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Prym Varieties and the Geodesic Flow on $SO(n)$

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0. Introduction

Completely integrable Hamiltonian systems, and in particular their interaction with algebraic geometry, have been recently studied rather intensively (see [12, 13] for a list of examples and references up to '78–'79); however, a systematization which would render the subject distinct from the collection of cases already treated is not yet complete. (See [1, 2, 7, 8] for three important contributions.)

The question addressed in this paper centers around the relationship between the space where the flow actually occurs (i.e., the invariant tori, into which the symplectic space decomposes), and the abelian variety, usually a Prym or a Jacobian variety, on which the flow is known to linearize. The starting point for this work was [3] and [9], where the specific case of the geodesic flow on $SO(4)$ was analyzed, and it was found that the affine variety obtained by fixing the values of the integrals of motion is an open subvariety of the abelian variety dual to the Prym of the corresponding spectral curve K. In particular this abelian variety is the Prym of the curve C, obtained as the intersection of Prym(K) and an appropriate translate of the Θ-divisor in Jac(K).

In an attempt to explain this result and understand what the correct generalization should be for any integrable case of the Euler-Arnold equations we were able:

1) to interpret the relationship between K and C as a singular (or alternatively, ramified) instance of the tetragonal construction [6];

2) to prove an algebrogeometric lemma for ramified double covers of hyperelliptic curves (similar in spirit to [6]), which amongst other things implied the duality result for $SO(4)$ *and* $SO(5)$ (contrast with [10, p. 278]), and as a bonus suggested that the variety $\mathrm{Prym}_i^*(K)$ defined in the proof as an intermediate step, is where the invariant tori naturally live; and finally,

3) to prove this in Sect. 4, by noting that for each X in $\mathfrak{so}(n)$, we can obtain not only a line bundle giving the eigenvector embedding of K in $\mathbf{P}^{n-1}$, but also n specified sections, which uniquely characterize X and can be used to construct an element of $\mathrm{Prym}_i^*(K)$ (as a line bundle in an appropriate singular curve). So in

particular we get a simple proof that the flow occurs in an open affine subset of an abelian variety (compare to the calculations in [3] and [9] or the direct argument in [15], in both cases for $n=4$).

This is a result that easily translates to Dubrovin's setting [matrices in $GL(n)$ with a fixed diagonal], where the flow ends up in a $\mathbb{C}_*$-extension of $\mathrm{Jac}(K)$ (corresponding to the points "at ∞" of K) even though K may be quite non-singular itself....

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1.1. The Euler-Arnold Equations

Let G be a semisimple Lie group, $\mathfrak{g}$ its Lie algebra [the standard example being $SO(n)$]. Recall the coadjoint representation of G in $\mathrm{End}(\mathfrak{g}^*)$

$$\begin{array}{ccc} \mathrm{Ad}^*: G & \to & \mathrm{End}(\mathfrak{g}^*) \\ \cup\!\!\!\!\,\shortmid & & \cup\!\!\!\!\,\shortmid \\ g & \longrightarrow & \mathrm{Ad}_g^* \end{array}$$

(Ad_g is the derivative of the group automorphism $R_{g^{-1}}L_g$ at the identity: an endomorphism of g. The "dual" map is Ad_g^*, i.e., the map induced on the cotangent space by $L_gR_{g^{-1}}$.) The orbits of the coadjoint representation are symplectic manifolds with the Kirillov symplectic structure, defined as follows:

Let $x\in\mathfrak{g}^*$, $\mathcal{O}(x)$ its orbit; $u,v\in T_x\mathcal{O}(x)$. The derivatives of Ad, Ad* at the identity give

$$\begin{array}{ccc} \mathrm{ad}: \mathfrak{g} & \to & \mathrm{End}(\mathfrak{g}) \\ \cup\!\!\!\!\,\shortmid & & \cup\!\!\!\!\,\shortmid \\ a & \longrightarrow & [a,\cdot] \end{array}$$

$$\begin{array}{ccc} \mathrm{ad}^*: \mathfrak{g} & \to & \mathrm{End}(\mathfrak{g}^*) \\ \cup\!\!\!\!\,\shortmid & & \cup\!\!\!\!\,\shortmid \\ a & \longrightarrow & \mathrm{ad}_a^* \end{array}$$

so that for all $x\in\mathfrak{g}^*$, $b\in\mathfrak{g}$ $x([a,b])=\mathrm{ad}_a^*x(b)$. Since ad* is the derivative of Ad* the tangent vectors u,v can be represented as $\mathrm{ad}_a^*(x)$, $\mathrm{ad}_b^*(x)$ for some $a,b\in g$ (think of u,v as elements of g^*). Then define the *Kirillov symplectic form* ω by

$$\omega(u,v)=x([a,b])=\mathrm{ad}_a^*x(b)=u(b)=-v(a)$$

(obviously skew symmetric and non-degenerate). ω is well defined, i.e., independent of the choices of a and b, and finally, closed:

$$d\omega(u_1,u_2,u_3)=\tfrac{1}{3}\sum_{i<j}(-1)^k\omega([u_i,u_j],u_k)=0$$

by the Jacobi identity. (For details see [18, p. 135].)

By the assumption of semisimplicity the Killing form $(\cdot,\cdot)$ induces an isomorphism $\mathfrak{g}^*\simeq\mathfrak{g}$.

A left invariant Riemannian metric on G can be thought of as a symmetric linear operator $\lambda:\mathfrak{g}\to\mathfrak{g}^*$, or equivalently as a linear operator symmetric with respect to the Killing form: $l:\mathfrak{g}\to\mathfrak{g}$. Now for a variable $X\in\mathfrak{g}$ the Euler-Arnold equations are

$$\dot{X}=[X,l(X)]; \tag{1}$$

expressing geodesic motion on G with respect to the given metric. This gives a Hamiltonian vector field on the orbits, and the corresponding Hamiltonian is

$$H=-(X,l(X)). \tag{2}$$

Note that:

1) l can be changed by adding a polynomial in X without affecting (1).

2) The complete integrability of the equations depends on the choice of metric l.

From now on $\mathfrak{g}$ will be $\mathfrak{so}(n)$ (over $\mathbb{C}$) and l will be a diagonal metric, thought of as a symmetric matrix (l_{ij}) (s.t. $l(X)_{ij}=l_{ij}X_{ij}$).

1.2. Completely Integrable Cases of Geodesic Flow on SO(n)

Manakov in [11] observed that for a metric l with

$$l_{ij}=\frac{\beta_i-\beta_j}{a_i-a_j}\left(A=\begin{pmatrix}a_1 & & 0\\ & \ddots & \\ 0 & & a_n\end{pmatrix},\ B=\begin{pmatrix}\beta_1 & & 0\\ & \ddots & \\ 0 & & \beta_n\end{pmatrix}\right)$$

and all a_i distinct (essentially the case of a n-dimensional free rigid body), the Euler-Arnold equations become

$$(X\dot{+}Ah)=[X+Ah,l(X)+Bh] \tag{1}$$

a so-called Lax equation with a parameter h, and are completely integrable.

The additional integrals come from considering the *spectral curve* K associated to (1) given by the characteristic polynomial

$$\det(Xu+Ah-zI)=0, \tag{2}$$

where $[z:h:u]$ is thought of as a point in $\mathbf{P}^2$. By (1), the curve given by (2) is independent of t, i.e., we have an *isospectral* flow. The adjoint orbit in $\mathfrak{so}(n)$ ($\simeq\mathfrak{so}(n)^*$) in which X is moving has dimension $\frac{n(n-1)}{2}-\left[\frac{n}{2}\right]$ in the generic case. Thus for $n=2k$ ($n=2k+1$) we need k^2 [respectively $k(k+1)$] integrals, $\left[\frac{n}{2}\right]=k$ of them, the traces of the even powers of X, defining the orbit (usually called trivial integrals); which is precisely the number of invariants we get from (2):

$$\det(Xu+Ah-zI)=\prod_{i=1}^{n}(a_ih-z)+u^2\{Q_1(X)z^{n-2}+\ldots+Q_{n-1}(X)h^{n-2}\}+\ldots \tag{3}$$

by setting $(Q_1(X),\ldots,Q_N(X))=(c_1,\ldots,c_N)=c$ (general point in $\mathbb{C}^N$). Note that for $n=2k$ $Q_{k^2}(X)=(\mathrm{Pf}(X))^2$ so if we set

$$A_c=\{X\in\mathfrak{so}(n)/Q_i(X)=c_i\}$$

we will get in that case $A_c=A_{c^+}\cup A_{c^-}$ the union of two disjoint isomorphic components ($c_{k^2}\neq 0$). So the usual thing is to fix $\mathrm{Pf}(X)=c_0$ and to restrict attention to the flow on A_{c^+} or A_{c^-}.

The integrals obtained are independent and involutive. They can all be expressed as quadrics in terms of Pfaffians of minors of X.

For any diagonal matrix B, let $\sigma_j(B)$ be the elementary symmetric polynomials in the entries of B. Let X_i be the minor of X for the i^{th} diagonal entry, x_{ij} the $(i,j)^{\text{th}}$ entry of X.

For

$$n=2k \quad \mathrm{Pf}_0^2(X)=\mathrm{Pf}^2(X), \qquad \mathrm{Pf}_2^2(X)=\sum_{i<j}\mathrm{Pf}^2(X_{ij}), \ldots$$

$$n=2k+1 \quad \mathrm{Pf}_1^2(X)=\sum_i \mathrm{Pf}^2(X_i), \ldots$$

$$\left(\mathrm{Pf}_{n-2}^2(X)=\sum_{i<j} x_{ij}^2,\ \mathrm{Pf}_n^2(X)=1\right).$$

Set

$$\sigma_{j-\varrho}(A) * \mathrm{Pf}_j^2(X)=\sum_{i_1<\ldots<i_j}\left\{\sum_{1\leqq k_1<\ldots<k_{j-\varrho}\leqq j} a_{i_{k_1}}\ldots a_{i_{k_{j-\varrho}}}\right\}\mathrm{Pf}^2(X_{i_1\ldots i_j}).$$

Then (3) can be written as

$$\det(Xu+Ah-zI)=\sigma_n(Ah-zI)+u^2\sigma_{n-2}(Ah-zI) * \mathrm{Pf}_{n-2}^2(X)+\ldots \tag{4}$$

or compactly as

$$\det(Xu+Ah-zI)=\sum_{\varrho+\tau+2j=n}(-1)^{\varrho}\sigma_\tau(A) * \mathrm{Pf}_{n-2j}^2(X)h^\tau z^\varrho u^{2j}. \tag{5}$$

Note. (1) The main thrust of [10] is to show that Manakov's metrics are the only "generic" example of completely integrable geodesic flow on $SO(n)$. A forthcoming paper by Adler and Van Moerbeke examines two other special metrics for which the $SO(4)$ case is completely integrable. Also several papers from the Soviet Union identify other cases for which the Euler-Arnold equations are integrable.

(2) The $n=3$ case is, of course, the contribution of Euler to the Euler-Arnold equations, and an early application of elliptic functions [K being an elliptic curve for $SO(3)$]. The complete integrability for $SO(4)$ predates [11], and was demonstrated in the early seventies by Dikii and Mischenko.

(3) Finally, as explained in [2], the geodesic flow on $SO(n)$ forms a special case of the so-called spinning-top type equations, another classical example of which is the Lagrange top. (Compare with Kowalewskaya's work on the integrable cases of the equations of a *heavy* rigid body.)

1.3. The Geometry of the Spectral Curve

Before concerning ourselves with the linearization of the flow on A_c, we will list for convenience some of the important elementary properties of K.

For a general c, K is a smooth plane curve of degree n and thus of genus $(n-1)(n-2)/2$. Its n points on the line $u=0$ are $p_i=[a_i:1:0]$, and will be considered as points "at ∞." Note that they are the same for all choices of c and depend solely on the metric l.

The geometry of K is quite special.[1]

$$i:\quad \begin{array}{ccc}\mathbf{P}^2 & \longrightarrow & \mathbf{P}^2\\ \cup & & \cup\\ {[z:h:u]} & \rightarrow & [z:h:-u]\end{array}$$

1 K however is quite general in another respect: any plane curve of degree n passing through $p_1, \ldots, p_n$ and symmetric w.r.t. to the origin can be obtained as a spectral curve for $SO(n)$ for appropriate c [see 1.2(3)]

gives an involution on the curve since

$$\det(Xu+Ah-zI)=\det((Xu+Ah-zI)^T)=\det(X(-u)+Ah-zI)$$

so

$$[z:h:u]\in K \leftrightarrow [z:h:-u]\in K.$$

[As we can also see from the expansion in 1.2 (3), all terms are of even degree in u.]

Factoring by i, K becomes a ramified double cover of a curve K_0 of genus $(k-1)^2$ [respectively $k(k-1)$], the ramification points being $p_1, \ldots, p_{2k}$ ($p_1, \ldots, p_{2k+1}$ together with the "origin" $p_0=[0:0:1]$). (The fixed points of i being the line at ∞ together with the origin.) Note that the tangents to K at the points at ∞ intersect at the origin, and that for $n=2k+1$ the curve has a flex at the origin.

Now, the natural projection

$$\begin{array}{ccc} K & \xrightarrow{\sigma_K} & \mathbf{P}^1 \\ \cup\!\!\!\cup & & \cup\!\!\!\cup \\ {[z:h:u]} & \to & {[z:h]} \end{array}$$

factors through $\pi_K: K \to K_0$

$$K \xrightarrow{\pi_K} K_0 \xrightarrow{\pi_{K_0}} \mathbf{P}^1$$

$$(\sigma_K[0:0:1]=\text{slope of tangent at origin for } n \text{ odd}).$$

The map π_{K_0} has degree k, so using the traditional terminology, K_0 is a k-gonal curve. The duality result in [9] and the similar result that can be proved for $SO(5)$ depend on the fact that $k=2$ for those cases and thus we're dealing with a hyperelliptic K_0 (see Sect. 3).

Finally it's worthwhile to note that the ramification points of π_{K_0} correspond to the bitangents of K passing through the origin. There are $2k(k-1)$ [respectively $2(k^2-1)$] such points from Riemann-Hurwitz (so the $n(n+1)$ tangents to K from the origin are the $2k$ tangents to points at ∞ and the $2k(k-1)$ bitangents counted twice; [respectively: $2k+1$ tangents, $2(k^2-1)$ bitangents, and the flex tangent at the origin with multiplicity 3]).

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2.1. *Some Facts From Prym Theory*

To any double cover of a curve one can associate its *Prym variety*. However the classical definition deals with the unramified case, while π_K is ramified. There are two ways to go: one, given in [14], has the disadvantage of giving a non-principally polarized abelian variety (which is thus not a good extension of the Prym map from $\mathscr{M}_g$ to the boundary); the second, in [5], *does* give a p.p.a.v. (namely an unramified cover of the first version), but is not unique for a given ramified double cover (depends on its representation as the normalization of an appropriate singular covering) giving distinct – but isogenous – p.p.a.v.s.

Thus we will have occasion to employ both: let $\mathrm{Prym}_i(K)$ stand for Mumford's version (together with a polarization $\varrho: \mathrm{Prym}_i(K) \to \widehat{\mathrm{Prym}_i}(K)$ of degree $2^{2g(K_0)}$: see [14]). Now $\mathrm{Prym}_i(K) \hookrightarrow \mathrm{Jac}(K) = \mathscr{J}$ is given by

$$\{D \in \mathscr{J}/D = D_0 - i^* D_0, \text{ for some } D_0 \in \mathscr{J}\}$$

where i^* is the involution induced by i on $\mathrm{Jac}(K)$. By [14, Sect. 3, Lemma], $\mathrm{Prym}_i(K) = \{D \in \mathscr{J}/D + i^* D \simeq 0\}$ $\left(\text{in contrast to the unramified case where } \ker\left\{\mathrm{Jac}(K) \xrightarrow{Nm} \mathrm{Jac}(K_0)\right\} \text{ is } \mathrm{Prym}_i(K) \times \mathbf{Z}_2\right)$. Beauville in [5] extended the Prym map (into the moduli space $\mathscr{A}_g$ of principally polarized abelian varieties) to the boundary; we shall briefly state his results in our case.

Let M be a curve with only ordinary double points such that

$$\eta: K \to M$$

is its normalization (e.g. M can be obtained from K by identifying the ramification points of π_K in pairs). M can be reducible but: 1) the nodes where the two branches are *not* exchanged by i are precisely the fixed points of the involution; and 2) the number of nodes exchanged by i equals the number of components exchanged (and both are equal to zero in the cases we will have use for).

Then i gives a well-defined involution on M whose only fixed points are the double points (k for $n=2k$, $k+1$ for $n=2k+1$). The quotient $M_0 = M/i$, has only ordinary double points also, and its normalization is K_0:

$$\begin{array}{ccc} K & \xrightarrow{\eta} & M \\ {\scriptstyle\pi_K}\downarrow & & \downarrow{\scriptstyle\pi_M} \\ K_0 & \xrightarrow[\eta_0]{} & M_0 \end{array}$$

Then Beauville defines the Prym: $\mathscr{P}_i(K)$ to be the abelian variety we obtain as we degenerate an unramified double cover of a non-singular curve to the double cover π_M of M.

1. *Definition.* $\mathscr{P}_i(K)$ [or $\mathscr{P}_i(M)$ since it depends on the choice of M] is the variety of line bundles in

$$\ker\left\{\mathrm{Pic}\, M \xrightarrow{Nm} \mathrm{Pic}\, M_0\right\}$$

of the form $\mathscr{L} \otimes i^* \mathscr{L}^{-1}$ with $\deg \mathscr{L} = 0$. [Nm is induced by the direct image map on the groups of Cartier divisors $(\pi_M)_*: \mathrm{Div}\, M \to \mathrm{Div}\, M_0$.]

Let now T be the group of divisors of degree 0 with singular support on M, T_2 the 2-torsion. Then the two Pryms are related by an exact sequence

$$0 \to T_2 \to \mathscr{P}_i(K) \times \mathbf{Z}_2 \to \mathrm{Prym}_i(K) \to 0 \qquad (1)$$

which gives an isogeny $g: \mathscr{P}_i(K) \to \mathrm{Prym}_i(K)$. ($T \cong \mathbf{C}_*^r$, $T_2 \cong \mathbf{Z}_2^r$ where $2r$ = number of ramification *pts* – for K, M irreducible.) $\mathscr{P}_i(K)$ is p.p.: In the generalized $\mathrm{Jac}(M)$ we define a theta-divisor $\Theta_{\mathscr{L}}$ for any line bundle of degree $g(M)-1$ on M:

2. *Definition.*

$$\Theta_{\mathscr{L}} = \{\mathscr{L}_0 \in \mathrm{Jac}(M)/h^0(\mathscr{L} \otimes \mathscr{L}_0) > 0\}$$

$\Theta_{\mathscr{L}}$ is algebraically equivalent to $(\eta^*)^{-1}(\Theta)$:

$$\begin{array}{c} \eta^*: \mathrm{Jac}(M) \to \mathrm{Jac}(K) \\ \phantom{\eta^*: \mathrm{Jac}(M) \to {}} \cup \\ \phantom{\eta^*: \mathrm{Jac}(M) \to {}} \Theta . \end{array}$$

Fix a line bundle $\mathscr{L}_0$ of degree $g(M_0)-1$ on M_0 with $\mathscr{L}_0^2 \cong \omega_{M_0}$ (ω_{M_0} is the sheaf of forms on K_0 regular with simple poles at the branch points, and equal and opposite residues at the points identified). Then we get

3. **Theorem.** *$\Theta_{\pi_M^* \mathscr{L}_0}|_{\mathscr{P}_i(K)}$ is algebraically equivalent to twice an ample divisor Ξ with $h^0\ (\mathcal{O}(\Xi))=1$ (i.e. $(\mathscr{P}_i(K), \Xi)$ is a* p.p.a.v.*). In fact, we can choose $\mathscr{L}_0$ such that $\Theta|_{\mathscr{P}_i(K)}=2\Xi$.*

(The results in Sect. 2.1 hold for an arbitrary ramified double cover $K \to K_0$.) We shall call all the coverings $M \to M_0$ which give an abelian variety in this fashion, *Beauville allowable covers.*

2.2. The $SO(4)$ and $SO(5)$ Cases: Definition of C

Before restricting attention to the cases of $SO(4)$ and $SO(5)$ for most of the rest of Sect. 2, we should point out that the counts in Sect. 1.2 and 1.3 agree for all n, i.e., $\dim \mathrm{Prym} = g(K) - g(K_0) = \dim A_c$. In fact, as it was shown in [2] the flow on A_c linearizes on $\mathrm{Prym}_i(K)$. (See Sect. 4 for details and a refinement.)

Now let $2(q_j+q_j^i)$; $1 \leqq j \leqq 4$ and $b^i \underset{\mathrm{def}^n}{=} i(b)$, $\forall b \in K$ [respectively $2(q_j+q_j^i)+p_0$] be the divisors corresponding to the bitangents for $n=4$ [$n=5$].

Let $\Theta_u = \mathrm{Im}\,\phi^{(2)}$ where $\phi^{(2)}: S^2K \to \mathrm{Jac}(K)$

$$x_1+x_2 \to [(x_1+x_2)-(q_1+q_1^i)]$$

Then *define*

$$C = \mathrm{Prym}_i(K) \cap \Theta_u \tag{1}$$

We then have $C = \{b_1+b_2-(q_1+q_1^i)/b_1+b_2+b_1^i+b_2^i = H; b_i \in K\}$

[respectively $C = \{b_1+b_2-(q_1+q_1^i)/b_1+b_2+b_1^i+b_2^i = H-p_0; b_i \in K\}$]

C is a curve of genus 3 [respectively 6] with properties similar to K:

a) There is a natural tetragonal map $C \xrightarrow{\sigma_C} \mathbf{P}^1$ sending $b_1+b_2-(q_1+q_1^i)$ to the point $[z:h]$ describing the line through $[0:0:1]$ on which b_1, b_2, b_1^i, and b_2^i lie.

b) A natural involution $(b_1+b_2)-(q_1+q_1^i) \to (b_1^i+b_2^i)-(q_1+q_1^i)$, also denoted by i, which is the restriction of $i^*: \mathscr{J} \to \mathscr{J}$ on C (the same as the involution given by $-1:(q_1+q_1^i)-(b_1+b_2) \cong (b_1^i+b_2^i)-(q_1+q_1^i)$). C/i is then an elliptic [respectively hyperelliptic] curve denoted C_0.

c) σ_C then factors as $C \xrightarrow{\pi_C} C_0 \xrightarrow{\pi_{C_0}} \mathbf{P}^1$, π_C is ramified at $(q_j+q_j^i)-(q_1+q_1^i)$ [all these are points of order 2 in $\mathrm{Prym}_i(K)$]. Thus π_{C_0} will send the branch locus of π_C down to the branch locus of π_{K_0} in $\mathbf{P}^1$, corresponding to the bitangent lines. And vice versa: π_{C_0} is ramified over points corresponding to lines touching K at infinity (i.e., at the p_j).

2.3. The Ramified Bigonal Construction: Relation of K and C

In [6] the tetragonal construction, which takes a tower

$$\tilde{C} \xrightarrow{\pi} C \xrightarrow{f} \mathbf{P}^1,$$

where π is an unramified double cover, and f a branched cover of degree 4, and gives two towers

$$\tilde{C}_0 \to C_0 \to \mathbf{P}^1, \quad \tilde{C}_1 \to C_1 \to \mathbf{P}^1$$

of the same type, was introduced because of the

1. **Proposition.** $\mathrm{Prym}(C, \tilde{C}) \simeq \mathrm{Prym}(C_0, \tilde{C}_0) \simeq \mathrm{Prym}(C_1, \tilde{C}_1)$.

We define a version of this type of construction where π itself is ramified, but f is of degree 2.

2. *Construction.* Start with $K \to K_0 \xrightarrow{\pi_{K_0}} \mathbf{P}^1$ (K_0 some hyperelliptic curve).

Construct a curve $(\pi_{K_0})_*(K)$ whose points correspond to the different ways of lifting the pair $\pi_{K_0}^{-1}(t)$, for $t \in \mathbf{P}^1$, to a pair in K: i.e.,

$$(\pi_{K_0})_*(K) = \{\mathscr{L} \in \mathrm{Pic}^{(2)}(K)/Nm\mathscr{L} = \pi_{K_0}^* \mathcal{O}_{\mathbf{P}^1}(1)\}$$

This is a 4-to-1 branched cover of $\mathbf{P}^1$ with an involution

$$\tau : (\pi_{K_0})_*(K) \to (\pi_{K_0})_*(K)$$

defined (as in [6]) by sending a lift to its complementary. We then call $(\pi_{K_0})_*(K)$ *bigonally related* to K.

A little thought convinces us that, up to a translation in $\mathrm{Jac}(K)$, the curve obtained in this fashion is just C in our case:

$$\begin{array}{ccccc} & \nearrow \mathrm{Pic}^{(2)}(K) \searrow & & & \\ (\pi_{K_0})_*(K) & \longrightarrow & \Theta_u & \hookrightarrow & \mathrm{Jac}(K) \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid & & \\ \mathscr{L} & \longrightarrow & [\mathscr{L}] - (q_1 + q_1^i) & & \end{array}$$

The image of $(\pi_{K_0})_*(K)$ in Θ_u is contained in $\mathrm{Prym}_i(K)$:

$$[\mathscr{L}] + i^*[\mathscr{L}] = H = [\sigma_K^* \mathcal{O}_{\mathbf{P}^1}(1)] \Rightarrow [\mathscr{L}] - (q_1 + q_1^i) + i^*[\mathscr{L}] - i^*(q_1 + q_1^i) \cong 0$$
$$[\text{respectively } [\mathscr{L}] + i^*[\mathscr{L}] = H - p_0].$$

Thus it is precisely $C = \Theta_u \cap \mathrm{Prym}_i(K)$.

Moreover the involution τ is thus identified with the involution on C.

Note. (1) The relation between K and C is very similar to the one coming from the tetragonal construction but Donagi's proposition above does not imply that $\mathrm{Prym}_i(K)$ and $\mathrm{Prym}_i(C)$ should be isomorphic. (It does, however, imply isogeny: see Sects. 2.4 and 3, where we show that they are dual.)

(2) A repetition of the construction would yield back a curve isomorphic to K: e.g., $(b_1 + b_2) + (b_1 + b_2^i) \in \mathrm{Pic}^{(2)}(C)$ would correspond to the "common point" $b_1 \in K$.

2.4. Doubling the Bigonal Construction

As it turns out, a simple way to exhibit the connection of K and C via the tetragonal construction with Beauville allowable covers $\left(2.3.1 \text{ holds for any allowable } \tilde{C} \xrightarrow{\pi} C\right)$, is to consider reducible curves whose normalizations give two copies of K and C respectively – instead of the singular curve M introduced in Sect. 2.1 to define $\mathscr{P}_i(K)$ (and its analogue for C).

As the results in this section and Sect. 3 are purely algebrogeometric we work in a more general framework (than 2.2):

Let $K \to K_0 \to \mathbf{P}^1$ give a ramified double cover of some hyperelliptic curve K_0 of genus g. Assume $g(K) = 2g + h$ and let the $2h+2$ ramification points be $p_1, \ldots, p_{2h+2}$. Then we can construct the bigonally related cover $C \to C_0 \to \mathbf{P}^1$ [respectively $g(C_0) = h$, $g(C) = 2h + g$, ramifications at $q_1, \ldots, q_{2g+2}$] as in 2.3.2.

1. *Construction.* Start with two copies of K: K^α, K^β say; touching at the points p_j $(1 \leqq j \leqq 2h+2)$. These form a double cover of two copies of $K_0 (K_0^\alpha, K_0^\beta)$, intersecting transversally. (Thus the fixed points are ordinary double points and the branches are not exchanged by the involution as required.)

A similar (reducible) curve can be built starting with C^α, C^β and identifying the points $q_j^\alpha \sim q_j^\beta$ $(1 \leqq j \leqq 2g+2)$. Denote these curves by $K \vee K$, $K_0 \vee K_0$ $(C \vee C$, $C_0 \vee C_0$ respectively).

2. **Proposition.** *The tetragonal construction applied to $K \vee K \to K_0 \vee K_0$ leads to:*
(1) *$C \vee C \to C_0 \vee C_0$; and*
(2) *an unramified double cover $\tilde{\Lambda} \to \Lambda$ (where Λ is a curve of genus $2g + 2h + 1$, $\cong K_0 \times_{\mathbf{P}^1} C_0$).*

Proof. The tetragonal construction gives the curve of different lifts of the points in a fibre of $K_0 \vee K_0 \to \mathbf{P}^1$ back to $K \vee K$. This is just $C^\alpha \times_{\mathbf{P}^1} C^\beta$: given two pairs of points in a general fibre, the lifts of one of them, coming from K_0^α say, will give (via 2.3.2) a copy of C; but with the following singularities introduced over the p_j:

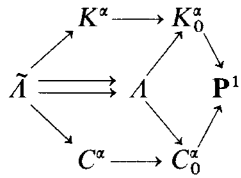
The two curves of the tetragonal construction (i.e., $\tilde{C}_0$ and $\tilde{C}_1$ in the notation of Proposition 2.3.1) will have four simple ramification points each, over each of the p_j. These are identified to form double points where the branches, one from each of the curves, touch in pairs.

Inside $C^\alpha \times_{\mathbf{P}^1} C^\beta$ we get two irreducible components isomorphic to C: the graphs of the identity and the involution of C (i.e., points coming from lifting the two pairs of points in a fibre in the same or "opposite" ways – given an identification of K^α with K^β). These are easily seen to intersect over the q_j in the same way as K^α and K^β do over the p_j.

Now, the curve of lifts coming from a tetragonal construction is always reducible [6, Sect. 6], and each of the components contains all the lifts in each fibre differing from each other by an even number of points. Thus, it follows that those two copies of C together form one of these two components.

One can check, by Riemann-Hurwitz, that the remaining component, $\tilde{\Lambda}$ say, is of the appropriate genus $(4g + 4h + 1)$, and that the involution sending a lift to its complementary has no fixed points on $\tilde{\Lambda}$.

To identify Λ we can embed $\tilde{\Lambda}$ in $K^\alpha \times_{\mathbf{P}^1} K^\beta$ (symmetrically, by starting from $C \vee C$ instead). We thus get a commutative diagram:



By the universal property of $C_0 \times_{\mathbf{P}^1} K_0$, there exists a unique map $\Lambda \to C_0 \times_{\mathbf{P}^1} K_0$ (making the appropriate diagram commutative) which is evidently (degree considerations) an isomorphism.

Note. The involutions of K and C induce isomorphisms on $\tilde{\Lambda}$ generating the dihedral group D_8: Λ is just the quotient by the rotation of order 2. $\tilde{\Lambda}$ is the "third" quotient of the $\mathbf{Z}_2^2$-action on $K \times_{\mathbf{P}^1} C$ – apart from $K \times_{\mathbf{P}^1} C_0$ and $K_0 \times_{\mathbf{P}^1} C$.

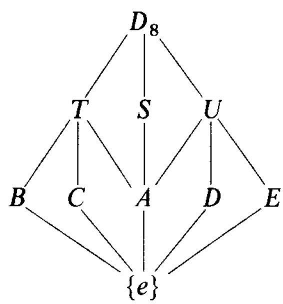
The immediate conclusions from 2.2.1 are:

3. **Corollary.** *$\mathscr{P}_i(K \vee K)$ and $\mathscr{P}_i(C \vee C)$ are isomorphic.*

4. **Corollary.** $\mathrm{Prym}_i(K)$ *and* $\mathrm{Prym}_i(C)$ *are isogenous.*

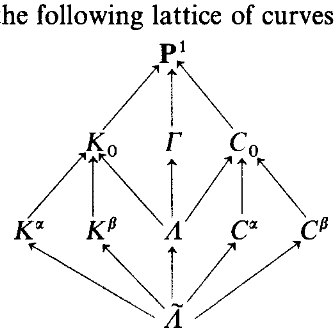
5. *Note.* I would like to thank David Mumford for suggesting the following way of looking at this situation:

Passing from curves and double covers to fields and field extensions of degree 2, then K and C over $\mathbf{P}^1$ have $\tilde{\Lambda}$ as their normal (or Galois) closure. The Galois group is D_8 and its subgroup lattice is of the form



where B, C, D, E are the non-normal subgroups and S is the group of rotations $\simeq \mathbf{Z}_4$.

This corresponds to the following lattice of curves



where Γ is the third quotient of $C_0 \times_{\mathbf{P}^1} K_0 \simeq \Lambda$ (double cover of $\mathbf{P}^1$ ramified at the points where either K_0 or C_0 is ramified).

This is exactly the same picture we would get examining the splitting field of x^4-2 over $\mathbf{Q}$ say. The Jacobian of $\tilde{\Lambda}$ (dim $=4g+4h+1$) splits up into $\mathrm{Prym}_i(K)$, $\mathrm{Prym}_i(C)$, $\mathrm{Jac}(K_0)$, $\mathrm{Jac}(C_0)$, and $\mathscr{J}(\Gamma)$ (genus of $\Gamma=g+h+1$). Finally, note that if we define

$$\tilde{\Lambda}_C = \{\text{ordered pairs } (p,q)/p,\ q\in C;\ \pi_C(p)=i\pi_C(q)\}$$

and similarly for $\tilde{\Lambda}_K$, $\tilde{\Lambda} \simeq \tilde{\Lambda}_C \simeq \tilde{\Lambda}_K$. $C\times_{\mathbf{P}^1} K$ gives then two copies of $\tilde{\Lambda}$.

3. The Prym Varieties of Bigonally Related Curves are Dual

Maintaining the same notation as in Sect. 2.4 we are going to prove that

1. **Proposition.** $\mathrm{Prym}_i(K)$ *and* $\mathrm{Prym}_i(C)$ *are dual abelian varieties.*

Proof. Let us first construct $K\vee K$, $K_0\vee K_0$ etc. as in Sect. 2.4, with the resulting isomorphism $\mathscr{P}(K\vee K) \xrightarrow{\sim} \mathscr{P}(C\vee C)$ of 2.4.3 and the square of short exact sequences (together with a similar one for C):

$$\begin{array}{ccccc} \mathbf{Z}_2^{2h+1} & \hookrightarrow & \mathbf{C}_*^{2h+1} & \longrightarrow & \mathbf{C}_*^{2h+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{P}(K\vee K)\times \mathbf{Z}_2 & \hookrightarrow & \mathscr{J}(K\vee K) & \longrightarrow & \mathscr{J}(K_0\vee K_0) \\ \downarrow & & \downarrow & & \downarrow \\ \{\mathrm{Prym}_i(K)\}^2 & \hookrightarrow & \mathscr{J}^2(K) & \longrightarrow & \mathscr{J}^2(K_0) \end{array}$$

(The last row of vertical maps being induced by normalization – see [5].) Restrict attention to the isogeny $\phi: \mathscr{P}(K\vee K) \to \{\mathrm{Prym}_i(K)\}^2$. Its kernel $\simeq \mathbf{Z}_2^{2h}$, can be thought of as the group of "twists" at an even number of the points $p_1, \dots, p_{2h+2}$ in the singular curve $K\vee K$. We will refer to its elements as *even twists*, and denote[2] them as

$$\Sigma - 1_{p_j}$$

(j in some appropriate subset of $\{1,\dots,2h+2\}$ with even number of elements).

In $\{\mathrm{Prym}_i(K)\}^2$ we have four distinguished abelian subvarieties: $\mathrm{Prym}_i(K)\times\{0\}$, $\{0\}\times\mathrm{Prym}_i(K)$; and, given a specified identification of the two copies of K, the "positive" and "negative" diagonals [consisting of elements (u,u), respectively $(u,-u)$, $\forall u\in\mathrm{Prym}_i(K)$]. Let $\mathscr{P}_K^\alpha$, $\mathscr{P}_K^\beta$, Δ_K^+, Δ_K^-; be their inverse images on $\mathscr{P}(K\vee K)$. Note that changing the given identification by composing with the involution of K would interchange Δ_K^+ and Δ_K^-, as it would the two copies of C inside $C^\alpha \times_{\mathbf{P}^1} C^\beta$. Of course, there are similar subvarieties corresponding to C, but they give nothing essentially new:

---

2 The same divisor is denoted as $\sum(-1,0,0)_{p_j}$ in [5]; think of a Cartier divisor supported on a singular point as an element in $\mathbf{C}_*\times\mathbf{Z}\times\mathbf{Z}$ after an appropriate choice of two valuations on the ring of rational functions; a simple viewpoint is to think of the possible extensions of a line bundle from the normalization to the singular curve as choices of the identifications of the fibres of the singular points

2. **Lemma.** *$\mathscr{P}_K^\alpha(\mathscr{P}_K^\beta;\mathscr{P}_C^\alpha,\mathscr{P}_C^\beta)$ are connected while $\Delta_K^+(\Delta_K^-;\Delta_C^+,\Delta_C^-)$ split entirely into 2^{2h} (respectively 2^{2g}) components (each mapped isomorphically onto its image by ϕ) and under the isomorphism $\mathscr{P}(K\vee K)\xrightarrow{\sim}\mathscr{P}(C\vee C)$ the zero components $(\Delta_K^+)_0$; $(\Delta_C^+)_0$ [respectively, $(\Delta_K^-)_0,(\Delta_C^-)_0$] are mapped onto $\mathscr{P}_C^\alpha;\mathscr{P}_K^\alpha$ (respectively, $\mathscr{P}_C^\beta,\mathscr{P}_K^\beta$).*

Proof. Recall that if we have a finite abelian cover of an algebraic variety, given say by a line bundle $\mathscr{L}$ such that $\mathscr{L}^{\otimes k}$ is the trivial bundle (i.e., k^{th} roots of unity in each fibre would form the cover); then the inverse image of a subvariety splits entirely into disjoint components if and only if $\mathscr{L}$ has a trivial restriction on that subvariety.

Now the first assertion follows from the obviously symmetric nature of the double points: the even twists are line bundles of order 2 and will induce equal bundles on the two copies of $\mathrm{Prym}_i(K)$, and thus the trivial bundle on the two diagonals.

To prove the second assertion recall, [6], that the tetragonal isomorphism is induced by a map $i_0: K\vee K\to \mathrm{Pic}^{(4)}(C\vee C)$ which we can describe as follows:

Restrict attention to a (generic) fibre over a point $\mathbf{P}^1$. Let $y_1,y_2,\bar{y}_1,\bar{y}_2$ be the points in $K_0\vee K_0$ ($^-$ denoting the given identification of the two copies of the curve); let $x_{ij};\bar{x}_{ij}$ be the points in $K\vee K$ ($j=0,1$; $i=1,2$, and $\pi_K^{-1}(y_i)=\{x_{i0},x_{i1}\}$). If the lift of $y_1,\ldots,\bar{y}_2$ to $x_{1j_1},\ldots,\bar{x}_{2j_2}$ (an element of $\mathrm{Pic}^{(4)}$) corresponds to a point in the fibre of $C\vee C$ denoted $\begin{bmatrix} j_1 & j_2\\ \bar{j}_1 & \bar{j}_2\end{bmatrix}$ the eight possible points are:

$$i_C\left(\begin{array}{c} \begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&1\end{bmatrix}\\ \hline \begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&0\\1&0\end{bmatrix}\end{array}\right. \qquad \begin{array}{c} \begin{bmatrix}0&0\\1&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\\ \hline \begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\end{array}$$

in C^α = graph of identity in C^β = graph of involution

(compare with Sect. 2.4)

Now $i_0(x_{ij})$ gives an element in $\mathrm{Pic}^{(4)}(C\vee C)$ by choosing from each column the point which has entry j in the i^{th} place. [Respectively for $i_0(\bar{x}_{ij})$ in the $\bar{i}^{\text{th}}$ place.] Thus $i_0(x_{ij})$ and $i_0(\bar{x}_{ij})$ give points in the two different diagonals: x_{ij} will give the "same" two pairs of points in C^α and C^β, while $\bar{x}_{ij}$ will give involute (by i_C) pairs.

This gives an isogeny from $\mathscr{P}_K^\alpha$ to $\phi(\Delta_C^+)$ which is of degree 1 and hence an isomorphism:

$\mathscr{P}_K^\alpha\cap\mathscr{P}_K^\beta$ has cardinality 2^{2h}, while $\Delta_C^+\cap\Delta_C^-$ has cardinality 2^{4g+2h} [the points of intersection of $\phi(\Delta_C^+)$ and $\phi(\Delta_C^-)$ being in a one-to-one correspondence to the points of order 2 in $\mathrm{Prym}_i(C)$]. Thus, since there are 2^{2g} components,

$$\#\{(\Delta_C^+)_0\cap(\Delta_C^-)_0\}=2^h.$$

Now by proving

3. **Lemma.** *The isomorphism $\mathrm{Prym}_i(K)\xrightarrow{\sim}(\Delta_K^+)_0\xrightarrow{\sim}\mathscr{P}_C^\alpha$ carries $\ker\varrho_K$, the kernel of the polarization map, to the group of even twists:*

$$\ker\{\mathscr{P}(C\vee C)\to\{\mathrm{Prym}_i(C)\}^2\}\cong\mathbf{Z}_2^{2g}.$$

We will have

4. **Corollary.** *The maps*

$$\mathrm{Prym}_i(C) \xrightarrow{\sim} \mathscr{P}_K^\alpha \twoheadrightarrow \mathrm{Prym}_i(K)$$
$$\mathrm{Prym}_i(K) \xrightarrow{\sim} \mathscr{P}_C^\alpha \twoheadrightarrow \mathrm{Prym}_i(C)$$

are the polarization maps, and in particular $\mathrm{Prym}_i(K)$ *and* $\mathrm{Prym}_i(C)$ *are dual.* (completing the proof of Proposition 3.1).

Proof of Lemma 3.3. As in the concluding argument in Lemma 3.2, it is enough to show that $(\Delta_K^+)_0 \cap (\Delta_K^-)_0$ consists precisely of the image of $\ker \varrho_K \hookrightarrow \mathrm{Prym}_i(K)$ in $(\Delta_K^+)_0$.

A point of order 2 in $\ker \varrho_K$ corresponds to a cycle σ of the hyperelliptic curve K_0 [14]. Then, assuming that the corresponding point of order 2 in $\mathrm{Jac}(K_0)$ is given by a divisor

$$\sum_j \mathfrak{q}_j^0 - \mathfrak{q}_j^1 ; \quad \mathfrak{q}_j^0, \mathfrak{q}_j^1 \in \{\mathfrak{q}_1, \ldots, \mathfrak{q}_{2g+2}\},$$

(where $\mathfrak{q}_j$ is the ramification point of $\pi_{K_0}: K_0 \to \mathbf{P}^1$ over the same point in $\mathbf{P}^1$ as the ramification point q_j of $\pi_C: C \to C_0$), we can define an even twist $h_\sigma = \sum_j (-1_{\mathfrak{q}_j^0}) + (-1_{\mathfrak{q}_j^1})$ in $\mathscr{P}(C \vee C)$. If $\pi_K^*(\sigma) = \tau + i^*\tau \in H_1(K)$, we claim in fact that

$$(i_0)_*: \mathscr{P}(K \vee K) \xrightarrow{\sim} \mathscr{P}(C \vee C)$$

will take the cycle $(\tau - i^*\tau) + (\bar{\tau} - i^*\bar{\tau})$ – where, as on 3.2, $\bar{\tau}$ is the "same" cycle as τ in the other copy of K and would be interchanged with $i^*\bar{\tau}$ if we picked the alternative identification – to the twist h_σ.

To prove the lemma, however, it is enough to check that $(i_0)_*$ carries $(\tau - i^*\tau) + (\bar{\tau} - i^*\bar{\tau})$ to $\mathscr{P}_C^\alpha$. [This, by virtue of 3.2, shows that it lies in $(\Delta_K^+)_0$; if we changed the identification of K^α and K^β, since the cycle $(\tau - i^*\tau) + (\bar{\tau} - i^*\bar{\tau})$ is not affected: $[2(\tau - i^*\tau) = 0]$, this would also show it lies in $\mathscr{P}_C^\beta \simeq (\Delta_K^-)_0$.]

Let $(i_0)_*(\tau) = u + v$; u a cycle in C^α, v in C^β (where in fact $v = \bar{u}$ for one of the two identifications). Then, from the description of i_0 in Lemma 2,

$$(i_0)_*(\bar{\tau}) = u + i^*v$$

Hence

$$(i_0)_*(\tau - i^*\tau + \bar{\tau} - i^*\bar{\tau}) = 2(u - i^*u)$$

[similarly for the other identification we would get

$$(\tau - i^*\tau + \bar{\tau} - i^*\bar{\tau}) \to 2(v - i^*v)] .$$

5. *The Variety* $\mathrm{Prym}_i^*(K)$

The abelian variety variously realized as $\mathscr{P}_K^\alpha$, $\mathscr{P}_K^\beta$, $(\Delta_C^+)_0$, $(\Delta_C^-)_0$ will be denoted as $\mathrm{Prym}_i^*(K)$. It can be defined for any double cover $K \to K_0$, but of course it will not be dual to $\mathrm{Prym}_i(K)$ if K_0 is not hyperelliptic. An element in $\mathrm{Prym}_i^*(K)$ can be thought of as a line bundle on K, together with an appropriate glueing of the fibres over $p_1, \ldots, p_{2h+2}$ with the corresponding fibres of the trivial line bundle on K. We will refer to elements in the kernel of $\mathrm{Prym}_i^*(K) \twoheadrightarrow \mathrm{Prym}_i(K)$ as even twists, and denote them as we did before in the case of hyperelliptic K_0.

4. A_c is an Affine Subvariety of $\mathrm{Prym}_i^*(K)$

We now return to the geodesic motion on $SO(n)$, in order to apply the material in Sects. 2 and 3. Recall that, in order to unify notation, for n even we take $\mathrm{Pfaff}(X)=c_0$ as one of the invariants in the definition of A_c.

Soon after Manakov's results in [11], Dubrovin [16] showed a wider class of examples[3] to be integrable, and he linearized the resulting flows on the Jacobian of the spectral curve. The kernel of this projection, say $A_c \to \mathrm{Jac}(K)$[4], is the group of (invertible) diagonal matrices acting by conjugation. Then Adler and Van Moerbeke obtained in [2], amongst other results, similar conclusions for the "spinning top type equations", but paid special attention to the $SO(n)$ case, where because of the skew-symmetry the linearization occurs in $\mathrm{Prym}_i(K)$. The kernel in this case consists of the elements of order 2 of the previously described group.

Finally in [9] Haine specified that A_c is an affine subvariety of $\mathrm{Prym}_i(C)$ for the $SO(4)$ case.

The aim of this section is to identify A_c in the general $SO(n)$ case. We have attempted to make the proof readable without necessitating continual reference to the earlier papers.

1. **Proposition.** *The Euler-Arnold equations on $SO(n)$ with a Manakov metric are completely integrable, and there is an isomorphism of affine varieites:*

$$\begin{array}{c} \mathfrak{so}(n) \\ \uparrow \\ A_c \xrightarrow{\sim} \mathrm{Prym}_i^*(K)\backslash\Theta^* \end{array}$$

carrying the flow on A_c onto a linear flow on $\mathrm{Prym}_i^*(K)$ *(where Θ^* is a divisor in* $\mathrm{Prym}_i^*(K)$*: see Lemma 4.3).*

Proof. We examine the affine variety A_c for a fixed but generic c. Let $Xu+Ah-zI$ be denoted as $M(X,p)$ for $X\in A_c$, $p\in \mathbf{P}^2$. $M(X,p)$ is singular iff $p\in K$.

For each $X\in A_c$ we define a pair of holomorphic embeddings of the spectral curve in projective space

$$f_X, f^X : K \to \mathbf{P}^{n-1}$$

called the *eigenvector mappings*, as follows:

For the general point of K, $\ker M(X,p)$ is one-dimensional, and thus gives a well-defined point in $\mathbf{P}^{n-1}$. Since K is smooth for c generic, the mapping extends to the whole of K. A similar mapping f^X can be defined starting with $(\mathrm{Im}\, M(X,p))^{\perp}$. The skew symmetry is equivalent to

$$f^X = f_X \circ i$$

($f^X = f_{X^T}$ in the absence of skew symmetry).

---

3 Compare with the class of Lax equations for which a geometric linearization condition is found in [8]

4 Where A_c is in the collection of polynomials with matrix coefficients (to be described briefly in 4.8)

[*Note.* This is just an instance of the following general situation: Think of $M(X,\cdot)$ as a bundle map (over $\mathbf{P}^2$). Then K is just the degeneracy locus of highest rank. The mappings f_X and f^X can now be thought of as the kernel and cokernel bundles respectively over K ($\setminus$ a finite number of points.).]

The general strategy (see for example [8]) is to extract information from f_X about X:

Let $\mathscr{L}_X = f_X^*(\mathcal{O}_{\mathbf{P}^{n-1}}(1))$ – the restriction of the hyperplane bundle to $f_X(K)$.

2. **Lemma** [2]. *$\mathscr{L}_X + i^*\mathscr{L}_X = (n-1)H$ (where H is the hyperplane bundle for K in $\mathbf{P}^2$.) and thus* $\deg \mathscr{L}_X = \dfrac{n(n-1)}{2}$.

Proof. Let $\Delta_{ij}(X,p)$ be the $(i,j)^{\text{th}}$ minor of $M(X,p)$, and $\det \Delta_{ij}(X,p) = d_{ij}(X,p)$.

The equations $d_{ij}=0$ define curves of degree $n-1$ in $\mathbf{P}^2$ which cut out divisors $D_{ij}(\simeq (n-1)H)$ on K.

Since $\dfrac{d_{ki}}{d_{kj}} = \dfrac{(f_X)_i}{(f_X)_j}$ and $\dfrac{d_{ik}}{d_{jk}} = \dfrac{(f^X)_i}{(f^X)_j}$ are independent of k, we can decompose the D_{ij} into

$$D_i(X) + D^j(X)$$

(where D_i is the divisor common to the i^{th} row of minors and D^j to the j^{th} column).

Now

$$\Delta_{ij}(X^T,p) = (\Delta_{ij}(X,p))^T = \Delta_{ij}(X,p^i),$$

hence

$$D^j(X) = D_j(X^T) = i^* D_j(X).$$

But

$$[D^j(X)] = \mathscr{L}_X \text{ for all } j;$$

since the zero divisor of $\dfrac{d_{kj}}{d_{ki}}$ is just $D^j(X)$ and represents the intersection of the coordinate hyperplane $z_j=0$ with $f_X(K)$ in $\mathbf{P}^{n-1}$. □

The immediate corollary of this lemma is that since $\mathscr{L}_X + i^*\mathscr{L}_X$ is independent of X we can define a mapping

$$\begin{array}{ccc} \lambda: A_c & \rightarrow & \mathrm{Pram}_i(K) \\ \cup\!\!\!| & & \cup\!\!\!| \\ X & \rightarrow & \mathscr{L}_X - \mathscr{L}_{X_0} \end{array}$$

for some fixed basepoint X_0 in the torus A_c.

Note that for all X, f_X sends the points $p_1, \ldots, p_n$ to n points in general position in $\mathbf{P}^{n-1}$, namely the coordinate points $[1:0:\!-\!\!-\!:0]$, —, $[0:\!-\!\!-\!:0:1]$. Thus in the linear system given by $\mathscr{L}_X$ there is a unique divisor $E^j(X) + p_1 + \ldots + \widehat{p_j} + \ldots + p_n = D^j(X)$, $\deg E^j = g(K)$, $p_j \notin E^j$ (i.e., the hyperplane section through $p_1, \ldots, \widehat{p_j}, \ldots, p_n$).

The definition of the map $A_c \rightarrow \mathrm{Prym}_i(K)$ in [2,9] uses the divisor $E^1(X)(E^j + i^*E^j = (n-3)H + 2p_j)$, but we think $\mathscr{L}_X$ is more natural.

3. **Lemma.** $\mathrm{Im}(\lambda) = \mathrm{Prym}_i(K) \setminus \Theta$ *where Θ is the intersection of* $\mathrm{Prym}_i(K)$ *with an appropriate translate of the Θ-divisor (depending on the choice of X_0).*

Proof. In a curve of genus g $S^{g-1}\times\{q_1+\ldots+q_{r+1}\}\xrightarrow{\mu}\mathrm{Pic}^g\xrightarrow{\sim}\mathrm{Jac}$ (where μ is the Abel map) gives a translate of the Θ divisor. The complement can be thought of as the complete linear systems g^n_{g+r} which do not pass through $q_1,\ldots,q_{r+1}$ [or as all the (general) divisors of that degree not containing $q_1+\ldots+q_{r+1}$].

To first show that $\mathrm{Im}(\lambda)\supset\mathrm{Prym}_i(K)\backslash\Theta$ we check that if $\mathscr{L}_X$ is a line bundle s.t.

1) $\mathscr{L}_X+i^*\mathscr{L}_X=(n-1)H$

2) $h^0(\mathscr{L}_X)=n$ and the corresponding map into $\mathbf{P}^{n-1}$ sends $p_1,\ldots,p_n$ into general position; then it can be obtained from $X\in A_c$.

This is done by simply picking n linearly independent sections

$$f_1,\ldots,f_n \text{ s.t. } f(p_1)=[1:\ldots:0],\ldots,f(p_n)=[0:\ldots:1].$$

Now $zf_j-\alpha_j hf_j=\left(\sum_{k\neq j}X_{jk}f_k\right)u$, since any section of $\mathscr{L}_X$ vanishing at p_j can be written as a linear combination of f_k, $k\neq j$.

Notice that since the f_j are determined up to a (non-zero) constant the matrix (X_{jk}) is determined up to conjugation by an invertible diagonal matrix. (For the inverse see the argument in [17] as applied in [2, p. 340]). □

For a given $X\in A_c$, let X^ν be $\nu\cdot X\cdot\nu$ where $\nu=\mathrm{diag}(\varepsilon_i^\nu)$ and $\varepsilon_i^\nu=\pm1$. (For n even, we assume $\Pi\varepsilon_i^\nu=1$ to remain in the same A_c.)

Of course $X^\nu=X^{-\nu}$, so we will often identify ν and $-\nu$.

The problem we face is that $\mathscr{L}_X$ and $\mathscr{L}_{X^\nu}$ are the same (f_X and f_{X^ν} are projectively equivalent embeddings) and we have to extract some more information. More precisely, $X^\nu u+Ah-zI=(Xu+Ah-zI)^\nu$ and thus if $f_X(p)$ is in the original kernel, $\nu\cdot f_X(p)=f_{X^\nu}(p)$ is in the new kernel: f_X and f_{X^ν} differ by a linear map $\mathbf{P}^{n-1}\xrightarrow{\nu}\mathbf{P}^{n-1}$.

Instead of a line bundle over K we wish to obtain one over $K\vee K$ such that its image under the normalization is the trivial bundle on the second copy of K.

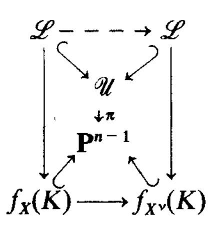
4. *Construction.* Let $\pi:\mathscr{U}\to\mathbf{P}^{n-1}$ be the universal bundle : $\mathscr{U}\hookrightarrow\mathbb{C}^n\times\mathbf{P}^{n-1}$.

Define $s_j(p)=\left(\frac{f_1}{f_j},\ldots,\frac{f_n}{f_j}\right)\in\pi^{-1}f_X(p)$. These are sections of the line bundle $-\mathscr{L}_X$: specifically $-(s_j)=D^j$.

5. **Claim.** *The mapping $X\to\{s_1(X),\ldots,s_n(X)\}$ is one to one.*

Proof. We need to compare $\{s_j(X)\}$ and $\{s_j(X^\nu)\}$, i.e., to show that they are the same iff $\nu=\pm1$.

Now $s_j(X^\nu,p)=\left(\frac{\varepsilon_1^\nu f_1}{\varepsilon_j^\nu f_j},\ldots,\frac{\varepsilon_n^\nu f_n}{\varepsilon_j^\nu f_j}\right)\in\pi^{-1}f_{X^\nu}(p)$ while $s_j(X,p)\in\pi^{-1}f_X(p)$. Consider the diagram



We can define a bundle isomorphism over the natural projective isomorphism between $f_X(K)$ and $f_{X^\nu}(K)$ in two ways: multiplying by ν or $-\nu$.
Either way $\{s_j(X)\}$ is different from $\{s_j(X^\nu)\}$:

$$\nu_* s(X) = (\pm \nu) \cdot s(X^\nu)$$

$$\langle \text{where } s(X) \text{ is the vector } (s_1(X), \ldots, s_n(X)).\rangle \quad \square$$

We are now in a position to finish the proof of 4.1:

Let us first consider the case n even.

We can construct a line bundle $\tilde{\mathscr{L}}_X$ in $\mathrm{Jac}(K \vee K)$ by taking a fixed section of the trivial bundle over one of the copies of K, say 1, where 1_p will denote the value at the fibre over $p \in K$. Then take $\mathscr{L}_X$ and identify

$$1_{p_j} \sim \frac{1}{s_j(X, p_j)}; \qquad 1 \leqq j \leqq n.$$

From 4.5 it is clear that the map is one to one, so it remains to check that $\tilde{\mathscr{L}}_X + i^* \tilde{\mathscr{L}}_X$ is independent of X. [$(\tilde{\mathscr{L}}_X - \tilde{\mathscr{L}}_{X_0}) + i^*(\tilde{\mathscr{L}}_X - \tilde{\mathscr{L}}_{X_0}) = 0$, and $X \to \tilde{\mathscr{L}}_X - \tilde{\mathscr{L}}_{X_0}$ would then give the required mapping into $\mathrm{Prym}_i^*(K)$.] Since $\mathscr{L}_X + i^* \mathscr{L}_X = (n-1)H$, this reduces to checking that the identifications over $p_1, \ldots, p_n$ are independent of X for each $\tilde{\mathscr{L}}_X + i^* \tilde{\mathscr{L}}_X$.

Now this is evidently the case if we compare at X and X^ν. [There $H^0(\mathscr{L}_X) = H^0(\mathscr{L}_{X^\nu})$ and

$$s_j(X^\nu, p) \otimes s_j((X^\nu)^T, p) = (\varepsilon_j^\nu)^2 s_j(X, p) \otimes s_j(X^T, p).]$$

More generally we can show that $s_j(X, p) \otimes s_j(X^T, p)$ gives an identification over the p_j independent of X by using the following linear algebra lemma:

6. **Lemma.** $\sum_j d_{jj}(X, p) = 0$ *gives the polar of* K *with respect to the point* $q_0 = [1:0:0]\,(=[z:h:u])$, i.e.

$$\sum_j d_{jj}(X, p) = -\frac{\partial}{\partial z} \det M(X, p)$$

and in particular is independent of $X \in A_c$.

If $\mathrm{Ad}(X, p) = (d_{ij}(X, p))^T$ is the adjoint of $M(X, p)$ we have $\mathrm{Ad}(X, p) = f_X(p) \cdot f^X(p)$.

If we think of the space of matrices equal to $\mathrm{Ad}(X, p)$ up to a constant factor as $\pi^{-1}(f_X(p)) \otimes \pi^{-1}(f^X(p))$ this is where the section $s_j(X, p) \otimes s_j(X^T, p)$ takes values. Dually $\sum_j d_{jj}(X, p) = \mathrm{Tr}(\mathrm{Ad}(X, p)) = f^X(p) \cdot f_X(p)$.

Now over the open set U where $\sum_j d_{jj}(X, p) \neq 0$ in $\mathbf{P}^2$ all the s_j can be trivialized simultaneously for all X

$$s_j(X, p) \otimes s_j(X^T, p) = \frac{d_{11}(X, p) + \ldots + d_{nn}(X, p)}{d_{jj}(X, p)}$$

(Note $p_j \in U \ \forall j$).

$$\left(\left(\frac{f_1}{f_j}, \ldots, \frac{f_n}{f_j} \right) \otimes \left(\frac{f^1}{f^j}, \ldots, \frac{f^n}{f^j} \right) = \frac{f_1 f^1 + \ldots + f_n f^n}{f_j f^j} = \frac{f^X(p) \cdot f_X(p)}{f_j f^j} \right)$$

where f^j is the j^{th} component of f^X.

Now by choosing say $d_{jj}(X, p_j) = 1 \quad \forall j, X$ we can think of d_{jj} as a specific section of $(n-1)H$ over K, depending on X, but whose values over the p_j are fixed. Now $\sum d_{jj}(X, p)$ is independent of X, so the values of $s_j(X, p) \otimes s_j(X^T, p)$ should be fixed over the p_j and $\tilde{\mathscr{L}}_X + i^* \tilde{\mathscr{L}}_X$ is a fixed line bundle.

For n odd we have the additional problem of identifying over p_0. Now for a fixed basepoint $X_0 \in A_c$ we can identify in an arbitrary fashion over p_0. This gives a unique way of identifying the fibres over p_0 for each X s.t. $\tilde{\mathscr{L}}_X - \tilde{\mathscr{L}}_{X_0}$ lies in $\mathrm{Prym}_i^*(K)$; (the mapping is then independent of the choice at X_0). [To be precise, there are two identifications s.t. $\pi_*(\tilde{\mathscr{L}}_X - \tilde{\mathscr{L}}_{X_0}) = 0$ one of which corresponds to an even twist:

$$\ker \pi_* = \mathscr{P}(K \vee K) \times \mathbb{Z}_2)$$

e.g., for X and $X^\vee$, $(\pm(\Pi \varepsilon_i^\vee, \varepsilon_1^\vee, \ldots, \varepsilon_n^\vee)$ gives the relation of the identifications giving $\tilde{\mathscr{L}}_X$ and $\tilde{\mathscr{L}}_{X^\vee}$; note that it is well defined in $\mathbb{Z}_2^{n+1}/\mathbb{Z}_2$ since n is odd.] □

Now Proposition 4.1, together with the results of Sect. 3 gives us

7. **Corollary.** *For the cases* $n = 4, 5$, *where* K_0 *is hyperelliptic,* A_c *is an affine subvariety of* $\mathrm{Prym}_i(C)$, *dual to* $\mathrm{Prym}_i(K)$ *(i.e., the mapping* λ *factors through the polarization map:*

$$\begin{array}{ccc} A_c & \xrightarrow{\lambda} & \mathrm{Prym}_i(K) \\ & \searrow & \uparrow \varrho \\ & & \mathrm{Prym}_i(C) \simeq \mathrm{Prym}_i^*(K).) \end{array}$$ □

8. The general spinning top type equation in the absence of skew symmetry. The analogue of Proposition 4.1 becomes simpler to prove when the skew symmetry is not present:

Following [16] we can define in a way similar to Sect. 1 a Lax equation where $Xu + Ah$ is replaced by

$$X_N u^N + X_{N-1} u^{N-1} h + \ldots + X_1 u h^{N-1} + A h^N = X(u, h).$$

(A as before a fixed diagonal matrix) (X_i $n \times n$ matrices, X_N with fixed diagonal) and the spectral curve K given by $\det(X(u,h) - zI) = 0$ has genus $\frac{N(n-1)n}{2} - (n-1)$, but lives in the rational ruled surface $\mathscr{F}_N$ (see also [8]).

As in Lemma 2 we can construct maps $f_X, f^X : K \to \mathbf{P}^{n-1}$ and thus in particular a divisor $\mathscr{L}_X$, finally getting an injection $A_c/\pi \to \mathrm{Jac}(K)$ where A_c is the space of $X(u, h)$ with fixed spectrum K and π is the group of (invertible) diagonal matrices acting by conjugation. The kernel of the action is given by the multiples of the identity; thus an element in π can be thought of as being in $\mathbb{C}_*^n/\mathbb{C}_*$.

Now we can follow the construction in 4.4 and the proof of 4.5 almost to the letter (exchanging $\mathbb{Z}_2$'s by $\mathbb{C}_*$'s). We get

9. **Corollary.** $A_c \hookrightarrow \mathrm{Jac}^*(K)$ *where* $\mathrm{Jac}^*(K)$ *is the* $\mathbb{C}_*^{n-1}$ *extension, corresponding to the* n *points at* ∞ *of* K, *of* $\mathrm{Jac}(K)$ □

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References

1. Adler, M., van Moerbeke, P.: Completely integrable systems, Euclidean Lie algebras, and curves. Adv. Math. **38**, 267–317 (1980)
2. Adler, M., van Moerbeke, P.: Linearization of Hamiltonian systems, Jacobi varieties and representation theory. Adv. Math. **38**, 318–379 (1980)
3. Adler, M., van Moerbeke, P.: The algebraic integrability of geodesic flow on $SO(4)$. Invent. Math. **67**, 297–331 (1982)
4. Arnold, V.I.: Mathematical methods of classical mechanics. Berlin, Heidelberg, New York: Springer 1978 (in Russian: Moscow: Nauka 1974)
5. Beauville, A.: Prym varieties and the Schottky problem. Invent. Math. **41**, 149–196 (1977)
6. Donagi, R.: The tetragonal construction. Bull. Am. Math. Soc., New Ser. **4**, 181–185 (1981)
7. Dubrovin, B.A.: Theta functions and non-linear equations. Usp. Mat. Nauk **36**, 11–80 (1981) [Russian Math. Surveys **36**, 11–92 (1981)]
8. Griffiths, P.A.: Linearizing flows and a cohomological interpretation of Lax equations (preprint)
9. Haine, L.: Geodesic flow on $SO(4)$ and Abelian surfaces. Math. Ann. **263**, 435–472 (1983)
10. Haine, L.: The algebraic complete integrability of geodesic flow on $SO(N)$. Commun. Math. Phys. **94**, 271–287 (1984)
11. Manakov, S.V.: A remark on the integration of the Euler equations of the dynamics of an n-dimensional rigid body. Funkts. Anal. Prilozh. **10**, 93–94 (1976) [Funct. Anal. Appl. **10**, 328–329 (1976)]
12. McKean, H.P.: Integrable systems and algebraic curves; global analysis. Lect. Notes Math. **755**, 83–200. Berlin, Heidelberg, New York: Springer 1979
13. Moser, J.: Various aspects of integrable Hamiltonian systems. In: C.I.M.E. Lectures: Bressanone, Italy, June 1978. Prog. Math. **8**, 233–289 (1980)
14. Mumford, D.: Prym varieties. I. In: Contributions to analysis. New York: Academic Press 1974
15. Mumford, D.: Appendix to [3]: Invent. Math. **67**, 297–331 (1982)
16. Dubrovin, B.A.: Completely integrable Hamiltonian systems associated with matrix operators and abelian varieties. Funkts. Anal. Prilozh. **11**, 28–41 (1977) [Funct. Anal. Appl. **11**, 265–277 (1977)]
17. van Moerbeke, P., Mumford, D.: The spectrum of difference operators and algebraic curves. Acta Math. **143**, 93–154 (1979)
18. Kirillov, A.A.: The characters of unitary representations of Lie groups. Funkts. Anal. Prilozh. **2**, 40–55 (1968) [Funct. Anal. Appl. **2**, 133–146 (1968)]

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