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Hölder and L^p-Estimates for the ∂ Equation in Some Convex Domains with Real-Analytic Boundary* **

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1. Introduction

Using the so-called Henkin-Ramirez reproducing kernel introduced in [2, 7], Grauert-Lieb and Henkin were able to define for a strictly pseudoconvex domain D an integral operator T satisfying $\bar{\partial}Tf=f$ for $\bar{\partial}$-closed forms f and for which a Hölder 1/2-estimate holds [1, 3, 5]. The problem arises whether something similar is true for weakly pseudoconvex domains.

Reproducing kernels can (trivially) be constructed in convex domains and so the operator T is also defined in this case (see Sect. 2). In [8], Range considered the domain

$$D=\left\{z:\sum_{i=1}^{n}|z_i|^{m_i}\leqq 1\right\},$$

where $m_1,\ldots,m_n$ are positive even integers, and proved that T satisfies a Hölder $1/m$-estimate, where $m=\max\{m_1,\ldots,m_n\}$. It is natural to ask whether this result holds for all convex domains with real analytic boundary.

In this paper we show that this is the case for domains $D=\{z:r(z)<0\}$ where r is a real-analytic, convex function of the type

$$r(z)=\sum_{i=1}^{n}s_i(|z_i|^2)-1,$$

and obtain as well L^p-estimates for the $\bar{\partial}$-equation. As in the strictly pseudoconvex case, the results depend on having a good control of the support function $F(\xi,z)$ $\xi\in bD$, $z\in D$, which appears in the denominator of the reproducing kernel (in the convex case is simply

$$F(\xi,z)=\sum_{i=1}^{n}\frac{\partial r}{\partial\xi_i}(\xi)(\xi_i-z_i)\Big).$$

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In the strictly pseudoconvex case this control is

$$2\operatorname{Re}F(\xi,z)\geqq -r(z)+c|\xi-z|^2,\qquad \xi\in bD, z\in D, |\xi-z| \text{ small}.$$

As in [8], the crucial point is to replace this by

$$2\operatorname{Re}F(\xi,z)\geqq -r(z)+c\,\mathrm{Lr}(\xi)\,(\xi-z)^2+c|\xi-z|^m \tag{$*$}$$

where $\mathrm{Lr}(\xi)$ denotes the complex Hessian of r at ξ and m is some positive integer. The motivation for $(*)$ comes from the fact, also pointed out in [4], that the eigenvalues of $\mathrm{Lr}(\xi)$ appear in the numerator of the kernels. Once $(*)$ is established, the proof of [8] essentially works and we slightly modify it so as to give as well the L^p-estimates.

The paper is organized as follows. We consider first a real version of $(*)$

$$2\operatorname{Re}F(\xi,z)\geqq -r(z)+c\,\mathrm{Hr}(\xi)\,(\xi-z)^2+c|\xi-z|^m,\quad \xi\in bD \tag{$**$}$$

where $\mathrm{Hr}(\xi)$ denotes the real hessian of r at ξ. Note that $(**)$ amounts to say that $\mathrm{Hr}(\xi)$ absorbs, even if it degenerates along certain directions, the remainder terms of the Taylor series of r at ξ. We are able to prove $(**)$ only if, all ξ_i, z_i are small (Sect. 2). Then in Sect. 3 we use it to obtain $(*)$ if $s_i(|z_i|^2)$ is strictly convex away from zero and in this case we show that the results hold for T (Theorems 3.3 and 3.4). In the general case (Sect. 4) we must construct another support function ϕ and consider a modification of T introduced in [9].

One final comment is in order: we feel that $(**)$ is true for all convex (bounded) domains with real analytic boundary. If this were the case, surely it could be applied to treat more general types of domains[1].

Finally, concerning notation, we use c to denote most of the constants and also employ $\simeq$ to mean that two quantities are of the same order of growth.

2. The Canonical Support Function and Its Estimate

2.1

From now on D will denote a bounded domain of the type $D=\{z: r(z)<0\}$ where

$$r(z)=\sum_{i=1}^{n} s_i(|z_i|^2)-1$$

is convex and real-analytic. We will denote by r_i the function of one complex variable defined by $r_i(w)=s_i(|w|^2)$. To be precise, the s_i are assumed to be real-analytic functions in an interval $[0,a_i]$ such that

(i) $s_i'(t)\geqq 0$, $s_i'(t)+2t\,s_i''(t)\geqq 0$ for $0\leqq t<a_i$ (i.e. r_i is convex)

(ii) $s_i(0)=0$ and $s_i(a_i)>1$ (i.e. D is bounded).

If $s_i(t)=b_k t^k+\dots$ is the development near 0, note that this implies $b_k>0$, hence $s_i'(t)>0$ for small $t\neq 0$, hence for all $0<t<a_i$ (because $r_i(t)$ is convex and $r_i'(t)=2t\,s_i'(t^2)$ increases). In particular, $s_i'(t)+t\,s_i''(t)>0$ if $0<t<a$. Since the complex Hessian of r_i at w is $s_i'(|w|^2)+|w|^2\,s_i''(|w|^2)$ this shows that r_i is strictly subharmonic if $w\neq 0$. To exclude the strictly pseudoconvex case we assume of course

(iii) $s_i'(0)=0$.

1 See the note at the end of the paper

Thus $\mathrm{Lr}_i(w)$ degenerates exactly at $w=0$, but $\mathrm{Hr}_i(w)$ may degenerate at some other points.

2.2

We define

$$F_i(w,v)=\frac{\partial r_i}{\partial w}(w)(w-v).$$

Using Taylor's development and the hypothesis on s_i it is easy to see that

$$g_i(w,v)\overset{\text{def}}{=}r_i(v)-r_i(w)+2\operatorname{Re}F_i(w,v)\geqq 0$$
$$\text{and is}\quad =0\quad\text{only if}\quad w=v. \tag{1}$$

The function $F(\xi,z)=\sum_i F_i(\xi_i,z_i)$ is the canonical support function for the domain D.

2.3

We shall obtain two different estimates of the function g_i in (1). In this subsection we drop out the index i and write r for r_i, g for g_i, F for F_i.

The first estimate is well known and is based on the following result [11]:

Theorem (Lojasiewicz). *Let g be a real analytic function in an open set U in $\mathbb{R}^n$ and let $z(g)$ denote the set of zeroes of g. Given a function $h\in C^\infty(U)$ vanishing on $z(g)$ and a compact set $K\subset U$, there exist a positive constant c and a positive integer m such that $|g(x)|\geqq c|h(x)|^m$, $x\in K$.*

By (1), we can apply the above to $g(w,v)$ in (1) with $h(w,v)=|w-v|^2$. Hence we get

$$g(w,v)\geqq c|w-v|^m \tag{2}$$

(this is the observation that all domains with real-analytic boundary are of finite type).

The second estimate we will need is the one contained in the following theorem. We use the notation

$$\mathrm{Hr}(w)(w-v)^2=(w-v,\bar w-\bar v)\begin{pmatrix} r_{w\bar w} & r_{ww}\\ r_{\bar w\bar w} & r_{w\bar w}\end{pmatrix}\begin{pmatrix}\bar w-\bar v\\ w-v\end{pmatrix}$$

Theorem. *There exists $\varepsilon>0$ and $c>0$ such that*

$$g(w,v)\geqq c\,\mathrm{Hr}(w)(w-v)^2\quad \textit{for}\quad |w|,|v|\leqq\varepsilon. \tag{3}$$

(Note that the theorem simply says that if r is a radial, convex and analytic function in a neighbourhood of 0 in the complex plane, then $\mathrm{Hr}(w)(w-v)^2$ controls all the other terms of higher order in the Taylor's development of $r(v)$ at w, for small v, w).

Proof. We regard both terms in (3) as functions of (w,v) in a neighbourhood of the origin in $\mathbb{C}^2$. First we prove the theorem for $r(z)=|z|^{2k}$. In this case by homogeneity we may suppose $(w, v) \in S^3$, the unit sphere in $\mathbb{C}^2$ and, because of (1), we may suppose as well that (w, v) is near the diagonal of S^3. But then w is far from zero and the result follows, for $\mathrm{Hr}(w)$ does not degenerate there.

The general case will be essentially reduced to this one. The main tool is the so-called "curve selection lemma", whose proof can be found in [6]:

Curve Selection Lemma. *Let U be an open set in $\mathbb{R}^m$ which is defined by inequalities between real-analytic functions. Then, if 0 is an accumulation point of U, there exists an analytic curve $\gamma: [0, \alpha) \to \mathbb{R}^m$ such that $\gamma(0) = 0$ and $\gamma(t) \in U$ for $0 < t < \alpha$.*

We use it in the following form:

Corollary. *Let G_1, G_2 be two analytic functions in a neighbourhood of 0 in $\mathbb{R}^m$. Then to prove that $G_1 \geqq G_2$ in some neighbourhood of 0 it is enough to prove that for each analytic curve $\gamma(t)$, with $\gamma(0) = 0$, there exists $\alpha > 0$ (which may depend on γ!) such that $G_1(\gamma(t)) \geqq G_2(\gamma(t))$ for $0 \leqq t < \alpha$.*

Hence it is enough to find $c > 0$ such that for any analytic curve $\gamma(t) = (w(t), v(t))$, $\gamma(0) = 0$, there exists $\alpha = \alpha(\gamma)$ such that

$$g(w(t), v(t)) \geqq c\,\mathrm{Hr}(w(t))\,(w(t) - v(t))^2$$

for $0 < t < \alpha$. Lets consider the developments of w, v

$$w(t) = w_0 t^p + o(t^p), \qquad v(t) = v_0 t^p + o(t^p),$$
$$w_0 \text{ or } v_0 \neq 0, \qquad p \geqq 1$$

and that of r, $r(z) = \sum_{m=k}^{\infty} b_m |z|^{2m}$ with $b_k > 0$. Here we write $r_k(z) = |z|^{2k}$, g_k for the corresponding function defined by (1) and let $c_k > 0$ be some constant corresponding to this case. Then, clearly,

$$g(w(t), v(t)) = b_k g_k(w_0, v_0) t^{2pk} + o(t^{2pk})$$
$$\mathrm{Hr}(w(t))\,(w(t) - v(t))^2 = b_k\,\mathrm{Hr}_k(w_0)\,(w_0 - v_0)^2 t^{2pk} + o(t^{2pk}).$$

Hence, if $w_0 \neq v_0$, t^{2pk} is the lower order term and we can choose any $c < c_k$.

If $w_0 = v_0$ we must find out which is the lower order term in $g(w(t), v(t))$. To do so we will make a change of parameter. Note that, due to the rotation invariance of (3) we may assume that w is real and take, for small values of t, w as a new parameter (assuming $w_0 > 0$). Let $x(t)$, $y(t)$ denote the real and imaginary parts of $v(t)$. Since now we are assuming that $v_0 = w_0$ is real

$$\begin{aligned} x(w) &= w + \varphi w^\alpha + o(|w|^\alpha), \\ y(w) &= \sigma w^\beta + o(|w|^\beta), \end{aligned} \tag{4}$$

with α, β rationals greater than 1. Lets denote as before $r_m(z) = |z|^{2m}$ and g_m for the corresponding function. A computation shows that

$$g_m(x(w), y(w), w)$$
$$= \sum_{k_1 + k_2 + k_3 = m}' \frac{m!}{k_1!\,k_2!\,k_3!} 2^{k_2} w^{2k_1 + k_2} (x - w)^{k_2} ((x - w)^2 + y^2)^{k_3}, \quad k_i \geqq 0 \tag{5}$$

where by the tilda we mean that the combinations $k_1=m$, $k_2=k_3=0$ and $k_1=m-1$, $k_2=1$, $k_3=0$ are omitted. When we use (4) we find that the k-th term of (5) is, if $\varrho=\min(\alpha,\beta)$

$$O(w^{2k_1+k_2+k_2\alpha+2k_3\varrho})+\text{higher order terms}.$$

Since $2k_1+k_2+k_2\alpha+k_3\varrho=2m+(\alpha-1)k_2+2(\varrho-1)k_3$, it turns out that

$$g_m=\text{const}\, w^{2m+2(\varrho-1)}+\text{higher order terms}$$

(in fact this is the contribution of the term $k_2=0$ $k_3=1$ and also that of $k_2=2$ and $k_3=0$ if $\alpha=\varrho$). This implies

$$g(x(w),y(w),w)=Aw^{2k+2(\varrho-1)}+o(|w|^{2k+2(\varrho-1)})$$

with $A>0$, i.e., what this analysis reveals is that the lower order term of $g(x(w),y(w),w)$ comes *only* from the first term $b_k|z|^{2k}$ of r (and A depends on b_k, k, φ^2, and σ^2). The same will happen with $\mathrm{Hr}(w)(r-w)^2$, which consists in the terms $k_2=0$, $k_3=1$ (with its corresponding coefficients) of (5). In fact we will have, of course,

$$\mathrm{Hr}(w)(v-w)^2=2Aw^{2k+2(\varrho-1)}+o(|w|^{2k+2(\varrho-1)}).$$

Therefore it suffices to take c also smaller than 1/2. This ends the proof of the theorem. □

Remark. We have been unable to obtain a more satisfactory proof of the theorem, in particular one not so heavily based on the radial character of r. It would be very interesting to find it, and for other types of functions. As said in the introduction we feel that the theorem holds for all real-analytic and convex functions[1].

For w real we already noticed that $\mathrm{Hr}(w)$ is a diagonal matrix with entries $2s'(w^2)+4w^2s''(w^2)$ and $2s'(w^2)$, and $\mathrm{Lr}(w)=s'(w^2)+w^2s''(w^2)$. Now it is clear that these three quantities are of the same order if w is near 0. Hence $\mathrm{Hr}(w)(w-v)^2\simeq\mathrm{Lr}(w)|w-v|^2$ if $|w|$ is small. Thus we can replace Hr by Lr in (3). Using then (2), we get

$$\begin{aligned} &r(v)-r(w)+2\operatorname{Re}F(w,v)\\ &\quad\geqq c\,\mathrm{Lr}(w)|w-v|^2+c|w-v|^m \quad \text{for} \quad |w|,|v|\leqq\varepsilon. \end{aligned} \tag{6}$$

3. The Results in a Particular Case

3.1

In this section we will prove our results with the auxiliary hypothesis

(iv) $$s_i'(t)+2ts_i''(t)>\text{for}\quad 0<t<a_i,$$

which means that $\mathrm{Hr}_i(w)$ only degenerates at $w=0$. As said in the introduction, (iv) is not necessary but will allow us to obtain the results for the canonical kernels associated with the canonical support function $F(\xi,z)$ in 2.2. To make more clear the exposition we prefer to treat first this case and then deal in the next section with the general case.

1 See the note at the end of the paper

Since now Hr_i and Lr_i only degenerate at $w=0$ it is clear that g_i will satisfy (3) with Hr_i replaced by Lr_i for all $|w| \geqq \varepsilon$ if $|w-v|$ is small. Thus (6) holds for r_i if $|w-v|$ is small, hence for all w, v because of (1), and adding on i we obtain the fundamental inequality

$$r(\xi)-r(z)+2\operatorname{Re}F(\xi,z) \geqq c\,\mathrm{Lr}(\xi)(\xi-z)^2+c|\xi-z|^m, \qquad \xi, z \in \bar{D} \tag{7}$$

which is the basis of all estimates.

3.2

Now we briefly recall the definition of the reproducing kernel $H(\xi, z)$ and the integral solution operator T. Set

$$w'(\xi)=\sum_{i=1}^{n}(-1)^{i-1}\xi_i \bigwedge_{j\neq i} d\xi_j, \qquad w(\xi)=d\xi_1 \wedge \ldots \wedge s\xi_n$$

$$P_i(\xi)=\frac{\partial r}{\partial \xi_i}(\xi), \qquad P(\xi)=(P_1(\xi), \ldots, P_n(\xi)).$$

For $u \in C^1(\bar{D})$ one has the decomposition formula

$$u(z)=c_n \int_{bD} u(\xi)H(\xi,z)+c_n \int_{bDx[0,1]} \bar{\partial}u(\xi)\wedge w'(\eta)\wedge w(\xi)$$

$$-c_n \int_{D} \bar{\partial}u(\xi)\wedge B(\xi,z)$$

where

$$\eta=(1-t)\frac{\bar{\xi}-\bar{z}}{|\xi-z|^2}+t\frac{P(\xi)}{F(\xi,z)}, \qquad c_n=(-1)^{n(n-1)/2}(n-1)!\,(2\pi i)^{-n}$$

and

$$H(\xi,z)=w'\left(\frac{P}{F}\right)\wedge w(\xi),$$

$$B(\xi,z)=w'\left(\frac{\bar{\xi}-\bar{z}}{|\xi-z|^2}\right)\wedge w(\xi)$$

are respectively the reproducing kernel and the Bochner-Martinelli kernel. The integral solution operator T is then, for a $(0,1)$ form f regular to the boundary,

$$Tf(z)=T_1f(z)+T_2f(z),$$

where

$$T_1f(z)=c_n \int_{bDx[0,1]} f(\xi)\wedge w'(\eta)\wedge w(\xi),$$

$$T_2f(z)=-c_n \int_{D} f(\xi)\wedge B(\xi,z).$$

Using the relation

$$(n-1)!\; w'(\eta)\wedge w(\xi)$$
$$=(-1)^{n(n-1)/2}\left(\sum_{i=1}^{n}\eta_i d\xi_i\right)\wedge\left(\sum_{i=1}^{n} d\eta_i\wedge d\xi_i\right)^{n-1}$$

and Newton's binomial formula for two forms, it is easily seen that the component in dt of $w'(\eta)\wedge w(\xi)$ to be considered is

$$\frac{(-1)^{n(n-1)/2}}{(n-2)!}\gamma_1\wedge\gamma_2\wedge(t\bar{\partial}_\xi\gamma_2+(1-t)\bar{\partial}_\xi\gamma_1)^{n-2}\wedge dt$$

where

$$\gamma_1=\frac{\partial_\xi|\xi-z|^2}{|\xi-z|^2}\quad\text{and}\quad\gamma_2=\frac{\partial r(\xi)}{F(\xi,z)}$$

Now, since

$$\bar{\partial}_\xi\gamma_1=\frac{\bar{\partial}_\xi\partial_\xi|\xi-z|^2}{|\xi-z|^2}+\gamma_1\wedge\ldots,\ \bar{\partial}_\xi\gamma_2=\frac{\bar{\partial}\partial r(\xi)}{F(\xi,z)}+\gamma_2\wedge\ldots$$

we finally obtain, performing the integration in t, that $T_1 f$ can be written

$$T_1=\sum_{k=0}^{n-2}c(n,k)H_k$$

where the $c(n,k)$ are constants and

$$H_k f(z)=\int_{bD} f(\xi)\wedge H_k(\xi,z)\,,\qquad z\in D$$

$$H_k(\xi,z)=|\xi-z|^{2(k+1-n)}F(\xi,z)^{-k-1}\partial_\xi|\xi-z|^2$$
$$\wedge\,\partial r(\xi)\wedge(\bar{\partial}\partial r(\xi))^k\wedge(\bar{\partial}\partial|\xi-z|^2)^{n-k-2}. \tag{8}$$

In the same manner the reproducing kernel $H(\xi,z)$ can be written

$$H(\xi,z)=\frac{(-1)^{n(n-1)/2}}{(n-1)!}\,\frac{\partial r(\xi)\wedge(\bar{\partial}\partial r(\xi))^{n-1}}{F(\xi,z)^n}. \tag{9}$$

3.3. Theorem. *If m is as in (7), for each $\alpha<1/m$ there exists $c(\alpha)>0$ such that*

$$|Tf(z)-Tf(w)|\leqq c(\alpha)\|f\|_\infty|z-w|^\alpha,\qquad z,w\in D. \tag{10}$$

Proof. Once one has (7) the proof of [8] can be applied. We include it here presented in a somewhat different way, because some aspects of the computation will be needed as well to obtain the L^p-estimates.

It is well known that $T_2 f$ has modulus of continuity $\delta|\log\delta|$. So, taking into account the expression of $T_1 f$ obtained above it is enough to prove

$$\left|\int_{bD} v_z H_k(\xi,z)\wedge f(\xi)\right|\leqq c(\alpha)|r(z)|^{\alpha-1}\quad v=\text{normal field} \tag{11}$$

for it is also well known that this implies that $H_k f$ satisfies (10). Now, there exists r_0 and δ_0 such that if $|r(z)|<r_0$ and $\frac{\partial r}{\partial z_j}(z) \neq 0$, say $j=1$, then

$$t_1 = r(\xi)-r(z), \qquad t_2 = \operatorname{Im} F(\xi,z),$$
$$t_{2j-1} = \operatorname{Re}(\xi_j - z_j), \qquad t_{2_j} = \operatorname{Im}(\xi_j - z_j) \qquad j=2,\dots,n$$

is a real coordinate system in the ball $B(z,\delta_0)$ such that $t(z)=0$, $|t|^2 \simeq |\xi - z|^2$ and $d\sigma(\xi) \simeq dt_2 dt_3 \dots dt_{2n}$ in $bD \cap B(z,\delta_0)$. Of course, in proving (11) we may assume that $|r(z)|<r_0$ and estimate only the contribution of $bD \cap B(z,\delta_0)$ in the integral. Hence we proceed to estimate $v_z H_k(\xi,z) \wedge f(\xi)$ as a multiple of $dt_2 \wedge \dots \wedge dt_{2n}$ in $B(z,\delta_0)$. From (8),

$$\begin{aligned} v_z H_k(\xi,z) = \Big\{ &(k+1-n) \frac{v_z|\xi-z|^2}{|\xi-z|^{2(n-k)}} \frac{\partial_\xi |\xi-z|^2}{F(\xi,z)^{k+1}} \\ &-(k+1)\frac{v_z F(\xi,z)}{F(\xi,z)^{k+2}} \frac{\partial_\xi|\xi-z|^2}{|\xi-z|^{2(n-k-1)}} \\ &+ \frac{v_z \partial_\xi |\xi-z|^2}{|\xi-z|^{2(n-1-k)} F(\xi,z)^{k+1}} \Big\} \\ &\wedge \partial r(\xi) \wedge (\bar\partial\partial r(\xi))^k \wedge (\bar\partial\partial|\xi-z|^2)^{n-k-2}. \end{aligned}$$

We use the notations

$$\alpha_i(\xi) = \frac{\partial^2 r}{\partial \bar\xi_i \, \partial \xi_i}(\xi), \qquad P_i(\xi) = \frac{\partial r}{\partial \xi_i}(\xi).$$

Using $\partial r, d\xi_2, \dots, d\xi_n, \bar\partial r, d\bar\xi_2, \dots, d\bar\xi$ as basis of 1-forms it is easy to see that $\partial r(\xi) \wedge (\bar\partial\partial r(\xi))^k$ equals the exterior product of $\partial r(\xi)$ with

$$\begin{aligned} &\left(\sum_{i=2}^n \alpha_i d\bar\xi_i \wedge d\xi_i\right)^k \\ &+ k\alpha_1 |P_1|^{-2} \left(\sum_{i=2}^n \alpha_i d\bar\xi_i \wedge d\xi_i\right)^{k-1} \wedge \sum_{i=2}^n \bar P_i d\bar\xi_i \wedge \sum_{i=2}^n P_i d\xi_i. \end{aligned}$$

This last expression is in turn a sum of forms of type (k,k) in $d\xi_2, \dots, d\xi_n$ whose coefficients are products of k different α's between the $\alpha_2, \dots, \alpha_n$ or products of $k-1$ different α's, $\alpha_{i_1}, \dots, \alpha_{i_{k-1}}$ between the $\alpha_2, \dots, \alpha_n$, with $k\alpha_1|P_1|^{-2}\bar P_j P_k$, j,k different from $1, i_1, \dots, i_{k-1}$.

Now observe that the hypothesis on r imply that

$$|P_j| \leqq c\alpha_j$$

To summarize, $v_z H_k(\xi,z) \wedge f(\xi)$ is in $bD \cap B(z,\delta_0)\, d\sigma(\xi)$ times a function which is a finite linear combination of functions of type

$$h^1_{i,i_1 \dots i_k}(\xi) \frac{f_i(\xi)\alpha_{i_1}(\xi) \dots \alpha_{i_k}(\xi)}{|\xi-z|^{2n-2k-2} F(\xi,z)^{k+1}} \tag{12}$$

or

$$h^2_{i,i_1 \dots i_k}(\xi) \frac{f_i(\xi)\alpha_{i_1}(\xi) \dots \alpha_{i_k}(\xi)}{|\xi-z|^{2n-2k-3} F(\xi,z)^{k+2}} \tag{12}$$

where $f=\sum f_i d\bar{\xi}_i$, the $i_1, \ldots, i_k$ are different indexes between $2, \ldots, n$ and the h's are bounded functions (with bounds independent of z).

From (7) follows that

$$\frac{\alpha_j(\xi)}{|F(\xi,z)|} \leqq c \frac{1}{|r(z)|+|\xi_j - z_j|^2}, \qquad \xi \in bD, \qquad z \in D \tag{13}$$

Writing t' for $(t_3, \ldots, t_{2n})$, observing that $t_1 = |r(z)|$ is the equation of bD and using (7) and (13) one is lead to the estimates (assuming $i_1 = 2, \ldots, i_k = k+1$)

$$\begin{aligned} I_1(k,t_1) &= \int_{\substack{|t_2| \leqq \delta_0 \\ |t'| \leqq \delta_0}} \cdot \frac{dt_2 dt'}{(t_1+|t_2|+|t'|)^{2n-2k-2}(t_1+|t_2|+|t'|^m) \prod_{j=2}^{k+1} (t_1 + t_{2-1}^2 + t_2^2)} \leqq c(\alpha) t_1^{\alpha-1} \\ I_2(k,t_1) &= \int_{\substack{|t_2| \leqq \delta_0 \\ |t'| \leqq \delta_0}} \cdot \frac{dt_2 dt'}{(t_1+|t_2|+|t'|)^{2n-2k-3}(t_1+|t_2|+|t'|^m)^2 \prod_{j=2}^{k+1} (t_1 + t_{2j-1}^1 + t_{2j}^2)} \leqq c(\alpha) t_1^{\alpha-1} \end{aligned} \tag{14}$$

which can be consulted in [8]. □

3.4. Theorem. *For each p, $1 \leqq p \leqq \infty$, there exists $c(p) > 0$ such that* $\|Tf\|_{L^p} \leqq c(p)\|f\|_{L^p}$.

Proof. Again it is enough to consider T_1, i.e., to prove the estimate for each H_k, $k=0, \ldots, n-2$. It is then convenient to have an expression of $H_k f$ as a volume integral. To do this we employ

$$A(\xi,z) = -r(\xi) + F(\xi,z)$$

as a continuation of $F(\xi,z)$ inside D. Note that $\operatorname{Im} A = \operatorname{Im} F$ and that (7) gives

$$2\operatorname{Re} A(\xi,z) \geqq -r(\xi) - r(z) + c\operatorname{Lr}(\xi)(\xi-z)^2 + c|\xi-z|^m \tag{15}$$

and so $A(\xi,z)$ never vanishes for $\xi, z \in D$. By the explicit expression (8) we see that

$$|H_k(\xi,z)| = 0(|\xi-z|^{2k+3-2n}),$$
$$|\bar{\partial}_\xi H_k(\xi,z)| = 0(|\xi-z|^{2k+2-2n}).$$

So, applying Stokes theorem in $D \backslash B(z,\varepsilon)$ and making $\varepsilon \to 0$ we obtain

$$H_k f(z) = \int_D f(\xi) \wedge \bar{\partial}_\xi H_k(\xi,z)$$

with H_k now given by (8) with F replaced by A. Clearly it is enough to show (denoting by dm the Lebesgue measure in D)

$$\int_D |\bar{\partial}_\xi H_k(\xi,z)| \mathrm{dm}(\xi) \leqq c, \qquad \int_D |\bar{\partial}_\xi H_k(\xi,z)| \mathrm{dm}(z) \leqq c.$$

To prove the first, by (15) we may assume, with the notations used in the proof of Theorem 3.3, that $|r(z)|<r_0$ and just estimate the integral over $D\cap B(z,\delta_0)$ where $dm(\xi)\simeq dt_1\dots dt_{2n}$. In a similar way as in Theorem 3.3, it turns out that the coefficients of $\bar\partial_\xi H_k(\xi,z)$ are in $D\cap B(z,\delta_0)$ of the type (12) with F replaced by A (and without the f_i's). Using then (15), one is lead as before to the estimates

$$\int_0^{\delta_0} I(k,t_1)dt_1\leqq c\,,\qquad \int_0^{\delta_0} I(k,t_1)dt_1\leqq c\,,\qquad k=0,\dots,n-2$$

which follow from the ones in (14). The second integral is evaluated exactly in the same way. □

Remark. As it can be easily seen from the proofs, Theorems 3.3 and 3.4 hold for all convex domains with real analytic boundary if $n=2$. This is so because in this case the estimate $r(z)-r(\xi)+2\operatorname{Re}F(\xi,z)\geqq c|\xi-z|^m$ is already sufficient (this is noticed in [8] for Theorem 3.3).

3.5

Finally we point out that in a similar way, with (9) and the basic inequalities (7) and (13) one can prove the following.

Theorem. *The reproducing kernel $H(\xi,z)$ satisfies*

$$\int_{bD} |H(\xi,z)|\,|\xi-z|d\sigma(\xi)\leqq c\,,\qquad z\in D\,.$$

Using this, one can extend to D a series of well known results for the strictly pseudoconvex domains as for instance, Cole-Range's Theorem on the structure of Henkin measures and its corollaries concerning the equivalence of the notions of zero set, peak set and peak-interpolation set for the algebra of holomorphic functions in D, continuous on $\bar D$ (see [10, p. 198]).

4. The Results in the General Case

4.1

In the absence of condition (iv) of 3.1. we cannot prove that the canonical support function satisfies (7) (because we are not able to prove (3) for all w, v near a point of degeneracy of Hr_i different from zero). Thus our first taks in the general case is to replace F by another support function ϕ for which (7) holds. This can easily be done using the fact that the r_i are strictly subharmonic away from zero.

We define

$$G_i(w,v)=F_i(w,v)+\frac{1}{2}\frac{\partial^2 r_i}{\partial w^2}(w)\,(w-v)^2\,.$$

Let $\varepsilon>0$ be as in (3). By strict subharmonicity in $\varepsilon/2<|w|<a_i^{1/2}$, one has

$$r_i(v)-r_i(w)+2\operatorname{Re}G_i(w,v)\geqq c\,\mathrm{Lr}_i(w)|w-v|^2\simeq|w-v|^2 \tag{16}$$

if $|w|\geqq\varepsilon/2$ and $|w-v|$ is small, say $|w-v|<\delta$.

Let now $\varphi(w)$ be a C^∞-function in the complex plane, $0 \leqq \varphi \leqq 1$ equal to 1 in $|w| < \varepsilon/2$ and zero outside $|w| < 2\varepsilon/3$. We define now

$$F'_i(w, v) = \varphi(w) F_i(w, v) + (1 - \varphi(w)) G_i(w, v) = P'_i(w, v)(w - v)$$

with

$$P'_i(w, v) = P_i(w) + (1 - \varphi(w)) \frac{1}{2} \frac{\partial^2 r_i}{\partial w^2}(w)(w - v).$$

From (6) (for r_i and F_i) and (16) it follows, reducing δ if necessary, that

$$r_i(v) - r_i(w) + 2\operatorname{Re} F'_i(w, v) \geqq c \operatorname{Lr}_i(w) |w - v|^2 + c|w - v|^m$$

for $|w - v| < \delta$. Now we define for $\xi, z \in \bar{D}$

$$F'(\xi, z) = \sum_{i=1}^{n} F'_i(\xi_i, z_i).$$

Then $F' \in C^\infty(\bar{D} \times \bar{D})$, is a polynomial in z and

$$r(\xi) - r(z) + 2\operatorname{Re} F'(\xi, z) \geqq c \operatorname{Lr}(\xi)(\xi - z)^2 + c|\xi - z|^m$$

holds for $|\xi - z| < \delta$.

4.2

To avoid the Henkin-Ramirez procedure and the division problem involved, we will use the simpler method of [4] and [9], which we briefly recall. Choose $\chi \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n)$ such that $0 \leqq \chi \leqq 1$, $\chi \leqq 1$ for $|\xi - z| \leqq \delta/2$, and $\chi = 0$ for $|\xi - z| \geqq \delta$. For $i = 1, \ldots, n$ define

$$P_i(\xi, z) = \chi P'_i(\xi_i, z_i) + (1 - \chi)(\bar{\xi}_i - \bar{z}_i)$$

$$\phi(\xi, z) = \sum_{i=1}^{n} P_i(\xi, z)(\xi_i - z_i) = \chi F'(\xi, z) + (1 - \chi)|\xi - z|^2.$$

Then there exists $\eta > 0$ such that $|\phi(\xi, z)| \geqq c > 0$ for $\xi \in bD$, $r(z) < \eta$ if $|\xi - z| \geqq \delta/2$, $\phi(\xi, z)$ and $P_i(\xi, z)$ are holomorphic in z in $|\xi - z| < \delta/2$ and the fundamental inequality

$$r(\xi) - r(z) + 2\operatorname{Re} \phi(\xi, z) \geqq c \operatorname{Lr}(\xi)(\xi - z)^2 + c|\xi - z|^m \tag{17}$$

remains for $|\xi - z| < \delta/2$. For $t \in [0, 1]$ and $\xi \in bD$, one sets

$$w_i(\xi, z, t) = t \frac{P_i(\xi, z)}{\phi(\xi, z)} + (1 - t) \frac{\bar{\xi}_i - \bar{z}_i}{|\xi - z|^2}$$

which is well defined for $z \in D \cup \{z : r(z) \leqq \eta, |z - \xi| \geqq \delta/2\}$, and $W = \sum_{i=1}^{n} w_j d\xi_i$.

Finally, the Cauchy-Fantappié form $\Omega_q(w)$ is defined, for $q = 0, 1$, by

$$\Omega_q(w) = (2\pi i)^{-n} \binom{n-1}{q} W \wedge (\bar{\partial}_{\xi, \lambda} W)^{n-q-1} \wedge (\bar{\partial}_z W)^q$$

Set now, for a (0, 1) form f

$$T_0 f = \int_{bD \times [0,1]} f \wedge \Omega_0 - \int_{D \times \{0\}} f \wedge \Omega_0$$

$$Ef = \int_{bD} f \wedge \mu * \Omega_1, \quad \mu(\xi) = (\xi, 1).$$

In [9] it is proved the following

Lemma. *If $f \in C^1_{(0,1)}(\bar{D})$, Ef is defined and of class C^∞ on $\bar{D}_\eta = \{z : r(z) \leqq \eta\}$. If $\bar\partial f = 0$, then $\bar\partial Ef = 0$ in D and $f = Ef + \bar\partial T_0 f$. Further*

$$\|Ef\|_{L^\infty(D^\eta)} \leqq c \|f\|_{L^\infty(D)}.$$

Thus the problem of solving $\bar\partial u = f$ in D is reduced to that of solving $\bar\partial u = Ef$. But now $Ef \in C^\infty_{(0,1)}(\bar{D})$ and if T_η denotes the canonical operator T of 3.2. for the (convex) domain $D_{\eta'}$ it follows that the operator

$$S = T_\eta E + T_0 \quad \text{satisfies} \quad \bar\partial S f = f$$

4.3. Theorem. *The operator S satisfies the estimates*

$$|Sf(z) - Sf(w)| \leqq c(\alpha) \|f\|_\infty |z - w|^\alpha, \qquad \alpha < 1/m$$

$$\|Sf\|_{L^p(D)} \leqq c(p) \|f\|_{L^p(D)}.$$

Proof. The estimates that an integral operator like E or T_0 satisfy only depend on the singularity of its kernel near the diagonal. The kernel $\mu * \Omega_1$ of E is zero for $|\xi - z| < \delta/2$ and so there is no problem with E. Clearly for T_η we have

$$\|T_\eta Ef\|_{L^p(D)} \leqq c(p) \|Ef\|_{L^p(D_\eta)}$$

$$\|T_\eta Ef\|_{L_{ip}\ D} \leqq c(\alpha) \|Ef\|_{L^\infty(D_\eta)}.$$

Hence everything is reduced to T_0. But, because of (17), we can argue with T_0 in the same manner we did for T in Theorems 3.3 and 3.4. The only change is that in (8) one must replace $\partial r(\xi) = \sum_{i=1}^{n} P_i(\xi) d\xi_i$ by $\sum_{i=1}^{n} P_i(\xi, z) d\xi_i$. Near the diagonal $P_i = P_i'$ and it is easily seen that also $|P_i'| \leqq c\alpha_i$. The rest of the proof is identical. □

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Note added in proof: Recently Prof. A. Nagel has sent us a nice proof of inequality (**) for all convex functions r for which there exists an integer m such that for all directions v and all points x, $\sum_{j=2}^{m} |D_v^j r(x)| > 0$. This can be used to extend our results to convex domains of finite type satisfying certain geometric conditions, the most relevant being the following one:

$$Hr(w)(v,v) \simeq Hr(w)(Jv, Jv),$$

for $w \in bD$ and $v \in T_w^c(bD)$, the complex tangent space at w. Here J denotes the operator of multiplication by i and it can be easily seen that this condition does depend on the defining function r.