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The Betti numbers of the Hilbert scheme of points on a smooth projective surface

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Dedicated to Hans Grauert

0. Introduction

Several authors have been interested in the Hilbert scheme $S^{[n]}:=\mathrm{Hilb}^n(S)$ parametrizing finite subschemes of length n on a smooth projective surface S. In [F 1] and [F 2] Fogarty shows that $S^{[n]}$ is smooth and there exists a natural birational morphism $\omega_n: S^{[n]}\to S^{(n)}$. Here $S^{(n)}$ is the n^{th} symmetric power of S. So $S^{[n]}$ is a natural desingularisation of $S^{(n)}$. He then computes the Picard group of $S^{[n]}$. In [I 1] the highest dimensional fiber $\mathrm{Hilb}^n(\mathrm{spec}(k[[x,y]]))$ of ω_n is studied, and in [I 2] it is shown, that the expectional locus of ω_n is an irreducible divisor. Fujiki has shown in [Fj] that $S^{[2]}$ is a symplectic variety, if S is a K 3 surface, thereby disproving a conjecture of Bogomolov. In [B] Beauville generalizes this result to arbitrary n. Several authors have worked on the cohomology of $S^{[n]}$: The case $n\leqq 2$ is trivial by the results of [F 2]. The homology groups of $\mathrm{Hilb}^3(\mathbf{P}_2)$ have been computed in [H]. Finally in [E–S] the homology groups of $\mathrm{Hilb}^n(\mathbf{P}_2)$, $\mathrm{Hilb}^n(\mathbf{A}^2)$, $\mathrm{Hilb}^n(\mathrm{spec}(k[[x,y]]))$, and $\mathrm{Hilb}^n(\Sigma_m)$ are computed. Here $\mathbf{P}_2$ is the projective plane, $\mathbf{A}^2$ is the affine plane and Σ_m is the m^{th} Hirzebruch surface. In this paper we will compute the Betti numbers of $S^{[n]}$ for an arbitrary smooth surface S using the Weil conjectures. We want to state our main result: Let $\mathbf{F}_q$ be a finite field with q elements, $\bar{\mathbf{F}}_q$ its algebraic closure. Let X be a smooth projective variety over $\mathbf{C}$ or over $\bar{\mathbf{F}}_q$. In the first case let $b_i(X)$ be the i^{th} Betti number of X. In the second case let $b_i(X)$ be the rank of the i^{th} l-adic cohomology group $H^i(X,\mathbf{Q}_l)$ of X. In both cases let $p(X,z)$ be the Poincaré polynomial $\sum_i b_i(X)z^i$ of X and $e(X):=\sum_i(-1)^i b_i(X)$ the Euler number of X. We put $\tilde{p}(X,z):=p(X,-z)$.

Theorem 0.1. *Let S be a smooth projective surface over $\mathbf{C}$ or over $\bar{\mathbf{F}}_q$. Then:*

$$\sum_{n=0}^{\infty}\tilde{p}(S^{[n]},z)t^n=\exp\left(\sum_{m=1}^{\infty}\frac{t^m}{m}\,\frac{\tilde{p}(S,z^m)}{1-z^{2m}t^m}\right),\tag{1a}$$

$$\sum_{n=0}^{\infty}p(S^{[n]},z)t^n=\prod_{m=1}^{\infty}\frac{(1+z^{2m-1}t^m)^{b_1(S)}(1+z^{2m+1}t^m)^{b_1(S)}}{(1-z^{2m-2}t^m)^{b_0(S)}(1-z^{2m}t^m)^{b_2(S)}(1-z^{2m+2}t^m)^{b_0(S)}},\tag{1b}$$

$$\sum_{n=0}^{\infty}e(S^{[n]})t^n=\prod_{m=1}^{\infty}(1-t^m)^{-e(S)}.\tag{2}$$

So the Betti numbers of $S^{[n]}$ depend only on the Betti numbers of S and the Euler number of $S^{[n]}$ depends only on the Euler number of S. According to [Md] the Betti numbers of $S^{(n)}$ are given by

$$\sum_n P(S^{(n)},z)t^n = \frac{(1+zt)^{b_1(S)}(1+z^3t)^{b_1(S)}}{(1-t)^{b_0(S)}(1-z^2t)^{b_2(S)}(1-z^4t)^{b_0(S)}},$$

so Theorem 0.1 reflects the close connection of $S^{[n]}$ and $S^{(n)}$.

Over an algebraically closed field k we have the following situation: The points of $S^{(n)}$ are the 0-cycles $\xi = \sum_i n_i \cdot x_i$ of degree n on S. $S^{(n)}$ is stratified by the partition of n given by the n_i (if the x_i are distinct). The open stratum contains the 0-cycles consisting of n different points. The fibre of the morphism ω_n over a point $\xi \in S^{(n)}$ depends only on the stratum in which ξ lies. $\omega_n^{-1}(n \cdot x)$ is the variety $V_{n,k} = \mathrm{Hilb}^n(\mathrm{spec}(k[[x,y]]))$ of subschemes of length n of a surface concentrated at a point. If $x_1, \ldots, x_r$ are distinct points of S, then

$$\omega_n^{-1}(n_1 \cdot x_1 + \ldots + n_r \cdot x_r) = V_{n_1,k} \times \ldots \times V_{n_r,k}.$$

Using this geometric description and keeping track of the action of the Frobenius we can count the points of $S^{[n]}$ over finite fields and so compute the Betti numbers via the Weil conjectures.

This paper is a simplified version of parts of my Diplom paper [Gö] at Bonn University written under the guidance of Andrew Sommese and Friedrich Hirzebruch. In the present paper we will use a result of [E–S]. The original proof, which is considerably longer, is independent of [E–S] and uses results of [I 1].

1. Notation and background material

Let $\mathbf{Z}$ denote the integers, $\mathbf{N}$ the positive integers, $\mathbf{N}_0$ the nonnegative integers, $\mathbf{Q}$ the rational numbers, $\mathbf{C}$ the complex numbers and $\mathbf{F}_q$ a finite field with q elements, where q is a prime power. If A is a ring and $X_1, \ldots, X_n$ are indeterminates, we denote by $A[X_1, \ldots, X_n]$ and $A[[X_1, \ldots, X_n]]$ the ring of polynomials and the ring of formal power series in $X_1, \ldots, X_n$ with coefficients in A respectively. For a set M we denote its cardinality by $\# M$. Let k be a field. We denote by $\mathbf{A}_k^n$ the affine n-space over k. We drop the k if it is understood. For k-schemes S, T we denote $S(T) := \mathrm{Hom}_k(T, S)$ the T-valued points of S. If $\bar{k}$ is an extension of k, we write $S(\bar{k})$ for $S(\mathrm{spec}(\bar{k}))$. We denote by $\mathrm{Gal}(\bar{k}/k)$ the Galois group of $\bar{k}$ over k, by $\bar{k}$ the algebraic closure of k and $\bar{S} := S \times_{\mathrm{spec}(k)} \mathrm{spec}(\bar{k})$. We denote by $\mathcal{O}_S$ the structure sheaf of S. If W is a T-scheme and $t \in T$, we denote the fiber of W over t by W_t. We denote by Z_{red} the reduced scheme of Z. If a group G operates on a set X, we denote by X^G the set of G-invariant elements of X. If $\mathcal{F}$ is a coherent sheaf on a scheme S, let $H^i(S, \mathcal{F})$ be the i^{th} cohomology group with coefficients in $\mathcal{F}$, let $h^i(S, \mathcal{F}) := rk(H^i(S, \mathcal{F}))$ and $\chi(\mathcal{F}) := \sum (-1)^i h^i(S, \mathcal{F})$. If S is smooth and projective over $\mathbf{C}$, let Ω_S^p be the sheaf of holomorphic p-forms on S and $h^{p,q}(S) := h^q(S, \Omega_S^p)$ the p, q^{th} Hodge number, $h(S,x,y) := \sum_{p,q} h^{p,q}(S) x^p y^q$ and $\bar{h}(S,x,y) := h(S,-x,-y)$. Then the χ_y genus is $\chi_y(S) := \sum_{p,q} (-1)^q h^{p,q}(S) y^p$, so we have $\chi(\mathcal{O}_S) = \chi_0(S)$, and $\chi_1(S)$ is the signature $\mathrm{sign}(S)$ of S.

We will use the Weil conjectures. Let X be a smooth projective variety of dimension d over $\mathbf{F}_q$. Let F_q be the geometric Frobenius of $\bar{X}$ over $\mathbf{F}_q$ and

$$Z_q(X,t):=\exp\left(\sum_{n>0} \# X(\mathbf{F}_{q^n})\frac{t^n}{n}\right)$$

the zeta function of X over $\mathbf{F}_q$. The operation of F_q on $X(\bar{\mathbf{F}}_q)$ is inverse to the operation of the Frobenius of $\mathbf{F}_q$, a topological generator of $\mathrm{Gal}(\bar{\mathbf{F}}_q/\mathbf{F}_q)$, so a point $x\in X(\bar{\mathbf{F}}_q)$ is in $X(\mathbf{F}_q)$ if and only if $x=F_q(x)$. We denote the action of F_q on the l-adic cohomology groups $H^i(X,\mathbf{Q}_l)$ by $F_q^*|_{H^i(X,\mathbf{Q}_l)}$.

Theorem 1.1. (Weil conjectures, [D]). (1) *$Z_q(X,t)$ is a rational function:*

$$Z_q(X,t)=\prod_{r=0}^{2d} Q_r(X,t)^{(-1)^{r+1}},$$

where $Q_r(X,t)=\det(1-tF_q^*|_{H^r(\bar{X},\mathbf{Q}_l)})$.

(2) $Q_r(X,t)\in\mathbf{Z}[t]$.

(3) *The eigenvalues $\alpha_{i,r}$ of $F_q^*|_{H^r(\bar{X},\mathbf{Q}_l)}$ have absolute value $|\alpha_{i,r}|=q^{r/2}$ with respect to any embedding into* $\mathbf{C}$.

(4) $$Z_q(X,1/q^d t)=\pm q^{\frac{1}{2}e(\bar{X})}t^{e(\bar{X})}Z_q(X,t).$$

(5) *If X is a good reduction of a smooth projective variety Y over $\mathbf{C}$, then $b_i(Y)=b_i(\bar{X})=\deg(Q_i)$ for $i=0,\dots,2d$.*

As a consequence of 1.1 (1), (3), and (5) we get:

Remark 1.2. Let $F(t,s_1,\dots,s_m)\in\mathbf{Q}[t,s_1,\dots,s_m]$. Let X and S be smooth projective varieties over $\mathbf{F}_q$, such that for all $n\in\mathbf{N}$ we have

$$\# X(\mathbf{F}_{q^n})=F(q^n,\# S(\mathbf{F}_{q^n}),\dots,\# S(\mathbf{F}_{q^{nm}})).$$

Then

$$\tilde{p}(\bar{X},z)=F(z^2,\tilde{p}(\bar{S},z),\dots,\tilde{p}(\bar{S},z^m)).$$

If both X and S are good reductions of smooth projective varieties $\tilde{X}$ and $\tilde{S}$ over $\mathbf{C}$, then

$$\tilde{p}(\tilde{X},z)=F(z^2,\tilde{p}(\tilde{S},z),\dots,\tilde{p}(\tilde{S},z^m)).$$

Let X be a smooth projective variety over a field k. The symmetric group $G(n)$ in n letters operates on X^n by permuting the factors. The n^{th} symmetric power $X^{(n)}$ of X is the geometric quotient $X^n/G(n)$. We can identify $X^{(n)}(\bar{k})$ with the set of effective 0-cycles $\sum_i n_i\cdot x_i$ of degree n on $\bar{X}$.

If k is a finite field $\mathbf{F}_q$, then the Frobenius F_q acts on $X^{(n)}(\bar{\mathbf{F}}_q)$ by $F_q\left(\sum_i n_i\cdot x_i\right)=\sum_i n_i\cdot F_q(x_i)$ and $X^{(n)}(\mathbf{F}_q)$ is the set of effective 0-cycles of degree n fixed under the action of F_q. A 0-cycle $\zeta=\sum_{i=0}^{r-1}F_q^i(x)$ with $x\in X(\mathbf{F}_{q^r})\backslash\left(\bigcup_{j<r}X(\mathbf{F}_{q^j})\right)$ will be called a *primitive 0-cycle* of degree r over $\mathbf{F}_q$ on X. We denote the set of these primitive

0-cycles by $P_r(X, \mathbf{F}_q)$. Any $\zeta \in X^{(n)}(\mathbf{F}_q)$ can be written in a unique way as a positive linear combination of distinct primitive 0-cycles. It is also clear from the definitions, that we have

$$\# X(\mathbf{F}_{q^n}) = \sum_{r|n} \# P_r(X, \mathbf{F}_{q^r}) \cdot r .$$

If we combine these facts, we can see that

$$Z_q(X,t) = \sum_{n=0}^{\infty} \# X^{(n)}(\mathbf{F}_q) t^n$$

i.e. the zeta function is the counting function for effective 0-cycles on X over $\mathbf{F}_q$.

Let X again be defined over an arbitrary field k. We want to use a natural stratification of $X^{(n)}$. Let $\Phi: X^n \to X^{(n)}$ be the quotient morphism. For $m \in \mathbf{N}$ let $\Delta_m \subset X^m$ be the diagonal. Let $\nu = (n_1, \ldots, n_r)$ be a partition of n. If ν is strictly finer than a partition μ of n, we will write $\nu < \mu$. Then we put

$$\overline{X_\nu^{(n)}} := \Phi(\Delta_{n_1} \times \ldots \times \Delta_{n_r})$$

and

$$X_\nu^{(n)} := \overline{X_\nu^{(n)}} \setminus \Big(\bigcup_{\nu < \mu} \overline{X_\mu^{(n)}} \Big).$$

This gives a stratification of $X^{(n)}$ into locally closed subschemes. We have

$$X_\nu^{(n)}(\bar{k}) = \{ n_1 \cdot x_1 + \ldots + n_r \cdot x_r \mid x_1, \ldots, x_r \text{ are distinct elements of } X(\bar{k}) \} .$$

We write $X_\nu^n := \Phi^{-1}(X_\nu^{(n)})$.

From now on let S be a smooth projective surface over k. Let $S^{[n]} := \mathrm{Hilb}^n(S)$ denote the component of the Hilbert scheme of S parametrising subschemes of length n of S. Let $Z^n(S) \subset S \times S^{[n]}$ be the universal subscheme. Then we have for any locally noetherian k-scheme T:

$$S^{[n]}(T) = \{ Z \subset S \times T \text{ closed subscheme, flat of degree } n \text{ over } T \} ,$$

$$Z^n(S)(T) = \{ (Z, \sigma) \mid Z \in S^{[n]}(T);\ \sigma: T \to Z \text{ section of } p_2|_Z \} .$$

According to [F 1] we have:

Theorem 1.3. (a) *There is a canonical morphism* $\omega_n: S^{[n]} \to S^{(n)}$, *which as a map of points is*

$$Z \mapsto \sum_{x \in S} \mathrm{length}(\mathcal{O}_{Z,x}) \cdot x .$$

(b) $S^{[n]}$ *is smooth.*

(c) ω_n *is birational.*

From the definitions and (b) it is clear that the following holds. If S over $\mathbf{F}_q$ is a good reduction of a smooth projective surface $\tilde{S}$ over $\mathbf{C}$, then $S^{[n]}$ is a good reduction of $\tilde{S}^{[n]}$. Using ω_n we get a natural stratification of $S^{[n]}$: We put $S_\nu^{[n]} := \omega_n^{-1}(S_\nu^{(n)})$, $Z_\nu^n(S) := p_2^{-1}(S_\nu^{[n]})$. (See for instance [F 2].)

Let $\mathbf{m}$ be the maximal ideal in $k[[x,y]]$ and

$$V_{n,k} := \mathrm{Hilb}^n(\mathrm{spec}(k[[x,y]]/\mathbf{m}^n)) .$$

$V_{n,k}$ represents a functor as $\mathrm{spec}(k[[x,y]]/\mathbf{m}^n) = \mathrm{spec}(k[x,y]/\mathbf{m}^n)$ is projective.

Definition 1.4 (See [Fu, Example 1.9.1]). Let X be a scheme over a field k. A *cell decomposition* of X is a filtration

$$X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$$

by closed subschemes, with each $X_i \backslash X_{i-1}$ a disjoint union of schemes $U_{i,j}$ isomorphic to affine spaces $A^{n_{i,j}}$. We call the $U_{i,j}$ the $n_{i,j}$-cells of the decomposition.

We will use the following two facts:

(1) If $f: Z \to T$ is a separated morphism of schemes, then any section $s: T \to Z$ of f is a closed immersion.

(2) Let X_0 be a scheme, $X_1 \subset X_0$ a closed subscheme defined by a nilpotent ideal sheaf and $q: Y \to X_0$ be an étale X_0-scheme. Then for every section $s_1: X_1 \to Y \times_{X_0} X_1$ of p_2 there is a unique section $s_0: X_0 \to Y$ of q, such that $s_0 \times 1_{X_1} = s_1$.

(1) is trivial and (2) follows for instance from [M, Theorem I.3.23].

2. Proof of the main theorem

In this section we shall only consider the Zariski topology. Let k be a field, S a smooth projective surface over k. There is an open affine cover $(U_i)_i$ of S and étale morphisms $f_i: U_i \to \mathbf{A}^2$. (See for instance [M, Proposition I.3.24]; [SGA 1, II].) Let $U \subset S$ be open and affine and $f: U \to \mathbf{A}^2$ an étale morphism.

The reader will notice that our argument below could be simplified, if one could see easily that taking the scheme theoretic direct image defines a morphism $U^{[n]}_{(n)} \to (\mathbf{A}^2)^{[n]}_{(n)}$.

Lemma 2.1. *There is a canonical morphism*

$$\psi_1 : Z^n_{(n)}(\mathbf{A}^2) \times_{A^2} U \to Z^n(U),$$

which induces a bijection

$$\psi_1(\bar{k}) : (Z^n_{(n)}(\mathbf{A}^2) \times_{A^2} U)(\bar{k}) \to Z^n_{(n)}(U)(\bar{k}).$$

Proof. We construct ψ_1 as a morphism of functors. Let T be a noetherian k-scheme. Let

$$(Z, \sigma, \tilde{\sigma}) \in (Z^n_{(n)}(\mathbf{A}^2) \times_{A^2} U)(T).$$

Then $(Z, \sigma) \in Z^n(\mathbf{A}^2)(T)$. Let $i: Z \to \mathbf{A}^2 \times T$ be the inclusion. Then we have $\omega_n \circ i \in (\mathbf{A}^2)^{(n)}_{(n)}(T)$ and $\tilde{\sigma}: T \to U \times T$ is a section of p_2, such that $i \circ \sigma = (f \times 1_T) \circ \tilde{\sigma}$. As $p_2|_Z$ is separated, σ is a closed immersion and induces an isomorphism $T \xrightarrow{\simeq} \Sigma := \sigma(T) \subset Z$.

We show next, that $\Sigma \subset Z$ is defined by a nilpotent ideal. For each $t \in T$ we have

$$\omega_n(Z_t) = \sum_{x \in (\mathbf{A}^2 \times T)_t} \operatorname{length}(\mathcal{O}_{Z_t, x}) \cdot x.$$

As ω_n is compatible with base change, we have

$$(\omega_n(Z))_t = \omega_n(Z_t) \in ((\mathbf{A}^2 \times T)_t)^{(n)}_{(n)}.$$

So the support $\operatorname{supp}(Z_t)$ of Z_t consists of a single point $x_t \in Z_t$ and $\operatorname{supp}(Z) = \{x_t \mid t \in T\}$. As σ is a section of $Z \to T$, we have $\operatorname{supp}(Z) = \operatorname{supp}(\Sigma)$, and so $\Sigma \subset Z$ is defined by a nilpotent ideal. We consider the following cartesian diagram.

$$\begin{array}{ccccc} U \times T & \xleftarrow{i^*} & Z_0 & \xleftarrow{\sigma^*} & Z_1 \\ \downarrow{\scriptstyle f \times 1_T} & \square & \downarrow{\scriptstyle f_0} & \square & \downarrow{\scriptstyle f_1} \\ \mathbf{A}^2 \times T & \xleftarrow{i} & Z & \xleftarrow{\sigma} & T. \end{array}$$

Let $s_2 := (\sigma, \tilde{\sigma}) : T \to Z_0$, $s_1 : (s_2, 1_T) : T \to Z_1$ and let $s_0 : Z \to Z_0$ be the unique lift of s_1. As f is separated, so is f_0, and so its section s_0 is a closed immersion. So $i^* \circ s_0$ is a closed immersion and hence induces an isomorphism $s : Z \to Z_{U,T} := (i^* \circ s_0)(Z)$. Let $\sigma_{U,T} := s \circ \sigma$. Then $\sigma_{U,T}$ is a section of $p_2|_{Z_{U,T}}$, and $\sigma_{U,T}(T) \subset Z_{U,T}$ is defined by a nilpotent ideal.

We define $\psi_1(Z, \sigma, \tilde{\sigma}) := (Z_{U,T}, \sigma_{U,T}) \in Z^n(U)(T)$. Using the uniqueness of the lift s_0 of s_1, it is trivial to verify that this definition is compatible with base change and so defines a morphism $\psi_1 : Z^n_{(n)}(\mathbf{A}^2) \times_{\mathbf{A}^2} U \to Z^n(U)$.

We want to prove that ψ_1 induces a bijection

$$\psi_1(\bar{k}) : (Z^n_{(n)}(\mathbf{A}^2) \times_{\mathbf{A}^2} U)(\bar{k}) \to Z^n_{(n)}(U)(\bar{k}).$$

Let $(Z, p, u) \in (Z^n_{(n)}(\mathbf{A}^2) \times_{\mathbf{A}^2} U)(\bar{k})$, then $f(u) = p$. We specialize the above diagram to $T = \operatorname{spec}(\bar{k})$ and define s_2, s_1, and s_0 as above. Then $\psi_1(Z, p, u) = (W, u)$ with $W := s_0(Z)$. W is a subscheme of length n in $\bar{U} := U \times \operatorname{spec}(\bar{k})$, such that $u \subset W$ is defined by a nilpotent ideal. So $\omega_n(W)$ is the 0-cycle $n \cdot u$. Thus we have $W \in U^{[n]}_{(n)}(\bar{k})$ and $(W, u) \in Z^n_{(n)}(U)(\bar{k})$. Now we want to define the inverse of $\psi_1(\bar{k})$. Let $(W, u) \in Z^n_{(n)}(U)(\bar{k})$, let $i_W : W \to \bar{U}$ the inclusion and $p := f(u)$.

Claim. $f \circ i_W : W \to \mathbf{A}^2_{\bar{k}}$ is a closed immersion.

Let $\hat{U}$ be the completion of $\bar{U}$ at u. Let $\hat{\mathbf{A}}^2$ the completion of $A^2_{\bar{k}}$ at p. Let $\hat{f} : \hat{U} \xrightarrow{\cong} \hat{\mathbf{A}}^2$ be the induced morphism and $\hat{W}_u$ the completion of W at u. Then we have $\hat{W}_u = W \times_{\bar{U}} \hat{U}$. As the support of W is u, it follows that $p_1 : \hat{W}_u \to W$ is an isomorphism. Let κ be its inverse. Then we have the following diagram:

$$\begin{array}{ccccc} \hat{\mathbf{A}}^2 & \xleftarrow{\hat{f}} & \hat{U} & \xleftarrow{\hat{i}_W} & \hat{W}_u \\ \downarrow & & \downarrow & \square & \downarrow{\scriptstyle \kappa^{-1}} \\ \mathbf{A}^2_{\bar{k}} & \xleftarrow{f} & \bar{U} & \xleftarrow{i_W} & W \end{array}$$

$\bar{f} := \hat{f} \circ \hat{i}_W \circ \kappa : W \to \hat{\mathbf{A}}^2$ is a closed immersion. $\hat{\mathbf{A}}^2$ is isomorphic to $\operatorname{spec}(\bar{k}[[x, y]])$, and each ideal in $\bar{k}[[x, y]]$ of colength n contains $\mathbf{m}^n$ (see for instance [I1, Lemma 1.1]) So $\bar{f}$ factorizes as:

$$W \xrightarrow{\kappa} \hat{W}_u \xrightarrow{\bar{i}_W} \operatorname{spec}(\bar{k}[[x, y]]/\mathbf{m}^n) \to \hat{\mathbf{A}}^2,$$

and $\bar{i}_W$ is a closed immersion. So $f \circ i_W$ factorizes as

$$W \xrightarrow{\bar{i}_W \circ \kappa} \operatorname{spec}(\bar{k}[[x, y]]/\mathbf{m}^n) \xrightarrow{i_n} \mathbf{A}^2_{\bar{k}},$$

where i_n is the closed immersion of $\bar{k}[[x,y]]/\mathbf{m}^n$ in $\mathbf{A}^2_{\bar{k}}$, which is induced by the canonical projection $\bar{k}[x,y]\to\bar{k}[[x,y]]/\mathbf{m}^n$. So $f\circ i_W$ is a closed immersion.

We define

$$\tilde{\varphi}_1(\bar{k}): Z^n_{(n)}(U)(\bar{k})\to(Z^n_{(n)}(\mathbf{A}^2)\times_{\mathbf{A}^2}U)(\bar{k})$$

by $(W,u)\mapsto(f(W),f(u),u)$. By the claim we have

$$(f(W),f(u),u)\in(Z^n_{(n)}(\mathbf{A}^2)\times_{\mathbf{A}^2}U)(\bar{k}).$$

It is easy to verify that $\tilde{\varphi}_1(\bar{k})=\psi_1(\bar{k})^{-1}$. □

Lemma 2.2. *There is a natural morphism*

$$\psi_2: V_{n,k}\times\mathbf{A}^2\to(\mathbf{A}^2)^{[n]},$$

which induces a bijection of geometric points

$$\psi_2(\bar{k}):(V_{n,k}\times\mathbf{A}^2)(\bar{k})\to(\mathbf{A}^2)^{[n]}_{(n)}(\bar{k}).$$

Proof. We construct ψ_2 as a morphism of functors. Let T be a noetherian k-scheme. Let $(Z,\sigma)\in(V_{n,k}\times\mathbf{A}^2)(T)$. Let again $i_n:\mathrm{spec}(k[[x,y]]/\mathbf{m}^n)\to\mathbf{A}^2$ be the closed immersion induced by the canonical map $k[x,y]\to k[[x,y]]/\mathbf{m}^n$. Let $V:=(i_n\times 1_T)(Z)$. Let plus: $\mathbf{A}^2\times\mathbf{A}^2\to\mathbf{A}^2$ be the morphism

$$((x_1,y_1),(x_2,y_2))\mapsto(x_1+x_2,y_1+y_2)$$

and minus: $\mathbf{A}^2\times\mathbf{A}^2\to\mathbf{A}^2$; $((x_1,y_1),(x_2,y_2))\mapsto(x_1-x_2,y_1-y_2)$. Let

$$\mathrm{plus}_\sigma:=(\mathrm{plus}\circ(p_1\times(p_1\circ\sigma\circ p_2)),p_2):\mathbf{A}^2\times T\to\mathbf{A}^2\times T,$$

$$\mathrm{minus}_\sigma:=(\mathrm{minus}\circ(p_1\times(p_1\circ\sigma\circ p_2)),p_2):\mathbf{A}^2\times T\to\mathbf{A}^2\times T$$

be the morphisms which add and subtract the section σ. Obviously both are isomorphisms and $(\mathrm{plus}_\sigma)^{-1}=\mathrm{minus}_\sigma$. Let $\psi_2(Z,\sigma):=\mathrm{plus}_\sigma(V)$. Then $\psi_2(Z,\sigma)$ is T-isomorphic to Z. It is easy to check that $(Z,\sigma)\mapsto\psi_2(Z,\sigma)$ is compatible with base change and so gives a morphism ψ_2.

We still have to prove the second statement. That $\psi_2(\bar{k})$ actually maps into $(\mathbf{A}^2)^{[n]}_{(n)}(\bar{k})$ is again clear. We will give the inverse map $\psi_2^{-1}(\bar{k})$: Let $W\in(\mathbf{A}^2)^{[n]}_{(n)}(\bar{k})$, $p:=(W_{\mathrm{red}})$ and $V=\mathrm{minus}_p(W)$. Then $\mathrm{supp}(V)$ is the origin $(0,0)$ of $\mathbf{A}^2$. The completion $\hat{V}$ of V at $(0,0)$ is isomorphic to V, so it is a subscheme of length n of $\mathrm{spec}(\bar{k}[[x,y]])$ defined by an ideal I of colength n in $\bar{k}[[x,y]]$. Then $I/\mathbf{m}^n$ defines a closed subscheme $\tilde{\varphi}_2(\bar{k})(W)$ of length n in $\mathrm{spec}(k[[x,y]]/\mathbf{m}^n)$. It is easy to see that $\tilde{\varphi}_2(\bar{k})=\psi_2^{-1}(\bar{k})$. □

Corollary 2.3. *There is a canonical bijection*

$$\psi(\bar{k}):(V_{n,k}\times U)(\bar{k})\to U^{[n]}_{(n)}(\bar{k})$$

commuting with the action of $\mathrm{Gal}(\bar{k}/k)$.

Proof. This follows from Lemma 2.1 and Lemma 2.2, as $p_{\mathbf{A}^2}:Z^n_{(n)}(\mathbf{A}^2)\to(\mathbf{A}^2)^{[n]}_{(n)}$ and $p_U:Z^n_{(n)}(U)\to U^{[n]}_{(n)}$ both induce bijections of geometric points. □

Lemma 2.4. *There is a (noncanonical) bijection*

$$\psi_{(n)}(\bar{k}): S^{[n]}_{(n)}(\bar{k}) \to (V_{n,k} \times S)(\bar{k})$$

commuting with the $\mathrm{Gal}(\bar{k}/k)$*-action.*

Proof. Let $(U_i)_i$ be a finite covering of S by open affine subschemes, such that there exist étale morphisms $f_i: U_i \to \mathbf{A}^2$. Let $\psi_i(\bar{k}): (V_{n,k} \times U_i)(\bar{k}) \to (U_i)^{[n]}_{(n)}(\bar{k})$ be the corresponding bijections according to Corollary 2.3. Let $W_1 := U_1$ and $W_j := U_j \setminus \left(\bigcup_{i<j} U_i\right)$ for $j > 1$. If $Z \in S^{[n]}_{(n)}(\bar{k})$, then there is a unique $i(Z)$, such that $Z_{\mathrm{red}} \in W_{i(Z)}$. We set $\psi_{(n)}(Z) := (\psi_{i(Z)}(\bar{k}))^{-1}(Z)$. All the W_i are defined over k, so for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$ we have $\sigma(Z)_{\mathrm{red}} = \sigma(Z_{\mathrm{red}}) \in W_{i(Z)}$ and thus $\psi_{(n)}(\bar{k})(\sigma(Z)) = \sigma(\psi_{(n)}(\bar{k})(Z))$. Furthermore for $Z \in S^{[n]}_{(n)}(\bar{k})$ we have $Z_{\mathrm{red}} \in W_i$ if and only if $\psi_i(\bar{k})(Z) \in (V_{n,k} \times W_i)(\bar{k})$. As all the $\psi_i(\bar{k})$ are bijective, so is $\psi_{(n)}(\bar{k})$. □

Let $\nu = (n_1, \ldots, n_r)$ be a partition of n. Then the quotient morphism $\Delta_{n_1}(S) \times \ldots \times \Delta_{n_r}(S) \to S^{(n)}_\nu$ factorizes as

$$\Delta_{n_1}(S) \times \ldots \times \Delta_{n_r}(S) \to S^{(n_1)}_{(n_1)} \times \ldots \times S^{(n_r)}_{(n_r)} \xrightarrow{s_\nu} S^{(n)}_\nu .$$

Let $S^{(n)}_{\nu *} := s_\nu^{-1}(S^{(n)}_\nu)$ and

$$S^{[n]}_{\nu *} := (\omega_{n_1} \times \ldots \times \omega_{n_r})^{-1}(S^{(n)}_{\nu *}) \subset S^{[n_1]}_{(n_1)} \times \ldots \times S^{[n_r]}_{(n_r)} .$$

Then we have

$$S^{[n]}_{\nu *}(\bar{k}) = \{(Z_1, \ldots, Z_r) \in S^{[n_1]}_{(n_1)} \times \ldots \times S^{[n_r]}_{(n_r)} \mid (Z_1)_{\mathrm{red}}, \ldots, (Z_r)_{\mathrm{red}} \text{ are distinct}\} .$$

The symmetric group $G(n)$ operates on $S^{[n]}_{\nu *} \subset S^{[n_1]}_{(n_1)} \times \ldots \times S^{[n_r]}_{(n_r)}$ by permuting the factors with the same n_i.

Lemma 2.5. *There is a natural morphism* $\psi_\nu: S^{[n]}_{\nu *} \to S^{[n]}$*, which induces a bijection*

$$\tilde{\psi}_\nu: S^{[n]}_{\nu *}(\bar{k})/G(n) \to S^{[n]}_\nu(\bar{k})$$

commuting with the $\mathrm{Gal}(\bar{k}/k)$*-action.*

Proof. Let T be a noetherian scheme, let $(Z_1, \ldots, Z_r) \in S^{[n]}_{\nu *}(T)$. We set

$$\psi_\nu(Z_1, \ldots, Z_r) := Z_1 \cup \ldots \cup Z_r \simeq Z_1 \dot{\cup} \ldots \dot{\cup} Z_r .$$

This is flat of degree n over T. ψ_ν is obviously compatible with base change and so defines a morphism. If $(Z_1, \ldots, Z_r) \in S^{[n]}_{\nu *}(\bar{k})$, then

$$\omega_n \circ \psi_\nu(\bar{k})(Z_1, \ldots, Z_r) = \omega_n(Z_1 \cup \ldots \cup Z_r) = \omega_{n_1}(Z_1) + \ldots + \omega_{n_r}(Z_r) .$$

So $\psi_\nu(\bar{k})(Z_1, \ldots, Z_r) \in S^{[n]}_\nu(\bar{k})$. $\tilde{\psi}_\nu$ is obviously well defined by $[(Z_1, \ldots, Z_n)] \mapsto Z_1 \cup \ldots \cup Z_n$, where $[\ ]$ denotes the class modulo $G(n)$. If $Z \in S^{[n]}_\nu(\bar{k})$, then it has a decomposition $Z = Z_1 \cup \ldots \cup Z_r$ into connected components and $(\tilde{\psi}_\nu)^{-1}(Z) = [(Z_1, \ldots, Z_r)]$. □

For the next two lemmas we introduce some notation. If $\bar{k}$ is an extension of k we write $V(\bar{k})$ for $\bigcup_{r=1}^{\infty} V_{r,k}(\bar{k})$. If $x \in V_{r,k}(\bar{k})$ we set $\mathrm{len}(x) = r$. If (M, f) is a pair consisting of a finite set $M \subset S(\bar{k})$ and a map $f: M \to V(\bar{k})$, we set $\mathrm{len}(f) = \sum_{m \in M} \mathrm{len}(f(m))$. $\mathrm{Gal}(\bar{k}/k)$ acts on these pairs by $\sigma(M,f) = (\sigma(M), \sigma f \sigma^{-1})$ for $\sigma \in \mathrm{Gal}(\bar{k}/k)$.

Lemma 2.6. *There is a bijection*

$$\phi: S^{[n]}(\bar{k}) \to \{(M,f) \mid M \subset S(\bar{k}) \text{ finite, } f: M \to V(\bar{k}) \text{ with } \mathrm{len}(f) = n\}$$

commuting with the $\mathrm{Gal}(\bar{k}/k)$*-action.*

Proof. Let $\nu = (n_1, \ldots, n_r)$ be a partition of n and $[(Z_1, \ldots, Z_r)] \in S^{[n]}_{\nu^*}$. We put

$$\phi([(Z_1, \ldots, Z_r)]) := (\{(Z_1)_{\mathrm{red}}, \ldots, (Z_r)_{\mathrm{red}}\}, f),$$

where $f: \{(Z_1)_{\mathrm{red}}, \ldots, (Z_r)_{\mathrm{red}}\} \to V(\bar{k})$ is defined by

$$f((Z_i)_{\mathrm{red}}) = p_1 \circ \psi_{(n_i)}(\bar{k})(Z_i)$$

(see Lemma 2.4). The result now follows from Lemma 2.4, Lemma 2.5 and the stratification of $S^{[n]}$. □

From now on let $k = \mathbf{F}_q$ and S be a smooth projective surface over $\mathbf{F}_q$. Let Q be some power of q. Let F_Q be the geometric Frobenius over $\mathbf{F}_Q$. We put $P(S, \mathbf{F}_Q) := \bigcup_{r>0} P_r(S, \mathbf{F}_Q)$. If $L \subset P(S, \mathbf{F}_Q)$ is a finite set and $g: L \to V(\bar{\mathbf{F}}_q)$, we say that g is *admissible* if $g(l) \in V(\mathbf{F}_{Q^r})$ whenever $l \in L \cap P_r(S, \mathbf{F}_Q)$, and we put

$$\mathrm{len}(g) := \sum_{l \in L} \deg(l)\, \mathrm{len}(g(l)).$$

Lemma 2.7.

$$\sum_{n=0}^{\infty} \# S^{[n]}(\mathbf{F}_Q) = \prod_{r=1}^{\infty} \left(\sum_{n=0}^{\infty} V_{n,\mathbf{F}_q}(\mathbf{F}_{Q^r}) t^{rn} \right)^{\# P_r(S, \mathbf{F}_Q)}.$$

Proof. (i) We set

$$M_{1,n} := \{(M,f) \mid M \subset S(\bar{\mathbf{F}}_q) \text{ finite, } f: M \to V(\bar{\mathbf{F}}_q),\ \mathrm{len}(f) = n,\ F_Q(M,f) = (M,f)\},$$

$$M_{2,n} := \{(L,g) \mid L \subset P(S, \mathbf{F}_Q) \text{ finite, } g: L \to V(\bar{\mathbf{F}}_q) \text{ admissible, } \mathrm{len}(g) = n\}.$$

We first want to show that there exists a bijection $\varrho: M_{1,n} \to M_{2,n}$. We choose an ordering $\leqq$ on $S(\bar{\mathbf{F}}_q)$. Let $(M,f) \in M_{1,n}$. Then

$$s := \sum_{m \in M} \mathrm{len}(f(m)) \cdot m \in S^{(n)}(\mathbf{F}_Q)$$

has a unique representation $s = \sum_{i=1}^{r} a_i \xi_i$ as positive linear combination of distinct primitive 0-cycles over $\mathbf{F}_Q$. Let $(b_i)_i$ be such, that $\xi_i \in P_{b_i}(S, \mathbf{F}_Q)$ for all i. Then we have $\xi_i = \sum_{j=0}^{b_i - 1} F_Q^j(x_i)$ for appropriate $x_i \in S(\mathbf{F}_{Q^{b_i}}) \setminus \left(\bigcup_{j < b_i} S(\mathbf{F}_{Q^j}) \right)$. We choose the x_i so that $x_i \leqq F_Q^j(x_i)$ for all $j \in \mathbf{Z}$. Then they are uniquely defined. Then we have $F_Q^{b_i}(f(x_i)) = f(F_Q^{b_i}(x_i)) = f(x_i)$, so we have $f(x_i) \in V(\mathbf{F}_{Q^{b_i}})$. We put $\varrho(M,f) := (\{\xi_1, \ldots, \xi_r\}, g)$, where $g: \{\xi_1, \ldots, \xi_r\} \to V(\bar{\mathbf{F}}_q)$ is defined by $g(\xi_i) := f(x_i)$.

The inverse mapping ϱ^{-1} is given as follows: Let $(\{\xi_1, \ldots, \xi_r\}, g) \in M_{2,n}$. Then there are $(b_i)_i$, such that $\xi_i \in P_{b_i}(S, \mathbf{F}_Q)$ for all i. For all i there is a unique $x_i \in S(\mathbf{F}_{Q^{b_i}}) \setminus \left(\bigcup_{j<b_i} S(\mathbf{F}_{Q^j}) \right)$, such that $\xi_i = \sum_{j=0}^{b_i - 1} F_Q^j(x_i)$ and $x_i \leqq F_Q^j(x_i)$ for all $j \in \mathbf{Z}$. We put

$$M = \bigcup_{i=1}^{r} \{x_i, F_Q(x_i), \ldots, F_Q^{b_i-1}(x_i)\}.$$

Then $\varrho^{-1}(\{\xi_1, \ldots, \xi_r\}, g)) = (M, f)$, where $f: M \to V(\bar{\mathbf{F}}_q)$ is defined by

$$f(F_Q^j(x_i)) := F_Q^j(g(x_i)).$$

(ii) For $i, j \in \mathbf{N}$ we put

$$N(i,j) := \#\left\{(M, m) \,\middle|\, M \subset P_i(S, \mathbf{F}_Q), f: M \to V(\mathbf{F}_{Q^i}), \sum_{m \in M} \operatorname{len}(f(m)) = j\right\}.$$

We observe that

$$\# M_{2,n} = \sum_{n_1 + 2n_2 + 3n_3 \ldots = n} \left(\prod_{s=1}^{\infty} N(s, n_s) \right).$$

On the other hand we have

$$\left(\sum_{n=0}^{\infty} V_{n, \mathbf{F}_q}(\mathbf{F}_{Q^r}) t^{rn} \right)^{\# P_r(S, \mathbf{F}_Q)} = \sum_{j=0}^{\infty} N(r,j) t^{rj}.$$

So the lemma follows. □

For $n, d \in \mathbf{N}$ we set $P(n,d) := \#\{\text{partitions of } n \text{ into parts } \leqq d\}$ as in [E–S]. We set $p(n,d) := \#\{\text{partitions of } n \text{ into } d \text{ parts}\}$. Then we have

$$\begin{aligned} p(n,d) &= \#\{\text{partitions of } n \text{ into parts the largest of which is } d\} \\ &= \#\{\text{partitions of } n-d \text{ into parts } \leqq d\} \\ &= P(n-d, d). \end{aligned}$$

We use the following result of [E–S].

Proposition 2.8 [E–S, Proposition 4.2]. *Let k be an algebraically closed field. Then $V_{n,k}$ has a cell decomposition, and for all $d \in \mathbf{N}_0$ the number of d-cells in the decomposition is $P(d, n-d)$.*

Ellingsrud and Strømme carry out their arguments only for $k = \mathbf{C}$ but state that they also hold over any algebraically closed field.

We denote by $\overset{t^l}{\equiv}$ the congruence modulo t^l in $\mathbf{Q}[[t]]$ or $\mathbf{Q}[[z,t]]$.

Lemma 2.9. *For all $n \in \mathbf{N}$ there is an $m_0 \in \mathbf{N}$, such that for all $M \in \mathbf{N}$, which are divisible by m_0, and $Q := q^M$ we have for all smooth projective surfaces S over $\mathbf{F}_Q$:*

$$\sum_{n=0}^{\infty} \# S^{[n]}(\mathbf{F}_Q) t^n \overset{t^l}{\equiv} \prod_{i=1}^{\infty} Z_Q(S, Q^{i-1} t^i) = \exp\left(\sum_{m=1}^{\infty} \frac{t^m}{m} \frac{\# S(\mathbf{F}_{Q^m})}{1 - Q^m t^m} \right).$$

Proof. Obviously we have

$$\prod_{i=1}^{\infty} \left(\frac{1}{1 - z^{i-1} t^i} \right) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} p(n, n-i) t^n z^i.$$

So we can reformulate Proposition 2.8 as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \#\{m\text{-dim. cells of } V_{n,k}\} t^n z^m = \prod_{i=1}^{\infty} \frac{1}{1-z^{i-1}t^i}.$$

Let $l \in \mathbf{N}$. Then there exists an $m_0 \in \mathbf{N}$ such that for all $n \leq l$ the cell decomposition of $V_{n,\bar{\mathbf{F}}_q}$ is already defined over $\mathbf{F}_{q^{m_0}}$. Let M be a multiple of m_0 and $r \in \mathbf{N}$. We get:

$$\sum_{n=0}^{\infty} \# V_{n,\bar{\mathbf{F}}_q}(\mathbf{F}_{Q^r}) t^{rn} \stackrel{t^l}{\equiv} \prod_{i=1}^{\infty} \frac{1}{1-Q^{r(i-1)}t^{ri}}.$$

So by Lemma 2.7 we have:

$$\begin{aligned}\sum_{n=0}^{\infty} \# S^{[n]}(\mathbf{F}_Q) t^n &\stackrel{t^l}{\equiv} \prod_{r=1}^{\infty}\prod_{i=1}^{\infty}\left(\frac{1}{1-Q^{r(i-1)}t^{ri}}\right)^{\# P_r(S,\mathbf{F}_Q)}\\ &= \exp\left(\sum_{i=1}^{\infty}\sum_{r=1}^{\infty}\sum_{h=1}^{\infty} \# P_r(S,\mathbf{F}_Q) Q^{hr(i-1)} t^{hri}/h\right)\\ &= \exp\left(\sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\left(\sum_{r\mid m} r\cdot \# P_r(S,\mathbf{F}_Q)\right) Q^{m(i-1)}\frac{t^{mi}}{m}\right)\\ &= \exp\left(\sum_{n=1}^{\infty}\frac{t^m}{m}\frac{\# S(\mathbf{F}_{Q^m})}{1-Q^m t^m}\right).\end{aligned}$$

The formula including the zeta function $Z_Q(S,t)$ follows by an easy calculation. □

Proof of Theorem 0.1. Let $n \in \mathbf{N}$. Let S be defined over $\bar{\mathbf{F}}_q$. Then there is a $\tilde{q} = q^l$ and a smooth projective surface S_0 over $\mathbf{F}_{\tilde{q}}$, such that $S = S_0 \times_{\mathbf{F}_{\tilde{q}}} \bar{\mathbf{F}}_q$. There exists a $Q = \tilde{q}^M$, such that for all $h \in \mathbf{N}$ the number $\# S^{[n]}(\mathbf{F}_{Q^h})$ is the coefficient of t^n in

$$\exp\left(\sum_{m=1}^{\infty}\frac{t^m}{m}\frac{\# S_0(\mathbf{F}_{Q^{hm}})}{(1-Q^{hm}t^m)}\right).$$

(1a) follows by Remark 1.2. (1b) follows by a trivial calculation and (2) by setting $z := -1$. If S is defined over $\mathbf{C}$, we consider a good reduction of S modulo q for an appropriate prime power q. □

Corollary 2.10. (a) *If* $e(S)=0$, *then for all* $n \in \mathbf{N}$ *we have* $e(S^{[n]})=0$.

(b) *If S is a K3 surface, then*

$$\sum_{n=1}^{\infty} e(S^{[n]}) q^n = \frac{q}{\Delta(\tau)},$$

where $q := e^{2\pi i\tau}$, $\tau \in \mathbf{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$ *and* Δ *is the cusp form of weight* 12 *for* $Sl_2(\mathbf{Z})$.

If S is an abelian variety then (a) is already known (see for instance [B, footnote on p. 769]).

Corollary 2.11. *Let S be a smooth irreducible surface over* $\mathbf{C}$. *Then*

$$P(S^{[n]},z) \stackrel{z^{n+1}}{\equiv} \prod_{m=1}^{\infty}\frac{((1+z^{2m-1})(1+z^{2m+1}))^{b_1(S)}}{(1-z^{2m})^{b_2+1}(1-z^{2m+2})}.$$

So the Betti numbers $b_l(S^{[n]})$ become stable for $n \geqq i$.

Proof. We set

$$G(z,t) := (1-t) \prod_{m=1}^{\infty} \frac{(1+z^{2m-1}t^m)^{b_1(S)}(1+z^{2m+1}t^m)^{b_1(S)}}{(1-z^{2m-2}t^m)(1-z^{2m}t^m)^{b_2(S)}(1-z^{2m+2}t^m)}.$$

Then we have to show that $P(S^{[n]}, z) \overset{z^{n+1}}{\equiv} G(z,1)$. For $f \in \mathbf{Q}[[z,t]]$ we denote by $a_{v,l}(f)$ the coefficient of $z^v t^l$ in f. If $v > l$, then $a_{v,l}(G(z,t)) = 0$. Let $v \leqq n$. Then we have by Theorem 0.1 (1b):

$$\begin{aligned} b_v(S^{[n]}) &= a_{v,n}\left(\left(\sum_{l=0}^{\infty} t^l\right) G(z,t)\right) \\ &= \sum_{l=0}^{n} a_{v,l}(G(z,t)) \\ &= \sum_{l=0}^{\infty} a_{v,l}(G(z,t)) \\ &= a_{v,0}(G(z,1)). \quad \square \end{aligned}$$

3. The Hodge numbers of $\mathrm{Hilb}^n(S)$

Let S be a smooth projective surface over $\mathbf{C}$. We conclude by giving a conjecture about the Hodge numbers of $S^{[n]}$ and proving a small part of it.

Conjecture 3.1.

$$\sum_{n=0}^{\infty} h(S^{[n]}, x, y)t^n = \prod_{k=1}^{\infty} \left(\frac{\prod_{p+q \text{ odd}} (1 + x^{p+k-1}y^{q+k-1}t^k)^{h^{p,q}(S)}}{\prod_{p+q \text{ even}} (1 - x^{p+k-1}y^{q+k-1}t^k)^{h^{p,q}(S)}} \right),$$

$$\sum_{n=0}^{\infty} \tilde{h}(S^{[n]}, x, y)t^n = \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n} \frac{\tilde{h}(S, x^n, y^n)}{1-(xyt)^n} \right), \tag{1}$$

$$\sum_{n=0}^{\infty} \chi_{-y}(S^{[n]})t^n = \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n} \frac{\chi_{-y^n}(S)}{1-(yt)^n} \right), \tag{2}$$

$$\sum_{n=0}^{\infty} \operatorname{sign}(S^{[n]})t^n = \prod_{k=1}^{\infty} \left(\frac{1+t^k}{1-t^k} \right)^{(-1)^{k+1}\operatorname{sign}(S)/2} (1-t^{2k})^{-e(S)/2} \tag{3}$$

(2) and (3) follow from (1) by a trivial computation as $\operatorname{sign}(S) = \chi_1(S)$ and $e(S) = \chi_{-1}(S)$. By the results of [E–S] the conjecture is true for the projective plane $\mathbf{P}_2$ and the Hirzebruch surfaces Σ_n. (1) is obtained from Lemma 1.9 by replacing Q by xy and $\# S(\mathbf{F}_{Q^i})$ by $\tilde{h}(S, x^i, y^i)$. Thus the conjecture is also true for all smooth surfaces S over $\mathbf{C}$, for which there exists a good reduction $\tilde{S}$ of S modulo q, such that for all $n \in \mathbf{N}$ the Newton polygon of $\tilde{S}^{[n]}$ coincides with the Hodge polygon of $S^{[n]}$ (see [Ma, (1.9)] for the definitions).

We will now compute the Hodge numbers $h^{p,0}$ and so prove a small part of the conjecture. We consider the complex topology.

$S^{(n)}$ is a V-manifold of dimension $2n$. We denote by ε the partition $(1, \ldots, 1)$. Then $S_\varepsilon^{(n)}$ is the smooth locus of $S^{(n)}$. Let $j: S_\varepsilon^{(n)} \to S^{(n)}$ be the inclusion. Following [S] we define $\tilde{\Omega}^p_{S^{(n)}} := j_* \Omega^p_{S_\varepsilon^{(n)}}$. As $\omega_n: S^{[n]} \to S^{(n)}$ is a resolution of $S^{(n)}$, we have by [S, Lemma 1.11]: $\tilde{\Omega}^p_{S^{(n)}} = (\omega_n)_* \Omega^p_{S^{[n]}}$.

Lemma 3.2. *For* $p = 0, \ldots, 2n$ *we have*

$$H^0(S^{[n]}, \Omega^p_{S^{[n]}}) \simeq H^0(S^n, \Omega^p_{S^n})^{G(n)}.$$

Proof. By the above $H^0(S^{[n]}, \Omega^p_{S^{[n]}}) \simeq H^0(S_\varepsilon^{(n)}, \Omega^p_{S_\varepsilon^{(n)}})$. Let $s \in H^0(S^n, \Omega^p_{S^n})^{G(n)}$. Let $(V_i)_i$ be an open cover of $S_\varepsilon^{(n)}$, such that for all i we have $\Phi^{-1}(V_i) = U_{i,1} \cup \ldots \cup U_{i,n}$, where the $(U_{i,j})_j$ are disjoint and $\Phi|_{U_{i,j}}: U_{i,j} \to V_i$ is an isomorphism. Then for some $j \in \{1, \ldots, n\}$ we let

$$s_i := ((\Phi|_{U_{i,j}})^{-1})^*(s|_{U_{i,j}}) \in H^0(V_i, \Omega^p_{V_i}).$$

s_i is independent of the choice of j, so for all i and k we have that s_i and s_k agree on $V_i \cap V_k$. So the s_i define a global section $g(s) \in H^0(S_\varepsilon^{(n)}, \Omega^p_{S_\varepsilon^{(n)}})$. This defines a homomorphism

$$g: H^0(S^n, \Omega^p_{S^n})^{G(n)} \to H^0(S_\varepsilon^{(n)}, \Omega^p_{S_\varepsilon^{(n)}}).$$

We shall now define the inverse homomorphism. Let $\tilde{s} \in H^0(S_\varepsilon^{(n)}, \Omega^p_{S_\varepsilon^{(n)}})$. Then we have $\Phi^*(\tilde{s}) \in H^0(S_\varepsilon^n, \Omega^p_{S_\varepsilon^n})^{G(n)}$. As $S^n \backslash S_\varepsilon^n$ has codimension 2 in S^n, we have by the theorem of Hartogs, that there is a unique $f(\tilde{s}) \in H^0(S^n, \Omega^p_{S^n})$, such that $f(\tilde{s})|_{S_\varepsilon^n} = \Phi^*(\tilde{s})$. By its uniqueness it satisfies $f(\tilde{s}) \in H^0(S^n, \Omega^p_{S^n})^{G(n)}$. Obviously we have $f = g^{-1}$. □

Proposition 3.3.

(a)
$$\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} h^{p,0}(S^{[n]}, z) t^n = \frac{(1+zt)^{h^{1,0}(S)}}{(1-t)(1-z^2 t)^{h^{2,0}(S)}},$$

(b)
$$\chi(\mathcal{O}_{S^{[n]}}) = \binom{\chi(\mathcal{O}_S) + n - 1}{n}.$$

Proof. Let $p_i: S^n \to S$ be the i^{th} projection. Then we have

$$H^0(S^n, \Omega^*_{S^n}) = p_1^*(H^0(S, \Omega^*_S)) \otimes \ldots \otimes p_n^*(H^0(S, \Omega^*_S)).$$

Let $\omega_1, \ldots, \omega_m$ be a homogeneous basis for $H^0(S, \Omega^*_S)$; i.e. there are $d_1, \ldots, d_m \in \{0, 1, 2\}$, such that $\omega_i \in H^0(S, \Omega^{d_i}_S)$. Then a basis of $H^0(S^n, \Omega^p_{S^n})$ is given by

$$\left\{ p_1^*(\omega_{i_1}) \wedge \ldots \wedge p_n^*(\omega_{i_n}) \,\middle|\, \sum_{j=1}^{n} d_{i_j} = p \right\}.$$

$\omega \in H^0(S^n, \Omega^p_{S^n})$ is $G(n)$-invariant if and only if $\omega = \sum_{\sigma \in G(n)} \sigma^*(\eta)$ for an appropriate $\eta \in H^0(S^n, \Omega^p_{S^n})$. For $(i_1, \ldots, i_n) \in \{0, \ldots, m\}^n$ we set

$$\langle i_1, \ldots, i_n \rangle := \sum_{\sigma \in G(n)} \sigma^*(\omega_{i_1} \wedge \ldots \wedge \omega_{i_n}) \in H^0(S^n, \Omega^p_{S^n})^{G(n)}.$$

Then $H^0(S^n, \Omega^p_{S^n})$ is generated by the set

$$\left\{ \langle i_1, \ldots, i_n \rangle \,\middle|\, (i_1, \ldots, i_n) \in \{0, \ldots, m\}^n, \sum_{j=1}^{n} d_{i_j} = p \right\}.$$

The form $\langle i_1, \ldots, i_n \rangle$ is independent up to sign of the ordering of $i_1, \ldots, i_n$ and so is determined up to sign by the multiplicity n_k with which the $k \in \{1, \ldots, m\}$ appear in $i_1, \ldots, i_n$. If $n_k > 1$ for a k with $d_k = 1$, then we have $\langle i_1, \ldots, i_n \rangle = 0$, as the wedge product of 1-forms is anticommutative. Apart from these relations the $\langle i_1, \ldots, i_n \rangle$ are linearly independent. So we get:

$$h^0(S^{[n]}, \Omega^p_{S^{[n]}}) = \# \left\{ (n_1, \ldots, n_m) \in \mathbb{N}_0^n \,\middle|\, \sum_i n_i = n,\ \sum_i n_i d_i = p,\ n_i \leqq 1 \text{ if } d_i = 1 \right\}$$

and

$$\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} h^{p,0}(S^{[n]}, z) t^n = \sum_{\substack{n_0, \ldots, n_m \\ n_i \leqq 1 \text{ if } d_i = 1}} t^{n_0 + \ldots + n_m} z^{n_0 d_0 + \ldots + n_m d_m}$$
$$= \frac{(1+zt)^{h^{1,0}(S)}}{(1-t)(1-z^2 t)^{h^{2,0}(S)}}.$$

This proves (a). (b) follows by setting $z := -1$. □

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