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Criteria for Membership of Bloch Space and its Subspace, BMOA

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Introduction

It is a commonplace in classical harmonic analysis on a locally compact group G that a function f belongs to a normed vector space V of functions on G if and only if certain weighted averages of f form a bounded subset of V [6, 15]. Such weighted averages are usually convolutions of f with the members of an approximate identity. This well-established principle has been exploited in the context of spaces of analytic functions on the open unit disc U, as well. In particular, it has been utilized in [8] to yield criteria for membership of the Hardy spaces H^p, $1 < p \leqq \infty$, of functions f analytic on U for which

$$\|f\|_p = \sup\left\{\left(1/2\pi \int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p} : 0 \leqq r < 1\right\} < \infty,$$

with the usual convention applying when $p = \infty$.

It is this principle which forms the central theme of this paper, which is made up of three separate, but interconnected parts.

In the first part, we apply the above principle to the class $\mathscr{B}$ of *Bloch functions* on U and derive criteria in terms of the Cesàro means. In the second part, we turn our attention to the subspace of $\mathscr{B}$ composed of functions with *bounded mean oscillation*, and give a characterization of the subspace in the same spirit. The third part is devoted to a discussion of *bounded Hankel operators* on H^2, and continues the work begun by Bonsall in [3] and continued in [4, 7, 14]. (Indeed, it was our investigations into Hankel operators that first suggested the possibility that Theorems 2 and 3, in particular, might be true.)

Part I

The Space of Bloch Functions

We recall some facts about $\mathscr{B}$.

First, $\mathscr{B}$ consists of functions f that are analytic on U whose derivatives f' are subject to the growth restriction

$$\sup\{(1-|z|^2)|f'(z)| : z \in U\} < \infty.$$

Equipped with the norm

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup\{(1-|z|^2)|f'(z)| : z \in U\},$$

$\mathscr{B}$ is a Banach space. It properly contains H^∞, and the inclusion $H^\infty \subset \mathscr{B}$ is continuous, as the easily derived inequality

$$\|f\|_{\mathscr{B}} \leqq 2\|f\|_\infty \quad (f \in H^\infty)$$

shows; it is invariant under the family M of disc automorphisms given by

$$M = \{\mu_\zeta : \zeta \in U\},$$

where, for each $\zeta \in U$,

$$\mu_\zeta(z) = (z+\zeta)/(1+\bar{\zeta}z) \quad (z \in U),$$

and it is the dual space of the space $\mathscr{I}$ of functions g that are analytic on U and such that the norm

$$\|g\|_{\mathscr{I}} = |g(0)| + 1/\pi \int_0^1 \int_0^{2\pi} |re^{i\theta} g'(re^{i\theta}) + g(re^{i\theta}) - g(0)| dr\, d\theta$$

is finite.

These and other basic properties of $\mathscr{B}$ were discovered, recorded and developed by Anderson et al. [1], which was the first systematic study of the space. (For another, more recent, account see [5].) For ease of reference, we state the form of the above mentioned duality that we use in this paper; we refer to [1] for a full discussion.

Given a pair of functions $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$, analytic on U, we write $f*g$ for their Hadamard product:

$$f*g(z) = \sum a_n b_n z^n \quad (z \in U).$$

(Here, and hereafter, sums written without limits are over the non-negative integers.)

In this notation, an examination of the proofs of Theorems 2.3 and 2.4 in [1] – with due regard paid to the different norms on $\mathscr{I}$ – reveals that we now have

Theorem A. (a) *Let $f \in \mathscr{B}$ and $g \in \mathscr{I}$. Then $f*g$ is continuous on the closure of U and*

$$\|f*g\|_\infty \leqq \|f\|_{\mathscr{B}} \|g\|_{\mathscr{I}}.$$

(b) *If ψ is a continuous linear functional on $\mathscr{I}$, then there is a unique function $f \in \mathscr{B}$ such that*

$$\psi(g) = \lim_{\varrho \to 1^-} f*g(\varrho) \quad (g \in \mathscr{I})$$

and

$$1/5\|f\|_{\mathscr{B}} \leqq \|\psi\| \leqq \|f\|_{\mathscr{B}}.$$

Criteria for Membership of $\mathscr{B}$

In preparation for our discussion, we introduce some standard notation that we will use throughout the paper.

Given

$$f(z)=\sum a_n z^n \quad (z\in U)$$

we write

$$s_n(f)(z)=\sum_{k=0}^{n} a_k z^k, \qquad n=0,1,2,\ldots,$$

and

$$\begin{aligned}\sigma_n(f)(z)&=\sum_{k=0}^{n} s_k(f)(z)/(n+1), \qquad n=0,1,2,\ldots\\ &=\sum_{k=0}^{n}(1-k/(n+1))a_k z^k, \qquad n=0,1,2,\ldots,\end{aligned}$$

for the partial sums and the Cesàro means of the power series for f.

With

$$k(z)=1/(1-z)=\sum z^n \quad (z\in U),$$

we see that $\sigma_n(f)=\sigma_n(k)*f$. Also, if $n\geqq 0$ and $z\in U$, then

$$\sigma_n(f)(z)=1/2\pi\int_0^{2\pi} K_n(\phi)f(ze^{-i\phi})d\phi,$$

where

$$\begin{aligned}K_n(\phi)&=\sum_{-n}^{n}(1-|k|/(n+1))e^{ik\phi}\\ &=\frac{1-\cos(n+1)\phi}{(n+1)(1-\cos\phi)},\end{aligned}$$

is Fejér's kernel. (For later reference, we note that $K_n\geqq 0$ and that

$$\left.\int_0^{2\pi} K_n(\phi)d\phi=2\pi, \quad (n=0,1,2,\ldots)\right).$$

Our first result is probably known to most workers in the field, and the equivalence of (i) and (iii) is explicitly recorded in [2] and [13]. We include it for completeness and give what seems to be a new derivation of this equivalence.

Theorem 1. *The following statements are equivalent.*

(i) $\qquad f\in\mathscr{B};$

(ii) $\qquad \|\sigma_n(f)\|_{\mathscr{B}}=O(1) \quad (n\to\infty);$

(iii) $\qquad \|\sigma_n'(f)\|_\infty=O(n) \quad (n\to\infty).$

Proof. (i) $\Rightarrow$ (ii). For, if $g\in\mathscr{I}$, then

$$\sigma_n(f)*g(z)=1/2\pi\int_0^{2\pi} K_n(\phi)f*g(ze^{-i\phi})d\phi \quad (z\in U).$$

Hence

$$|\sigma_n(f)^*g(z)| \leqq 1/2\pi \int_0^{2\pi} K_n(\phi)\,\|f^*g\|_\infty d\phi$$
$$= \|f^*g\|_\infty$$
$$\leqq \|f\|_{\mathscr{B}} \|g\|_{\mathscr{I}},$$

by part (a) of Theorem A. Applying part (b), we deduce that

$$\|\sigma_n(f)\|_{\mathscr{B}} \leqq 5\|f\|_{\mathscr{B}},$$

whence (ii) holds.

(ii) $\Rightarrow$ (iii). To see this, let $p_n = \sigma_n'(f)$ and $z = r\zeta$, where $r = |z| < 1$. Then

$$|p_n(\zeta) - p_n(r\zeta)| = \left|\int_r^1 p_n'(t\zeta)\zeta dt\right|$$
$$\leqq \int_r^1 \max_{|w|=t} |p_n'(w)| dt$$
$$\leqq (1-r)\,\|p_n'\|_\infty$$
$$\leqq n(1-r)\,\|p_n\|_\infty,$$

by Bernstein's theorem on polynomials [15, p. 118, Vol. I]. Hence

$$|p_n(\zeta)| \leqq |p_n(z)| + n(1-r)\|p_n\|_\infty.$$

Choose $r = 1 - 1/2n$. Then

$$|p_n(\zeta)| \leqq 2n \sup\{(1-|z|^2)|p_n(z)| : z \in U\} + 1/2\|p_n\|_\infty,$$

whence it follows that

$$\|p_n\|_\infty \leqq 4n\|\sigma_n(f)\|_{\mathscr{B}} = O(n) \quad (n \to \infty)$$

and (iii) holds.

(iii) $\Rightarrow$ (i). For, if $f(z) = \sum a_n z^n$, and $z = r\zeta$ as before, then

$$zf'(z) = \sum k a_k z^k = (1-r)^2 \sum (n+1)\zeta\sigma_n'(f)(\zeta) r^n$$

after two summations by parts. It follows that

$$r|f'(z)| \leqq (1-r)^2 \sum (n+1)\,\|\sigma_n'(f)\|_\infty r^n$$
$$= O(1) r/(1-r),$$

as $r \to 1^-$ and so $f \in \mathscr{B}$.

This completes the proof.

Theorem 2. *The following statements are equivalent.*

(i) $$f \in \mathscr{B};$$

(ii) $$\|(n+m+1)\sigma_{n+m}(f) - (n+1)\sigma_n(f) - (m+1)\sigma_m(f)\|_\infty = O(\sqrt{(nm)}) \quad (n, m \to \infty);$$

(iii) $$\|\sigma_{2n}(f) - \sigma_n(f)\|_\infty = O(1) \quad (n \to \infty).$$

Proof. (i) $\Rightarrow$ (ii). Noting that $\sigma_n(f)=\sigma_n(k)^*f$, we see that

$$(n+m+1)\sigma_{n+m}(f)-(n+1)\sigma_n(f)-(m+1)\sigma_m(f)\equiv g_{n,m}{}^*f,$$

where

$$g_{n,m}=g_{m,n}=(n+m+1)\sigma_{n+m}(k)-(n+1)\sigma_n(k)-(m+1)\sigma_m(k).$$

Now

$$\sigma_n(k)(z)=\sum_0^n(1-k/(n+1))z^k, \qquad n=0,1,2,\dots,$$
$$=\frac{-z(1-z^{n+1})}{(n+1)(1-z)^2}+\frac{1}{1-z}.$$

Hence

$$g_{n,m}(z)=\frac{z(1-z^{n+1})(1-z^{m+1})}{(1-z)^2}.$$

To establish the assertion, it suffices, in view of part (a) of Theorem A, to show that

$$\pi\|g_{n,m}\|_{\mathscr{J}}=\int_0^1\int_0^{2\pi}|re^{i\theta}g'_{n,m}(re^{i\theta})+g_{n,m}(re^{i\theta})|dr\,d\theta=O(\sqrt{(nm)})$$

as $n,m\to\infty$. With this in mind we observe that, if $n,m=1,2,\dots$, then

$$zg'_{n-1,m-1}(z)+g_{n-1,m-1}(z)$$
$$=\frac{d}{dz}zg_{n-1,m-1}(z)$$
$$=\frac{2z(1-z^n)(1-z^m)}{(1-z)^3}-\frac{nz^{n+1}(1-z^m)}{(1-z)^2}-\frac{mz^{m+1}(1-z^n)}{(1-z)^2}.$$

Hence, appealing to the Cauchy-Schwarz inequality, it is enough to show that the following estimates hold:

$$I_1(n,m)=n\int_0^1\int_0^{2\pi}|(1-z^m)/(1-z)^2|r^{n+1}dr\,d\theta=O(\sqrt{(nm)})$$

and

$$I_2(n)=\int_0^1\int_0^{2\pi}|(1-z^n)/(1-z)^{3/2}|^2r\,dr\,d\theta=O(n).$$

To deal with $I_1(n,m)$, another application of the Cauchy-Schwarz inequality shows that

$$I_1(n,m)\leq n\int_0^1 r^{n+1}\sqrt{\left(\int_0^{2\pi}|(1-z^m)/(1-z)|^2d\theta\int_0^{2\pi}1/|1-z|^2d\theta\right)}\,dr$$
$$\leq 2\pi n\sqrt{m}\int_0^1 r^{n+1}(1-r^2)^{-1/2}dr$$
$$=\pi n\sqrt{m}\Gamma(1+n/2)\Gamma(1/2)/\Gamma((n+3)/2)$$
$$=O(\sqrt{(nm)})$$

as $n,m\to\infty$.

To handle $I_2(n)$, let

$$\frac{1}{(1-z)^{3/2}} = \sum \binom{-3/2}{k} (-1)^k z^k \equiv \sum c_k z^k,$$

so that, defining $c_s = 0$ if $s < 0$, we have

$$\frac{(1-z^n)}{(1-z)^{3/2}} = \sum (c_k - c_{k-n}) z^k.$$

Hence

$$\begin{aligned} I_2(n) &= 2\pi \int_0^1 \sum |c_k - c_{k-n}|^2 r^{2k+1} dr \\ &= \pi \sum |c_k - c_{k-n}|^2/(k+1) \\ &= \pi \sum_0^{n-1} |c_k|^2/(k+1) + \pi \sum_n^\infty |c_{k-n} - c_k|^2/(k+1) \\ &\equiv \sum_1 + \sum_2 . \end{aligned}$$

Now

$$c_k = \gamma(k), \qquad k = 0, 1, 2, \ldots,$$

where, for $t \geqq 0$,

$$\gamma(t) = \frac{\Gamma(t+3/2)}{\Gamma(3/2)\Gamma(t+1)}.$$

Standard properties of the gamma function [12, p. 149] show that

$$\begin{aligned} \gamma'(t) &= \gamma(t) \left\{ \frac{\Gamma'(t+3/2)}{\Gamma(t+3/2)} - \frac{\Gamma'(t+1)}{\Gamma(t+1)} \right\} \\ &= \gamma(t) \sum_1^\infty \frac{1/2}{(n+t)(n+t+1/2)} \\ &\leqq \gamma(t) \sum_1^\infty \frac{1/2}{(n+t-1/2)(n+t+1/2)} \\ &= \gamma(t)/(1+2t). \end{aligned}$$

It follows that $\gamma(t)/\sqrt{(2t+1)}$ is decreasing on $[0, \infty)$, and hence that the following inequalities are true: for $t \geqq 0$

$$\gamma(t) \leqq \sqrt{(1+2t)} \quad \text{and} \quad \gamma'(t) \leqq 1/\sqrt{(1+2t)}.$$

In particular, then, $c_k \leqq \sqrt{2(1+k)}$, and hence $\sum_1 = O(n)$ $(n \to \infty)$.

To deal with $\sum_2$, we note that if $k \geqq n$, then

$$0 \leqq c_k - c_{k-n} = \int_{k-n}^{k} \gamma'(t)dt$$
$$\leqq \int_{k-n}^{k} \frac{dt}{\sqrt{(1+2t)}}$$
$$= \sqrt{(1+2k)} - \sqrt{(1+2k-2n)}$$
$$\leqq 2n/\sqrt{(1+2k)}.$$

It follows at once that

$$\sum_2 \leqq 4\pi n^2 \sum_{k=n}^{\infty} (k+1)^{-2} = O(n) \quad (n \to \infty).$$

Collecting our results we see that (ii) holds.

(ii) $\Rightarrow$ (iii). Taking $m=1$ in (ii) to start with, we easily obtain the rough estimate $\|\sigma_n(f)\|_\infty = O(n)$ as $n \to \infty$. Using this and applying (ii) with $m=n$, we deduce (iii).

(iii) $\Rightarrow$ (i). Suppose (iii) obtains. Since $\sigma_{2n}(f) - \sigma_n(f)$ is a polynomial of degree at most $2n$, another application of Bernstein's theorem on polynomials shows that

$$\|\sigma'_{2n}(f) - \sigma'_n(f)\|_\infty = O(n) \quad (n \to \infty).$$

Taking $n = 2^j$ here and summing on j from $j=0$ to $j=m$, we infer that

$$\|\sigma'_{2^m}(f)\|_\infty = O(2^m) \quad (m \to \infty).$$

Let

$$V_m = (2+2^{-m})\sigma_{2^{m+1}}(k) - (1+2^{-m})\sigma_{2^m}(k).$$

Then

$$V_m * f = (2+2^{-m})\sigma_{2^{m+1}}(f) - (1+2^{-m})\sigma_{2^m}(f).$$

Hence

$$\|(V_m * f)'\|_\infty = O(2^m) \quad (m \to \infty).$$

Noting that μ_0 is the identity function on U, it is easy to verify that

$$\sigma_n(\mu_0 f') = \mu_0 \sigma'_n(f).$$

Also, it is classical that $\|\sigma_n(h)\|_\infty \leqq \|h\|_\infty$, whenever $h \in H^\infty$.

Furthermore, the first 2^m Taylor coefficients of f and $V_m * f$ are equal.

We put these three facts together to prove that $\|\sigma'_n(f)\|_\infty = O(n)$ as $n \to \infty$. So, let $n \geqq 1$ and choose m so that $2^{m-1} \leqq n < 2^m$. Then

$$\|\mu_0 \sigma'_n(f)\|_\infty = \|\sigma_n(\mu_0 f')\|_\infty = \|\sigma_n(\mu_0 (V_m * f)')\|_\infty$$
$$\leqq \|(V_m * f)'\|_\infty$$
$$= O(2^m)$$
$$= O(n)$$

as $n \to \infty$.

A direct appeal to Theorem 1 now completes the proof of the implication (iii) $\Rightarrow$ (i).

Corollary 1. *If $f \in \mathscr{B}$, then*

$$\|\sigma_n(f)\|_\infty = O(\log n) \quad (n \to \infty).$$

The converse is false.

Proof. By part (iii) of Theorem 2, $\|\sigma_{2n}(f) - \sigma_n(f)\|_\infty = O(1)$, whence

$$\|\sigma_{2^m}(f)\|_\infty = O(m) \quad (m \to \infty),$$

and so

$$\|V_m * f\|_\infty = O(m).$$

If now $2^{m-1} \leqq n < 2^m$, then, as above,

$$\begin{aligned} \|\sigma_n(f)\|_\infty &= \|\sigma_n(V_m * f)\|_\infty \\ &\leqq \|V_m * f\|_\infty \\ &= O(m) \\ &= O(\log n) \end{aligned}$$

as $n \to \infty$.

One way to see that the converse is false is to use the example

$$f(z) = \sum a_n z^n = \sum 2^k z^{n(k)},$$

where $n(k) = 2^{2^k}$, $k = 0, 1, 2, \ldots$. It is clear that

$$\|s_n(f)\|_\infty = \sum_1^n a_m = O(\log n) \quad (n \to \infty),$$

and so too that

$$\|\sigma_n(f)\|_\infty = \sigma_n(f)(1) = O(\log n) \quad (n \to \infty).$$

But f does not belong to $\mathscr{B}$. This can be deduced directly as follows. If $0 < r < 1$, then

$$\begin{aligned} rf'(r) &= \sum m a_m r^m \\ &> s'_{n(k)}(f)(r) \\ &> r^{n(k)} \sum_0^k 2^j n(j) \\ &> r^{n(k)} 2^k n(k) \end{aligned}$$

for any k. It results from this that $(1-r)f'(r) \to \infty$ if $r \to 1^-$ on the sequence $r = r(k) = 1 - 1/n(k)$ and $k \to \infty$. In other words, f is not a Bloch function.

Corollary 2. *Let $f(z) = \sum a_n z^n$ $(z \in U)$. If*

$$\|s_{2n}(f) - s_n(f)\|_\infty = O(1) \quad (n \to \infty),$$

then $f \in \mathscr{B}$. The converse is true if $a_n \geqq 0$, for $n = 0, 1, \ldots$; but false in general.

Proof. Arguments similar to those used in the proof of the implication (iii) $\Rightarrow$ (i) show that $\|s_n'(f)\|_\infty = O(n)$, whence the conclusion of the first part follows readily.

If, on the other hand, $f \in \mathscr{B}$ and its Taylor coefficients are non-negative, then

$$\sum k a_k r^k = r f'(r) = O(1/(1-r)) \quad (r \to 1)$$

and so, in a standard manner,

$$0 \leqq s_{2n}(f) - s_n(f) = \sum_{n+1}^{2n} a_k \leqq (1/(n+1)) \sum_{0}^{2n} k a_k = O(1) \quad (n \to \infty).$$

We defer the construction of an appropriate example showing that the full converse is false to Part II.

Corollary 3. *Let $\mathscr{P}_n$ denote the collection of analytic polynomials p of degree at most 2n whose first n+1 Taylor coefficients vanish. Let $\mathscr{P} = \cup\{\mathscr{P}_n : n \geqq 1\}$. Then the Bloch norm and the H^∞ norm are equivalent on $\mathscr{P}$.*

Proof. Since $\mathscr{P} \subset H^\infty$, $\|p\|_{\mathscr{B}} \leqq 2\|p\|_\infty$ whenever $p \in \mathscr{P}$. On the other hand, if $p \in \mathscr{P}_n$, for some n, then

$$\begin{aligned} np &= (3n+1)\,\sigma_{3n}(p) - (2n+1)\sigma_{2n}(p) - (n+1)\sigma_n(p) \\ &= g_{n,2n} * p\,, \end{aligned}$$

in the notation used above in the proof of Theorem 2. Hence

$$\begin{aligned} n\|p\|_\infty &\leqq \|g_{n,2n}\|_{\mathscr{I}} \|p\|_{\mathscr{B}} \\ &\leqq Kn\|p\|_{\mathscr{B}} \end{aligned}$$

for some absolute constant K. Thus $\|p\|_\infty \leqq K\|p\|_{\mathscr{B}}$, and we are finished.

Theorem 3. *$f \in \mathscr{B}$ if and only if*

$$\sup\{\sqrt{(1-|u|^2)}\sqrt{(1-|v|^2)}|(f(u)-f(v))/(u-v)| : u, v \in U\} < \infty .$$

Proof. If $u, v \in U$, the difference quotient $(f(u)-f(v))/(u-v)$ is seen to be the Hadamard product of f and $g_{u,v}$ evaluated at 1, where

$$\begin{aligned} g_{u,v}(z) &= \frac{k(uz)-k(vz)}{u-v}, \quad (z \in U) \\ &= \frac{z}{(1-uz)(1-vz)}. \end{aligned}$$

The stated result follows therefore from Theorem A if it can be shown that

$$\sup\{\sqrt{(1-|u|^2)}\sqrt{(1-|v|^2)}\,\|g_{u,v}\|_{\mathscr{I}} : u, v \in U\} < \infty .$$

But

$$\begin{aligned} z g_{u,v}'(z) + g_{u,v}(z) &= \frac{d}{dz} z g_{u,v}(z) \\ &= \frac{2z}{(1-uz)(1-vz)} + \frac{uz^2}{(1-uz)^2(1-vz)} + \frac{vz^2}{(1-uz)(1-vz)^2} \\ &= \frac{z}{(1-uz)^2(1-vz)} + \frac{z}{(1-uz)(1-vz)^2} \end{aligned}$$

and an application of the Cauchy-Schwarz inequality shows that

$$\begin{aligned}
&\int_0^1 \int_0^{2\pi} |1/(1-uz)^2(1-vz)| r dr\, d\theta \\
&\leqq \int_0^1 \left(\sqrt{\int_0^{2\pi} |1/(1-uz)^2|^2 d\theta} \sqrt{\int_0^{2\pi} |1/(1-vz)|^2 d\theta} \right) r dr \\
&= 2\pi \int_0^1 \sqrt{\left(\sum (n+1)^2 |u|^{2n} r^{2n} / (1-|v|^2 r^2)\right)} r dr \\
&\leqq 2\pi \sqrt{2} \left(\int_0^1 (1-|u|^2 r^2)^{-3/2} r dr \right) \Big/ \sqrt{(1-|v|^2)} \\
&\leqq 2\sqrt{2\pi} / \sqrt{(1-|u|^2)(1-|v|^2)} .
\end{aligned}$$

Putting these estimates together, we see that

$$\sup\{\sqrt{(1-|u|^2)(1-|v|^2)}\, \|g_{u,v}\|_{\mathscr{I}} : u, v \in U\} \leqq 4\sqrt{2},$$

and the conclusion of the theorem follows.

Part II

The Space of Functions of Bounded Mean Oscillation

The space of analytic functions on U of bounded mean oscillation, denoted by BMOA, consists of functions f in H^2 for which

$$\sup\{\|f_\zeta\|_2 : \zeta \in U\} < \infty,$$

where, for each $\zeta \in U$,

$$f_\zeta = f \circ \mu_\zeta - f(\zeta), \quad \mu_\zeta \in M.$$

BMOA is a Banach space under the norm given by

$$\|f\|_* = |f(0)| + \sup\{\|f_\zeta\|_2 : \zeta \in U\}.$$

Since

$$\begin{aligned}
f_\zeta(z) &= f(\mu_\zeta(z)) - f(\zeta), \quad (z \in U) \\
&= (1-|\zeta|^2) f'(\zeta) z + \ldots,
\end{aligned}$$

BMOA $\subset \mathscr{B}$; and the inclusion is strict.

An alternate characterization of BMOA can be given in terms of the Poisson kernel. Indeed, noting by a change of variable that

$$\|f_\zeta\|_2^2 = \int_0^{2\pi} |f(e^{it}) - f(\zeta)|^2 P(\zeta e^{-it}) \frac{dt}{2\pi},$$

where

$$P(w) = \frac{1-|w|^2}{|1-w|^2} \quad (w \in U)$$

is Poisson's kernel, we see that an analytic function f belongs to BMOA if and only if it belongs to H^2 and

$$\sup\left\{\int_0^{2\pi} |f(e^{it})-f(\zeta)|^2 P(\zeta e^{-it})dt : \zeta \in U\right\} < \infty .$$

(Other descriptions of BMOA are given in [11].)

We proceed to give a characterization of BMOA in the spirit of the principle enunciated in our opening remarks.

Theorem 4. $f \in$ BMOA if and only if

$$\sup\{\|\sigma_n(f)\|_* : n=1,2,\ldots\} < \infty .$$

Proof. Suppose $f \in$ BMOA. Let $z \in U$, $t \in \mathbb{R}$. Then

$$\sigma_n(f)(e^{it}) - \sigma_n(f)(z) = \int_0^{2\pi} [f(e^{i(t-\phi)}) - f(ze^{-i\phi})] K_n(\phi) \frac{d\phi}{2\pi}$$

and hence

$$|\sigma_n(f)(e^{it}) - \sigma_n(f)(z)|^2 \leqq \int_0^{2\pi} |f(e^{i(t-\phi)}) - f(ze^{-i\phi})|^2 K_n(\phi) \frac{d\phi}{2\pi} .$$

It follows from this that

$$\int_0^{2\pi} |\sigma_n(f)(e^{it}) - \sigma_n(f)(z)|^2 P(ze^{-it}) \frac{dt}{2\pi}$$

$$\leqq \int_0^{2\pi} K_n(\phi) \int_0^{2\pi} |f(e^{i(t-\phi)}) - f(ze^{-i\phi})|^2 P(ze^{-it}) \frac{dt}{2\pi} \frac{d\phi}{2\pi} ,$$

i.e.,

$$\|\sigma_n(f)_z\|_2^2 \leqq \int_0^{2\pi} K_n(\phi) \, \|f_{ze^{-i\phi}}\|^2 \frac{d\phi}{2\pi} \leqq \sup\{\|f_z\|_2^2 : z \in U\} .$$

Thus

$$\|\sigma_n(f)\|_* \leqq \|f\|_* , \quad \text{for} \quad n=1,2,\ldots ,$$

and the necessity part of the theorem follows.

Suppose next that

$$M = \sup\{\|\sigma_n(f)\|_* : n=1,2,\ldots\} < \infty .$$

Noting first that $\sigma_n(f)_0 = \sigma_n(f) - f(0)$ and that

$$\|\sigma_n(f)_0\|_2 \leqq \|\sigma_n(f)\|_* \leqq M ,$$

we see that $\|\sigma_n(f)\|_2 \leqq M$, for $n=1,2,\ldots$. Hence $f \in H^2$ [8].

Noting next that, if $z \in U$ and $0<r<1$, then

$$f_r(z) \equiv f(rz) = (1-r)^2 \sum_0^{\infty} (n+1) \sigma_n(f)(z) r^n ,$$

we see that $f_r \in \mathrm{BMOA}$ and that

$$\begin{aligned}\|f_r\|_* &\leqq (1-r)^2 \sum_0^\infty (n+1)\,\|\sigma_n(f)\|_*\, r^n \\ &\leqq (1-r)^2 \sum_0^\infty (n+1) M r^n \\ &= M\,.\end{aligned}$$

We deduce that

$$\int_0^{2\pi} |f(re^{it}) - f(rz)|^2 P(ze^{-it}) \frac{dt}{2\pi} \leqq (M - |f(0)|)^2$$

if $z \in U$ and $0 < r < 1$. Since $f(re^{it}) \to f(e^{it})$ a.e. as $r \to 1^-$, we can let $r \to 1^-$ in the last displayed inequality and conclude that $f \in \mathrm{BMOA}$; and that $\|f\|_* \leqq M$. This finishes the sufficiency part of the theorem, and concludes the proof.

Remark. A perusal of the proof shows that

$$\sup\{\|\sigma_n(f)\|_* : n = 0, 1, 2, \ldots\} = \|f\|_*\,.$$

Since, as we have already mentioned, $\mathrm{BMOA} \subset \mathscr{B}$, statement (iii) of Theorem 1 and statements (ii) and (iii) of Theorem 2, in particular, are true for functions f in BMOA; and so too is Corollary 1 to Theorem 2. But are these the best we can say about functions in BMOA? The example

$$f(z) = \log(1/(1-z)) = \sum_1^\infty z^n/n\,,$$

which belongs to BMOA and for which

$$\sigma_n(f)(1) = \sum_1^n 1/k - n/(n+1)\,,$$

and

$$\sigma_n'(f)(1) = n/2\,,$$

shows that we cannot improve on these estimates within BMOA.

Next, we show that the converse of Corollary 2 is false for functions in BMOA. Fejér's technique [12] for constructing a continuous function whose Fourier series diverges at a point can be adapted for this purpose. With this in mind, let m, n be positive integers, and

$$\phi(n, m, z) = z^m \left\{ \sum_1^n \frac{z^k}{2n - 2k + 1} - \sum_1^n \frac{z^{n+k}}{2k-1} \right\}.$$

Then $\phi(n, m, 1) = 0$, $\deg \phi(n, m, z) = m + 2n$ and the first term in the Taylor expansion of $\phi(n, m, z)$ is $z^{m+1}/(2n-1)$. Moreover [12, pp. 42, 417]

$$\sup\{\|\phi(n, m, \ldots)\|_\infty : m, n = 1, 2, \ldots\} < \infty\,.$$

Let now m_k, n_k be sequences of positive integers chosen so that in the first instance $m_k + 2n_k < m_{k+1} + 1$ and let

$$f(z) = \sum_1^\infty \phi(n_k, m_k, z)/k^2 \quad (z \in U)\,.$$

Then f is analytic on U and continuous on the closure of U. Moreover, since the polynomials $\phi(n_k, m_k, .)$ have disjoint support, the power series expansion of f is obtained by opening out the last displayed series and discarding brackets. It follows that

$$s_{m_k+2n_k}(f)(1)=\sum_1^k \phi(n_j, m_j, 1)=0, \quad k=1,2,\dots.$$

Let $n_k=k^{k^2}$, $k=1,2,\dots,m_1=0$, $m_k=2\sum_1^{k-1} n_j$, $k=2,3,\dots$; so that $m_k+2n_k=m_{k+1}$, and $m_k/n_k\to 0$ as $k\to\infty$.

Consider the partial sum $s_N(f)$, where $N=N(k)=\sum_1^k n_j$. An easy argument establishes that $s_N(f)(1)$ is close to $s_{n_k}(f)(1)$ which in turn is close to

$$k^{-2}\sum_1^{n_k} 1/(2j-1)\sim k^{-2}2^{-1}\log n_k\sim 2^{-1}\log k$$

as $k\to\infty$.

It follows that

$$\|s_{2N}(f)-s_N(f)\|_\infty\to\infty$$

as $k\to\infty$. Hence the converse of Corollary 2 is false for BMOA functions, and *a fortiori* for Bloch functions.

Of course, the necessity part of Theorem 3 holds for functions in BMOA. In Part III, we present an alternative proof of this and statement (ii) of Theorem 2.

Part III

Hankel Operators on H^2

Given $f(z)=\sum a_n z^n\in H^2$, the Hankel matrix $[a_{i+j}]$ determines a linear transformation A on the subspace of H^2 spanned by $\{\chi_n : n=0,1,2,\dots\}$, where

$$\chi_n(z)=z^n, \quad n=0,1,2,\dots \quad (z\in U),$$

and

$$A\chi_j=\sum_{i=0}^{\infty} a_{i+j}\chi_i, \quad j=0,1,2,\dots.$$

If A admits of a bounded extension to all of H^2, we call the extension a Hankel operator – which we continue to denote by A – and denote the class of all Hankel operators on H^2 by $\mathscr{S}$. Necessary and sufficient conditions for A to be a Hankel operator were given first by Nehari – see [10] for the basic facts about Hankel operators. For our purposes, it suffices to note that Nehari's condition is equivalent to the requirement that the analytic function f generating A belongs to BMOA.

The "symbol function" associated with A is the harmonic function ϕ defined by

$$\phi(z)=\bar{z}f(\bar{z}) \quad (z\in U).$$

One connection between A and ϕ is given by the identity

$$\|Av(z)\|_2^2=\int_0^{2\pi}|\phi(e^{it})-\phi(z)|^2P(ze^{-it})\frac{dt}{2\pi}\quad(z\in U),$$

which we borrow from Bonsall's treatment (see [3, formula (5)]). Here, for each $z\in U$, $v(z)$ is the unit vector in the Hilbert space H^2 given by

$$v(z)(w)=\sqrt{(1-|z|^2)}k(\bar{z}w)\quad(w\in U),$$

where k is as before. Taking account of the definition of ϕ, we can rewrite the above identity in the form

$$\|Av(\bar{z})\|_2^2=\int_0^{2\pi}|(e^{it}-z)f(e^{it})+z(f(e^{it})-f(z))|^2P(ze^{-it})\frac{dt}{2\pi},$$

and conclude that

$$\begin{aligned}\|Av(\bar{z})\|_2&\leqq\sqrt{(1-|z|^2)}\|f\|_2+\|f_z\|_2\quad(z\in U)\\&\leqq 2\|f\|_*,\end{aligned}$$

if $f\in$ BMOA.

In much the same way, it can be seen that

$$\begin{aligned}\|f_z\|_2&\leqq\|Av(\bar{z})\|_2+\sqrt{(1-|z|^2)}|f(z)|\quad(z\in U)\\&\leqq\|A\|+\|f\|_2,\end{aligned}$$

and so

$$\|f\|_*\leqq 3\|A\|,$$

(since $|f(0)|\leqq\|f\|_2=\|A\chi_0\|_2\leqq\|A\|$) if $A\in\mathscr{S}$.

These inequalities, coupled with Corollary 3 in [3], tell us that there is a positive real constant M such that

$$1/3\|f\|_*\leqq\|A\|\leqq 2M\|f\|_*.$$

We shall refer below to the constant M as Bonsall's *constant.*

A Criterion for Membership of $\mathscr{S}$

We consider two sequences, (s_n) and (σ_n), of linear operators from $\mathscr{S}$ to the finite rank Hankel operators defined as follows: if $A\in\mathscr{S}$ and is generated by $f(z)=\sum a_nz^n$, we set

$$s_n(A)\chi_j=\sum_{i=0}^{n-j}a_{i+j}\chi_i,\quad 0\leqq j\leqq n;\quad s_n(A)\chi_j=0,\quad j>n$$

and

$$\sigma_n(A)=(1/(n+1))\sum_0^n s_k(A),$$

for $n=0,1,2,\ldots$. Thus the polynomials $s_n(f)$ and $\sigma_n(f)$ generate $s_n(A)$ and $\sigma_n(A)$, respectively, in the same way that f generates A.

As a final illustration of the central theme of the paper, we record the following theorem, which supplements the criteria given by Bonsall [3] and ourselves [7].

Theorem 5. *$A\in\mathscr{S}$ if and only if* $\sup\{\|\sigma_n(A)\|: n=0,1,2,\ldots\}<\infty$.

Proof. Combining Theorem 4 with the remarks just preceding this section, the stated result is an immediate consequence of the following chain of equivalences: if f generates A, then

$$\begin{aligned} A\in\mathscr{S} &\Leftrightarrow f\in\text{BMOA}\\ &\Leftrightarrow \|\sigma_n(f)\|_*=O(1) \quad (n\to\infty)\\ &\Leftrightarrow \|\sigma_n(A)\|=O(1) \quad (n\to\infty).\end{aligned}$$

Remark. A different proof of the theorem involving a more direct appeal to Nehari's theorem shows, in fact, that

$$\sup\{\|\sigma_n(A)\|: n=0,1,2,\ldots\}=\|A\|.$$

Some Identities

In this section, we indicate how we were led to Theorems 2 and 3.

Let $z,w\in U$. Let $f(z)=\sum a_n z^n$ generate A. Expanding the inner-product $\langle Av(\bar z), v(w)\rangle$, we see that

$$\begin{aligned}\langle Av(\bar z), v(w)\rangle &= \sqrt{(1-|z|^2)}\sqrt{(1-|w|^2)}\sum_{i,j}^{\infty} a_{i+j}z^i w^j\\ &= \sqrt{(1-|z|^2)}\sqrt{(1-|w|^2)}\sum_0^{\infty} a_i\sum_0^{i} w^j z^{i-j}\\ &= \sqrt{(1-|z|^2)}\sqrt{(1-|w|^2)}\,(zf(z)-wf(w))/(z-w).\end{aligned}$$

With $\phi(\bar z)=zf(z)$, we deduce that

$$\sup\left\{\sqrt{(1-|z|^2)}\sqrt{(1-|w|^2)}\left|\frac{\phi(z)-\phi(w)}{z-w}\right| : z,\ w\in U\right\}\leqq\|A\|.$$

We remark in passing that Bonsall drew the second author's attention to the boundedness of the quantity on the left side of this inequality for the symbol function of the Hankel operator A, and wondered if the converse were true; Theorem 3 provides the answer.

To highlight the source of Theorem 2, we borrow another piece of notation from [3], namely,

$$u_n(\zeta)=1/\sqrt{(n+1)}\,\{\chi_0+\bar\zeta\chi_1+\ldots\bar\zeta^n\chi_n\},\qquad n=0,1,2,\ldots.$$

For each ζ on the unit circle, $u_n(\zeta)$ is a unit vector in H^2.

Given $A \in \mathscr{S}$, with generator $f(z) = \sum a_n z^n$, an easy piece of algebraic manipulation shows that, for any pair of integers $n, m \geqq 0$,

$$\begin{aligned}&\sqrt{(n+1)}\sqrt{(m+1)}\langle Au_n(\bar{\zeta}), u_m(\zeta)\rangle \\ &= \sum_0^n \sum_0^m a_{i+j}\zeta^{i+j} \\ &= \sum_{j=0}^n s_{m+j}(f)(\zeta) - \sum_{j=0}^{n-1} s_j(f)(\zeta) \\ &= (n+m+1)\sigma_{n+m}(f)(\zeta) - n\sigma_{n-1}(f)(\zeta) - m\sigma_{m-1}(f)(\zeta),\end{aligned}$$

whence it follows that

$$\|(n+m+1)\sigma_{n+m}(f) - n\sigma_{n-1}(f) - m\sigma_{m-1}(f)\|_\infty \leqq \sqrt{(n+1)}\sqrt{(m+1)}\,\|A\|.$$

We end this section by proving an analogue of Corollary 3 for BMOA functions; we retain the same terminology.

Theorem 6. *The* BMOA *norm and the H^∞ norm are equivalent on $\mathscr{P}$.*

Proof. Since

$$\|f_\zeta\|_2^2 = \int_0^{2\pi} |f(e^{it})|^2 P(\zeta e^{-it})\frac{dt}{2\pi} - |f(\zeta)|^2$$

for every $f \in H^2$ and $\zeta \in U$, we see that $\|f_\zeta\|_2 \leqq \|f\|_\infty$, if $f \in H^\infty$, and hence that

$$\|p\|_* \leqq 2\|p\|_\infty,$$

whenever $p \in \mathscr{P}$.

If $p \in \mathscr{P}_n$, then by taking $m = 2n+1$ in the above identity involving the Cesàro means, we find that

$$\begin{aligned}(n+1)p(\zeta) &= (3n+2)\sigma_{3n+1}(p)(\zeta) - (2n+1)\sigma_{2n}(p)(\zeta) - n\sigma_{n-1}(p)(\zeta) \\ &= (n+1)\sqrt{2}\langle Au_n(\bar{\zeta}), u_{2n+1}(\zeta)\rangle,\end{aligned}$$

whence it results that

$$\|p\|_\infty \leqq \sqrt{2}\|A\| \leqq 2\sqrt{2}M\|p\|_*,$$

where M is Bonsall's constant mentioned at the end of the previous section.

This completes the proof.

Some Examples

Motivated by the results [7, 14], and encouraged by the result just established in Theorem 5, we now raise the question: does the boundedness of A follow from the uniform boundedness of the sequence of functions $\{\|\sigma_n(A)u_n(\zeta)\|_2\}$?

We will answer this question in the negative with the help of the example constructed in [7]; and take the opportunity also to show that the apparently stronger assumption that $\{\|s_n(A)u_n(\zeta)\|_2\}$ is uniformly bounded does not force A to belong to $\mathscr{S}$ either. (This was inadvertently left out in [7].)

We commence by noting that if $a_n \geqq 0$ for all n, then

$$\begin{aligned}\|\sigma_n(A)u_n(\zeta)\|_2^2 &\leqq \|\sigma_n(A)u_n(1)\|_2^2\\ &=(1/(n+1))\sum_{i=0}^{n}\left(\sum_{j=i}^{n}(1-j/(n+1))a_j\right)^2\\ &\leqq (1/(n+1))\sum_{i=0}^{n}\left(\sum_{j=i}^{n}a_j\right)^2\\ &=\|s_n(A)u_n(1)\|_2^2 .\end{aligned}$$

For each $m \geqq 0$, let

$$E_m=\{2^m+2^j-1 : 0\leqq j\leqq m\} \quad \text{and} \quad E=\bigcup_0^\infty E_m .$$

Define

$$\begin{aligned} a_n &= \frac{1}{(m+1)\log(m+2)}, \quad \text{if} \quad n\in E_m,\\ &=0, \quad \text{if} \quad n\notin E .\end{aligned}$$

As was observed in [7], the function $f(z)=\sum a_n z^n$ belongs to $\mathscr{B}$, but is not in BMOA. At the same time, if $0\leqq l\leqq 2^p-1$ and $p<m$, then

$$\sum_{j=2^p+l}^{2^m-1} a_j \leqq \sum_{k=p}^{m-1}\sum_{j=2^k}^{2^{k+1}-1} a_j=\sum_{k=p}^{m-1} 1/\log(k+2)\leqq 2(m-p),$$

whence, if $n=2^m-1$, then

$$\begin{aligned}(n+1)\|s_n(A)u_n(1)\|_2^2 &= \sum_{i=0}^{n}\left(\sum_{j=i}^{n}a_j\right)^2\\ &=\left(\sum_{j=1}^{n}a_j\right)^2+\sum_{i=1}^{n}\left(\sum_{j=i}^{n}a_j\right)^2\\ &\leqq 2\sum_{p=0}^{m-1}\sum_{l=0}^{2^p-1}\left(\sum_{j=2^p+l}^{2^m-1}a_j\right)^2\\ &\leqq \sum_{p=0}^{m-1}2^{p+1}(2(m-p))^2\\ &=2^{m+3}\sum_{s=1}^{m}2^{-s}s^2\\ &=O(2^m).\end{aligned}$$

A standard argument finishes the proof that, for this example,

$$\|s_n(A)u_n(1)\|_2=O(1) \quad (n\to\infty),$$

and so also

$$\|\sigma_n(A)u_n(1)\|_2=O(1) \quad (n\to\infty);$$

yet A is not a Hankel operator.

It was shown in [14] that Theorem 5 above is false in general if $\sigma_n(A)$ is replaced by $s_n(A)$. We conclude this paper by reproving this fact with the help of an explicit construction, which is based on an observation outlined to the second author by V. Peller at the recent Lancaster Conference on Operators and Function Theory. We note too that Barry Johnson has discovered another approach to the problem [9], which he has very kindly communicated to us.

Theorem 7. $0<\limsup_{n\to\infty}\frac{\|s_n\|}{\log n}<\infty$.

Proof. Here, $\|s_n\|$ is the norm of the operator $s_n:\mathcal{S}\to\mathcal{S}$ defined in the preamble to Part III, i.e.,

$$\|s_n\|=\sup\{\|s_n(A)\|:A\in\mathcal{S},\ \|A\|\leqq 1\}.$$

Let

$$q_n(z)=\sum_1^n z^k/k,\qquad n=1,2,\ldots,\quad (z\in U).$$

Note [12, p. 42] that

$$\sup\{\|\mathrm{Im}\, q_n\|_\infty : n=1,2,\ldots\}=C<\infty,$$

where, here and in what follows, C is an absolute constant, but not always the same one.

Define the sequence of analytic polynomials p_n by

$$p_n(z)=z^{2n+1}\{q_n(z)-q_n(1/z)\},\qquad n=1,2,\ldots,\quad (z\in U).$$

Let A_n be the finite rank Hankel operator generated by p_n. Then

$$\|A_n\|\leqq 2M\|p_n\|_*\leqq 4M\|p_n\|_\infty=8M\|\mathrm{Im}\, q_n\|_\infty\leqq C.$$

Also, since the first $n+1$ Taylor coefficients of p_n vanish, $s_{2n}(p_n)\in\mathcal{P}_n$, in the notation of the previous theorem, and so, by that theorem,

$$\|s_{2n}(p_n)\|_\infty\leqq\sqrt{2}\|s_{2n}(A_n)\|\leqq C\|s_{2n}\|.$$

But

$$q_n(1)=\sum_1^n 1/k\sim\log n\quad (n\to\infty),$$

and $\|s_{2n}(p_n)\|_\infty\geqq q_n(1)$. Hence

$$0<\limsup_{n\to\infty}\|s_n\|/\log n.$$

We refer to [14] for the relatively easy proof of the finiteness of the lim sup.

This completes the proof.

References

1. Anderson, J.M., Clunie, J.G., Pommerenke, Ch.: On Bloch functions and normal functions. J. Reine Angew. Math. **270**, 12–37 (1974)
2. Bennett, G., Steganga, D.A., Timoney, R.M.: Coefficients of Bloch and Lipschitz functions. Ill. J. Math. **25**, 520–531 (1981)

3. Bonsall, F.F.: Boundedness of Hankel operators. J. Lond. Math. Soc. **29**, 289–300 (1984)
4. Bonsall, F.F.: Criteria for boundedness and compactness of Hankel operators. Proc. Conf. Banach Algebras and Several Complex Variables. Contemp. Math. **32**, 83–95 (1984)
5. Cima, J.: The basic properties of Bloch functions. Int. J. Math. Math. Sci. **2**, 369–413 (1979)
6. Cooper, J.L.B.: Fourier transforms and inversion formulae for L^p functions. Proc. Lond. Math. Soc. **14**, 271–298 (1963)
7. Holland, F., Walsh, D.: Boundedness criteria for Hankel operators. Proc. R.I.A. **84** A, 141–154 (1984)
8. Hoffman, K.: Banach spaces of analytic functions. Englewood Cliffs: Prentice-Hall 1962
9. Johnson, B.: Written communication
10. Power, S.C.: Hankel operators on Hilbert space. Research notes in Mathematics, 64. London: Pitman 1982
11. Sarason, D.: FunctionTheory on the unit circle, Lecture notes, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 1978
12. Titchmarsh, E.C.: Theory of functions, 2^{nd} ed. Oxford: University Press 1939
13. Van Castern, J.: Boundedness properties of resolvents and semigroups. Preprint, Universiteit Antwerpen, 1978
14. Walsh, D.: On partial sum Hankel matrices. J. Lond. Math. Soc. **30**, 288–294 (1984)
15. Zygmund, A.: Trigonometric Series. I, II. Cambridge: University Press 1968

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