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The Gromov Norm of the Kaehler Class of Symmetric Domains

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Let D be bounded symmetric domain of rank p, and let ω be the Kähler form of its Bergmann metric, where the metric is normalized so that the minimum holomorphic sectional curvature is -1. If X is a compact manifold whose universal cover is D, ω defines a cohomology class $[\omega] \in H^2(X, \mathbb{R})$. The purpose of this paper is to compute its sup norm in the sense of Gromov [5]. We show $\|[\omega]\|_\infty = p\pi$. Strictly speaking, we only prove this for three of the four classes of classical domains (cf. Sect. 2).

This theorem has the following topological corollary. Let S be a Riemann surface of genus $g>1$ and $f: S \to X$ a continuous map. Then $\left|\int_S f^*\omega\right| \leq 4p(g-1)\pi$. This was proved in [8] for the case $p=1$, i.e., $D =$ unit ball in $\mathbb{C}^n$, by different methods: harmonic mappings and Bochner's formula. The original motivation for this paper was to extend these methods to higher rank. This turned out to be impossible, in the sense that the pointwise inequality needed to apply Bochner's formula is actually false for $p>1$. We thus find it quite interesting that Gromov's methods can be used to derive the topological corollary.

In [8] the case of equality was also settled: it can only hold for mappings homotopic to a totally geodesic holomorphic (or anti-holomorphic) immersion. For higher rank we find that certain complex geodesics give equality, but that (at least in certain "reducible" situations) these need not be the only extremal homotopy classes.

We point out the relation of this paper to bounded cohomology. One would like to prove that if $D = G/K$ is a symmetric space of non-compact type and ω is a G-invariant form on D, then for any compact manifold X covered by D, the corresponding cohomology class $[\omega]$ is bounded. One would then want to find the precise value of $\|[\omega]\|_\infty$. What is known in the literature is the following:

1) If ω is a characteristic class, $\|[\omega]\|_\infty < \infty$ [5].

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2) If ω is the volume form of real hyperbolic space, $\|[\omega]\|_\infty < \infty$, and its value is known [5, 6].
3) If ω is the volume form of $SL(n, \mathbb{R})/SO(n)$, then $\|[\omega]\|_\infty < \infty$ [7].
4) If ω is as in this paper, explicit upper bounds are given in [3, 9].

Since the form ω considered in this paper is a characteristic class, $\|[\omega]\|_\infty < \infty$ follows from (1). Coarse but explicit upper bounds have been derived by Dupont in [3]. For the symplectic group Turaev [9] gives as an upper bound our value of $\|[\omega]\|_\infty$, but does not prove its sharpness. Our contribution is to compute the precise value $\|[\omega]\|_\infty = p\pi$. This is the only case, other than (2), where the precise value of $\|[\omega]\|_\infty$ is known.

1. Reduction of the Problem

We first recall the definitions and basic properties of Gromov norms. Let X be a topological space and c a singular cochain on S. The sup norm of c is defined by

$$\|c\|_\infty = \sup\{|c(\sigma)| : \sigma \text{ a singular simplex in } X\}.$$

If $\alpha \in H^*(X, \mathbb{R})$, the sup norm of α is defined by

$$\|\alpha\|_\infty = \inf\{\|c\|_\infty : c \text{ a singular cocycle representing } \alpha\}.$$

This defines a pseudo-norm on $H^*(X, \mathbb{R})$ – all properties of a norm except that it can take on the value ∞.

If $z = \sum a_i \sigma_i$ is a singular chain in X, i.e., $a_i \varepsilon \mathbb{R}$, σ_i singular simplices in X, its L^1-norm is defined by $\|z\|_1 = \sum |a_i|$, and if $x \in H_*(X, \mathbb{R})$, its L_1 norm is defined by $\|x\|_1 = \inf\{\|z\|_1 : z \text{ a singular cycle representing } x\}$. This defines a pseudo-norm on $H_*(X, \mathbb{R})$ – all properties of a norm except that it can have the value zero on non-zero homology classes.

The basic properties of these norms that we will need are:

1) $$|\alpha(x)| \leqq \|\alpha\|_\infty \|x\|_1 .$$

2) If $f : X \to Y$ is continuous, then

$$\|f^*\alpha\|_\infty \leqq \|\alpha\|_\infty \quad \text{and} \quad \|f_* x\|_1 \leqq \|x\|_1 .$$

3) If X is a complete manifold of non-positive curvature, we can consider, in the definition of the norms, only the class of geodesic simplicies, and not change the value of the norm. A geodesic simplex in S is a singular simplex σ obtained in the following way. Let $\tilde{X}$ be the universal covering of X, let $P_0, \ldots, P_k$ be points of $\tilde{X}$, and let

$$\tilde{\sigma} : \Delta^k \to \tilde{X}$$

be the map

$$(\lambda_0, \ldots, \lambda_k) \to \text{center of gravity of the mass distribution } \lambda_0 \delta_{P_0} + \ldots \lambda_k \delta_{P_k}, \text{ where } \delta_Q = \text{Dirac measure at } P.$$

Then let $\sigma = \pi \cdot \tilde{\sigma}$, where $\pi : \tilde{X} \to X$ is the covering projection. Note that the asserted center of gravity exists by a well-known theorem of Cartan. Also note that if X is not of constant curvature this is not the only definition of "geodesic simplex" that has been used in this context, but the particular choice is immaterial. The only point for our purposes is that geodesic k-simplicies are parametrized by $\tilde{X}^{k+1}/\pi_1(X)$, and that their edges are geodesic segments.

4) If X is a Riemann surface of genus $g>1$ and $[X]$ its fundamental integral homology class, then $\|[X]\|_1=4(g-1)$.

Assertions (1) and (2) are clear from the definitions; for details on (3) and (4) see [5].

Now let D be a bounded symmetric domain with its Bergmann metric. We assume the situation is normalized so that the minimum holomorphic sectional curvature is -1, and thus the image of the holomorphic sectional curvature function of D is the closed interval $[-1, -1/p]$, $p=\operatorname{rank}(D)$ (corresponding to the values of the holomorphic sectional curvature on a maximal totally geodesic polydisk, which is a product of p disks each with the Poincaré metric of curvature -1). Let ω be the Kähler form of this metric, and let X be a manifold with universal cover D. By (3) above, to give an upper bound for $\|[\omega]\|_\infty$, where $[\omega]\in H^2(X,\mathbb{R})$ is the two-dimensional cohomology class determined by ω, it is enough to give an estimate for $\int_\Delta \omega$, Δ a geodesic triangle in D. We will prove.

Theorem 1. *Let D, p, and ω be as above, and Δ a geodesic triangle in D. Then*

$$\left|\int_\Delta \omega\right| \leqq p\pi .$$

Corollary 1. *Let X be covered by D, and $[\omega]\in H^2(X,\mathbb{R})$ be the class of ω. Then*

$$\|[\omega]\|_\infty \leqq p\pi .$$

Corollary 2. *Let G be the group of biholomorphic automorphisms of D, let $G_d=G$ with the discrete topology, and let $[\omega]\in H^2(G_d,\mathbb{R})$ be the cohomology class of the Eilenberg-MacLane cochain*

$$\langle g_0, g_1, g_2\rangle \to \int_{\Delta(g_0,g_1,g_2)} \omega ,$$

where $\Delta(g_0,g_1,g_2)=$ geodesic triangle with vertices $g_0 0$, $g_1 0$, $g_2 0$ for some fixed point $0\in D$. Then $\|[\omega]\|_\infty \leqq p\pi$.

Corollary 2 is a "universal formulation" of Corollary 1. It is useful in studying characteristic classes of flat G-bundles, e.g., representations π_1 (surface) $\to G$ as in [4].

We now start the proof of Theorem 1, with the main computation deferred to the next section. We first observe that if $P\in D$, there exists a (unique) potential ϱ_P for the Bergmann metric, i.e., a function ϱ_P much that $dd^{\mathbb{C}}\varrho_P=\omega$, with the following properties:

A) $\varrho_P(P)=0$

B) ϱ_P is invariant under $K_P=$ isotropy group of P,

C) $d^{\mathbb{C}}\varrho_P=0$ on geodesics through P.

To see this, take any potential satisfying (A), average it so it satisfies (B), and observe that (C) follows: Let $\gamma(t)$ be a geodesic with $\gamma(0)=P$. γ is contained in a maximal flat F that is a totally real subspace of D which at each of its points is orthogonal to the K_P – orbit through that point. Thus if J denotes the complex structure, $J\gamma'(t)$ is orthogonal to F, hence tangent to $K_P\gamma(t)$. But then $d^{\mathbb{C}}\varrho_P(\gamma'(t)) = d\varrho_P(J\gamma'(t)) = 0$ since ϱ_P is K_P-invariant.

Using these properties we can begin to compute. Let Δ be a geodesic simplex in D, with vertices P, Q, R. Then

$$\int_{\Delta} \omega = \int_{\Delta} dd^{\mathbb{C}}\varrho_P = \int_{\partial\Delta} d^{\mathbb{C}}\varrho_P = \int_{\gamma(Q,R)} d^{\mathbb{C}}\varrho_P,$$

where $\gamma(Q, R)$ = geodesic segment from Q to R, and the last equality follows from (C).

Now $h_{PQ} = \varrho_P - \varrho_Q$ is a pluri-harmonic function on D (i.e., $dd^{\mathbb{C}}h_{PQ} = 0$), thus $h_{PQ} = \operatorname{Re} H_{PQ}$, where H_{PQ} is holomorphic on D. Let $k_{PQ} = \operatorname{Im} H_{PQ}$. Then

$$\int_{\gamma(Q,R)} dk_{PQ} = k_{PQ}(R) - k_{PQ}(Q),$$

where the second equality follows from (C) applied to ϱ_Q. Thus we have to show that

$$|k_{PQ}(R) - k_{PQ}(Q)| < p\pi .$$

We observe that both sides are additive under products, so it is enough to prove the inequality for irreducible domains. The irreducible case is discussed in the next section.

2. The Domain $D_{p,q}$

Recall that the irreducible bounded symmetric domains are classified as follows: [2]

$I_{p,q}$: $D_{p,q} = \{Z \in M_{p,q}(\mathbb{C}) : Z^*Z < 1\}$. We may (and do) assume $p \leqq q$.

II_p: $\{Z \in D_{p,p} : Z^t = -Z\}$

III_p: $\{Z \in D_{p,p} : Z^t = Z\}$

IV_n: $\{(Z_1, \ldots, Z_n) \in C^n : |Z_1^2 + \ldots + Z_n^2| < 1$ and $1 + |Z_1^2 + \ldots + Z_n^2|^2 > 2(|Z_1|^2 + \ldots + |Z_n|^2\}$

V, VI: Exceptional domains in dimensions 16, 27.

The ranks are as follows: $I_{p,q}$ and III_p have rank p, II_p has rank $[p/2]$, IV_n has rank 2. III_p is the "unit ball model" of the Siegel upper half-plane, IV_n is the non-compact dual of the complex quadric hypersurface in $\mathbb{C}P^{n+1}$.

We complete the proof of the estimate promised in Sect. 1 for the domain $D_{p,q}$. This domain is the non-compact dual of the Grassmannian $G(p, q)$ of complex p-planes in $\mathbb{C}^{p+q} = \{(u, v) : u \in \mathbb{C}^p, v \in \mathbb{C}^q\}$, in fact it is the open submanifold of $G(p, q)$ on which the Hermitian form $|u|^2 - |v|^2$ is positive definite, the correspondence being given by

$$Z \in D_{p,q} \to \{(u, Zu) : u \in \mathbb{C}^p\} \in G(p, q).$$

From this it is easy to see that its group of automorphisms is

$$SU(p,q)=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in SL(p+q,\mathbb{C}):\right.$$
$$\left.\begin{pmatrix}A^* & C^*\\ B^* & D^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}$$

$SU(p,q)$ acts on $D_{p,q}$ by $Z\to(AZ+B)(CZ+D)^{-1}$.

Proposition. *Let Δ be the geodesic triangle in $D_{p,q}$ with vertices O, Z_0, Z. Then*

$$\int_\Delta \omega=\arg\frac{\overline{\det(1-Z_0^*Z)}}{\det(1-Z_0^*Z)}.$$

"arg" *means the continuous branch that vanishes when $Z=Z_0$.*

Proof. The potential of the Bergmann metric centered at O is easily seen to be

$$\varrho_0=-\log\det(1-Z^*Z).$$

To obtain ϱ_{Z_0} we observe that the following is an element of $SU(p,q)$ that takes Z_0 to 0:

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}(1-Z_0Z_0^*)^{-1/2} & -(1-Z_0Z_0^*)^{-1/2}Z_0\\ -Z_0^*(1-Z_0Z_0^*)^{-1/2} & (1-Z_0^*Z_0)^{-1/2}\end{pmatrix}.$$

Then

$$\begin{aligned}\varrho_{Z_0}&=-\log\det(1-[(AZ+B)(CZ+D)^{-1}]^*(AZ+B)(CZ+D)^{-1})\\&=-\log\det((CZ+D)^{*-1}(CZ+D)^*(CZ+D)(CZ+D)^{-1}\\&\quad-(CZ+D)^{*-1}(AZ+B)^*(AZ+B)(CZ+D)^{-1})\\&=\log|\det(CZ+D)|^2\\&\quad-\log\det((CZ+D)^*(CZ+D)-(AZ+B)^*(AZ+B))\\&=\log|\det(CZ+D)|^2+\varrho_0\end{aligned}$$

since, by the defining equations for

$$SU(p,q),\quad (CZ+D)^*(CZ+D)-(AZ+B)^*(AZ+B)=1-Z^*Z.$$

Thus the pluriharmonic function $h_{0_{Z_0}}$ as in Sect. 1 is given by the formula

$$h_{0_{Z_0}}=-\log|\det(CZ+D)|^2=\mathrm{Re}(\log\det(CZ+D)^{-2}).$$

By the formula at the end of Sect. 1

$$\begin{aligned}\int_\Delta\omega&=\arg\det(CZ+D)^{-2}\Big|_{Z_0}^{Z}\\&=\arg\frac{\overline{\det(CZ+D)}}{\det(CZ+D)}\Bigg|_{Z_0}^{Z}=\arg\frac{\overline{\det(1+D^{-1}CZ)}}{\det(1+D^{-1}CZ)}\Bigg|_{Z_0}^{Z}.\end{aligned}$$

Now, using the explicit formulas for C, D in terms of Z_0

$$D^{-1}CZ=-(1-Z_0^*Z_0)^{1/2}Z_0^*(1-Z_0Z_0^*)^{-1/2}Z=-Z_0^*Z$$

[since $Z_0^* f(Z_0 Z_0^*) = f(Z_0^* Z) Z_0^*$ for an analytic function f]. Hence

$$\int_\Delta \omega = \arg \frac{\overline{\det(1-Z_0^*Z)}}{\det(1-Z_0^*Z)}\Bigg|_{Z_0}^{Z} = \arg \frac{\det\overline{(1-Z_0^*Z)}}{\det(1-Z_0^*Z)},$$

as was to be proved.

Theorem 1 for $D_{p,q}$ follows immediately from the proposition, since Z_0, $Z \in D_{p,q} \Rightarrow Z_0^* Z \in D_{p,p}$, hence the eigenvalues $\lambda_1, \ldots, \lambda_p$ of $Z_0^* Z$ lie in the unit circle, so

$$\left|\int \omega\right| = \left|\arg \frac{1-\bar{\lambda}_1}{1-\lambda_1} + \ldots + \arg \frac{1-\bar{\lambda}_p}{1-\lambda_p}\right| < p\pi$$

because $\left|\arg \frac{1-\bar{\lambda}}{1-\lambda}\right| < \pi$ for $|d| < 1$.

For a domain of type III_p, since it has rank p and is totally geodesically embedded in $D_{p,p}$, the same argument goes through to prove Theorem 1.

For a domain of type II_p, its rank is $[p/2]$ and is totally geodesically embedded in $D_{p,p}$ with minimum holomorphic sectional curvature $-\frac{1}{2}$. Let $\tilde{\omega}$ be the Kähler form induced from $D_{p,p}$. One checks that, in the notation above, $Z_0^* Z$ has rank $\leqq 2[p/2]$, hence we get

$$\left|\int_\Delta \tilde{\omega}\right| \leqq 2[p/2]\pi .$$

If ω is the Kähler form of the normalized Bergmann metric, $\tilde{\omega} = 2\omega$, hence

$$\left|\int \omega\right| \leqq [p/2]\pi$$

which is Theorem 1 in this case.

The case of a domain of type IV requires separate treatment, which we do not present. We have not attempted to try the domains V, VI.

3. Computation of the Norm

So far we have proved the inequality $\|[\omega]\|_\infty \leqq p\pi$. We now prove an equation:

Theorem 2. *Let D, p, ω be as in Sect. 1, and let X be a compact manifold covered by D (thus $X = \Gamma \backslash D$, $\Gamma \subset G$ co-compact and torsion-free). Then $\|[\omega_X]\|_\infty = p\pi$, where $[\omega_X] \in H^2(X, \mathbb{R})$ is the class of ω.*

Corollary. *Let $[\omega]$ be as in Corollary 2, Sect. 1. Then $\|[\omega]\|_\infty = p\pi$.*

Proof of Theorem 2. We know that universally (as in Sect. 1, Corollary 2), $\|[\omega]\|_\infty \leqq p\pi$. We only need to find, for each D, a discrete, torsion-free, co-compact Γ in G so that $\|[\omega_x]\|_\infty = p\pi$ for $X = \Gamma \backslash D$: The proportionality principle [4, Sect. 2.3] says that $\|[\omega_X]\|_\infty$ is the same for all compact X with universal cover D.

We construct an example for the domain $D_{p,q}$. The other domains are treated in a similar way.

We use the following fact from the theory of arithmetic groups, cf. last section of [1]: In the Grassmannian model of $D_{p,q}$, change the form to $|u|^2 - \sqrt{2}|v|^2$, i.e., represent $D_{p,q}$ as the open subset of $G(p,q)$ on which $|u|^2 - \sqrt{2}|v|^2$ is positive definite. Let O be the ring of integers in $Q(i,\sqrt{2})$ and $\Gamma = SL(p+q,O) \cap SU(H)$, where $H = |u|^2 - \sqrt{2}|v|^2$. Then Γ is co-compact in $SU(H) \approx G$.

Now represent the unit disk $D_{1,1} = \{Z \in \mathbb{C} : |z| < 1/\sqrt{2}\}$ in the same way: lines in $\mathbb{C}^2$ on which the form $H_1 = |u|^2 - \sqrt{2}|v|^2$ is positive. The mapping

$$D_{1,1} \to D_{p,q},$$

$$z \to \left(\begin{array}{c} z \quad \\ \ddots \\ \quad z \\ \hline 0 \end{array} \right)$$

is a totally geodesic mapping whose image has curvature $-1/p$ in the normalized Bergmann metric. It is stabilized by a group isomorphic to $SU(H_1)$. Let $\Gamma_1 = SL(2,0) \cap SU(H_1)$. Then Γ_1 is co-compact in $SU(H_1)$. Let Γ_1', Γ' be torsion free subgroups of finite index in Γ_1, Γ respectively, let $S = \Gamma_1' \backslash D_{1,1}$, $X = \Gamma' \backslash D_{p,q}$. Then S is a compact Riemann surface of some genus $g > 1$, X is a compact manifold covered by $D_{p,q}$, and the above mapping gives a mapping $f : S \to X$ with the property that $f^*\omega$ is the Kähler form ($=$ area form) of a metric of constant curvature $-1/p$ on S. Therefore

$$\int_S f^*\omega = p \int_S 1/p f^*\omega = -p \int_S -K dA = p 4\pi(g-1)$$

by the Gauss-Bonnet formula. By the properties of the norm [(1), (2), and (4) of Sect. 1], and Theorem 1,

$$|\int f^*\omega| = |f^*[\omega_X]([S])| \leqq \|[\omega_X]\| \, \|[S]\|_1 \leqq p\pi 4(g-1).$$

Since equality holds, $\|[\omega_X]\|_\infty = p\pi$, as was to be proved.

As pointed out in the introduction, Theorem 1 has the following topological consequence:

Corollary. *Let S be a Riemann surface of genus $g > 1$ and X, ω as above. Let $f : S \to X$ be a continuous map. Then*

$$\left| \int_S f^*\omega \right| \leqq 4p(g-1)\pi.$$

In the proof of Theorem 2 we saw that totally geodesic complex curves of curvature $-1/p$ give extremals for these inequality, and in [8] it was proved that for $p=1$ these are, up to homotopy, the only extremals. If the rank $p > 1$ there can be other extremals. For example, if S is a Riemann surface, $X = S \times S$ and f is the graph of any orientation-preserving diffeomorphism, then equality also holds. It would be interesting to find geometric conditions on X that insure that all extremals are homotopic to geodesics.

Finally we point out that the use of Bochner's formula, as in [8], easily gives that the only holomorphic extremals are geodesics.

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