On the multiple holomorphs of a group.

By

G. A. Miller of Urbana, U. S. A.

If any group K is represented as a regular substitution group its holomorph H has been defined as the group composed of all the possible substitutions which transform K into itself and involve no letters except those found in K. The holomorph has also been defined as the abstract group which is simply isomorphic with this substitution group. In the present article the former of these two definitions will be employed. If H is invariant under a larger substitution group on the same letters all the substitutions of this group which are not in H must transform K into another invariant subgroup under H. Moreover, if H involves an invariant subgroup which is similar to K without being identical with K, the substitutions which transform K into this invariant subgroup transform H into itself. In this case K is said to have a multiple holomorph, the degree of multiplicity being the number of the different invariant subgroups of H which are similar to K.

Since all the substitutions*) which are commutative with every substitution of the regular group K constitute a group K' similar to K, it follows that any non-abelian group has a multiple holomorph. Any substitution which transforms K into K' must also transform K' into K since each of these subgroups contains all the substitutions which are commutative with the other. Hence such a substitution transforms H into itself and has its square in H. The group generated by all the substitutions which transform K either into itself or into K' has been called the *double holomorph* of the non-abelian group K. The object of the present paper is to examine the abelian groups with respect to multiple holomorphs. We shall prove that if an abelian group has a multiple holomorph the degree of multiplicity is either 2 or 4. In

*) It is assumed throughout this article that the substitutions under consideration involve no letters besides those found in K.

particular, we shall prove that an abelian group whose order is not divisible by 8 cannot have a multiple holomorph. From this it follows that the assumption made by Burnside*) that an abelian group of odd order be a characteristic subgroup of its holomorph is unnecessary.

From the known fact that the group of isomorphisms of an abelian group is the direct product of the groups of isomorphisms of its Sylow subgroups, it follows that the holomorph of an abelian group is the direct product of the holomorphs of its Sylow subgroups. It is known that the holomorph of a cyclic group of odd order is a complete group and that the holomorph of the cyclic group K of order 2^m contains just one other invariant subgroup which is similar to K, whenever $m > 2$. Hence it follows that the necessary and sufficient condition that a cyclic group has a multiple holomorph is that its order is divisible by 8. If its order is divisible by 8 it has a double holomorph, and the degree of multiplicity of the holomorph of a cyclic group cannot exceed 2.

§ 1.

Abelian groups of odd order.

Let K be an abelian group of odd order and suppose that its holomorph H involves another invariant abelian subgroup K_1 which is of the same type as K. Since the substitutions which are common to K and K_1 form a characteristic subgroup of K they cannot involve any of the operators of highest order in K**). The group J which is composed of all the substitutions of H which omit a given letter is simply isomorphic with the group of isomorphisms of K and it involves an invariant substitution i_0 which transforms each operator of K into its inverse and hence is of order 2. As i_0 transforms K_1 into itself and is not commutative with any substitution of H except those of J it follows that each division of K_1 with respect to the substitutions which it has in common with K must involve a substitution of J since i^0 transforms each of these divisions into itself and the division involves an odd number of substitutions.

The commutator subgroup of the group $\{K, K_1\}$ generated by K and K_1 is composed of invariant operators under $\{K, K_1\}$. Hence any operator of J which is also in $\{K, K_1\}$ is of the same order as some commutator of $\{K, K_1\}$ and therefore its order divides the order of some operator which is common to K and K_1. As the order of the

*) Theory of groups of finite order, 1897, p. 238.

**) American Journal of Mathematics, vol. 27 (1905), p. 15.

product of such an operator into an operator of K which is not of highest order could not be equal to that of an operator of highest order in K, it follows that some operators of K_1 must be the product of an operator of J and an operator of highest order in K. As i_0 would transform such a product into itself multiplied by an operator of highest order in K while K and K_1 cannot have an operator of highest order in common, we have arrived at a contradiction by assuming that H involves another invariant subgroup which, is of the same type as K. Hence K is a characteristic subgroup of H and we have proved the theorem that *the holomorph of any abelian group of odd order is a complete group.* In particular, an abelian group of odd order cannot have a multiple holomorph and as the groups of orders 2 and 4 do not have a multiple holomorph it results that *if an abelian group has a multiple holomorph its order is divisible by* 8, and there is at least one abelian group of order $8k$, k being an arbitrary integer, which has a multiple holomorph.

§ 2.

Abelian groups of order 2^m which involve only one independent generator of highest order.

When K involves only one independent generator t of highest order 2^α all its operators of this order may be obtained by multiplying t by operators of lower order in K and t can be transformed into all its conjugates under H by means of substitutions of J which are commutative with each one of the other independent generators of K*). Suppose that H contains another invariant subgroup K_1 which is similar to K and let i be an operator of J which transforms the operators of K in the same way as an operator of highest order in K_1. The commutator subgroup C of $\{K, K_1\}$ is composed of operators which are common to K and K_1, and these are invariant under $\{K, K_1\}$. It is easy to prove that C cannot involve any operator of order 4 whenever $\alpha > 3$. If an operator of order 2^α in K_1 may be obtained by multiplying i into an operator of highest order in K this product is transformed by i_0 into itself multiplied by the square of an operator of highest order in K. In this case the common operators of K, K_1 include the square of all the operators of K since they constitute a characteristic subgroup and include the square of one operator of highest order in K. As the square of every operator of K_1 would be in K and the commutators are invariant, C must be composed of operators of order 2 in this case.

*) Annals of Mathematics, vol. 6 (1904), p. 1.

If an operator of highest order in K_1 can be obtained by multiplying i into an operator of order $2^{\alpha-1}$ in K this operator is transformed by i into itself multiplied by an operator of order $2^{\alpha-1}$ in C. As C is a characteristic subgroup of K and involves an operator of order $2^{\alpha-1}$, it involves the fourth power of every operator of K. It must therefore also involve the fourth power of every operator of K_1. Hence $2^{\alpha-1} < 2^3$; i. e. $\alpha < 4$. In other words, when $\alpha > 3$ it is not possible to obtain operators of order 2^α by multiplying i into operators of order $2^{\alpha-1}$ in K. As the products of i into the operators of lower order in K cannot be of order 2^α it has been proved that C does not involve any operator of order 4 whenever $\alpha > 3$. In what follows it will be assumed that α satisfies this condition.

It will now be proved that C is of order 2. Suppose that t was so selected that ti is in K_1 and that i transforms some operator of K into itself multiplied by a non-characteristic operator of order 2. There is an operator in J which is commutative with t and also with every operator of the quotient group of $\{K, K_1\}$ with respect to the squares of all the operators of K, while this operator transforms any given noncharacteristic operator of order 2 which is common to K and K_1 into itself multiplied by the characteristic operator of order 2 in K. This operator must therefore be non-commutative with i. As t has as many conjugates under J as ti can have and none of the operators of J are commutative with ti unless they are commutative with both t and i it follows that J cannot contain an operator which is commutative with t but not with i. Hence i cannot transform an operator of K into itself multiplied by a non-characteristic operator and we have proved the theorem: *If an abelian group K of order 2^m contains only one largest invariant exceeding 2^3 and if its holomorph H contains another invariant subgroup K_1 which is similar to K then will K and K_1 generate a group whose commutator subgroup is of order* 2.

Since ti can be transformed into all its conjugates under H by operators of J which transform into themselves all the independent generators of K besides t it follows that all the operators of highest order in K_1 transform the operators of K which are not of highest order in the same manner and hence each of the operators which is not of highest order in K_1 is commutative with every operator of K which is not of highest order. The operators of K_1 whose order is less than 2^α must therefore either be commutative with all the operators of K or transform each of these operators into its $2^{\alpha-1}+1$ power. In the former case there is always one K_1 which has just half of its operators in common with K. The latter case can present itself only when K contains a characteristic subgroup of one-fourth its own order, which includes t^2. This is possible only when the group generated by the remaining independent generators of the group composed of all the operators of K whose orders divide $2^{\alpha-1}$ has a characteristic subgroup of half its order; i. e. this group can have only one independent generator of highest order. When this condition is satisfied H contains exactly three subgroups which can be used for K_1 since the operators of highest order in K_1 may transform the operators of highest order in K in two distinct ways. Hence the theorem: *If an abelian group K contains only one largest invariant equal to 2^α but more than one second largest invariant its holomorph contains only one other invariant subgroup which is similar to K. When K contains only one largest invariant equal to 2^α and only one second largest invariant its holomorph contains three other invariant subgroups which are similar to K.*

This theorem may also be expressed as follows: If an abelian group contains only one largest invariant equal to 2^α, but more than one second largest invariant its holomorph is the holomorph of just one other similar subgroup and hence this holomorph is invariant under a group of twice its order on the same letters but under no larger group. When it contains only one largest and only one second largest invariant its holomorph is the holomorph of three other similar subgroups and it is invariant under a group of four times its own order on the same letters but under no larger group. Under this multiple holomorph the four similar subgroups in question are transformed according to the non-cyclic transitive group of order four. The operators which are common to these four similar subgroups are composed of the squares of all their operators together with all their operators whose orders are less than the second largest invariant. These four subgroups may be divided into two pair, each pair having a subgroup of half the order of the group in common.

§ 3.

Abelian groups of order 2^m in which all the invariants are equal to each other.

We begin with the case when K contains only two independent generators of order 2^α, $\alpha > 3$. From the proof in the preceding section it follows that if the holomorph of K contains a second invariant subgroup K_1 similar to K the common operators of K, K_1 are allthe operators of K whose orders divide $2^{\alpha-1}$. With respect to these common operators K_0 the quotient group of K is evidently the four-group. Hence the substitutions of $\{K, K_1\}$ may be written as follows:

$$K_0 + K_0 t_1 + K_0 t_2 + K_0 t_3 + K i_1 + K i_2 + K i_3$$

where t_1, t_2, t_3 are operators of order 2^α in K and i_1, i_2, i_3 are commutative operators of order 2 in J. Suppose that i_1, i_2, i_3 have been so selected as to satisfy the following conditions:

$$i_1 t_1 i_1 = t_1 t_1^{2^{\alpha-1}}, \quad i_1 t_2 i_1 = t_2 t_3^{2^{\alpha-1}}, \quad i_1 t_3 i_1 = t_3 t_2^{2^{\alpha-1}},$$
$$i_2 t_1 i_2 = t_1 t_3^{2^{\alpha-1}}, \quad i_2 t_2 i_2 = t_2 t_2^{2^{\alpha-1}}, \quad i_2 t_3 i_2 = t_3 t_1^{2^{\alpha-1}},$$
$$i_3 t_1 i_3 = t_1 t_2^{2^{\alpha-1}}, \quad i_3 t_2 i_3 = t_2 t_1^{2^{\alpha-1}}, \quad i_3 t_3 i_3 = t_3 t_3^{2^{\alpha-1}}.$$

The four-group $1, i_1, i_2, i_3$ is invariant and the three operators i_1, i_2, i_3 are conjugate under J. The group

$$K' \equiv K_0 + K_0 t_1 i_1 + K_0 t_2 i_2 + K_0 t_3 i_3$$

is abelian

$$t_1 i_1 \cdot t_2 i_2 = t_2 i_2 \cdot t_1 i_1 = t_1 t_2 t_3^{2^{\alpha-1}} i_3.$$

Hence it is similar to K. It is invariant under H since

$$K_0 t_\beta i_\beta \ (\beta = 1, 2, 3)$$

is composed of all the operators of order 2^α in $K i_\beta$ which are both transformed into their $2^{\alpha-1}+1$ powers by operators of order 2^α in K and also transform these operators into the same powers. Hence it has been proved that the holomorph of K contains at least one other invariant subgroup which is similar to K. That is, the holomorph of K is also the holomorph of K'. We proceed to prove that H does not contain another invariant subgroup which is similar to K.

There are 15 operators of order 2 in J which transform each operator of K_0 into itself and the remaining operators of K either into themselves or into themselves multiplied by an operator of order 2 in K_0. Three of these have been considered and it has been proved that they give rise to at least one invariant subgroup K' which can be used for K_1. That they cannot lead to more than one such subgroup follows from the fact that J contains operators which transform t_2 into itself and transform i_1 into i_3. Hence $t_2 i_1$ has more conjugates under J than t_2 has. Similarly it may be observed that $t_3 i_1$ has more conjugates under J than t_3 has. That is: i_1, i_2, i_3 lead to only one invariant subgroup which is similar to K and not identical with K. If three others would lead to such an invariant subgroup they would form a complete set of conjugates under J. It is readily seen that no such set exists; for 6 of them are separately commutative with half the operators of K and transform half of the remaining operators of K into their $2^{\alpha-1}+1$ powers. As these form a complete set of conjugates they may be directly rejected.

Three of the remaining 6 operators of J are separately commutative with half the operators of K and transform the remaining operators of K into themselves multiplied by the $2^{\alpha-1}$ power of one of these invariant operators of order 2^α. If we call these operators i_1, i_2, i_3, the operators of the group generated by them and K can be arranged in exactly the same way as in the case considered above, and there is again one invariant subgroup which is conformal with K while the other two are conjugate under J. As this invariant subgroup is non-abelian these three conjugate-operators of J do not give rise to a group which may be used for K.

Twelve of the possible 15 operators have now been considered. As one of the remaining ones is invariant under J there cannot be another set of three conjugates under J and we have therefore proved the following theorem: *The holomorph of an abelian group K generated by two independent operators of order 2^α, $\alpha > 3$, contains one and only one other invariant subgroup which is similar to K.* Hence the holomorph of K is invariant under a substitution group of twice its own order but not under any larger group on the same letters.

Having disposed of the case when K contains 2 equal invariants we proceed to the consideration of the general case when K contains $n > 2$ such invariants and begin with the hypothesis that some operator of order 2^α in K_1 is commutative with no operator of K except those of K_0; i. e., the operators whose orders are less than 2^α. The operator of J which transforms the operators of K in the same manner as the given operator of K_1, will be denoted by i. As the group K/K_0 is composed of operators of order 2 besides the identity, the isomorphisms under consideration may be associated with the isomorphisms of the abelian group of order 2^n and of type $(1, 1, 1, \cdots)$, the operator of J which transforms every operator of K into the same power corresponding to the identity. If K_1 exists it must correspond to a system of $2^n - 1$ conjugates which form a complete system under the group of isomorphisms (J_0) of the group of type $(1, 1, 1, \cdots)$ and of order 2^n. It is therefore only necessary to prove that J_0 cannot have such a complete system of conjugates when $n > 2$.

To prove this theorem we may first observe that J_0 cannot have a complete system of $2^n - 1$ operators of odd order. If such a system existed each of its operators would be invariant under a Sylow subgroup of order 2^λ in J_0. Such a Sylow subgroup is composed of one transitive constituent of each of the orders $2, 2^2, 2^3, \cdots, 2^{n-1}$. As a substitution s cannot be commutative with every substitution of a transitive group on the same letters whose order is a power of a prime unless the order of s is a power of the same prime, and as the transitive constituents of the

Sylow group under consideration cannot be permuted, it has been proved that every operator of odd order in J_0 has an even number of conjugates under J_0. To complete the proof of the theorem under consideration it is only necessary to prove that every operator of order 2 has more than $2^n - 1$ conjugates under J_0. This follows almost directly from the fact that such an operator must be commutative with more than 2 operators of the group of type $(1, 1, 1, \cdots)$ since J_0 is positive, being simple. The operator must therefore have more than $2^n - 1$ conjugates under J_0 since the number of subgroups of order 2^α, $1 < \alpha < n - 1$, in a group of the given type is greater than $2^n - 1$. If this operator of order 2 is commutative with 2^{n-1} operators of the given group it evidently has more than $2^n - 1$ conjugates.

It remains to consider the cases when i is commutative with some operator of order 2^α in K. When i is commutative with two or more independent generators these could be selected in at least $2^n - 1$ ways and as this selection would not determine i it would have more than $2^n - 1$ conjugates under J. Similarly i would have more than $2^n - 1$ conjugates if it were commutative with only one of the independent generators of K. Hence we have established the theorem: *If an abelian group K of order 2^m is generated by $n > 2$ independent operators of order 2^α, $\alpha > 3$, its holomorph cannot involve another invariant subgroup which is similar to K.*

§ 4.

Conclusion.

When K contains two equal largest invariants which are equal to 2^α, $\alpha > 3$, together with one or more smaller invariants it may be proved just as in § 2 that the commutator subgroup of $\{K, K_1\}$ is the characteristic subgroup of order 4 in K. If K_0 represents all the operators which are not of highest order in K, the considerations employed at the beginning of the preceding section prove that H contains at least one subgroup which may be used for K_1. To prove that it cannot contain more than one such subgroup it is only necessary to observe that the operators of J which would transform the operators of K in the same manner as those of highest order in such a subgroup, could not form a complete set of conjugates under J. This statement follows almost directly from the fact that an operator of order 2^α in K can be transformed into one-third of the operators of order 2^α by means of operators of J which are commutative with each of the independent generators of K except those of order 2^α and are also commutative with each operator of the characteristic subgroup of order 4 in K.

In what precedes it was assumed that the largest invariant in K exceeds 8. It is not difficult to prove that all of the results are true even when the largest invariant is 8. In fact, the reason for assuming this largest invariant greater than 8 was to make it easier to prove that each of the commutators of $\{K, K_1\}$ must be of order 2 if it is not the identity. We shall now prove that the order of such a commutator could not exceed 2 even when the largest invariant in K is 8. In this case the J of K would involve only two invariant operators whose degree is equal to the order of K diminished by 2. One of these transforms each operator of K into its inverse while the other transforms each of these operators into its third power. As an operator which transforms K into K_1 may be so selected as to transform J into itself, and, as J is also the group of isomorphisms of K_1, it follows that the two given invariant operators of J are either commutative with this operator or are transformed among themselves by it. As each of these invariant operators of J transforms operators of order 8 in K into themselves multiplied by operators of order 4 while it transforms operators of order 4 in K into themselves multiplied by operators of order 2, it must have the same effect on the operators of K_1. Hence it results that an operator of order 8 in K_1 is obtained by multiplying an operator of J into an operator of order 8 in K. That is, the square of every operator of K and of K_1 is among the common operators of K and K_1 and hence all the commutators of $\{K, K_1\}$ are of order 2.

It remains yet to consider the case when K contains operators of order 4 but of no higher order. In this case J contains only one invariant operator i_0 and hence i_0 must also transform each operator of K_1 into its inverse. As i_0 has exactly as many conjugates under H as there are independent generators of order 4 in K, it follows that the operators of K and K_1 have the same squares. Since the common operators of $\{K, K_1\}$ are of order 2, besides the identity, the operators of J contained in $\{K, K_1\}$ are all of order 2. Let i be such an operator and suppose that ti is in K_1, t being an operator of order 4 in K. As i_0 is not commutative with ti it must transform it into its inverse. That is, ti is of order 4 and $t^{-1}i = (ti)^{-1} = it^{-1}$. Hence t and i are commutative. Moreover, all the operators of order 4 in K_1 are obtained by multiplying some i into an operator of order 4 in K.

When K has independent generators of order 2 it contains two characteristic subgroups besides the identity but it contains only one such subgroup when it does not involve any independent generator of order 2*).

*) American Journal of Mathematics, vol. 27 (1905), p. 15.

As the common operators of K and K_1 form a characteristic subgroup of K and as i is commutative with an operator of order 4 it follows that the common operators of K, K_1 cannot involve all the operators of order 2 in K when K contains only one highest invariant. Moreover, it follows from the preceding section that K could not contain two or more invariants equal to 4 and that the operators of order 2 in K_1 are commutative with the operators of this order in K. These results lead to the theorem: *If an abelian group K of order 2^m involves no operators whose orders exceed 4 its holomorph cannot involve any other invariant subgroup which is similar to K except when K is of type* (2, 1). In this special case H contains only one subgroup which can be used for K_1. There is another invariant subgroup in H, which is of the same type as K but it does not occur in the form of a regular group and hence is not similar to K as a substitution group.

If we bear in mind that the holomorph of any abelian group is the direct product of the holomorphs of its Sylow subgroups and that the holomorph of an abelian group of odd order is a complete group we may state the preceding results as follows: If the holomorph of any abelian group K contains four invariant subgroups which are similar to K then the Sylow subgroup of order 2^m in K contains just one invariant equal to 2^α, $\alpha > 2$, and just one second largest invariant. The holomorph of K cannot contain more than four invariant subgroups which are similar to K nor can it contain exactly three such subgroups. The holomorph of K contains exactly one other invariant subgroup similar to K when one of the following conditions is satisfied: 1) The Sylow subgroup of order 2^m in K is of type (2, 1); 2) This Sylow subgroup is cyclic and $m > 2$; 3) The Sylow subgroup of order 2^m contains only one largest invariant exceeding 4 but more than one second largest invariant; 4) This Sylow subgroup contains exactly two largest invariants which exceed 4. In all other cases the holomorph of K contains no subgroup which is invariant and similar to K except K itself.