Math. Ann. 270, 63–68 (1985)

Mathematische Annalen

Strongly Plurisubharmonic Exhaustion Functions on 1-Convex Spaces

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1. Introduction

In [3] the first author has proved the following:

Theorem 1.1. *Let X be a complex space which carries a strongly plurisubharmonic exhaustion function $\phi: X \to [-\infty, \infty)$. Then X is holomorphically convex and is obtained from a Stein space by blowing up finitely many points.*

The conclusion "obtained from a Stein space by blowing up finitely many points" means precisely that there exist:

– a compact analytic set $S \subset X$ with $\dim_x S > 0$ for any $x \in S$;

– a Stein space Y, a finite set $A \subset Y$ and a proper holomorphic map $p: X \to Y$ inducing a biholomorphism $X \setminus S \cong Y \setminus A$ and which satisfies $p_* \mathcal{O}_X \cong \mathcal{O}_Y$.

Following a customary terminology X is called an 1-convex space, S its exceptional set and Y the Remmert reduction of X.

Theorem 1.1. was conjectured by Fornaess-Narashimhan in [5].

In this paper we prove the converse:

Theorem 1.2. *Let X be an 1-convex space. Then X carries a strongly plurisubharmonic exhaustion function $\phi: X \to [-\infty, \infty)$. Moreover, ϕ can be chosen $-\infty$ exactly on the exceptional set S of X and real analytic outside S.*

This theorem together with Theorem 1.1. and the results of Andreotti-Grauert [1] and Narasimhan [9] gives the following characterization of 1-convex spaces:

Theorem. *Let X be a complex space. Then the following statements are equivalent:*

(i) *X is an 1-convex space.*

(ii) $\dim_{\mathbb{C}} H^q(X, \mathcal{F}) < \infty$ *for any* $q > 0$ *and any coherent analytic sheaf* $\mathcal{F}$ *on X.*

(iii) *X carries a continuous exhaustion function which is strongly plurisubharmonic outside a compact set.*

(iv) *X carries a strongly plurisubharmonic exhaustion function* $\phi: X \to [-\infty, \infty)$.

(i)⇔(iv) could be called the Levi problem with discontinuous functions.

The key ingredients of the proof of Theorem 1.2 are the identity between plurisubharmonic functions and weakly plurisubharmonic functions ([5], Theorem 5.3.1) and the analytic version of Chow's lemma.

2. Preliminaries

Unless otherwise specified, all complex spaces are assumed to be reduced and countable at infinity.

An upper semicontinuous function on a complex space $\phi: X \to [-\infty, \infty)$ is said to be plurisubharmonic if for every holomorphic map $f: D \to X$ (D = the unit disc in $\mathbb{C}$) it follows that $\phi \circ f$ is subharmonic on D (possibly $\equiv -\infty$). ϕ is called strongly plurisubharmonic if for any C^∞ real-valued function θ with compact support in X there is an $\varepsilon_0 > 0$ such that $\phi + \varepsilon\theta$ is plurisubharmonic for $|\varepsilon| \leqq \varepsilon_0$.

It is known (Fornaess-Narashimhan [5], Theorem 5.3.1) that a (strongly) plurisubharmonic function is locally the restriction of a (strongly) plurisubharmonic function in an open set in some $\mathbb{C}^N$ in which X is locally embedded, i.e. the above definition coincides with the usual one as given in [9].

Proposition 2.1. *Let X, Y be complex spaces and $p: X \to Y$ a proper, surjective, holomorphic map. Let $\phi: Y \to [-\infty, \infty)$ be an upper semicontinuous function such that $\phi \circ p$ is plurisubharmonic on X. Then ϕ is plurisubharmonic on Y.*

Proof (sketch). In case ϕ is real-valued and continuous but p is supposed only holomorphic and surjective this is exactly Proposition 1.3 in Borel and Narasimhan [2]. In the general case the proof is a slight modification of theirs. For the sake of completeness we indicate the necessary modifications to be done.

Exactly as in [2] we may assume that Y is the unit disc in $\mathbb{C}$. Since p is proper, for any irreducible component X_0 of X it follows that $p|_{X_0}$ is constant or $p(X_0) = Y$. Therefore we may assume X irreducible. This hypothesis together with the maximum principle yields the following equality:

$$\varlimsup_{\substack{y \to y_0 \\ y \neq y_0}} \phi(y) = \phi(y_0) \quad \text{for any} \quad y_0 \in Y.$$

Now the proof can easily be concluded using the following two remarks:

a) Let $x_0 \in X$ be such that $p(x_0) = y_0$. Then there exists an analytic curve C in a neighbourhood of x_0 in X such that $p|_C$ is a ramified covering of a neighbourhood N of y_0 (see for example Fischer [4], 3.3).

b) Let $N \subset \mathbb{C}$ be a domain, $y_0 \in N$ and $\phi: N \to [-\infty, \infty)$ an upper semicontinuous function such that:

(i) $\phi|_{N \setminus \{y_0\}}$ is subharmonic

(ii) $\varlimsup_{\substack{y \to y_0 \\ y \neq y_0}} \phi(y) = \phi(y_0)$.

Then ϕ is subharmonic on N (see for instance Grauert and Remmert [6], Satz 5).

Remark. Simple examples show that the assumption that p is proper cannot be dropped in Proposition 2.1.

Corollary 2.2. *Let X, Y be complex spaces and $p: X \to Y$ a proper, surjective, holomorphic map. Let $\phi: Y \to [-\infty, \infty)$ be an upper semicontinuous function such* *that $\phi \circ p$ is strongly plurisubharmonic on X. Then ϕ is strongly plurisubharmonic on Y.*

Applying Proposition 2.1 to $\pi:\hat{X}\to X$ (the normalisation of X) one easily verifies:

Corollary 2.3. *Let X be a complex space and $\phi:X\to[-\infty,\infty)$ an upper semicontinuous function.*

Then ϕ is (strongly) plurisubharmonic on X iff restricted to any irreducible component of X is (strongly) plurisubharmonic.

Another result which will be used in the proof of Theorem 1.2 is the following:

Lemma 2.4. *Let Y be a Stein space and $\mu:Y\to[-\infty,\infty)$ a function which is continuous outside a compact set containing $\{\mu=-\infty\}$. Then one can find a real analytic strongly plurisubharmonic function $\lambda:Y\to\mathbb{R}$ such that $\tau=\lambda+\mu$ is an exhaustion function on Y, i.e. $Y_c=\{\tau<c\}\subset\subset Y$ for any $c\in\mathbb{R}$.*

The proof is obtained slightly modifying the proof of the well known fact that any Stein space carries a real analytic strongly plurisubharmonic exhaustion function (see for instance Narasimhan [8]). λ can be chosen as a convergent series $\sum_{j\in\mathbb{N}}|f_j|^2$, $f_j\in\Gamma(Y,\mathcal{O}_Y)$.

Finally, to construct strongly plurisubharmonic functions on a given 1-convex space we shall use the following analytic version of Chow's lemma:

Lemma of Chow (Hironaka [7]). *Let X be an 1-convex space, $S\subset X$ its exceptional set and $p:X\to Y$ the Remmert reduction of X. Suppose that S is rare.*

Then there exist a coherent ideal $\mathcal{I}\subset\mathcal{O}_Y$ such that $\operatorname{supp}(\mathcal{O}_Y/\mathcal{I})=p(S)$ and a commutative diagram:

$$\begin{array}{ccc} Y^* & \xrightarrow{f} & X \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y & \end{array}$$

where $\pi:Y^*\to Y$ is the blowing-up of Y with the center $(p(S),(\mathcal{O}_Y/\mathcal{I})|_{p(S)})$ and f is holomorphic, proper and surjective.

As a general reference for the construction and basic properties of analytic blowing-up we refer to Fischer [4].

3. Proof of Theorem 1.2

From now on X will be an 1-convex space, S its exceptional set, $p:X\to Y$ the Remmert reduction and $A=p(S)$. Recall that Y is Stein, p is proper, holomorphic, surjective and A is finite.

The proof will be divided into several steps:

(i) S is rare (i.e. S does not contain any irreducible component of X) and $\dim X<\infty$.

(ii) S is rare and no assumption on $\dim X$.

(iii) the general case.

Step i) Consider $\mathscr{I} \subset \mathcal{O}_Y$ the ideal such that $A = \operatorname{supp}(\mathcal{O}_Y/\mathscr{I})$ given by Chow's lemma and $\pi: Y^* \to Y$ the blowing-up of Y with center $(A, (\mathcal{O}_Y/\mathscr{I})|_A)$.

The idea is to construct on Y a strongly plurisubharmonic exhaustion function $\tau: Y \to [-\infty, \infty)$ such that $\tau = -\infty$ on A and τ is real analytic outside A, in such a way that $\tau \circ \pi$ is strongly plurisubharmonic on Y^*. Then, using Corollary 2.2, $\phi = \tau \circ p$ is strongly plurisubharmonic on X. The other required properties of ϕ are easily verified.

The construction of τ goes as follows.

Since Y is a Stein space of bounded dimension, a well known argument which uses theorem B of Cartan shows that there exist $h_1, \ldots, h_l \in \Gamma(Y, \mathscr{I})$ such that:

$$A = \{h_1 = \ldots = h_l = 0\}.$$

Choose $h_{l+1}, \ldots, h_s \in \Gamma(Y, \mathscr{I})$ which generate the fiber $\mathscr{I}_y$ for any $y \in A$ (A is finite!).

Then the germs $h_{1,y}, \ldots, h_{s,y}$ generate $\mathscr{I}_y$ for any $y \in Y$. Hence $h = (h_1, \ldots, h_s): Y \to \mathbb{C}^s$ is a holomorphic map such that:

$$h^{-1}(0) = (A, (\mathcal{O}_Y/\mathscr{I})|_A).$$

According to Lemma 2.4. we can choose a real analytic strongly plurisubharmonic function $\lambda: Y \to \mathbb{R}$ such that $\tau = \lambda + \log\left(\sum_{j=1}^{s} |h_j|^2\right)$ is an exhaustion function on Y.

It is clear that τ is strongly plurisubharmonic on Y, $\{\tau = -\infty\} = A$ and τ is real analytic outside A.

Let $\phi = \tau \circ p: X \to [-\infty, \infty)$. We claim that ϕ has the required properties. In fact we only have to check that ϕ is strongly plurisubharmonic because the other properties are obviously verified.

According to Corollary 2.2, it is enough to show that $\tau \circ \pi$ is strongly plurisubharmonic on Y^*. To do this we need the explicit description of analytic blowing-up.

Let $m \subset \mathcal{O}_{\mathbb{C}^s}$ be the sheaf of ideals of the origin. There is an exact sequence on $\mathbb{C}^s$:

$$\mathcal{O}_{\mathbb{C}^s}^{\binom{s}{2}} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{C}^s}^s \to m \to 0 \qquad (*)$$

where α is given by the holomorphic $s x \binom{s}{2}$-matrix

$$\begin{bmatrix} x_2 & & \vdots & & \vdots \\ -x_1 & & \vdots & & \vdots \\ & \cdots & x_j & \cdots & \vdots \\ & & \vdots & & \vdots \\ & & -x_i & & \vdots \\ & & \vdots & & x_s \\ & & \vdots & & -x_{s-1} \end{bmatrix}$$

and $x_1, \ldots, x_s$ denote the coordinate functions on $\mathbb{C}^s$.

Since h^* (the analytic inverse image) is right exact we get an exact sequence on Y:

$$\mathcal{O}_Y^{\binom{s}{2}} \xrightarrow{h^*\alpha} \mathcal{O}_Y^s \to h^* m \to 0. \qquad (**)$$

Let $\xi_1 : \mathbb{P}(\mathscr{I}) \to Y$ and $\xi_2 : \mathbb{P}(h^*\mathscr{m}) \to Y$ be the projective varieties over Y associated to $\mathscr{I}$, respectively to $h^*\mathscr{m}$ (in general they are not reduced). The canonical epimorphism $h^*\mathscr{m} \to \mathscr{I}$ yields an embedding $\mathbb{P}(\mathscr{I}) \hookrightarrow \mathbb{P}(h^*\mathscr{m})$.

Since, by the construction of the analytic blowing-up, Y^* is a closed subspace of $\mathbb{P}(\mathscr{I})$, in order to verify that $\tau \circ \pi$ is a strongly plurisubharmonic, it will be enough to test that $\tau \circ \xi_2 : \mathbb{P}(h^*\mathscr{m}) \to [-\infty, \infty)$ is strongly plurisubharmonic (note that $\pi = \xi_2|_{Y^*}$).

From (**) $\mathbb{P}(h^*\mathscr{m}) \hookrightarrow Yx\mathbb{P}_{s-1}$ is given by the equations:

$$h_j(y)z_i - h_i(y)z_j = 0\,, \qquad 1 \leqq i < j \leqq s$$

where $(z_1 : \ldots : z_s)$ are the homogeneous coordinates on $\mathbb{P}_{s-1}$.

Set $U_i = \{(y,z) \in Yx\mathbb{P}_{s-1}/z_i \neq 0\}$, $\alpha_i : U_i \to Yx\mathbb{C}^{s-1}$

$$\alpha_i(y,z) = (y, z_1/z_i, \ldots, z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_s/z_i)$$

and define $\psi_i : Yx\mathbb{C}^{s-1} \to [-\infty, \infty)$ by:

$$\psi_i = \lambda + \log\left(1 + \sum_{j=1}^{s-1} |t_j|^2\right) + \log(|h_i|^2)$$

where $(t_1, \ldots, t_{s-1})$ are the affine coordinates on $\mathbb{C}^{s-1}$.

Then ψ_i is strongly plurisubharmonic on $Yx\mathbb{C}^{s-1}$ and $\tau \circ \xi_2 = \psi_i \circ \alpha_i$ on $U_i \cap \mathbb{P}(h^*\mathscr{m})$.

This proves that $\tau \circ \xi_2$ is strongly plurisubharmonic on $\mathbb{P}(h^*\mathscr{m})$ thus ending the proof of Step i).

Step ii) We drop the assumption that X has finite dimension.

This is done by carefully analysing the arguments given above.

Let $U \subset\subset Y$ be a relatively compact open neighbourhood of A and $V = p^{-1}(U) \subset\subset X$. Let X_0 be an analytic subset of X having finite dimension and containing V. Then $Y_0 = p(X_0)$ is an analytic subset of Y and $U \subset Y_0$. Finally, let $\mathscr{I}$ be the ideal on Y given by Chow's lemma.

Using again Cartan's theorem B we find (since Y_0 is of bounded dimension) $h_1, \ldots, h_l \in \Gamma(Y, \mathscr{I})$ such that:

$$A = \{y \in Y_0/h_1(y) = \ldots = h_l(y) = 0\}\,.$$

Choose $h_{l+1}, \ldots, h_s \in \Gamma(Y, \mathscr{I})$ which generate $\mathscr{I}_y$ for any $y \in A$. Then the germs $h_{1,y}, \ldots, h_{s,y}$ generate $\mathscr{I}_y$ for any $y \in Y_0$. Next choose a countable set $\{g_k\}_{k \in \mathbb{N}} \subset \Gamma(Y, \mathcal{O}_Y)$ such that $Y_0 = \bigcap_{k \in \mathbb{N}} \{g_k = 0\}$.

We may suppose that the series $\sum_{k \in \mathbb{N}} |g_k|^2$ converges uniformly on compact subsets of Y.

By Lemma 2.4 we find a real analytic strongly plurisubharmonic function $\lambda : Y \to \mathbb{R}$ such that

$$\tau = \lambda + \log\left(\sum_{j=1}^{s} |h_j|^2 + \sum_{k \in \mathbb{N}} |g_k|^2\right)$$

is an exhaustion function on Y.

Clearly $\tau = -\infty$ exactly on A and is real analytic on $Y \setminus A$. Moreover $\tau|_U = \lambda + \log\left(\sum_{j=1}^{s} |h_j|^2\right)$ since $g_k = 0$ on $Y_0 \supset U$.

For that reason if we set $\phi = \tau \circ p$ the same arguments as in Step i) prove that ϕ is a strongly plurisubharmonic exhaustion function on X, $S = \{\phi = -\infty\}$ and ϕ is real analytic outside S.

Remark. Under the assumptions of Step ii) one can prove that given a finite set $B \subset X \setminus S$ there is a strongly plurisubharmonic exhaustion function $\phi: X \to [-\infty, \infty)$ such that $\phi = -\infty$ exactly on $S \cup B$ and ϕ is real analytic outside $S \cup B$.

The proof is straightforward and so will be omitted.

Step iii) S is not necessarily rare. Let $\tilde{X}$ be the union of those irreducible components of X not contained in S. Being a closed subspace of an 1-convex space, $\tilde{X}$ is itself 1-convex and is clear that its exceptional set is rare. However $\tilde{S} = \tilde{X} \cap S$ contains the exceptional set of $\tilde{X}$ and (eventually) a finite set. Anyhow by the Remark ending Step ii) there is a strongly plurisubharmonic exhaustion function $\tilde{\phi}: \tilde{X} \to [-\infty, \infty)$ such that $\tilde{S} = \{\tilde{\phi} = -\infty\}$ and $\tilde{\phi}$ is real analytic outside $\tilde{S}$.

Define $\phi: X \to [-\infty, \infty)$ by $\phi = \tilde{\phi}$ on $\tilde{X}$ and $\phi = -\infty$ on S. Then ϕ is an upper semicontinuous exhaustion function on X, real analytic outside S and $S = \{\phi = -\infty\}$. Because of Corollary 2.3. ϕ is strongly plurisubharmonic and we are done.

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Received April 7, 1984