The Minimum Value of Quadratic Forms, and the Closest Packing of Spheres.

Von

H. F. Blichfeldt in Stanford University (Cal., U.S.A.).

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1. The minimum value of a positive definite quadratic form of given determinant D, under the condition that the variables assume only positive or negative integral values or zero (not *all* zero), has been the subject of several papers from the time of Gauss[1]). If the form has n variables, the minimum value is $\leqq \gamma_n D^{1/n}$, where γ_n is a function of n only. The author obtained the following superior limit to γ_n: $\frac{2}{\pi}\left[\Gamma\left(1+\frac{n+2}{2}\right)\right]^{2/n}$.[1])

The determination of the quadratic forms in n variables for which γ_n has the least permissible value, will also solve the problem of the closest packing, in regular layers, of equal spheres in a large cube in space of n dimensions R_n. (By packing in "regular layers" we mean that we place the centers of the spheres in the points obtained by subjecting to a linear transformation the lattice points (points whose coordinates are positive or negative integers or zero) in R_n.) In fact, in a regular packing the ratio ϱ of the volume occupied by the spheres to the whole volume of the large cube containing them, is

$$(1) \qquad \varrho \leqq \frac{\left(\frac{\pi \gamma_n}{4}\right)^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)} = \varrho_0, \quad \text{say.}$$

The ratio for the *closest* regular packing is the number ϱ_0.

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[1]) See "Bulletin of the American Mathematical Society" **25** (1919), pp. 449–453, for references; as well as L. E. Dickson, "History of the Theory of Numbers" **3**, pp. 234–252. — See also article by Remak, "Mathematische Zeitschrift" **26** (1927), pp. 694–699.

It is the object of this article to find by elementary geometrical processes a number ϱ_1 such that if equal spheres be packed in a large cube, whether in regular layers or not, then the ratio ϱ_2 of the total volume occupied by the spheres to the volume of the cube is

$$\varrho_2 < \varrho_1 .$$

Since evidently $\varrho_0 < \varrho_1$, we derive from (1) that

$$\gamma_n < \frac{4}{\pi}\left[\varrho_1 \Gamma\left(1+\frac{n}{2}\right)\right]^{2/n}. \tag{2}$$

2. Using rectangular coordinates in R_n, let the spheres $S_1, \ldots$ have their centers at $(a_1, b_1, \ldots, k_1), \ldots,$ and radii all equal to unity. The conditions of the problem require that the distance between every pair of centers is at least 2:

$$(a_i - a_j)^2 + (b_i - b_j)^2 + \ldots + (k_i - k_j)^2 \geqq 4 .$$

Adding these inequalities for $i, j = 1, 2, \ldots, m$, we obtain

$$m \sum (a_i^2 + b_i^2 + \ldots + k_i^2) - (\sum a_i)^2 - (\sum b_i)^2 \ldots \geqq 2m(m-1), \tag{3}$$

the summation extending from 1 to m. Hence, representing by r_i the distance from the point $T = (0, 0, \ldots, 0)$ to the center of the sphere S_i, we have

$$\sum r_i^2 \geqq 2(m-1). \tag{4}$$

3. Leaving the centers fixed, we now replace the given geometrical spheres by physical spheres of superposable matter, all of radius $\sqrt{2}$. Furthermore, we assume the density of the matter in each sphere to be $2 - r^2$ at the distance r from its center. These new spheres will to a certain extent overlap; but the total density of the matter from the spheres will at no point in the otherwise empty R_n be greater than 2. For, if just m spheres overlap at the point T, we have from (4):

$$\sum (2 - r_i^2) \leqq 2m - 2(m-1) = 2 .$$

4. Now consider a large cube of edge E, containing say k of the original spheres. The new spheres will be wholly contained in the cube whose edges are longer than the edges E by $2(\sqrt{2}-1)$. Now the total matter in the extended cube is less than twice its volume. Hence, if $K = \dfrac{\pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)}$ is the volume of a sphere of radius 1, then

$$2\left(E + 2(\sqrt{2}-1)\right)^n > kK \int_0^{\sqrt{2}} (2 - r^2)\, dr^n = \frac{4kK\,2^{n/2}}{n+2},$$

from which we derive

$$\varrho_2 = \frac{kK}{E^n} < \frac{n+2}{2^{\frac{n+2}{2}}}\left(1 + \frac{2\sqrt{2}-2}{E}\right)^n = \varrho_1 .$$

Taking E sufficiently large, the value of ϱ_1 is as close to $\frac{n+2}{2^{\frac{n+2}{2}}}$ as we wish. This limiting value, substituted in (2), gives the superior limit to γ_n already found by the author (§ 1).

5. Elaborating the method of § 2 under the condition that the radii of the spheres are $\leqq \sqrt{2}$, we may obtain the following inequality, stronger than (4):

$$\sum_{i=1}^{m} r_i \geqq \sqrt{2m(m-1)}. \tag{5}$$

Namely, we start by passing the X_1-axis through the center of the sphere S_1, so that $a_1^2 = r_1^2$, $b_1 = \ldots = k_1 = 0$. It is now easily proved that if $r_1, \ldots, r_m$ are all $\leqq \sqrt{2}$, then $a_2, \ldots, a_m$ all have the same sign, say $+$, opposite to that of a_1. For, from the condition that the distance between the centers of S_1 and S_i is at least 2, we get

$$2 r_1 a_i \geqq 4 - r_1^2 - r_i^2 \geqq 0. \tag{6}$$

From the condition (3), applied to the subscripts $i = 2, \ldots, m$, follows

$$(m-1)\sum r_i^2 \geqq \left(\sum a_i\right)^2 + 2(m-1)(m-2).$$

Hence, by (6),

$$(m-1)\sum r_i^2 \geqq \frac{1}{4r_1^2}\left[(m-1)(4-r_1^2) - \sum r_i^2\right]^2 + 2(m-1)(m-2),$$

which transforms into

$$8m(m-1)r_1^2 \geqq \left[(m-1)(4+r_1^2) - \sum r_i^2\right]^2.$$

Since $(m-1)(4+r_1^2) > \sum r_i^2$, we therefore deduce

$$\sum r_i^2 \geqq (m-1)(4+r_1^2) - 2r_1\sqrt{2m(m-1)}.$$

Adding all the inequalities corresponding to this by replacing in turn r_1 by $r_2, \ldots, r_m$, we finally get (5).

6. Reverting to the physical spheres of § 3, we shall now assume their density to be $2 - r^2$ if $1 \leqq r \leqq \sqrt{2}$, $(2-r)^2$ if $2 - \sqrt{2} \leqq r \leqq 1$, and 2 if $0 \leqq r \leqq 2 - \sqrt{2}$. The arguments of § 4 will be unchanged, except that the integral will now become

$$\frac{4kK}{n+2}\left\{2^{n/2} + \frac{1}{n+1} - (2-\sqrt{2})^{n+1}\left(\frac{\sqrt{2}}{2} + \frac{1}{n+1}\right)\right\} = \frac{4kK\,2^{n/2}}{n+2}(1+g),$$

say, giving $\varrho_1 = \dfrac{n+2}{2^{\frac{n+2}{2}}(1+g)}$.

Making the corresponding changes in (2) we then have

$$\gamma_n < \frac{2}{\pi}\left[\frac{\Gamma\left(1+\frac{n+2}{2}\right)}{(1+g)}\right]^{2/n}.$$

The exact (least) values of γ_n for $n = 2, 3, 4, 5$ are respectively $\frac{2}{\sqrt{3}}$, $\sqrt[3]{2}$, $\sqrt{2}$, $\sqrt[5]{8}$. The author has recently proved that for $n = 6, 7, 8$ the values are respectively $\sqrt[6]{\left(\frac{64}{3}\right)}$, $\sqrt[7]{64}$, and 2, which he hopes to publish soon. By the above formula, $\gamma_8 < 2.104$.

7. The closest packing of equal circles in the plane is regular ("shot-pile" arrangement). But in R_3 the closest packing of equal spheres may not be regular. The shot-pile packing gives $\varrho = .74..$, while the formula above makes $\varrho_1 = .843\ldots$ But a more elaborate use of the relations (5) and (6) should give better results. Thus the author has obtained $\varrho_1 = .835\ldots$ Hence, in no case can equal spheres be packed in a large cube in space of three dimensions such that the space occupied by the spheres is as much as $\frac{835}{1000}$ of the volume of the cube.

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