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The Fibre of the Prym Map in Genus Three*

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If $\pi:\tilde{C}\to C$ is an ètale double cover of a smooth curve of genus g and $\pi_*:J\tilde{C}\to JC$ is the induced morphism of jacobians one defines the Prym variety as the connected component of the identity in $\mathrm{Ker}(\pi_*)$:

$$P(\tilde{C}, C)=(\mathrm{Ker}\,\pi_*)^0;$$

$\dim(P(\tilde{C}, C))=g-1$ and $P(\tilde{C}, C)$ has a natural principal polarization $\Xi=\frac{1}{2}(P\cdot\theta)$, θ being the theta divisor on $J\tilde{C}$, [12, p. 342]. We denote by

$$\mathscr{R}_g, \mathscr{A}_g$$

respectively the coarse moduli spaces for the pairs $(\tilde{C}, C)$ and for principally polarized abelian varieties of dimension g; then the morphism

$$P:\mathscr{R}_g\to\mathscr{A}_{g-1}$$

which sends the class of $(\tilde{C}, C)$ in the class of $P(\tilde{C}, C)$ is by definition the Prym map. P extends to a morphism $\bar{P}$: $\bar{\mathscr{R}}_g\to\mathscr{A}_{g-1}$; where $\bar{\mathscr{R}}_g$ contains $\mathscr{R}_g$ as a dense open subset and is the moduli space for allowable double covers of stable genus g curves, [2], [8, Definition 2.1].

$\bar{P}$ is known to be generically injective for $g>6$, [9], and dominant for $g=6$, [8]. For $g=5$ it is announced in [7] that the fibre of $\bar{P}$ is a double cover of the Fano surface of the lines of a cubic threefold; while, for $g=4$, it is known that the fibre of $\bar{P}$ is a 3-dimensional Kummer variety, [16].

In this paper we study the (extended) Prym map:

$$\bar{P}:\bar{\mathscr{R}}_3\to\mathscr{A}_2.$$

It turns out that, in this case too, the fibre of $\bar{P}$ is rich of geometry; our main result is the following:

Let $\tilde{S}$ be an abelian surface, s its moduli point, $\theta\subset\tilde{S}$ a symmetric theta divisor; assume s general [i.e. $\mathrm{Aut}(\theta)=\mathbb{Z}/2\mathbb{Z}$], then

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(1) $\bar{P}^{-1}(s)$ is obtained from the Siegel modular quartic threefold V by a sequence of two blowing up's σ_1 and σ_2. $\sigma_1 : \hat{V} \to V$ blows up V in a point v, while $\sigma_2 : \bar{P}^{-1}(s) \to \hat{V}$ is centered along a curve B isomorphic to $\theta/\mathrm{Aut}(\theta)$. The exceptional divisor of $\sigma_1 \cdot \sigma_2$ is the union of two surfaces $\mathscr{I}_s$ and $\mathscr{E}_s$. The points of $\mathscr{I}_s = (\sigma_1 \cdot \sigma_2)^{-1}(v)$ correspond to double covers $\pi : \tilde{C} \to C$ such that $\tilde{C}$ is hyperelliptic. $\mathscr{E}_s = \sigma_2^{-1}(B)$ is entirely contained in $\bar{\mathscr{R}}_3 - \mathscr{R}_3$, its points correspond to allowable double covers $\pi : \tilde{C} \to C$ such that both $\tilde{C}$ and C are elliptic tails, [7, (3) 2.3 and Sect. 3 (3.16)].

The Siegel modular quartic threefold $V \subset \mathbb{P}^4$ is considered by Van der Geer in [19]. In this paper the author shows that, via theta functions, V is biregular to $\bar{\mathscr{A}}_2^2$; where $\mathscr{A}_2^2$ is the moduli space for principally polarized abelian surfaces endowed with a level two structure and $\bar{\mathscr{A}}_2^2$ is a minimal compactification of it. Thus we can rephrase our result in this way:

(2) $\bar{P}^{-1}(s)$ is obtained from $\bar{\mathscr{A}}_2^2$ by a sequence of two blowing up's σ_1 and σ_2 as above.

Since σ_1 has its center in a point $v \in \mathscr{A}_2^2$, we are also able to deduce from [19] that $f(v) = s$; $f : \mathscr{A}_2^2 \to \mathscr{A}_2$ being the forgetful map.

The proof of our theorem relies on the results in [13, 15] and the classical geometry of Kummer surfaces, [4, 11]: If $\tilde{C}$ is a stable curve in the linear system $|2\theta|$ then the quotient morphism $\pi : \tilde{C} \to \tilde{C}/\langle -1 \rangle$ is allowable and the associated Prym variety is $\tilde{S}$. Since $|2\theta| = \mathbb{P}^3$ this defines a rational map $\phi : \mathbb{P}^3 \to \bar{P}^{-1}(s)$ sending $\tilde{C}$ in class of $(\tilde{C}, \tilde{C}/\langle -1 \rangle)$ in $\bar{\mathscr{R}}_3$. We describe the curve $T = \{\tilde{C} \in |2\theta| / \tilde{C}$ is not stable$\}$ which is the fundamental locus of ϕ (Sect. 2) and show that the complementary set of $\phi(\mathbb{P}^3 - T)$ in $\bar{P}^{-1}(s)$ contains exactly the classes of pairs $(\tilde{C}, C)$ such that $\tilde{C}$ is hyperelliptic or both $\tilde{C}$ and C are elliptic tails.

On the other hand the group G of the translation on $\tilde{S}$ by elements of order 2 acts linearly on $|2\theta|$; We use the geometry of Kummer surfaces to show that ϕ is the restriction to $\mathbb{P}^3 - T$ of the quotient morphism $p : \mathbb{P}^3 \to \mathbb{P}^3/G$. By [4, Vol. III, p. 210], $\mathbb{P}^3/G$ is biregular to the Siegel modular quartic threefold V, hence $\bar{P}^{-1}(s)$ is birational to V. To complete the description of $\bar{P}^{-1}(s)$ we show in Sect. 3 how to extend ϕ to a finite surjective morphism $\tilde{\phi}$ on a suitable birational model $\hat{\mathbb{P}}^3$ of $\mathbb{P}^3$ such that $\bar{P}^{-1}(s) = \tilde{\phi}(\hat{\mathbb{P}}^3) = \hat{\mathbb{P}}^3/G = (\sigma_1 \cdot \sigma_2)^{-1}(V)$.

We work over the complex field. Nevertheless our result, as it is stated in (1), seems to hold in any characteristic $\neq 2$.

Further Comments and Notations

Two allowable double covers $\pi_i : \tilde{C}_i \to C_i$, $(i = 1, 2)$, are said to be isomorphic iff there exists isomorphisms $\tilde{\beta} : \tilde{C}_1 \to \tilde{C}_2$ and $\beta : C_1 \to C_2$ such that $\beta \cdot \pi_1 = \pi_2 \cdot \tilde{\beta}$.

If $\pi : \tilde{C} \to C$ is an allowable double cover $[\tilde{C}, C]$ stands for its isomorphism class in $\bar{\mathscr{R}}_g$. Conversely, $\forall [\tilde{C}, C] \in \bar{\mathscr{R}}_g$, the corresponding double cover will be denoted by π.

We recall that to give $\pi : \tilde{C} \to C$ is equivalent to give a pair (C, η) where η is a non zero order two element in $\mathrm{Pic}(C)$, [2, Lemma 3.2]; if C is smooth these elements are $2^{2g} - 1$.

If S is a principally polarized abelian variety, $\dim(S) = g$, then $[S]$ stands for its class in $\mathscr{A}_g$. $-1 : S \to S$ is the involution sending p in $-p$; if $Z \subset S$ then $-Z$ is its image by -1. Z is said to be symmetric if $Z=-Z$. We denote by θ a symmetric theta divisor on S and by S_n the set of the elements of order n on S.

If C is a curve JC stands for its jacobian, ω_C for the dualizing sheaf.

If V is a vector or projective space $V\hat{}$ denotes its dual.

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Let $\tilde{S}$ be a principally polarized abelian surface, s its class in $\mathscr{A}_2$, $\theta \subset \tilde{S}$ a symmetric theta divisor; the 3-dimensional linear system $|2\theta|$ will be denoted by

$$\mathbb{P}^3.$$

In this section we construct a dominant rational map

$$\phi: \mathbb{P}^3 \to \bar{P}^{-1}(s)$$

and study its fundamental locus; ($\bar{P}: \bar{\mathscr{R}}_3 \to \mathscr{A}_2$ is the extended Prym map: see Definition 2.2). With this purpose, let us review some known facts about the geometry of Kummer and abelian surfaces:

Proposition 2.1. (a) *Every element $\tilde{C}$ of $\mathbb{P}^3$ is a symmetric curve of arithmetic genus 5. $\mathbb{P}^3$ is a base point free linear system and its general element is a smooth, irreducible curve.*

Let $g: \tilde{S} \to \mathbb{P}^{3\hat{}}$ be the morphism associated to $|\tilde{C}|$ and $R = g(\tilde{S})$, then:

(b) *Either* $\deg(g)=2$ *and R is the Kummer quartic surface of $\tilde{S}$ or* $\deg(g)=4$ *and R is a smooth quadric.*

(c) $\operatorname{Deg}(g)=2 \Leftrightarrow \tilde{S}=J\theta$, *where θ is a smooth, irreducible genus* 2 *curve.* $\operatorname{Deg}(g)=4 \Leftrightarrow \tilde{S}=E\times F$, *with E, F elliptic curves, $\theta=\{e\}\times F+E\times\{f\}$, ($e, f$ being the identities on E, F).*

Proof. Cf. [3, p. 129, pp. 139–142]; [10, Chap. 6]; [11].

We want to describe the elements $\tilde{C}$ of $\mathbb{P}^3$ according to their singularities and their stability, therefore let us consider the following subvarieties of $\mathbb{P}^3$, [which is canonically identified to its bidual $(\mathbb{P}^{3\hat{}})\hat{}$]:

(2.1)
$$R\hat{} = \text{dual surface of } R,$$
$$Z_\tau = \{\tilde{C} \in \mathbb{P}^3 / \tau \in \tilde{C}\}, \quad \text{with} \quad \tau \in \tilde{S},$$
$$Z = \cup Z_\tau, \quad \tau \in \tilde{S}_2,$$
$$T = Z \cap R\hat{}.$$

$R\hat{}$ is birational to R; if $\deg(R)=2$ then $R\hat{}$ is a smooth quadric. If $\deg(R)=4$ it is well known that $R\hat{}$ is a Kummer quartic surface. Z_τ is just a plane in $\mathbb{P}^3$.

The following fact is a standard consequence of the duality between $\tilde{S}$ and $\tilde{S}\hat{} = \operatorname{Pic}^0(\tilde{S})$ (cf. [14, II.8]):

Let $\delta: \tilde{S} \to \tilde{S}\hat{}$ be the duality isomorphism sending $x \in \tilde{S}$ in $\tilde{\delta}(x) = x\hat{} =$ class of $\theta_x - \theta_{-x}$ in $\operatorname{Pic}^0(\tilde{S})$. The principal polarization of $\tilde{S}\hat{}$ is $\theta\hat{} = \tilde{\delta}(\theta) = \{x\hat{}/x \in \theta\}$; moreover, by the square theorem on an abelian variety, $\theta_x + \theta_{-x}$ is linearly equivalent to 2θ. Let $g\hat{}: \tilde{S}\hat{} \to \mathbb{P}^3$ be the morphism sending $x\hat{}$ in $\theta_x + \theta_{-x}$, $g\hat{}$ is associated to $|2\theta^{\frown}|$ and the following diagram commutes:

$$(2.2)\qquad \begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\delta}} & \tilde{S}^{\wedge} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g^{\wedge}} \\ R & \xrightarrow{\delta} & R^{\wedge} \end{array}$$

where δ is the birational map sending R in its dual surface.

Proposition 2.2. (a) *$\tilde{C} \in Z \cup R^{\wedge} \Leftrightarrow \tilde{C}$ is a singular curve.*

(b) *$\tilde{C} \in R^{\wedge} \Leftrightarrow \tilde{C} = \theta_a + \theta_{-a}$ for some $a \in \tilde{S}$.*

(c) *$\tilde{C} \in Z - T \Leftrightarrow \tilde{C}$ is reduced, irreducible with $m \leqq 3$ ordinary nodes and* $\mathrm{Sing}(\tilde{C}) = \tilde{S}_2 \cap \tilde{C}$.

Proof. (a) Let $S = \tilde{S}/\langle -1 \rangle$ be the Kummer surface of $\tilde{S}$, $\varrho: \tilde{S} \to S$ the quotient morphism; the ramification set of ϱ is $\tilde{S}_2$ and $\varrho(\tilde{S}_2) = \mathrm{Sing}(S)$. Moreover $g = \sigma\varrho$ and either $R = S$, $\sigma = \mathrm{id}_S$ or $\deg(R) = 2$, $\sigma: S \to R$ is a finite morphism of degree 2. Therefore $\tilde{C}$ singular $\Leftrightarrow \frac{1}{2}\varrho_* \tilde{C}$ singular $\Leftrightarrow$ either $(1/\deg(g))\, g_* \tilde{C}$ is a singular plane section of R or $g(\tau) \in g(\tilde{C})$ for some $\tau \in \tilde{S}_2 \Leftrightarrow \tilde{C} \in Z \cup R^{\wedge}$.

(b) Let $g^{\wedge}: \tilde{S}^{\wedge} \to \mathbb{P}^3$ be as above, by (2.2) $g(\tilde{S}^{\wedge}) = R^{\wedge}$. This implies (b).

(c) The "if" part follows from the equivalences in (a) and (b). Conversely, assume $\tilde{C} \in Z - T$, then, by definition of $R^{\wedge}$, $\tilde{C} \notin R^{\wedge} \Rightarrow \mathrm{Sing}(\tilde{C}) \subset \tilde{S}_2$. Let us show that $\tilde{S}_2 \cap \tilde{C} \subset \mathrm{Sing}(\tilde{C})$ and that every singular point of $\tilde{C}$ is a node. Let $\tau \in \tilde{S}_2 \cap \tilde{C}$, up to translating by an order 2 element we can assume $\tau = o =$ identity on $\tilde{S}$ and $o \in \theta$. Notice that $\tilde{S}_2 \cap \theta$ is the set of the six Weierstrass points of θ and that every translate θ_j of θ, $(j = 1, \ldots, 6)$, by such a point is a symmetric theta divisor containing o. Moreover $2\theta_j \in |2\theta|$ and any three elements of the set $\{2\theta_j\}, j = 1, \ldots, 6$ are linearly independent in $|2\theta|$. Let us fix $\theta_1, \theta_2, \theta_3$ together with local coordinates u, v at o; let $p_i = p_i(u, v) = 0$, $(i = 1, 2, 3)$, be the local equation of θ_i at o, then every element of Z_0 has a local equation: $x_1 p_1^2 + x_2 p_2^2 + x_3 p_3^3 = 0$; where $x = (x_1 : x_2 : x_3) \in \mathbb{P}^2$. If $\tilde{S}$ is not a product of two elliptic curves θ is smooth and it is immediate to check that θ_i and θ_j are transversal at o, $(i \neq j;\ i, j = 1, 2, 3)$. Therefore we can assume $p_1 = u +$ higher degree terms, $p_2 = v + \ldots$, $p_3 = au + bv + \ldots$, $(a, b \neq 0)$. If $\tilde{S} = E \times F$ then $o = (e, f)$, $(e, f$ being the identities on the elliptic curves $E, F)$ and we can choose θ_1, θ_2 smooth and transversal at o; $\theta_3 = \theta = \{e\} \times F + E \times \{f\}$. The local equations are as above but for the third which has $a = b = 0$. In both cases $\tilde{C}$ is given by setting $0 = x_1 u^2 + x_2 v^2 + x_3(au + bv)^2 + \ldots$. Notice that $x_1 = x_2 = 0 \Rightarrow \tilde{C} = 2\theta \Rightarrow \tilde{C} \in T$. Hence $\tilde{C}$ has a double point at o and $\mathrm{Sing}(\tilde{C}) = \tilde{S}_2 \cap \tilde{C}$.

To check that o is a node for $\tilde{C}$ we describe explicitely the curve $B \subset Z_0$ of the elements for which this is not true. The equation of B is clearly the discriminant of the quadratic form in $u, v: x_1 u^2 + x_2 v^2 + x_3(au + bv)^2$ and B is a conic. Notice that the differential of the -1 involution on $\tilde{S}$ acts as the identity on the tangent space $T_{\tilde{S},o} = H^0(\theta, \omega_\theta)$; hence $o \in \theta_x \cap \theta_{-x} \Rightarrow \theta_x, \theta_{-x}$ are not transversal at o. Moreover $o \in \theta_x \cap \theta_{-x} \Leftrightarrow x \in \theta$, so that $\{\theta_x + \theta_{-x},\ x \in \theta\} \subset B$. It is easy to check that the previous inclusion is actually a bijection; hence $B \subset T$ and o is a node for $\tilde{C}$. In particular:

$$(2.3)\qquad B = \{\theta_x + \theta_{-x}, x \in \theta\} = \theta/\langle -1 \rangle .$$

Let $\tilde{C} \in Z - T$, to complete the proof observe that: (1) $\tilde{C} \in Z - T \Rightarrow \tilde{C}$ has finitely many singular points $\Rightarrow \tilde{C}$ is reduced.

(2) Let $D \subset \tilde{C}$ be an irreducible component of $\tilde{C}$, $\tau \in D \cap \tilde{S}_2$; as for the case $\tau = o$ the differential of the -1 involution of $\tilde{S}$ acts as the identity on $T_{\tilde{S},\tau}$; hence $-D$ is not transversal to D at τ. Since $-D \subset \tilde{C}$ and τ is a node for $\tilde{C}$ the only possibility is $D = -D$. Moreover D is the unique irreducible component of $\tilde{C}$ through τ. Since $\tilde{C}$ is connected and $\mathrm{Sing}(\tilde{C}) = \tilde{C} \cap \tilde{S}_2$ it follows $D = \tilde{C}$ and $\tilde{C}$ is irreducible.

(3) Assume $\tilde{C}$ has $m \geqq 4$ nodes, then $C = \tilde{C}/\langle -1 \rangle$ is a curve of arithmetic genus 3 with at least 4 singular points. Hence C splits and also $\tilde{C}$ splits: contradiction.

We recall that a curve C is said to be *stable* if it is connected with only nodes as its singularities and if every smooth, irreducible, rational component contains three nodes of C.

Definition 2.1. Let $\tilde{S}$, θ, $g : \tilde{S} \to R$ be as above, then, for every $\tau \in \tilde{S}_2$, the curve $g(\theta_\tau)$ is said to be a trope of R, [11].

Every trope is a conic and it is biregular to $\theta/\langle -1 \rangle$, [1], [10, Chap. 6].

Corollary 2.1. (a) *Let $\tilde{C} \in \mathbb{P}^3$, then $\tilde{C}$ not stable $\Leftrightarrow \tilde{C} \in T \Leftrightarrow \tilde{C} = \theta_{x+\tau} + \theta_{-x+\tau}$, for some $x \in \theta$, $\tau \in \tilde{S}_2$.*

(b) *T is the union of the tropes of $R\check{}$. $\mathrm{Sing}(T) = g\check{}(\tilde{S}_2)$. In particular: $\tilde{C} \in \mathrm{Sing}(T) \Leftrightarrow \tilde{C} = 2\theta_\tau$ for some $\tau \in \tilde{S}_2$.*

Proof. (a) follows from Proposition 2.2 and the proof of its statement (c). (b): from (2.3) and the commutative diagram (2.2) it follows $T =$ union of the tropes of $R\check{}$, $\mathrm{Sing}(T) = g\check{}(R\check{})$.

We quote from the theory of Prym varieties some basic results and definitions:

Definition 2.2. Let $\pi : \tilde{C} \to C$ be a double cover of stable curves of arithmetic genera $2g-1$ and g. Let $i : \tilde{C} \to \tilde{C}$ be the involution induced by π, $i_* : J\tilde{C} \to I\tilde{C}$, $\pi_* : J\tilde{C} \to JC$ the induced homomorphisms of (generalized) jacobians. Consider $P(\tilde{C}, C) = \mathrm{Im}(\mathrm{id} - i_*) = \mathrm{Ker}(\pi_*)^0 : \pi$ is said to be allowable if the following equivalent conditions hold:

(1) $P(\tilde{C}, C)$ is a $g-1$ dimensional abelian variety;

(2) the only fixed points of i are nodes where the two branches are not exchanged. The number of nodes exchanged by i equals the number of irreducible components exchanged by i.

The equivalence of (1) and (2) is shown in [2, Lemma 5.1]. $P(\tilde{C}, C)$ is principally polarized by $\frac{1}{2}(\theta \cdot P(\tilde{C}, C))$, ($\theta =$ theta divisor of $J\tilde{C}$).

Let $\bar{\mathscr{R}}_g$ be the coarse moduli space for allowable double covers of genus g curves: $\bar{\mathscr{R}}_g$ exists, is irreducible and contains $\mathscr{R}_g$ as an open subset; moreover the map

$$\bar{P} : \bar{\mathscr{R}}_g \to \mathscr{A}_{g-1}$$

sending $[\tilde{C}, C] \in \bar{\mathscr{R}}_g$ in $[P(\tilde{C}, C)]$ is a proper morphism, [2, 8]. $\bar{P}$ is called the extended Prym map; of course $\bar{P}/\mathscr{R}_3 = P =$ Prym map for etale double covers.

Definition 2.3. Let $\pi : \tilde{C} \to C$ be an allowable double cover, $X = P(\tilde{C}, C)$, $j : \tilde{C} \to J\tilde{C}$ the Abel-Jacobi map. The composition morphism

$$(\mathrm{id} - i_*) \cdot j = \alpha : \tilde{C} \to X$$

is called the Abel-Prym map.

Proposition 2.3. *Let $\pi:\tilde{C}\to C$ be an allowable double cover, $\alpha:\tilde{C}\to X=P(\tilde{C},C)$ the Abel-Prym map, Ξ the theta divisor of X. Then $\alpha_*\tilde{C}$ is a symmetric curve representing the homology class $(2/(d-2)!)\Xi^{d-2}$, $(\dim(X)=d)$. Moreover, if $\alpha_*(\tilde{C})$ is stable, then π is the quotient map $\alpha_*(\tilde{C})\to\alpha_*(\tilde{C})/\langle -1\rangle$.*

Proof. Cf. [15, Lemma 3.2].

The converse of this proposition can be stated in the following way:

Proposition 2.4. *Let X be a d-dimensional abelian variety, Ξ its principal polarization, $\tilde{C}\subset X$ a symmetric curve representing the homology class $(2/(d-2)!\,\Xi^{d-2}$. If the quotient morphism $\pi:\tilde{C}\to\tilde{C}/\langle -1\rangle$ is allowable then $X=P(\tilde{C},C)$ as a principally polarized abelian variety.*

Moreover, up to translating by an order 2 element, the inclusion $\tilde{C}\subset X$ is the Abel-Prym map.

Proof. Cf. [15, 5.3], [8, 4.4], [2].

It is easy to apply these results to the situation we considered at the beginning of this section: let $\tilde{S}$, θ be as above, $\mathbb{P}^3=|2\theta|$. Consider the quotient morphism $\varrho:\tilde{S}\to\tilde{S}/\langle -1\rangle$, then, for every $\tilde{C}\in\mathbb{P}^3$, if the double cover $\varrho/\tilde{C}:\tilde{C}\to C=\tilde{C}/\langle -1\rangle$ is allowable it follows from Proposition 2.4 that $P(\tilde{C},C)=\tilde{S}$. Hence $[\tilde{C},C]\in\bar{P}^{-1}(s)$.

Proposition 2.5. *Let $\tilde{C}\in\mathbb{P}^3$, then $\varrho/\tilde{C}$ is allowable if and only if $\tilde{C}$ is stable.*

Proof. It follows from the description of stable elements of $\mathbb{P}^3$ given in Proposition 2.2, Corollary 2.1: $\tilde{C}$ stable $\Leftrightarrow\tilde{C}\in\mathbb{P}^3-T$; assume $\tilde{C}$ stable, let $i:\tilde{C}\to\tilde{C}$ be the involution induced by $\varrho/\tilde{C}$. o is fixed by i iff o is a node and $o\in\tilde{S}_2\cap\tilde{C}$, since the differential of -1 is the identity on $T_{\tilde{S},o}$ the two branches of $\tilde{C}$ at o are not interchanged. If $\tilde{C}$ is reducible then $\tilde{C}=\theta_x+\theta_{-x}$ and it is immediate to check that the number of nodes exchanged by i equals the number of irreducible components exchanged by i.

By Propositions 2.4, 2.5 there exists a map

$$\phi:\mathbb{P}^3-T\to\bar{P}^{-1}(s)$$

sending $\tilde{C}\in\mathbb{P}^3$ in $[\tilde{C},\tilde{C}/\langle -1\rangle]=$ class of $\varrho/\tilde{C}$.

Let $\tilde{\mathscr{C}}=\{(\tilde{C},x)\in(\mathbb{P}^3-T)\times\tilde{S}/x\in\tilde{C}\}$ be the universal curve over $\mathbb{P}^3-T$, $\mathscr{C}=\{(\tilde{C},y)\in(\mathbb{P}^3-T)\times\tilde{S}/\langle -1\rangle/y\in\tilde{C}/\langle -1\rangle\}$ the universal quotient curve, $R:\tilde{\mathscr{C}}\to\mathscr{C}$ the obvious quotient morphism; then there exists a commutative diagram

$$\text{(2.4)}\qquad\begin{array}{ccc}\tilde{\mathscr{C}} & \xrightarrow{R} & \mathscr{C}\\ {\scriptstyle\tilde{p}}\downarrow & & \downarrow{\scriptstyle p}\\ \mathbb{P}^3-T & \xrightarrow{\text{id}} & \mathbb{P}^3-T\end{array}$$

($\tilde{p},p$ being the natural projections). If $z\in\mathbb{P}^3-T$ and $\tilde{p}^{-1}(z)=\tilde{C}$ then $p^{-1}(z)=\tilde{C}/\langle -1\rangle$ and $R/\tilde{p}^{-1}(z)=\varrho/\tilde{C}$. Hence, by the universal property of $\bar{\mathscr{R}}_3$, ϕ is a morphism.

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Let $\phi:\mathbb{P}^3-T\to\bar{P}^{-1}(s)$ be as above: ϕ is a rational map from $\mathbb{P}^3$ in $\bar{P}^{-1}(s)$ having T as its fundamental locus. In this section we show that ϕ extends to a morphism

$$\tilde{\phi}=\phi\cdot\sigma:\hat{\mathbb{P}}^3\to\bar{P}^{-1}(s)$$

where $\sigma=\sigma_2\cdot\sigma_1:\hat{\mathbb{P}}^3\to\mathbb{P}^3$, $\sigma_2=$ blowing up of $\mathbb{P}^3$ at Sing(T) and σ_1 blows up the strict transform of the conics which are components of T.

For sake of simplicity, unless explicitely mentioned, we assume $\tilde{S}$ is not a product (i.e. θ smooth, irreducible). If $\tilde{S}$ is a product the results are analogous (cf. Remark 3.1).

If $[\tilde{C}, C]\in\bar{\mathcal{R}}_3$ we will denote by $\pi:\tilde{C}\to C$ the corresponding allowable double cover, by α the Abel-Prym map and by i the involution induced by π on $\tilde{C}$.

A stable curve C, $(p_a(C)>1)$, will be said *hyperelliptic*, [2, Lemma 4.7] iff there exist $\mathcal{L}\in\mathrm{Pic}(C)$ such that $h^0(\mathcal{L})=\deg(\mathcal{L})=2$ on every irreducible component of C. If C is hyperelliptic then $\mathcal{L}$ is unique.

By Proposition 2.3, for every $[\tilde{C}, C]\in\bar{P}^{-1}(s)$, $\alpha_*\tilde{C}\in\mathbb{P}^3$; therefore we need to know for which $[\tilde{C}, C]\,\alpha_*\tilde{C}\in T$.

Let $\sigma_1:\hat{\mathbb{P}}^3\to\mathbb{P}^3$ be the blowing up of $\mathbb{P}^3$ at Sing(T). Assume $t\in$ Sing(T) is the point corresponding to the divisor 2θ (or $2\theta_\tau$, $\tau\in\tilde{S}_2$), *let*

$$E=\sigma_1^{-1}(t)$$

be the exceptional plane over t. Consider the exact sequence

$$0\to H^0(\mathcal{O}_{\tilde{S}}(\theta))\to H^0(\mathcal{O}_{\tilde{S}}(2\theta))\xrightarrow{r} H^0(\mathcal{O}_\theta(2\theta))\to 0:$$

the restriction homomorphism r defines a natural $1:1$ correspondence between E and $\mathbb{P}H^0(\mathcal{O}_\theta(2\theta))$. Indeed:

(3.1) $$E=\{\text{pencils } P\in|2\theta| \text{ containing } 2\theta \text{ as an element}\},$$

thus, to every $P=\lambda s_0+\mu s_1$, $((\lambda:\mu)\in\mathbb{P}^1;\ s_0, s_1\in H^0(\mathcal{O}_S(2\theta)),\ \mathrm{div}(s_0)=2\theta)$, it corresponds the one dimensional vector space $V_P\subset H^0(\mathcal{O}_\theta(2\theta))$ generated by $r(s_1)=s_1/\theta$. For every $P=\lambda s_0+\mu s_1\in E$ we denote by

(3.2) $$b_P$$

the zero scheme of s_1/θ and by

(3.3) $$\alpha_P:\tilde{C}_P\to\theta$$

the double covering of θ branched on b_P. Notice that $\mathcal{O}_\theta(2\theta)=\omega_\theta^2=$ bicanonical sheaf of θ. Moreover $\tilde{C}_P$ has arithmetic genus 5 and it is not difficult to see that $\tilde{C}_P$ is stable but for the following exception:

(3.4) $$b_P=4\tau, \text{ where } \tau \text{ is a Weierstrass point of } \theta.$$

For every trope (Definition 2.1) $B\subset R\hat{}$ let $\hat{B}$ its strict transform by σ_1 and

(3.5) $$\hat{T}\subset\hat{\mathbb{P}}^3$$

the union of the curves $\hat{B}$, then:

Proposition 3.1. *Assume $P\in E=\sigma_1^{-1}(t)\subset\sigma_1^{-1}(\mathrm{Sing}(T))=$ exceptional divisor of σ_1; then $\tilde{C}_P$ is stable if and only if $P\notin\hat{T}$.*

Proof. As it is well known θ contains exactly six points of order 2 so that there are six tropes $B_\tau=Z_\tau\cap R\hat{}$, $\tau\in\tilde{S}_2\cap\theta$, through the point t. As in the proof of

Proposition 2.2 we can choose local parameters u, v at τ and three symmetric theta divisors $\theta_1, \theta_2, \theta_3 = \theta$ through τ whose equations are given respectively by the polynomials $p_1(u,v) = u + \ldots, p_2(u,v) = v + \ldots, p_3(u,v) = au + bv + \ldots$. Then the curves of Z_τ are given by $x_1 p_1^2 + x_2 p_2^2 + x_3 p_3^2 = 0$, $((x_1 : x_2 : x_3) \in \mathbb{P}^2)$ and the equation of B_τ is the discriminant of the quadratic form $x_1 u^2 + x_2 v^2 + x_3(au+bv)^2$, (cf. proof of Proposition 2.2).

Therefore the tangent line to B_τ at t is the pencil $P = \{b^2 x_1 + a^2 x_2 = 0\}$ and the curve $C = \{a^2 p_1 - b^2 p_2 = 0\}$ belongs to P. Since $\theta = \{au + bv + \ldots = 0\}$ C is tangent to θ and $C \cdot \theta = 3\tau + q$. On the other hand $b_p \in |\omega_\theta^2|$ so that the only possibility is $\tau = q$, $b_P = 4\tau$. Hence $\tilde{C}_P$ is not stable. This implies the statement.

Proposition 3.2. *Assume $P \in \sigma_1^{-1}(\mathrm{Sing}(T))$, $P \notin \hat{T}$. Then there exists a unique admissible double cover $\pi : \tilde{C}_P \to C_P$ such that $[\tilde{C}_P, C_P] \in \bar{P}^{-1}(s)$ and $\alpha_P : \tilde{C}_P \to \theta$ is the Abel-Prym map. Moreover $\tilde{C}_P$ is hyperelliptic.*

Proof. Since $\varrho : \theta \to \theta/\langle -1 \rangle$ is the canonical map and $b_P \in |\omega_\theta^2|$ it follows that $\varrho(b_P)$ is a degree 2 divisor. Let $\beta : N \to \theta/\langle -1 \rangle$ be the double cover of $\theta/\langle -1 \rangle$ branched on $\varrho(b_P)$, consider the fibre product:

$$(3.6) \qquad \begin{array}{ccc} \tilde{C} & \xrightarrow{\alpha'} & \theta \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\varrho} \\ N & \xrightarrow{\beta} & \theta/\langle -1 \rangle \end{array}$$

clearly α' is branched on b_P; hence $\tilde{C}_P = \tilde{C}$, $\alpha' = \alpha_P$. Since $p_a(N) = 0$ $\tilde{C}_P$ is hyperelliptic.

Let a and g be the involutions induced by α' and γ; by (3.6) $\tilde{C}_P$ carries a third involution $i = g \cdot a = a \cdot g$. It is not difficult to check that the quotient morphism $\pi : \tilde{C}_P \to C_P$, $(\tilde{C}_P = \tilde{C}_P/\langle -1 \rangle)$, is admissible. Let α be the Abel-Prym map of π: since $\tilde{C}_P$ has a unique hyperelliptic linear series $i(x + g(x)) \sim x + g(x)$ for every $x \in \tilde{C}_P$. Hence $x - i(x) = \alpha(x) = i(g(x)) - g(x) = \alpha(i(g(x))) = \alpha(a(x))$ so that α factors through α' and $\alpha_*(\tilde{C}_P)$ is not reduced. Thus $\alpha_*(\tilde{C}_P)$ has to be twice the theta divisor of the Prym of π. This implies $\deg(\alpha) = 2$, $\alpha = \alpha'$, $\alpha_*(\tilde{C}_P) = 2\theta$ so that $[\tilde{C}_P, C_P] \in \bar{P}^{-1}(s)$. The uniqueness of the construction is clear.

By the previous proposition there exists a well defined map

$$\hat{\phi} : \hat{\mathbb{P}}^3 - \hat{T} \to \bar{P}^{-1}(s)$$

such that $\hat{\phi}/(\hat{\mathbb{P}}^3 - \sigma_1^{-1}(T)) = \phi \cdot \sigma_1$ and, for every $P \in \sigma_1^{-1}(\mathrm{Sing}\, T)$, $P \notin \hat{T}$

$$\hat{\phi}(P) = [\tilde{C}_P, C_P] \text{ as in Proposition 3.2.}$$

Proposition 3.3. *$\hat{\phi} : \hat{\mathbb{P}}^3 - \hat{T} \to \bar{P}^{-1}(s)$ is a morphism.*

Proof. To show the theorem it is not restrictive to assume σ_1 = blowing up of $\mathbb{P}^3$ in just one of the sixteen points of $\mathrm{Sing}(T)$. Assume also that this point is t and corresponds to the divisor 2θ.

Let us consider as in (2.4) the universal curves $\tilde{p} : \tilde{\mathscr{C}} \to \mathbb{P}^3$ in $\tilde{S} \times \mathbb{P}^3$, $p : \mathscr{C} \to \mathbb{P}^3$ in $\tilde{S}/\langle -1 \rangle \times \mathbb{P}^3$, together with the universal double cover $R : \tilde{\mathscr{C}} \to \mathscr{C}$. Let s_0, s_1, s_2, s_3 be a basis for $H^0(\mathcal{O}_{\tilde{S}}(2\theta))$ such that $\mathrm{div}(s_0) = 2\theta$; assume $p_i(u,v) = 0$ is the equation for $\mathrm{div}(s_i)$, $i = 1, 2, 3$, on an open subset of $\tilde{S}$ with local parameters u, v. Then $\tilde{\mathscr{C}}$ is locally given in $\mathbb{P}^3 \times \tilde{S}$ by

(3.7) $$z_1p_1 + z_2p_2 + z_3p_3 + p_0^2 = 0,$$

where $p_0(u, v) = au + bv + \ldots$ is a local equation for θ; z_1, z_2, z_3 are affine coordinates on $\mathbb{A}^3 \subset \mathbb{P}^3$. After blowing up $\mathbb{P}^3$ at $t = (0, 0, 0)$, $\mathscr{C}$ pulls back to a smooth, flat family $\tilde{d}: \tilde{\mathscr{D}} \to \hat{\mathbb{P}}^3$ which is locally given by

(3.8) $$z_3x = z_1, z_3y = z_2, z_3(p_1 + xp_2 + yp_3) + p_0^2 = 0.$$

After the local base change $\alpha: B \to \sigma_1^{-1}(\mathbb{A}^3)$ given by

$$\xi^2 = z_3$$

$\tilde{\mathscr{D}}$ pulls back again to the family $\beta: \tilde{\mathscr{B}} \to B$ which is given by

(3.9) $$\begin{gathered} \xi^2 x = z_1, \xi^2 y = z_2, \xi^2 = z_3 \\ \xi^2(p_1 + xp_2 + yp_3) + p_0^2 = 0 \end{gathered}$$

in local parameters $(\xi; x, y; z_1, z_2, z_3; u, v)$ on $\mathbb{A}^1 \times \mathbb{A}^2 \times \mathbb{A}^3 \times \tilde{S}$. Since α is unramified on $z_3 \neq 0$ and $\tilde{\mathscr{D}}$ is smooth it follows that $\tilde{\mathscr{B}} - \{z_3 = 0\}$ is smooth.

On the other hand it is clear from (3.9) that

$$\mathrm{Sing}(\tilde{\mathscr{B}}) = \{z_3 = 0\}.$$

Let q be any point of $\mathrm{Sing}(\tilde{\mathscr{B}})$, we can always choose u, v so that $u(q) = v(q) = 0$, then the tangent cone to $\tilde{\mathscr{B}}$ at q is

(3.10) $$\xi^2(\bar{p}_1 + \bar{x}\bar{p}_2 + \bar{y}\bar{p}_3) + (au + bv)^2 = 0, \; z_1 = z_2 = z_3 = 0,$$

(where $\bar{p}_i$, $\bar{x}$, $\bar{y}$ are the values of p_i, x, y at q). Moreover, if $\tilde{\beta}(q) = P''$, then $P = \alpha(P'') \in E = \sigma_1^{-1}(t)$ corresponds, via the bijection in (3.1), to the pencil

$$\{\lambda p_0^2 + \mu(p_1 + \bar{x}p_2 + \bar{y}p_3) = 0\}$$

and, obviously, $\tilde{\beta}^{-1}(P'') = \theta$. Then, by 3.10, it follows that

(3.11) $\tilde{\mathscr{B}}$ has two branches at $q \Leftrightarrow \bar{p}_1 + \bar{x}\bar{p}_2 + \bar{y}\bar{p}_3 = 0 \Leftrightarrow q \in b_P$,

(where b_P is defined as in 3.2). Therefore the normalization

$$\tilde{p}': \tilde{\mathscr{C}}' \to B$$

of $\tilde{\mathscr{B}}$ is a family of curves such that, if $P'' \in B$ and $\alpha(P'') = P \in E$

$$\tilde{p}'^*(P'') = \tilde{C}_P$$

as in (3.3). Otherwise $\tilde{p}'^*(P'') = \tilde{p}^*(\sigma_1(\alpha(P'')))$.

Consider now the universal quotient curve $p: \mathscr{C} \to \mathbb{P}^3$, one can construct in exactly the same way, families $d: \mathscr{D} \to \mathbb{P}^3$ and $\beta: \mathscr{B} \to B$ from $\mathscr{C}$. In this case $q \in \mathscr{B}$ is a not normal double point if and only if $\beta(q) = P''$ and $\alpha(P'') = P \in E$. Moreover, if so, $\beta^{-1}(P'') = \theta/\langle -1 \rangle$. Observe also that $\mathscr{C} \subset \mathbb{P}^3 \times \tilde{S}/\langle -1 \rangle$ is singular (with normal singularities) along $\mathscr{C} \cap (\mathbb{P}^3 \times \mathrm{Sing}(\tilde{S}/\langle -1 \rangle))$ and $\mathscr{D}$ is singular along the pull back of it. In particular, if $P \in E$, $d^{-1}(P) = \theta/\langle -1 \rangle$ intersects $\mathrm{Sing}(\mathscr{D})$ along the six branch points of $\varrho: \theta \to \theta/\langle -1 \rangle$. Using this remark and the equation for the tangent cone to $\mathscr{B}$ at q as in (3.10) it turns out that

(3.12) $\mathscr{B}$ has two branches at $q \Leftrightarrow q \in \varrho(b_P)$ or q is a branch point for ϱ.

Let $\tilde{p}': \mathscr{C}' \to B$ be the normalization of $\mathscr{B}$: it is not difficult to see that the universal double covering $R: \tilde{\mathscr{C}} \to \mathscr{C}$ induces a commutative diagram

$$\text{(3.13)} \qquad \begin{array}{ccc} \tilde{\mathscr{C}}' & \xrightarrow{\ R'\ } & \mathscr{C}' \\ & {\scriptstyle \tilde{p}'}\searrow \quad \swarrow{\scriptstyle p'} & \\ & B & \end{array}$$

If $\alpha(P'') = P$ and $P \in E$ then the morphism of fibres $R_{P''}: \tilde{\mathscr{C}}'_{P''} \to \mathscr{C}'_{P''}$ is exactly the allowable double covering considered in Proposition 3.2 (provided that $P \notin \hat{T}$). If $\alpha(P'') \notin E$ then $R_{P''} = R_P: \tilde{\mathscr{C}}_P \to \mathscr{C}_P$. Thus, by the universal property of $\bar{\mathscr{R}}_3$ and Proposition 3.2, (3.13) defines a morphism $\psi: B \to \bar{P}^{-1}(s)$ such that $\hat{\phi} \cdot \alpha = \psi$. Hence $\hat{\phi}$ is a morphism.

Consider in $\bar{P}^{-1}(s)$ the subloci

$$\mathscr{I}_s, \mathscr{T}_s, \mathscr{E}_s,$$

which are defined in the following way:

(3.14) $[\tilde{C}, C] \in \mathscr{I}_s$ if and only if $\tilde{C}$ is a hyperelliptic curve.

(3.15) (see [8, (2) 2.3], [2]): $[\tilde{C}, C] \in \mathscr{T}_s$ if and only if $\tilde{C} = \theta_1 \cup \theta_2$ is the union of two copies θ_j, $(j = 1, 2)$, of θ intersecting transversally at two points and i/θ_1 is the identity isomorphism from θ_1 to θ_2, (i being the involution induced by $\pi: \tilde{C} \to C$).

(3.16) (*Elliptic tails:* see [8 (3) 2.3], [2]): $[\tilde{C}, C] \in \mathscr{E}_s$ if and only if the following conditions hold:

(i) $\tilde{C} = \theta_1 \cup \tilde{E} \cup \theta_2$ is the union of two copies θ_j, $(j = 1, 2)$, of θ and a curve $\tilde{E}$ of arithmetic genus 1. θ_j intersects $\tilde{E}$ transversally in a point p_j; $p_1 - p_2$ is a non zero order 2 element on $\mathrm{Pic}^0(\tilde{E})$;

(ii) $i(\theta_1) = \theta_2$ and i/θ_1 is the identity isomorphism between the two copies of θ; $i(\tilde{E}) = \tilde{E}$ and $i/\tilde{E}$ induces the translation by $p_1 - p_2$ on $\mathrm{Pic}^0(\tilde{E})$;

(iii) $C = \theta \cup E$; $\tilde{E} = E/\langle i \rangle$ and $p_a(E) = 1$. $E \cap \theta = \{p\}$, $p = \pi(p_1) = \pi(p_2)$.

Since $\tilde{S}$ is not a product it follows easily that, for every $[\tilde{C}, C] \in \bar{P}^{-1}(s)$,

(3.17) $\tilde{C}$ is reducible if and only if $[\tilde{C}, C] \in \mathscr{T}_s \cup \mathscr{E}_s$.

Furthermore:

(3.18) (i) $\mathscr{T}_s \cap \mathscr{E}_s = \emptyset$,

(ii) $\mathscr{T}_s \cap \mathscr{I}_s = \{[\tilde{C}, C] \in \mathscr{T}_s / \tilde{C}$ is obtained from two copies of θ by glueing the point $x \in \theta$ to $-x$, $(x \neq -x)\}$,

(iii) $\mathscr{E}_s \cap \mathscr{I}_s = \{[\tilde{C}, C] \in \mathscr{E}_s / \tilde{C} = \theta_1 \cup \tilde{E} \cup \theta_2$ as in (3.16) and θ_j, $(j = 1, 2)$, is glued to $\tilde{E}$ in the Weierstrass point $\tau \in \theta \cap \tilde{S}_2\}$.

Proposition 3.4. *$\hat{\phi}(\hat{\mathbb{P}}^3 - \hat{T}) = \bar{P}^{-1}(s) - \mathscr{E}_s$. In particular:*

(1) Let R' be the strict transform of $R^{\wedge}$ by σ_1, then $\hat{\phi}(R' - (R' \cap \hat{T})) = \mathscr{T}_s$;

(2) Let E be an exceptional plane for σ_1, then $\hat{\phi}(E - (E \cap \hat{T})) = \mathscr{I}_s - (\mathscr{I}_s \cap \mathscr{E}_s)$.

Proof. $[\tilde{C}, C] \in \hat{\phi}(\hat{\mathbb{P}}^3 - \hat{T}) \Rightarrow \alpha(\tilde{C}) \in |2\theta|$ or $\tilde{C}$ is a finite double cover of θ. In both cases $\tilde{C}$ cannot split as in (3.16); hence $\hat{\phi}(\hat{\mathbb{P}}^3 - \hat{T}) \subset \bar{P}^{-1}(s) - \mathscr{E}_s$. Conversely let $[\tilde{C}, C] \in \bar{P}^{-1}(s) - \mathscr{E}_s$, assume $\tilde{C}$ is irreducible: If $\deg(\alpha) = 1$ then, by Corollary 2.1, $\alpha_*(\tilde{C})$ is stable; hence, by Proposition 2.3, $[\tilde{C}, C] = [\alpha_*(\tilde{C}), \alpha(\tilde{C})/\langle -1 \rangle]$, so that $[\tilde{C}, C] \in \phi(\mathbb{P}^3 - T)$. If $\deg(\alpha) > 1$ consider $x, y \in \tilde{C}$, $(x \neq y)$, such that $\alpha(x) = \alpha(y)$. Then $x - i(x) \sim y - i(y)$ so that $x + i(y) \sim y + i(x)$ and $\tilde{C}$ is hyperelliptic. Thus $\alpha_*(\tilde{C}) = (\deg \alpha)\, \alpha_*(\tilde{C}) = 2\theta$ and the following diagram commutes:

$$(3.19)\qquad \begin{array}{ccc} \tilde{C} & \xrightarrow{\ \alpha\ } & \theta \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \varrho} \\ C & \xrightarrow{\ \beta\ } & \theta/\langle -1 \rangle \end{array} .$$

Since π is admissible $\pi^* \omega_C = \omega_{\tilde{C}}$, [2, Lemma 5.1]; hence it easily follows that β is branched on $w + b$; where $w = \sum \tau$, $\tau \in \theta \cap \tilde{S}_2$ is a Weierstrass point and $\deg(b) = 2$. Therefore $\varrho^* b \in |\omega_\theta^2|$ and $\varrho^* b = b_P$ for some $P \in E$. Moreover α is branched on b_P so that $\tilde{C} = \tilde{C}_P$ and, being $\tilde{C}$ stable, $P \notin E \cap \hat{T}$. By Proposition 3.3 $\hat{\phi}(P) = [\tilde{C}, C]$. In the end assume $[\tilde{C}, C] \in \bar{P}^{-1}(s) - \mathscr{E}_s$, $\tilde{C}$ reducible; then $[\tilde{C}, C] \in \mathscr{T}_s$. Looking to the Abel-Prym map one sees that either $\alpha_*(\tilde{C}) = \tilde{C} = \theta_x + \theta_{-x} \in R^\wedge$ or α is a double cover of θ branched on $2(x + y)$, $(x \neq y)$.

This implies $\hat{\phi}(\hat{\mathbb{P}}^3 - \hat{T}) = \bar{P}^{-1}(s) - \mathscr{E}_s$ and the statement (1).

To show (2) observe that, since $\tilde{S}$ is not a product, $|2\theta|$ cannot contain elements which are hyperelliptic curves. Hence $[\tilde{C}, C] \in \mathscr{I}_s - (\mathscr{I}_s \cap \mathscr{E}_s) \Rightarrow \alpha_*(\tilde{C}) = 2\theta \Rightarrow [\tilde{C}, C] \in \hat{\phi}(E)$.

Let us consider now $\mathscr{E}_s$:

Lemma 3.1. $\forall s \in \mathscr{A}_2, \mathscr{E}_s = (\theta/\mathrm{Aut}(\theta)) \times \bar{\mathscr{R}}_1$.

Proof. It follows from the more general Lemma 1.31 of [8]; in fact: to give $[\tilde{C}, C] \in \mathscr{E}_s$ is equivalent to give $x \in \theta (\mathrm{mod}\, \mathrm{Aut}(\theta))$ such that $\{x\} = \theta \cap E$, where $C = \theta \cup E$ as in (3.16), together with $[\tilde{E}, E] \in \bar{\mathscr{R}}_1$.

We recall that

$$\bar{\mathscr{R}}_1 = \mathbb{P}^1 .$$

Assume $[\tilde{C}, C] \in \mathscr{E}_s$, $\tilde{C} = \theta_1 \cup \tilde{E} \cup \theta_2$; since $i/\tilde{E}$ is induced by the translation by a non zero order two element in $\mathrm{Pic}^0(\tilde{E})$ we have:

$$(3.20)\qquad \alpha(\tilde{E}) = \text{one point } \tau \in \tilde{S}_2 .$$

On the other hand α/θ_i maps θ_i isomorphically onto a copy of θ and clearly

$$\alpha(\theta_1) \cap \alpha(\theta_2) = \{\tau\}$$

or $\alpha(\theta_1) = \alpha(\theta_2)$. Hence, by the description of reducible curves of $|2\theta|$ given in Sect. 2, $\alpha_*(\tilde{C}) = \theta_{\tau + x} + \theta_{\tau - x}$ for some $x \in \theta$. That is:

$$(3.21)\qquad \alpha_*(\tilde{C}) \in B \subset T ,$$

where B is a trope. We know that $B = \theta/\langle -1 \rangle = \mathbb{P}^1$, ($\tilde{S}$ not a product), and that B is a conic passing through six fundamental points of σ_1. In particular $B = Z_\tau \cap Q$, (where Q is a quadric), and its strict transform is the complete intersection

$$\hat{B} = \hat{Z}_\tau \cap \hat{Q}$$

(with $\hat{Z}_\tau, \hat{Q}$ = strict transforms of Z_τ, Q). The ideal $\mathscr{F}$ of $\hat{B}$ in $\hat{\mathbb{P}}^3$ is $\mathcal{O}_{\hat{\mathbb{P}}}(-\hat{Z}) \oplus \mathcal{O}_{\hat{\mathbb{P}}}(-\hat{Q})$; moreover $\hat{B}^2 = -2$ in $\hat{Z}_\tau$ and $\hat{B}^2 = -4$ in $\hat{Q}$ so that the conormal bundle of $\hat{B}$ in $\hat{\mathbb{P}}^3$ is:

$$\mathscr{F}/\mathscr{F}^2 = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4).$$

Let $\mathscr{G} \subset \mathscr{F}$ be the ideal $\mathcal{O}_{\hat{\mathbb{P}}^3}(-2\hat{Z}) \oplus \mathcal{O}_{\hat{\mathbb{P}}^3}(-\hat{Q})$ having $\hat{B}$ as its support: after blowing up $\hat{\mathbb{P}}^3$ along the ideal $\mathscr{G}$ the exceptional divisor is

$$\mathbb{P}(\mathscr{G}/\mathscr{G}^2) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(4)) = \hat{B} \times \mathbb{P}^1. \tag{3.22}$$

Since all the $\hat{B}$'s are two by two disjoint we can blow up all of them as above, in this way we obtain a birational morphism

$$\sigma_2 : \tilde{\mathbb{P}}^3 \to \hat{\mathbb{P}}^3$$

having as exceptional divisor the union of 16 copies of $\theta/\langle -1 \rangle \times \mathbb{P}^1$. Thus, for every $\hat{B}$, we can identify $\sigma_2^{-1}(\hat{B})$ to $\hat{B} \times \mathscr{R}_1$. This defines in a natural way a map

$$\tilde{\phi} : \tilde{\mathbb{P}}^3 \to \bar{P}^{-1}(s) \tag{3.23}$$

which is surjective and extends $\hat{\phi}$. By definition $\tilde{\phi} = \sigma_2 \cdot \hat{\phi}$ on $\tilde{\mathbb{P}}^3 - \sigma_2^{-1}(\hat{T})$; on the other hand, if $P = (\pm x; [\tilde{E}, E]) \in \hat{B} \times \mathscr{R}_1 \subset \sigma_2^{-1}(\hat{T})$, we set $\tilde{\phi}(P) = [\tilde{C}, C]$; where $\tilde{C}$ is obtained by glueing to the copies θ_x, θ_{-x} of θ the curve $\tilde{E}$ in such a way that $\theta_x \cap \tilde{E} = \{x\}$, $\theta_{-x} \cap \tilde{E} = \{-x\}$. Moreover the difference of these two points on $\tilde{E}$ defines an order two translation j whose quotient is E and $C = \tilde{C}/\langle i \rangle$, where $i/\tilde{E} = j$ and $i/(\theta_x \cup \theta_{-x})$ is the -1 involution of $\tilde{S}$. It is clear that, (up to automorphisms), the quotient map $\pi : \tilde{C} \to C$ is defined as in (3.16) and that $[\tilde{C}, C] \in \mathscr{E}_s$.

In particular

$$\tilde{\phi}(\tilde{\mathbb{P}}^3) = \bar{P}^{-1}(s), \; \phi(\sigma_2^{-1}(\hat{B})) = \mathscr{E}_s.$$

By Lemma 3.1, if $\mathrm{Aut}(\theta) = \mathbb{Z}/2\mathbb{Z}$ then $\tilde{\phi}/(\sigma_2^{-1}(\hat{B}))$ is an isomorphism.

After some tedious computations, similar to the ones we produced in Proposition 3.3, one shows the following

Proposition 3.5. *The map $\tilde{\phi} : \tilde{\mathbb{P}}^3 \to \bar{P}^{-1}(s)$ is a surjective morphism.*

Remark 3.1. In particular, by Proposition 3.5, $\bar{P}^{-1}(s)$ is irreducible. If $\tilde{S}$ is a product this is no longer true: $\bar{P}^{-1}(s)$ contains a 3-dimensional component $\mathscr{F}_s$ such that $\mathscr{F}_s \cap P^{-1}(s) = \emptyset$, $(P = \bar{P}/\mathscr{R}_3)$. The elements of $\mathscr{F}_s$ are defined in [9, Lemma 12, Proposition 1.5]. Nevertheless one can extend the results of this section by constructing a morphism $\tilde{\phi} : \tilde{\mathbb{P}}^3 \to \bar{P}^{-1}(s)$ also when $\tilde{S}$ is a product. In this case it turns out that $\tilde{\phi}(\tilde{\mathbb{P}}^3)$ = Zariski closure of $P^{-1}(s)$ in $\bar{P}^{-1}(s)$.

4

To complete the construction of $\bar{P}^{-1}(s)$ we describe now the morphism $\tilde{\phi}$. We assume again that $\tilde{S}$ is not a product (see Remark 4.1). Let G be the group of the translations on $\tilde{S}$ by elements of order two. G is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Since $2\theta \sim 2\theta_\tau$, $\tau \in \tilde{S}_2$, G acts linearly (and faithfully) on $\mathbb{P}^3 = |2\theta|$. Observe that:

(1) The sixteen points of $\mathrm{Sing}(T)$ form the orbit of the element 2θ;

(2) G acts transitively on the sixteen tropes of $R^{\wedge}$.

Then, since $\sigma_1: \hat{\mathbb{P}}^3 \to \mathbb{P}^3$ is a blowing up centered in Sing(T), the action of G extends to an action on $\hat{\mathbb{P}}^3$ and G acts transitively on the strict transforms of the sixteen tropes of $R\hat{}$. On the other hand $\sigma_2: \hat{\hat{\mathbb{P}}}^3 \to \hat{\mathbb{P}}^3$ is a blowing up centered on $\hat{T}$ = union of the strict transforms of the tropes of $R\hat{}$. Hence the action of G extends to an action on $\hat{\hat{\mathbb{P}}}^3$.

Proposition 4.1. *Assume* $\mathrm{Aut}(\theta) = \mathbb{Z}/2\mathbb{Z}$ *then:* $\forall P_1, P_2 \in \hat{\hat{\mathbb{P}}}^3$

$$\tilde{\phi}(P_1) = \tilde{\phi}(P_2) \Leftrightarrow P_1 \text{ and } P_2 \text{ are in the same } G\text{-orbit}.$$

Proof. Let $\tilde{\phi}(P_i) = [\tilde{C}_i, C_i]$, $(i = 1, 2)$; assume $\tilde{\phi}(P_1) = \tilde{\phi}(P_2)$. This implies that the allowable double covers $\pi_1: \tilde{C}_1 \to C_1$ and $\pi_2: \tilde{C}_2 \to C_2$ are isomorphic. Hence there exist isomorphisms $\tilde{\sigma}: \tilde{C}_1 \to \tilde{C}_2$, $\sigma: C_1 \to C_2$, such that $\pi_2 \cdot \tilde{\sigma} = \sigma \cdot \pi_1$ and $i_2 \cdot \tilde{\sigma} = \sigma \cdot i_1$, ($i_j$, $j = 1, 2$, being the involution induced by π_j). $\tilde{\sigma}$ extends to an isomorphism $\tilde{\sigma}_*: J\tilde{C}_1 \to J\tilde{C}_2$ such that $i_{2*} \cdot \tilde{\sigma}_* = \sigma_* \cdot i_{1*}$. Therefore $\tilde{\sigma}_*/\mathrm{Im}(\mathrm{id} - i_{1*}) = t$ is an isomorphism from $\mathrm{Im}(\mathrm{id} - i_{1*})$ to $\mathrm{Im}(\mathrm{id} - i_{2*})$. In other words t is an automorphism of $\tilde{S}$ as a principally polarized abelian surface. Since we can choose Abel-Jacobi embeddings $\tilde{C}_1 \subset J\tilde{C}_1$, $\tilde{C}_2 \subset J\tilde{C}_2$ such that $\tilde{\sigma}_*(\tilde{C}_1) = \tilde{C}_2$ it follows that $t(\alpha_1(\tilde{C}_1)) = \alpha_2(\tilde{C}_2)$, ($\alpha_j$ being the Abel-Prym map for $\tilde{C}_j$). Hence $|2\theta|$ is changed in itself by t. On the other hand t is induced by an element of $\mathrm{Aut}(\theta)$; since $\mathrm{Aut}(\theta) = \mathbb{Z}/2\mathbb{Z}$ it follows $t = \mathrm{id}$ or $t = -1$ so that t acts as the identity on $|2\theta|$.

Since the Abel-Prym map is defined up to translating by an element of $\tilde{S}_2$, we have $\alpha_1(\tilde{C}_1) = \alpha_2(\tilde{C}_2)$ modulo such a translation. Hence P_1, P_2 are in the same G-orbit. Conversely assume $P_1 = P_2 \bmod(G)$, then $\alpha_1(\tilde{C}_1)_\tau = \alpha_2(\tilde{C}_2)$, $(\tau \in \tilde{S}_2)$. If $\tilde{\phi}(P_1) \notin \mathscr{I}_s \cup \mathscr{E}_s$ then $\alpha_{i*}\tilde{C}_i$ is stable and $[\tilde{C}_i, C_i] = [\alpha_{i*}\tilde{C}_i, \alpha_{i*}\tilde{C}_i/\langle -1 \rangle]$.

Hence, obviously, $[\tilde{C}_1, C_1] = [\tilde{C}_2, C_2]$. If $P_1 \in \mathscr{I}_s$ then $\alpha_1(\tilde{C}_1) = \theta$ and α_1 is branched on b_{P_1} [which is defined as in (3.3)]. Thus $\alpha_2(\tilde{C}_2) = \theta_\tau$. Let b_{P_2} be the branch divisor of α_2, since $P_1 = P_2 \bmod(G)$ it follows $b_{P_2} = (b_{P_1})_\tau$ and $\tilde{C}_1 = \tilde{C}_2$. By Proposition 3.2 this implies $[\tilde{C}_1, C_1] = [\tilde{C}_2, C_2]$. A similar argument applies if $P_1 \in \mathscr{E}_s$.

Corollary 4.1. *Assume* $\mathrm{Aut}(\theta) = \mathbb{Z}/2\mathbb{Z}$, *then* $\hat{\hat{\mathbb{P}}}^3/G = \bar{P}^{-1}(s)$.

The group G and its representation as a subgroup of the projective linear group $PGL(4)$ are classically well known, [4, Vol. III], [11]: there exist finitely many system of coordinates $(x_1 : x_2 : x_3 : x_4)$ on $|2\theta|$, [finitely many bases of $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2\theta))$], such that the equation of $R\hat{}$ is

$$\begin{aligned} E(x_1^4 + x_2^4 + x_3^4 + x_4^4) + 2D(x_1x_2x_3x_4) + A(x_1^2x_2^2 + x_3^2x_4^2) \\ + B(x_1^2x_3^2 + x_2^2x_4^2) + C(x_1^2x_4^2 + x_2^2x_3^2) = 0, \end{aligned} \tag{4.1}$$

where $(A : B : C : D : E) \in \mathbb{P}^4$ and satisfies the cubic equation

$$E(4E^2 - A^2 - B^2 - C^2 - D^2) + ABC = 0. \tag{4.2}$$

It is also known that to fix $(x_1 : x_2 : x_3 : x_4)$ as above is equivalent to fix a level two structure on $\tilde{S}$. Moreover, if the equation of $R\hat{}$ is normalized as in (4.1), then G acts linearly on $\mathbb{P}^3 = |2\theta|$ as the group generated by the involutions sending $(x_1 : x_2 : x_3 : x_4)$ respectively in

$$(x_4 : -x_3 : x_2 : x_1), \quad (x_4 : x_3 : -x_2 : -x_1), \quad (x_3 : x_4 : -x_1 : -x_2),$$
$$(-x_2 : x_1 : x_4 : -x_3), \quad (x_2 : -x_1 : x_4 : -x_3).$$

We denote by $V^{\wedge}$ the cubic threefold defined by (4.2) and by

$$V \subset \mathbb{P}^4$$

its dual hypersurface; it is well known [4, Vol. VII], [11] that $V^{\wedge}$ is the so called Segre primal containing ten nodes as its only singularities. On the other hand V is a (rational) quartic threefold having as its singular locus 15 double lines intersecting three by three in 15 triple points. In the end the space $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))^G$ of the G-invariant quartic forms is 5-dimensional and a basis for it is

$$(x_1^4+x_2^4+x_3^4+x_4^4), (x_1x_2x_3x_4), (x_1^2x_2^2+x_3^2x_4^2), (x_1^2x_3^2+x_2^2x_4^2), (x_1^2x_4^2+x_2^2x_3^2).$$

We denote by

$$\mu : \mathbb{P}^3 \to \mathbb{P}^4$$

the morphism associated to $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))^G$.

Proposition 4.2. (1) $\mu(\mathbb{P}^3) = V$,
(2) $V = \mathbb{P}^3/G$ *and* $\mu : \mathbb{P}^3 \to V$ *is the quotient map.*

Proof. This is shown in [4, Vol. III, p. 210].

Lemma 4.1. (1) $\mu(\mathrm{Sing}(T)) = o$, *where o is a smooth point of* V,
(2) $\mu(R^{\wedge}) = R^{\wedge}$,
(3) $\mu(R^{\wedge}) = H_0 \cap V$ *where H_0 is the tangent hyperplane to the quartic threefold V at the point o,*
(4) $\mu(T) = Q$, *where Q has an ordinary sixtuple point in o and the normalization of Q in o is isomorphic to* $\theta/\langle -1 \rangle$.

Proof. (1) By Proposition 4.2 $\deg(\mu) = 16$, since $\#\mu^{-1}(o) = \#\mathrm{Sing}(T) = 16$ o is not in the branch locus of μ hence o is smooth. (2) It is clear that the following diagram is commutative

$$\begin{array}{ccc} \tilde{S}^{\wedge} & \xrightarrow{\mu_2} & \tilde{S}^{\wedge} \\ \downarrow & & \downarrow \\ R^{\wedge} & \xrightarrow{\mu} & R^{\wedge} \end{array}$$

where μ_2 is the multiplication by 2 on the dual $\tilde{S}^{\wedge}$ of $\tilde{S}$ and the vertical arrows are the quotient map from $\tilde{S}^{\wedge}$ to $R^{\wedge} = \tilde{S}^{\wedge}/\langle -1 \rangle$. Hence $\mu(R^{\wedge}) =$ Kummer surface of $\tilde{S}^{\wedge} = R^{\wedge}$ [recall also that $R^{\wedge}$ is G-invariant: for instance its Eq. (4.1) is given by an invariant quartic form]. (3) Since the equation of $R^{\wedge}$ is given by an element of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))^G$ it follows that $\mu(R^{\wedge})$ is a hyperplane section of V. The general hyperplane section of V is a quartic surface with 15 nodes; on the other hand $R^{\wedge}$ is a quartic surface with 16 nodes, $\mathrm{Sing}(R^{\wedge}) = \mathrm{Sing}(T)$ and $\mu(\mathrm{Sing}(R^{\wedge})) = o$. Hence o is singular for $\mu(R^{\wedge})$. Since o is smooth for V the result follows. (4) Let $B \subset T$ be a trope, then $\mu(B) = \mu(T)$. It is immediate to check that $\mu(B) = Q$: μ/B is $1:1$ except for the six points of $B \cap \mathrm{Sing}(T)$ which are contracted to o. This implies (4).

Theorem 4.1. *Assume* $\mathrm{Aut}(\theta) = \mathbb{Z}/2\mathbb{Z}$, *then, with the same notations as above,* $\bar{P}^{-1}(s) = (\zeta_1 \cdot \zeta_2)^{-1}(V)$; *where* $\zeta_1 : \hat{V} \to V$ *blows up the point* $o = \mu(\mathrm{Sing}(T))$ *and* $\zeta_2 : \bar{P}^{-1}(s) \to \hat{V}$ *is a blowing up of $\hat{V}$ centered in the strict transform of the curve* $Q = \mu(T)$.

Proof. By Proposition 4.1 $\bar{P}^{-1}(s)=\hat{\mathbb{P}}^3/G$ and $\tilde{\phi}:\hat{\mathbb{P}}^3\to\bar{P}^{-1}(s)$ is the quotient map. Hence we have a commutative diagram:

$$
\begin{array}{ccc}
\hat{\mathbb{P}}^3 & \xrightarrow{\sigma_1\cdot\sigma_2} & \mathbb{P}^3 \\
\downarrow{\scriptstyle\tilde{\phi}} & & \downarrow{\scriptstyle\mu} \\
\bar{P}^{-1}(s) & \xrightarrow{\zeta} & V.
\end{array}
\tag{4.3}
$$

σ_1,σ_2 are defined as in Sect. 3: $\sigma_1:\hat{\mathbb{P}}^3\to\mathbb{P}^3$ blows up $\mathrm{Sing}(T)$, $\sigma_2:\hat{\mathbb{P}}^3\to\hat{\mathbb{P}}^3$ is a blowing up centered on $\hat{T}=$ union of the strict transforms of the irreducible components of T. Since μ is not ramified along T it follows that $\zeta=\zeta_1\cdot\zeta_2$.

Corollary 4.2. *Let $\zeta=\zeta_1\cdot\zeta_2$ be as above, then $\mathscr{I}_s=\zeta^{-1}(o)$; $\mathscr{E}_s=\zeta_2^{-1}(Q')$, where Q' is the strict transform of Q by ζ_1; $\mathscr{T}_s=$ strict transform of $\mu(R^\frown)$ by ζ.*

Proof. It follows from Proposition 3.4, Proposition 2.2 (b) and the commutativity of the diagram (4.3).

It is interesting to remark that $\mathbb{P}^3/G$ is well known from another point of view, [19]:

Let $\mathscr{H}_2$ be the Siegel upper half space of degree 2 and $\Gamma_2(2)$ the 2-congruence subgroup of the symplectic group $\mathrm{Sp}(4,Z)$. The quotient $\mathscr{H}_2/\Gamma_2(2)$ is the moduli space for principally abelian surfaces endowed with a level 2 structure. Its minimal compactification $\bar{\mathscr{A}}_2^2$ is biregular, via modular forms, to the quartic threefold $V=\mathbb{P}^3/G$. For this reason V is also called Siegel modular quartic threefold.

By the results of [19] the condition $R^\frown=T_{V,o}\cap V$ of Lemma 4.1 implies that, under the previous isomorphism $g:V\to\bar{\mathscr{A}}_2^2$, the point $g(o)$ is one of the isomorphism classes of $\tilde{S}$ endowed with a level two structure. Notice also that the orbit of $g(o)$ by $\mathrm{Aut}(\bar{\mathscr{A}}_2^2)$ is exactly the set of all such isomorphism classes. Hence we can reformulate the results of Theorem 4.1 and Corollary 4.2 in the following way:

Theorem 4.2. *Let $s\in\mathscr{A}_2$, assume $\mathrm{Aut}(\theta)=\mathbb{Z}/2\mathbb{Z}$; fix a point $o\in\bar{\mathscr{A}}_2^2$ corresponding to the isomorphism class of $\tilde{S}$ and a level two structure on it. Then there exists a sequence of blowing up's*

$$\bar{P}^{-1}(s)\xrightarrow{\zeta_2}\hat{\mathscr{A}}\xrightarrow{\zeta_1}\bar{\mathscr{A}}_2^2$$

such that: (1) *ζ_1 is the blowing up of $\bar{\mathscr{A}}_2^2$ in o.* (2) *The exceptional divisor of ζ_2 is $\mathscr{E}_s=\theta/\langle-1\rangle\times\bar{\mathscr{R}}_1$ and $\zeta_2(\mathscr{E}_s)=\theta/\langle-1\rangle$.* (3) *$(\zeta_1\cdot\zeta_2)^{-1}(o)=\mathscr{I}_s$.*

Remark 4.1. If $\tilde{S}$ is not a product $\mathrm{Aut}(\theta)=\mathbb{Z}/2\mathbb{Z}$ but for finitely many exceptions, [5]. If $\tilde{S}$ is a product we have $\mathrm{Aut}(\theta)=(\mathbb{Z}/2\mathbb{Z})^2$ for general $\tilde{S}$. Under this assumption the results of this section extend to products (with modified proofs) if one considers instead of $\bar{P}^{-1}(s)$ the Zariski closure of $P^{-1}(s)$, $(P=\bar{P}/\bar{\mathscr{R}}_3)$; (cf. Remark 3.1).

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