^2$.

The inequality part of Theorem 2 was first proved in [M] using algebraic geometry. The advantage of our differential geometric proof of Theorem 2, whose original is Yau's method in the proof of Calabi's conjecture [Y1], gives us a point of view to regard the inequality as the precise deviation of the canonical Einstein-Kähler V-metric from the ball-metric.

1. Remarks and Examples

1. If $C=\emptyset$ and D contains only (-2)-curves, Theorem 1 implies the existence of the canonical Einstein-Kähler V-metric on the canonical model X of $\bar{X}$, where $\bar{X}$ is a minimal surface of general type (cf. [K1]). The first Chern class $c_1(\bar{X})$ of $\bar{X}$ is represented by the Kähler form of the canonical Einstein-Kähler V-metric in the current sense. In this case, the inequality in Theorem 2 becomes

$$0 \leqq 3\left\{\sum_p e(D_p)-(1/|G_p|)\right\} \leqq 3c_2(\bar{X})-c_1^2(\bar{X}),$$

where p runs over the rational double points on X and D_p is the corresponding exceptional set. The values of the invariant $k(p)=e(D_p)-(1/|G_p|)$ is $n(n+2)/(n+1)$, $(4m^2-4m-9)/4(m-2)$, 167/24, 383/48, 1079/120, according as p (or D_p) is of type

A_n, D_m, E_6, E_7, E_8, respectively. Therefore the first inequality becomes equality if and only if $K_{\bar{X}}$ is ample and the second inequality becomes equality if and only if there exists a discrete group $\Gamma \subset PSU(2,1)$ which acts on B^2 properly discontinuously with at worst isolated fixed points such that $\bar{X}$ is biholomorphic to the minimal resolution of $\Gamma \backslash B^2$. The following example is due to Hirzebruch: Let X' be a surface in $P_4(\mathbb{C})$ defined by $\sum_{i=0}^{4} Z_i^5=0$ and $\sum_{i=0}^{4} Z_i^{15}=0$. X' has 50 singularities each of which is resolved in a smooth curve of genus 6 with self-intersection number -5 and 1875 rational double points of type A_4. Let X be the partial minimal resolution over the former 50 singularities and $\bar{X}$ the full minimal resolution. Then $\bar{X}$ is of general type and X is its canonical model. It is shown that $3c_2(\bar{X})-c_1^2(\bar{X})=27{,}000$ and $k(A_4)=24/5$. The middle term of the above inequality is $3\times 1875\times(24/5)=27{,}000$. Hence we obtain X by taking the quotient variety of B^2 with respect to some discrete subgroup Γ of $PSU(2,1)$ with isolated fixed points of type A_4.

2. In [H2], Hirzebruch constructed the following example. There is a sequence $X_n (n=2,3,\ldots)$ of minimal surfaces of general type with the following properties: $c_2(X_n)=n^7$, $3c_2(X_n)-c_1^2(X_n)=4n^5$. X_n carries $4n^4$ smooth disjoint elliptic curves C_n. The pair $(X,C)=(X_n,C_n)$ satisfies the assumptions in Theorem 2 and the equality

$$(K_{X_n}+C_n)^2=3e(X_n-C_n)=3n^7$$

holds. Therefore the universal covering of X_n-C_n is biholomorphic to B^2.

3. Hirzebruch presented the following example in his lecture at Osaka University, 1984. Let $L=\bigcup_{j=1}^{21} L_j$ be a line configuration on $P_2(\mathbb{C})$ such that L has 28 triple points and 21 quadruple points. Write $l_j=0$ for the defining equation of L_j. Let X' be a surface obtained by blowing up all r-ple points ($r\geqq 3$) in the line configuration. Let $\bar{X}\to X$ be the Abelian covering of order 2^{20} obtained from the Abelian extension

$$\mathbb{C}(P_2(\mathbb{C}))(\sqrt{l_2/l_1},\ldots,\sqrt{l_{21}/l_1})/\mathbb{C}(P_2(\mathbb{C})).$$

It is shown that $\bar{X}$ contains $28\cdot 2^{17}$ disjoint (-2)-curves D_i and $21\cdot 2^{16}$ disjoint elliptic curves C_i with $C_i\cdot C_i=-4$. Let X be obtained by contracting these $28\cdot 2^{17}$ (-2)-curves. Then $((X,C_i),\bar{X},D_i)$ satisfies the assumptions in Theorem 2. The numerical invariants of $\bar{X}$ are

$$3c_2(\bar{X})-c_1^2(\bar{X})=21\cdot 2^{20} \quad \text{(using Höfer's formula [Hö])}$$

$$\sum_i C_i\cdot C_i=(-4)\cdot 21\cdot 2^{16},$$

$$\sum_i e(D_i)-(1/|G_i|)=(2-1/2)\cdot 28\cdot 2^{17}=21\cdot 2^{18}.$$

Therefore

$$\Big(K_{\bar{X}}+\sum_i C_i\Big)^2-3\Big\{c_2(\bar{X})-\sum_i e(C_i)-\sum_i e(D_i)+(1/|G_i|)\Big\}$$
$$=-21\cdot 2^{20}+4\cdot 21\cdot 2^{16}+3\cdot 21\cdot 2^{18}=0.$$

It follows from Theorem 2 that $\bar{X} - C_i$ is biholomorphic to the minimal resolution of $\Gamma\backslash B^2$ for some discrete subgroup $\Gamma \subset PSU(2,1)$ which acts on B^2 properly discontinuously with isolated fixed points of type A_1.

2. Zariski Decomposition

Let D be a rational divisor $\in \mathrm{Div}(X)\otimes\mathbb{Q}$ on a smooth surface X. D is a formal sum $\sum_{i=1}^{\infty} d_i D_i$, where $d_i \in \mathbb{Q}$ and $\{D_i\}_{i\in\mathbb{Z}}$ a locally finite sequence of irreducible curves on X. We call D *effective* if all d_i's are non-negative and not all zero. Assume $D \in \mathrm{Div}(X)\otimes\mathbb{Q}$ is effective. Then the Zariski Decomposition Theorem is:

Theorem A (Theorem 7.7 in [Z]). *There exists one and only one effective rational divisor N having the following properties:*

(1) *Either $N=0$ or the intersection matrix of the curve in* Supp(N) *is negative definite;*

(2) *$D-N$ is numerically effective, i.e., the intersection number with every irreducible curve is non-negative;*

(3) $(D-N)\cdot N=0$;

Furthermore we have necessarily $P:=D-N$ is an effective rational divisor.

The decomposition $D=P+N$ is called the *Zariski decomposition*. We call P and N the *positive* and *negative* part, respectively. We shall use the following result due to Zariski (cf. p. 612 of [Z]):

Theorem B [Z]. *Let D be an effective divisor with the Zariski decomposition $D=P+N$. If B_n denotes the fixed component of $|nD|$, then under the assumption that* $\dim|hD|>0$ *for some h, the divisor $\bar{B}_n = B_n - nN$ is effective and is bounded from above in n, i.e., the number of irreducible curves which appears in $\bar{B}_n$ is bounded in n and the coefficients of these curves are bounded in n. If in addition D is numericallly effective, then B_n is bounded above in n.*

We collect some Lemmas on effective divisors on a compact smooth surface.

Lemma 1. *Let L be an effective divisor and $L=P+N$ the Zariski decomposition. Then if $\kappa(L)\geqq 1$, we have $\kappa(L)=\kappa(P)$.*

Proof. Let B_m be the fixed component of $|mL|$. It follows from Theorem B that $B_m = mN + \bar{B}$, where $\bar{B}$ is an effective divisor. Thus we get a natural isomorphism $H^0(X, \mathcal{O}(mL)) \cong H^0(X, \mathcal{O}(mP))$ if mP is an integral divisor. Q.E.D.

Lemma 2. *If L is an effective divisor and B the fixed component of $|L|$. Then $(L-B)\cdot B \geqq 0$.*

Proof. Let $B=\sum_{i=1}^{r} B_i$ be the decomposition into irreducible components. For every i, there is a holomorphic section s_i of $[L-B]$ such that B_i is not a component of (s_i). Pick a point p_i from $B_i \backslash \left(\bigcup_{j\neq i} B_j\right)$ so that $s_i(p_i) \neq 0$. Clearly we can choose complex numbers a_i for $i=1,\dots,r$ such that $\sum_{i=1}^{r} a_i s_i(p_j) \neq 0$ for every j. Let $E=\left(\sum_{i=1}^{r} a_i s_i\right) \in |L-B|$. Then no B_i is a component of E. Thus $(L-B)\cdot B = E\cdot B \geqq 0$. Q.E.D.

For an effective divisor L we define a rational map $\Phi_{|L|}$ of X into $P_N(\mathbb{C})$ by $\Phi_{|L|}(x)=(\sigma_0(x):\ldots:\sigma_N(x))\in P_N(\mathbb{C})$, where $\{\sigma_0,\ldots,\sigma_N\}$ is a $\mathbb{C}$-basis for $H^0(X,\mathcal{O}(L))$ and $N=\dim|L|+1$.

Lemma 3. *Let L be a divisor which is effective and numerically effective. Then $\kappa(L)=2$ if and only if $L^2>0$.*

Proof. If part follows from the Riemann-Roch formula. To prove only-if part, let B_m be the fixed component of mL. It follows from Theorems A and B that B_m is bounded above in m. Since $h^0(X,\mathcal{O}(mL-B_m))=h^0(X,\mathcal{O}(mL))\to\infty$ as $m\to\infty$, we must have $(mL-B_m)^2\geqq 0$. Since L is numerically effective, we have $L\cdot B_m\geqq 0$. It follows from Lemma 2 that $(mL-B_m)\cdot B_m\geqq 0$. Now suppose $L^2=0$. Then $0=(mL)^2=(mL)\cdot B_m+(mL-B_m)^2+(mL-B_m)\cdot B_m\geqq 0$. Hence we must have $(mL-B_m)^2=0$. This implies that $|mL-B_m|$ has no base points. Therefore $\Phi_{|mL-B_m|}$ is a holomorphic mapping into some $P_N(\mathbb{C})$. Since $\kappa(mL-B_m)=\kappa(mL)=2$, we see that $(mL-B_m)^2=$ "the volume of $\Phi_{|mL-B_m|}(X)$ with respect to the Fubini-Study metric" >0. Contradiction. Hence we get $L^2>0$. Q.E.D.

Now let X, C, $\bar{X}$, D be as in Theorem 1. We assume the conditions (1) to (4) in Theorem 1. Furthermore we assume $\kappa(K_{\bar{X}}+D+C)=2$, i.e., $h^0(\bar{X},\mathcal{O}(K_{\bar{X}}+D+C))$ grows like m^2. Let $K_{\bar{X}}+D+C=P+N$ be the Zariski decomposition. We decompose D into irreducible components D_i $(1\leqq i\leqq s)$. It is well known that each D_i is $P_1(\mathbb{C})$ (cf. [B]). Let $\mu_i\in\mathbb{Q}$ be defined by requiring $K_{\bar{X}}+\sum_{i=1}^{s}\mu_iD_i$ should be trivial near D. μ_i's are uniquely determined and $0\leqq\mu_i<1$. It is justified in the following argument: Let $\pi:(Y,D)\to(X,p)$ be the minimal resolution of a quotient singularity $(X,p)=(G\backslash B, G\cdot 0)$ where B is a ball and G is a finite subgroup of $U(2)$ with only one fixed point 0. Set $h=|G|$. Then $(dx\wedge dy)^h$ is a G-invariant h-ple holomorphic 2-form on B. Thus there exists a h-ple holomorphic 2-form ω on $Y-D$. On the other hand, since $h^1(Y,\mathcal{O})=\dim R^1\pi_*\mathcal{O}=0$ and $H^2(Y,\mathcal{O})=0$, we have $H^1(Y,\mathcal{O}^*)=H^2(Y,\mathbb{Z})=\mathbb{Z}^s$. Thus every element of $H^1(Y,\mathcal{O}^*)$ can be written as $\sum_{i=1}^{s}\lambda_iD_i$ for some rational numbers λ_i. Hence ω must be a h-ple meromorphic 2-form on Y. Since ω is locally $L^{2/h}$ integrable at D, the order of the pole along D is at most $h-1$. Hence, if $(\omega)=\sum_{i=1}^{s}\lambda_iD_i$, we have $-h<\lambda_i$. μ_i's corresponding to $K_{\bar{X}}$ are determined in the following way: Set $a_i=K_{\bar{X}}\cdot D_i$, $a_{ij}=D_i\cdot D_j$. Then $a_i=\sum_{j=1}^{s}a_{ij}\lambda_j$. It is easily shown that if every $a_i\geqq 0$, then every $\lambda_i\leqq 0$. Here, we have used the fact that (a_{ij}) is symmetric and negative definite with $a_{ij}>0$ for $i\neq j$ and $a_{ii}<0$. Since every D_i is a rational curve with self-intersection number -2, we have $a_i\geqq 0$. Thus $-1<\lambda_i/h=\mu_i\leqq 0$.

Lemma 4. *Let $K_{\bar{X}}+D+C=P+N$ be the Zariski decomposition. Then*

$$P=K_{\bar{X}}+\sum_{i=1}^{s}\mu_iD_i+C$$

and

$$N=\sum_{i=1}^{s}(1-\mu_i)D_i,$$

where every μ_i is a non-negative rational number less than one and is determined uniquely by requiring that $K_{\bar{X}} + \sum_i \mu_i D_i$ should be trivial near D.

Proof. Set $N = \sum_{i=1}^{s} (1-\alpha_i)D_i + \sum_{j=1}^{r} \beta_j B_j$, where α_i, β_j are rational numbers with $\alpha_i \leqq 1$, $\beta_j \geqq 0$ and B_j are irreducible curves on $\bar{X}$. We rewrite this as $N = \sum_{i=1}^{s} (1-\mu_i)D_i + E - F + G$. Then $P = K_{\bar{X}} + \sum_{i=1}^{s} \mu_i D_i + C - E + F - G$, where E, F, G are defined by $\sum_{\mu_k > \alpha_k} (\mu_k - \alpha_k)D_k$, $\sum_{\mu_l < \alpha_l} (\alpha_l - \mu_l)D_l$, $\sum_{j=1}^{r} \beta_j B_j$, respectively. Since P is numerically effective, we have $P \cdot \left(\sum_i (1-\mu_i)D_i + E + G\right) \geqq 0$. Since $K_{\bar{X}} + \sum_i \mu_i D_i \sim 0$ near D, $C \cap D = \emptyset$ and the intersection matrix of $\mathrm{Supp}(D)$ is negative definite, we get $P \cdot F \leqq F^2 \leqq 0$. But $P \cdot \left(\sum_i (1-\mu_i)D_i + E - F + G\right) = P \cdot N = 0$. Thus we get $\mu_i \geqq \alpha_i$ for all i. Hence P can be written as $P = K_{\bar{X}} + \sum_i \mu_i D_i + C - \sum_i (\mu_i - \alpha_i)D_i - \sum_j \beta_j B_j$ with $\alpha_i \leqq \mu_i$ and $\beta_j \geqq 0$. We claim that $P' = K_{\bar{X}} + \sum_i \mu_i D_i + C$ is numerically effective. Note that P' is effective. If E is a curve contained in $\mathrm{Supp}(C+D)$, we have $P' \cdot E \geqq 0$ by our assumption. If E is a curve which is not a component of $\mathrm{Supp}(C+D)$, then $P' \cdot E$ must be non-negative. Indeed, if otherwise, we have $K_{\bar{X}} \cdot E < 0$ and $E^2 < 0$. Thus E is a (-1)-curve with $C \cdot E = 0$ which has been excluded by our assumption. Therefore P' is numerically effective. From the uniqueness of the Zariski decomposition, $P = P'$. Q.E.D.

Proposition 1. *Under the assumption of Theorem 1, P has the following properties:*

(1) $P^2 > 0$;

(2) *P is numerically effective;*

(3) *For every irreducible curve E on $\bar{X}$, $P \cdot E = 0$ holds only if E is a component of* $\mathrm{Supp}(D+C)$.

Proof. From Lemmas 1, 3, and 4, we have $P^2 > 0$. (2) is clear. If E is an irreducible curve which is not a component of $\mathrm{Supp}(D+C)$ and $P \cdot E = 0$, then $K_{\bar{X}} \cdot E \leqq 0$ and $E^2 < 0$. Thus E is a (-1)-curve with $C \cdot E \leqq 1$ or a (-2)-curve which is not contained in $\mathrm{Supp}(D)$. But both have been excluded by our assumption. Q.E.D.

Let m be a positive integer such that mP is an integral divisor. We write $\mathscr{E}$ for the union of all irreducible curves E with $P \cdot E = 0$, and $\mathscr{E} = \sum \mathscr{E}_\nu$ the decomposition into connected components in the usual topology.

Proposition 2 (cf. Theorem (5.8) in [S]). *Under the assumption of Theorem 1, there exists a positive number $n_0 = n_0(\bar{X}, D, C)$ with the following properties: for any integer $n > n_0$,* (i) *$|nmP|$ has no base points;* (ii) *If $N+1 = h^0(\bar{X}, \mathcal{O}(nmP))$ and $\{\sigma_0, \ldots, \sigma_N\}$ is a $\mathbb{C}$-basis for $H^0(\bar{X}, \mathcal{O}(nmP))$, then the map $\Phi = \Phi_{|nmP|} : \bar{X} \to P_N(\mathbb{C})$ is a holomorphic map whose restriction to $\bar{X} - \mathscr{E}$ is a biholomorphic map onto its image and $\Phi^{-1}\Phi(z)$ is equal to either z or $\mathscr{E}_\nu$ according to $z \in \bar{X} - \mathscr{E}$ or $z \in \mathscr{E}_\nu$, respectively.*

Proof. The proof is almost the same as that of Theorem (5.8) in [S]. In 105–110 of [S], Sakai proved a sequence of Lemmas concerning $m\Gamma$ where Γ is the adjoint divisor of a semi-stable curve C. It is easy to modify these Lemmas into suitable forms to our purposes. Roughly speaking, these Lemmas remain valid if we replace Γ by our mP, where $P=K_{\bar{X}}+\sum_i \mu_i D_i+C$. Q.E.D.

3. Proof of Theorem 1

From Proposition 2, we get the following:

Lemma 5. *Under the assumption of Theorem 1, there exists a real closed* $(1,1)$ *form* γ *representing* $c_1(mP)$ *in the de Rham cohomology having the following properties:*

(i) γ *is positive definite outside of* $\mathscr{E}$; (ii) *For any irreducible component* E *of* $\mathscr{E}$, $i_E^*\gamma$ *vanishes where* i_E *is the inclusion of* E *into* $\bar{X}$.

Proof. Set $n\gamma=\Phi^*$ [the Fubini-Study metric form on $P_N(\mathbb{C})$] for large m. Then γ has the required properties. Q.E.D.

From here on, we regard X as a Satake V-surface.

Definition. Let p be a singular point of X, and G the finite subgroup of $U(2)$ corresponding p. There is a strictly pseudo-convex neighborhood U of p such that $(U,p)\cong(G\backslash B, G\cdot 0)$, where B is a ball with center 0. We call the quotient map $B\to U$ with respect to G a *local uniformization* around p. Let T be a smooth tensor field on the regular part of X. We call T a $V-C^\infty$ tensor field if we obtain T by pushing down a C^∞ tensor field in terms of local uniformizations.

Let $D=\sum_\mu D_\mu$, $C=\sum_\beta C_\beta$ be the decomposition into connected components, and $D=\sum_i D_i$, $C=\sum_m C_m$ the decomposition into irreducible components. For each i, m, let s_i and t_m be the holomorphic sections of $[D_i]$ and $[C_m]$ such that $(s_i)=D_i$ and $(t_m)=C_m$. There exist a smooth volume form Ω on $\bar{X}$, Hermitian metrics for $[D_i]$ and $[C_m]$ such that the singular volume form $\Omega'=\Omega\Big/\prod_{i,m}|s_i|^{2\mu_i}|t_m|^2$ satisfies $\gamma=-\mathrm{Ric}\,\Omega'$, where γ is as in Lemma 5 and $\mathrm{Ric}\,V=-\sqrt{-1}\,\partial\bar\partial\log v$ for a volume form $V=v\prod_i(\sqrt{-1}\,dz^i\wedge d\bar{z}^i)$. Now let $(U,p)\cong(G\backslash B, G\cdot 0)$ be a neighborhood of a singular point of X and $(\tilde{U}, D_\mu)$ its minimal resolution. Since the usual flat volume element is $U(2)$ invariant, we can push this down to $U-\{p\}=\tilde{U}-D_\mu$. We write v_μ for the resulting volume form on $U-\{p\}$. From the arguments just before Lemma 4, we see that there is a C^∞ function $\tilde{h}_\mu$ defined in $\tilde{U}$ such that $\Omega'=\tilde{h}_\mu v_\mu$ on $\tilde{U}$. We define a C^∞ function h_μ on $B-\{0\}$ by $\pi^*(\tilde{h}_\mu|U-\{p\})$, where $\pi: B\to U$ is the quotient map. On the other hand, let $\hat{h}_\mu$ be a local Kähler potential for the Fubini-Study form on $P_N(\mathbb{C})$. Thus we get a C^∞ function $h'_\mu=\pi^*\Phi^*\hat{h}_\mu$ on B. Since $\partial\bar\partial h_\mu=\partial\bar\partial h'_\mu$ on $B-\{0\}$, $h'_\mu\in C^\infty(B)$, and $h_\mu\in C^\infty(B-\{0\})$, we must have $h_\mu\in C^\infty(B)$. Summing up, we get the following:

Lemma 6. $\Omega'=\Omega\Big/\prod_{i,m}|s_i|^{2\mu_i}|t_m|^2$ *is a* $V-C^\infty$ *volume form on* $X-C$.

The connected components $\mathscr{E}_\nu$ of $\mathscr{E}$ are classified in the following:

Lemma 7 (cf. Lemma 1 of [K2]). *Each $\mathscr{E}_\nu$ is one of the following five types:*
(1) *$\mathscr{E}_\nu = D_\mu$ for some μ;*
If $\mathscr{E}_\nu$ is contained in C, $\mathscr{E}_\nu$ is one of the followings:
(2) *a smooth elliptic curve with negative self-intersection number;*
(3) *a cycle of smooth rational curves with self-intersection numbers $\leqq -2$ and some of them $\leqq -3$;*
(4) *a rational curve with a node with negative self-intersection number;*
(5) *a chain of smooth rational curves with self-intersection numbers $\leqq -2$.*

Proof. See the proof of Lemma 1 in [K2]. Q.E.D

Lemma 8. *Let $(Y, A) \to (X, p)$ be the minimal resolution of an isolated 2-dimensional singularity where A is a smooth elliptic curve with self-intersection number $-b<0$. Then by replacing X by an appropriate neighborhood of p if necessary, we can express $X-\{p\}$ as an orbit space in the following way: There exist a positive number a, real numbers α, β and a lattice $\mathscr{L}$ in $\mathbb{C}$ defined by $\{m+n\omega;\ m, n \in \mathbb{Z}, \operatorname{Im}(\omega)=a\}$ such that*

$$Y-A = X-\{p\} \cong \Gamma \backslash W,$$

where $W = \{(u,v) \in \mathbb{C}^2;\ \operatorname{Im}(u) - |v|^2 > L\}$ for some positive number L,

$$\Gamma = \left\{ \begin{pmatrix} 1 & 2i\bar{\gamma} & i|\gamma|^2 - 2h(\gamma) \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix};\ \begin{matrix} \gamma \text{ runs over } \mathscr{L}, \text{ and for each } \gamma = m+n\omega, \\ h(\gamma) \text{ runs over the class of } m\alpha + n\beta - mna \\ \text{modulo } (2a/b)\mathbb{Z}. \end{matrix} \right\}.$$

The action of the above matrix is to send (u,v) into $(u+2i\bar{\gamma}v + i|\gamma|^2 - 2h(\gamma),\ v+\gamma)$.

Proof. See the proof of Lemmas 4 and 5 in [K2]. Q.E.D.

Since $\mathscr{S} = \{(u,v) \in \mathbb{C}^2;\ \operatorname{Im}(u) - |v|^2 > 0\}$ is biholomorphic to B^2, $\mathscr{S}$ admits a complete Kähler metric form with constant holomorphic curvature. The Kähler potential of this metric form is given by $\log F$, where $F(u,v) = (\operatorname{Im}(u) - |v|^2)^{-1}$. Since $F(u,v)^{-1}$ is invariant under the action of Γ, this projects down to a function f_A on $X-\{p\}$. Thus we obtain in a canonical way the function f_A having the property that $-\sqrt{-1}\,\partial\bar{\partial}\log f_A$ is the ball metric near A. Let $\{\mathscr{E}_{\nu(1)}\}$ be the union of connected components of $\mathscr{E}$ of type (2) in Lemma 7. For each $\nu(1)$, we have got in a canonical way the function $f_{\nu(1)}$ defined in a neighborhood of $\mathscr{E}_{\nu(1)}$ minus $\mathscr{E}_{\nu(1)}$, whose $-\sqrt{-1}\,\partial\bar{\partial}\log$ is the ball-metric near $\mathscr{E}_{\nu(1)}$.

Lemma 9. *Let $(Y, B) \to (X, p)$ be the minimal resolution of an isolated 2-dimensional singularity where B is a cycle $\sum_{i=1}^{r-1} B_i$ of $\mathbb{P}^1$'s with $-B_i \cdot B_i = b_i \geqq 2$, some $b_i \geqq 3$ and $r \geqq 2$ or a rational curve B_0 with a node with $-B_0 \cdot B_0 = b_0 \geqq 1$. Let w_k be an irrational quadratic number defined by $\lim_{s\to\infty}\{q_k - (q_{k-1} - (\ldots - (q_{s-1} - q_s^{-1})^{-1}\ldots)^{-1})^{-1}\}$, where $\{q_k\}$ is a periodic sequence with period r defined by $q_k = b_k$ for $0 \leqq k \leqq r-1$ when $r \geqq 2$ or $q_0 = b_0 + 2$ when $r=1$. Let $\{R_k\}_{k\in\mathbb{Z}}$ be a sequence defined by $R_{-1} = w_0$, $R_0 = 1$, $q_k R_k = R_{k-1} + R_{k+1}$ for $k \in \mathbb{Z}$. Let M be a free $\mathbb{Z}$-module of* rank 2 *generated* *by* 1 *and* w_0, V *the infinite cyclic multiplicative group generated by* R_r. *Then by replacing* X *by an appropriate neighborhood of* p *if necessary, we obtain*

$$Y - B = X - \{p\} \cong \Gamma \backslash W,$$

where $W = \{(z_1, z_2) \in \mathbb{C}^2; y_1 > 0, y_2 > 0, y_1 y_2 > L$, *where* $z_i = x_i + \sqrt{-1}\, y_i\}$, *for some positive number* L, *and*

$$\Gamma = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix}; \varepsilon \in V, \mu \in M \right\}.$$

The action of the above matrix is to send (z_1, z_2) *into* $(\varepsilon z_1 + \mu, \varepsilon' z_2 + \mu')$, *where* $'$ *means to take the conjugate over* $\mathbb{Q}$.

Proof. See [H1]. Q.E.D.

The product metric of the Poincaré metric on $H^2 = \{(z_1, z_2) \in \mathbb{C}^2; \operatorname{Im}(z_1) > 0, \operatorname{Im}(z_2) > 0\}$ is given by $\underline{i}\,\partial\bar{\partial} \log F$, with $F(z_1, z_2) = (\operatorname{Im}(z_1) \operatorname{Im}(z_2))^{-1}$ which is invariant under the action of Γ, we obtain in a canonical way the function f_B defined on $X - \{p\}$ whose $\underline{i}\,\partial\bar{\partial} \log$ is the H^2-metric near B. Let $\{\mathscr{E}_{v(2)}\}$ be the union of $\mathscr{E}_v$ of type (3) or (4) in Lemma 7. For each $v(2)$, we have got in a canonical way the function $f_{v(2)}$ defined in a neighborhood of $\mathscr{E}_{v(2)}$ minus $\mathscr{E}_{v(2)}$ whose $-\underline{i}\,\partial\bar{\partial} \log$ is the H^2-metric near $\mathscr{E}_{v(2)}$.

Lemma 10. *Let* $(Y, E) \to (X, p)$ *be the minimal resolution of an isolated 2-dimensional singularity where* $E = \sum_{k=1}^{r} E_k$ *is a chain of* $\mathbb{P}^1$*'s with* $b_k = -E_k \cdot E_k \geqq 2$. *Let* n/q *be the irreducible representation of* $b_1 - (b_2 - (\ldots(b_{r-1} - b_r^{-1})^{-1} \ldots)^{-1})^{-1}$ *and* Γ *the cyclic group generated by* $\begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix}$, *where* $\xi = \exp(2\pi \underline{i}/n)$. *Then have* $(X, p) \cong (\Gamma \backslash B, \Gamma \cdot 0)$ *by replacing* X *by an appropriate neighborhood of* x *if necessary.*

Proof. See for example [B]. Q.E.D.

The product metric of the Poincaré metrics on the unit punctured disk on D^{*2} has a Kähler potential $F(z_1, z_2) = (\log|z_1|^{-2})^{-1} (\log|z_2|^{-2})^{-1}$, which is invariant under the action of Γ. Thus we get in a canonical way the function f'_E whose $-\underline{i}\,\partial\bar{\partial} \log$ is the D^{*2}-metric near E. Let E_0 and E_{r+1} be the irreducible components of $C - \mathscr{E}_{v(3)}$ intersecting E_1 and E_r, respectively, where $E = \mathscr{E}_{v(3)}$, one of the $\mathscr{E}_v$ of type (5) in Lemma 7. We can extend f'_E to the function f_E defined in a deleted neighborhood of $E_0 \cup E \cup E_{1+r}$ which is written as $\log|\sigma_0|^2$ and $\log|\sigma_{r+1}|^2$ in the newly added domain of definition of f_E near E_0 and E_{r+1}, respectively.

Definition. Let V be a domain in $\mathbb{C}^m$. Let X be an m-dimensional complex manifold and ϕ a holomorphic map of V into X. ϕ is called a *quasi-coordinate map* if ϕ is of maximal rank everywhere. In this case, (V, ϕ) is called a *local quasi-coordinate* of X.

Definition. Let X be a complex V-manifold with at worst isolated quotient singularities and ω a complete Kähler V-metric on X. We say that (X, ω) has *bounded geometry* if the following conditions hold: There exist a system of quasi-coordinates $\mathscr{V} = \{(V_\alpha, v_\alpha^i)\}$ and a neighborhood $U(p)$ of each quotient singularity p

such that $(U(p),p)\cong(G_p\backslash B, G_p\cdot 0)$ and $U(p)\cap U(q)=\emptyset$ having the following properties:

(i) $\bigcup_\alpha(\text{Image of } V_\alpha)\cup\bigcup_p U(p)=X$;

(ii) there are positive numbers ε and δ independent of α such that $B(\varepsilon)\subset V_\alpha\subset B(\delta)$;

(iii) there are positive constants c and $\mathscr{A}_k$ $(k=0,1,\dots)$ such that $c^{-1}(\delta_{ij})<(g_{\alpha i\bar{j}})<c(\delta_{ij})$ and

$$|\partial^{|p|+|q|}g_{\alpha i\bar{j}}/\partial v_\alpha^p\,\partial\bar{v}_\alpha^q|<\mathscr{A}_{|p|+|q|}$$

for all multi-indices p, q where $\omega=\underline{i}g_{\alpha i\bar{j}}dv_\alpha^i\,d\bar{v}_\alpha^j$ in terms of (v_α^i) for all α, and the same inequalities hold for all lifted metrics from $U(p)$'s.

Definition. Let u be a function of class $V-C^\infty$. For a nonnegative integer k and $\alpha\in(0,1)$ the $V-C^{k,\alpha}(X)$ *norm* of u is defined by

$$|u|_{V,k,\alpha}=\max(A,B)$$

with

$$A=\sup\Big\{\sup_{\zeta\in B}\sum_{|p|+|q|\le k}|\partial^{|a|+|b|}\pi_p^*u(\zeta)/\partial\zeta^a\,\partial\bar{\zeta}^b|$$
$$+\sup_{\substack{\zeta,\zeta'\in B\\ \zeta\neq\zeta'}}\sum_{|a|+|b|=k}|\zeta-\zeta'|^{-\alpha}$$
$$\cdot|(\partial^k\pi_p^*u(\zeta)/\partial\zeta^a\,\partial\bar{\zeta}^b)-(\partial^k\pi_p^*u(\zeta')/\partial\zeta^a\,\partial\bar{\zeta}^b)|\Big\},$$

$$B=\sup_\alpha\Big\{\sup_{z\in V_\alpha}\sum_{|a|+|B|\le k}|\partial^{|a|+|b|}u(z)/\partial v_\alpha^a\,\partial\bar{v}_\alpha^b|$$
$$+\sup_{\substack{z,z'\in V_\alpha\\ z\neq z'}}\sum_{|a|+|b|=k}|z-z'|^{-\alpha}$$
$$\cdot\Big|\Big(\partial^k u(z)/\partial v_\alpha^a\,\partial\bar{v}_\alpha^b\Big)-\Big(\partial^k u(z')/\partial v_\alpha^a\,\partial\bar{v}_\alpha^b\Big)\Big|\Big\},$$

where $\pi_p: B\to U(p)$ is the quotient map around p.

Definition. For a nonnegative integer k and $\alpha\in(0,1)$, the function space $V-C^{k,\alpha}(\bar{X}-C)$ is the Banach space obtained as the completion of $\{u\in V-C^\infty(X); |u|_{V,k,\alpha}<\infty\}$ with respect to the norm $|\cdot|_{V,k,\alpha}$.

Let $\mathscr{E}'_v$ be defined by:

(1) If $\mathscr{E}_v$ is of type (1), then $\mathscr{E}'_v=\emptyset$;

(2) If $\mathscr{E}_v$ is of type (2) or (3) or (4), then $\mathscr{E}'_v=\mathscr{E}_v$;

(3) If $\mathscr{E}_v$ is of type (5), then $\mathscr{E}'_v$ is the union of $\mathscr{E}_v$ and the irreducible curves of C intersecting $\mathscr{E}_v$.

Lemma 11. *There exist a smooth volume form Ω on $\bar{X}$, Hermitian metrics for $[D_i]$ and $[C_m]$, a $V-C^\infty$ function h on $\bar{X}$ and $V-C^\infty$ functions q_v on $\bar{X}-\mathscr{E}'_v$ such that*

$$\Psi=h\Omega\Big/\prod_{i,m}|s_i|^{2\mu_i}|t_m|^2\prod{}'g_v\prod{}''(\log|s_m|^{-2})^2$$

(where $\prod'$ means to take the product over v such that $\mathscr{E}_v$ is contained in C and $\prod''$ means to take the product over m such that C_m is disjoint from $\mathscr{E}$). Is a $V-C^\infty$ volume form on $\bar{X}-C$ with the following properties:

(i) *$\omega=-\mathrm{Ric}\,\Psi$ is a complete Kähler V-metric on $\bar{X}-C$ with finite total volume;*

(ii) *$(\bar{X}-C,\omega)$ has a quasi-coordinate system with bounded geometry;*

(iii) *The function $\log(\Psi/\omega^2)$ is of class $V-C^{k,\alpha}(\bar{X}-C)$ for any k,α with finite $V-C^{k,\alpha}$ norm.*

Sketch of the Proof (detailed discussion is given in Lemma 5 of [K2]). If p is a singular point of X with (X,p) $(G\backslash B, G\cdot 0)$, then h should be the push down of $1-|Z|^2$ defined in B. Hence $\omega=-\mathrm{Ric}\,\Psi$ is essentially the push down of the restriction of the Bergman metric of B^2 to B. Let $\mathscr{E}_v$ be a connected component of $\mathscr{E}$ contained in C. We have shown the existence of the canonical function f_v whose $-\partial\bar{\partial}\log$ gives an Einstein-Kähler metric coming from the ball-metric or the H^2-metric in a deleted neighborhood of $\mathscr{E}_v$. Let g_v be a $V-C^\infty$ function on $\bar{X}-\mathscr{E}'_v$ which is equal to f_v^3 or f_v^2 according as $\mathscr{E}_v$ is of type (2) or of types (3) and (4) near $\mathscr{E}_v$. Then ω is essentially the canonical metric coming from the ball-metric or the H^2-metric. Thus to consider $-\mathrm{Ric}\,\Psi$ is to make $-\mathrm{Ric}\,\Omega'$ more positive by adding the ball-metric or the H^2-metric of cusps near $\mathscr{E}_v$ and by adding the Poincaré metric of the transversal punctured disk to C_m disjoint from $\mathscr{E}$. It is a straightforward computation to verify that $\log(\Psi/\omega^2)$ belongs to $V-C^{k,\alpha}(\bar{X}-C)$ for any k,α. Q.E.D.

We expect that ω is a good approximation of the desired Einstein-Kähler metric. Hence we deform ω into $\omega+\underline{i}\,\partial\bar{\partial}u=\tilde{\omega}$ such that $\tilde{\omega}$ is Einstein-Kähler.

Proof of Theorem 1. Theorem 1 is a direct consequence of Lemma 11 and the following general result:

Theorem C. *Let X be a complex n-dimensional V-manifold with a $V-C^\infty$ volume form Ψ such that $\omega=-\mathrm{Ric}\,\Psi$ is a complete Kähler V-metric with bounded geometry and the function $f=\log(\Psi/\omega^n)$ belongs to $V-C^{k,\alpha}(X)$ for any k,α. Then there is a unique solution u of*

$$(\omega+\sqrt{-1}\,\partial\bar{\partial}u)^n=\exp(u+f)\omega^n$$

belonging to $\mathscr{U}=\{u\in V-C^\infty; c^{-1}\omega<\omega+\sqrt{-1}\,\partial\bar{\partial}u<c\omega$ for some positive constant $c\}$. The resulting $\tilde{\omega}=\omega+\underline{i}\,\partial\bar{\partial}u$ is a Ricci-negative complete Einstein-Kähler V-metric on X.

Proof. This is a direct consequence of the following:

Theorem D. *Let M be a V-manifold endowed with a complete Kähler V-metric of class $V-C^{k,\alpha}$ with $k\geqq 5$. If (M,ω) has bounded geometry, then for any f $V-C^{k-2,\alpha}(M)$, there is a unique solution u to*

$$(\omega+\sqrt{-1}\,\partial\bar{\partial}u)^n=\exp(u+f)\omega^n$$

belonging to $\{u\in V-C^{k,\alpha}(M); c^{-1}\omega<\omega+\underline{i}\,\partial\bar{\partial}u<c\omega$ for some positive number $c\}$.

Proof. See [K1, K2] and [C, Y]. Q.E.D.

4. Inequality for Chern Numbers

Let M be a V-surface with at worst isolated quotient singularities. Let g be a Riemannian V-metric. Let $\bar{M}$ be the minimal resolution of M and g_0 a smooth Riemannian metric on $\bar{M}$. Then we have

$$\int_M \mathrm{e}(g) = \mathrm{e}(\bar{M}) - \sum_p (\mathrm{e}(E_p) - 1/|G_p|),$$

where $\mathrm{e}(g)$ is the Euler form of g and p runs over the singularities of M. To prove this, we recall the Gauss-Bonnet Theorem for manifolds with boundary:

Gauss-Bonnet Theorem. *Let (M, g) be a 2k-dimensional compact oriented Riemannian manifold with boundary N and assume that, near N, it is isometric to a product of N and an interval. Then*

$$\int_M \mathrm{e}(g) = \mathrm{e}(M).$$

Let U be $G\backslash B$ where G is a finite subgroup of $U(2)$ and g a Riemannian metric on U which is isometric to a product near the boundary and is a flat V-metric near the singularity $G \cdot 0$. Let $\bar{U}$ be the minimal resolution of U and g_0 a Riemannian metric which is isometric to a product near the boundary and is smooth near E, where E is the exceptional set. Then we apply the Gauss-Bonnet to get

$$|G|^{-1} = \int_U \mathrm{e}(g) = \int_{\bar{U}} \mathrm{e}(g_0) + \lim_{r \to 0} \int_{\partial W(r)} Q,$$

$$\int_{\bar{U}} \mathrm{e}(g_0) = \mathrm{e}(\bar{U}) = \mathrm{e}(\bar{U} \setminus E) + \mathrm{e}(E) = \mathrm{e}(E),$$

where $W(r) = \bar{U} - \pi(B(r))$ and π is the projection. Q is the universal polynomial of the difference of the connection forms and the respective curvature forms. Thus we have

$$\lim_{r \to 0} \int_{\partial W(r)} Q = -(\mathrm{e}(E) - |G|^{-1}).$$

To prove the desired formula, let g be a V-metric on M which is flat near the singularities and g_0 a smooth Riemannian metric on $\bar{M}$. Then we have

$$\int_M \mathrm{e}(g) - \int_{\bar{M}} \mathrm{e}(g_0) = \lim_{r \to 0} \int_{\partial W(r)} Q = -\sum_p (\mathrm{e}(E_p) - |G_p|^{-1}).$$

Let X, C, $\bar{X}$, D be as in Theorem 1. Then there exists a unique complete Einstein-Kähler V-metric $\tilde{\omega}$ on $X - C$ such that $\mathrm{Ric}(\tilde{\omega}) = -\tilde{\omega}$. Let $\tilde{c}_i$ be the i-th Chern form of the Hermitian connection of $\tilde{\omega}$. Set $\omega = -\mathrm{Ric}\,\Psi$, where Ψ is as in Lemma 11.

Lemma 12. *Under the above situation, the following equalities hold:*

$$\int_X \tilde{c}_1^2 = \left(K_{\bar{X}} + \sum_i \mu_i D_i + C\right)^2,$$

$$\int_X \tilde{c}_2 = \mathrm{e}(\bar{X}) - \mathrm{e}(D) - \mathrm{e}(C) + \sum_p |G_p|^{-1}.$$

Proof. The proof is the same as that of Lemma 9 in [K2] except determining the contribution of the singularities of X. Since the proof of the first equality is exactly the same as the corresponding part of the proof of Lemma 9 in [K2], we prove the second equality. Let g_1, g_2, and g_3 be three Riemannian metrics on the regular part of X. They are distinguished from each other according to their behavior at the singularities and C. g_1 is equal to the canonical cusp metric near C and is a flat V-metric near each singular point. g_2 is a smooth Hermitian metric for $\Omega_{\bar{X}}^1(\log C)^*$ and is a smooth flat V-metric near the singularities. g_3 is a smooth Hermitian metric for $\Omega_{\bar{X}}^1(\log C)^*$. Let $W(r) = X - \left(\bigcup_p \pi_p(B(r))\right)$, where π_p is the projection from B onto a neighborhood of a singular point p. Then we have

$$\begin{aligned}\int_X \tilde{c}_2 &= \int_X c_2(g_1) = \int_X c_2(g_2) \quad \text{(by the proof of Lemma 9 in [K2])}\\ &= \lim_{r\to 0} \int_{W(r)} c_2(g_3) + dQ\\ &= c_2(\Omega_{\bar{X}}^1(\log C)) + \lim_{r\to 0} \int_{\partial W(r)} Q\\ &= \mathrm{e}(\bar{X}) - \mathrm{e}(C) - \mathrm{e}(D) + \sum_p |G_p|^{-1}. \quad \text{Q.E.D.}\end{aligned}$$

Proof of Theorem 2. Since $\tilde{\omega}$ is Einstein-Kähler, we have a point-wise inequality $0 \leq 3\tilde{c}_2 - \tilde{c}_1^2$. Integrating this inequality over X and applying Lemma 11 yields the desired inequality. Since $3\tilde{c}_2 - \tilde{c}_1^2$ measures the point-wise deviation of $\tilde{\omega}$ from the ball-metric, the equality in Theorem 2 occurs if and only if $\tilde{\omega}$ is the ball-metric. The rest is the same as the corresponding part of Theorem 2 in [K2]. Q.E.D.

As an application of Theorem 2, we prove a rigidity theorem. Let $\Gamma \subset PSU(2,1)$ be a discrete group of automorphisms of B^2 acting freely. Assume the volume of $X'' = \Gamma \backslash B^2$ is finite. Then X'' is compactified to X' if we add a finite number of cusp points. Let X be the minimal resolution of X' over the cusp points and C the exceptional curve. C consists of mutually disjoint elliptic curves C_i with $C_i^2 < 0$.

Theorem 3. *Let (X, C) be as above and let (Y, D) be a pair of a compact smooth surface Y and a reduced curve D with normal crossings. If there is an oriented homotopy equivalence between (X, C) and (Y, D), then $Y - D$ is biholomorphic to $X - C$.*

Proof. It follows from the assumption that the tubular neighborhood of D is diffeomorphic to that of C. We can identify the connected component D_i of D, which is a torus, with the corresponding component C_i. We have also $K_Y \cdot D_i = D_i^2 = C_i^2 = K_X \cdot C_i$. Since $\mathrm{e}(Y) = \mathrm{e}(X)$ and $\mathrm{Sign}(Y) = \mathrm{Sign}(X)$, we have $K_Y^2 = 2\mathrm{e}(Y) + 3\,\mathrm{Sign}(Y) = 2\mathrm{e}(X) + 3\,\mathrm{Sign}(X) = K_X^2$. Therefore $(K_Y + D)^2 = (K_X + C)^2 = 3(\mathrm{e}(Y) - \mathrm{e}(D))$. We want to show that the logarithmic Kodaira dimension $\bar{\kappa}$ of (Y, D) is 2. To prove this, we need to recall the notion of the minimal model of (Y, D) (see pp. 89–90 of [S] for precise definitions). If C is a (-1)-curve on Y which intersects D at most one point, C is called a *D-exceptional curve*. Since $X - C$ is a $K(\Gamma, 1)$-space, so is $Y - D$. Since $(K_Y + D)^2 > 0$, Y is projective algebraic. Therefore, there are no rational curves on $Y - D$. If E is a D-exceptional curve with $D \cdot E = 1$ and $p : (Y, D) \to (Y', D')$ is the blowing down of E, then $p^*(L') = L$, where $L = K_Y + D$

and $L' = K_{Y'} + D'$. By a finite number of blowing downs of D-exceptional curves, we arrive at the minimal model (Y_0, D_0) of (Y, D), which has no D_0-exceptional curves. If $\bar{\kappa} = -\infty$, it follows from Sakai [S] that (Y_0, D_0) must be (elliptic ruled surface, a section). Thus $\Gamma = \pi_1(X - C) = \pi_1(Y - D) = \mathbb{Z} \oplus \mathbb{Z}$. This is a contradiction, since Γ is not abelian. If $\bar{\kappa} = 0$ or 1, then L_0 must be numerically effective. Therefore, since $L_0^2 = (K_X + C)^2 > 0$, we have $h^2(Y_0, \mathcal{O}(-mL_0 - D_0)) = 0$ for $m > 0$. By the Riemann-Roch formula, $h^0(Y_0, \mathcal{O}(mL_0)) = \mathcal{O}(m^2)$ when $m \to \infty$. But this contradicts the assumption. Therefore we deduce $\bar{\kappa} = 2$. Since (Y_0, D_0) satisfies the condition of Theorem 1, we get the following inequality:

$$L^2 = L_0^2 \leqq 3(\mathrm{e}(Y_0) - \mathrm{e}(D_0)) = 3(\mathrm{e}(Y) - b - \mathrm{e}(D)),$$

where b is the number of the blow downs in $Y \to Y_0$. Hence we have $b = 0$, i.e., $(Y, D) = (Y_0, D_0)$. It follows from Theorem 2 that the universal cover of $Y - D$ is biholomorphic to B^2. By the rigidity theorem of Gerland-Raghunathan-Prasad [P], $Y - D$ is in fact biholomorphic to $X - D$. Q.E.D.

Note. In [Y2], Yau proved the rigidity theorem for compact quotients of B^2: *Let X be a compact smooth surface that is covered by B^2. Then any compact smooth surface that is oriented homotopic to X is biholomorphic to X.*

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