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Solutions of Minimal Period for a Class of Convex Hamiltonian Systems

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0. Introduction

In this paper we deal with the existence of periodic solutions of Hamiltonian systems with n degree of freedom:

$$(H)\quad \dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad \dot{q}_k = \frac{\partial H}{\partial p_k}, \quad k = 1, \dots, n.$$

Recently Rabinowitz [16] has shown that if H is "superquadratic" then for all $T>0$ (H) has nontrivial T-periodic solutions. He left open the question to find a solution having T as *minimal* period.

The results we are able to prove here are just a contribution to the solution of such a problem. One of these results is valid for a class of convex, superquadratic Hamiltonians, to which belong, for example, the H homogeneous of degree greater than 2: for every $T>0$ (H) possesses a solution having T as minimal period. A second result is concerned with the case in which $H''(0)$ is a positive definite symmetric matrix. This situation is different from the previous one because it is known by examples (cf. [16]) that the minimal period of the solutions of (H) cannot be arbitrarily large. We show that it is possible to find a solution of minimal period T, provided $T < \frac{2\pi}{\omega_n}$, the smallest fundamental pulsation of the linear part. This result is proven in the particular case in which (H) can be transformed in a second-order system. Even if we expect the result to be true for (H), we are not able to handle such general case.

The method of the proof consists in two steps. First, using a device introduced by Clarke [21], we transform the original system (H) into an equivalent one by means of the Legendre-transform G of H. This new dual system is solved using the calculus of variation in the large, namely finding the stationary points of a functional f on a Banach space E. Unfortunately, since H is superquadratic, f is not bounded from above nor from below and the stationary points of f are saddle points (cf. Ekeland [11], where is given an elegant proof of the Rabinowitz result).

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On the other side, it is natural to expect that solutions of minimal period should correspond to some "minimal" property of these stationary points. Then, unlike [11], we use the following device: the search of the stationary points of f on E is substituted with an *equivalent* problem of critical points of f on a suitable manifold M: it turns out that f has a minimum on M, which allows to find a solution of (H) having T as minimal period. Such a device has been used by several people (Nehari [14], Coffman [10], Hempel [12, 13], [1, 3]) for the purpose of the existence of solutions of elliptic equations. While those results can be found in greater generality using direct variational arguments (see, for example Clark [7], [5, 2]), the new feature here is that the method is employed to obtain stationary points satisfying additional properties (for another application to a free boundary problem arising in fluid dynamics, see [4]).

The paper is divided into 6 sections. Section 1 contains the abstract framework, while in Sect. 2, 4, and 6 are stated the existence results. The relationships between the periods and the amplitudes of the orbits are investigated in Sect. 3. Section 5 is devoted to obtain wider classes to which apply the above results: with perturbation arguments we find solutions of large minimal period and small amplitude.

While many papers deal with the existence of periodic solutions of Hamiltonian systems (see, for example [16, 21, 11], Amann and Zehnder [19] and references therein), on the contrary, we know only one result concerning the existence of solutions of (H) with prescribed minimal period: the work by Clarke and Ekeland [8]. They assume H convex and *subquadratic*: in this case, unlike then in our situation, the dual functional is bounded from below and has a minimum to which correspond a solution with given minimal period.

1. Abstract Setting: Minimal Critical Points

Let E be a real Banach space with dual E'; denote by $\|\cdot\|$ the norm in E and by $\langle\cdot,\cdot\rangle$ the pairing between E and E'. Let us consider a functional $f: E \to \mathbb{R}$, which throughout in the following will be assumed to possess a continuous Frechét derivative $f'(u)$. Motivated by the applications in the following sections, we take f of the form:

$$f(u) = -\tfrac{1}{2}a(u,u) + b(u),$$

where $a\colon E\times E \to \mathbb{R}$ and $b\colon E \to \mathbb{R}$ satisfy to

(I_1) a is a continuous, symmetric, bilinear form;

(I_2) there exist α, $1<\alpha<2$ and constants c_1, $c_2>0$ such that

$$c_1\|u\|^\alpha \leqq \langle b'(u),u\rangle \leqq c_2\|u\|^\alpha \quad \forall u\in E;$$

(I_3) there exists $\theta>\frac{1}{2}$ such that

$$b(u) \geqq \theta\langle b'(u),u\rangle \quad \forall u\in E;$$

(I_4) set $d(u)=\langle b'(u),u\rangle$, then $d\in C^1(E,\mathbb{R})$ and

$$\langle d'(u),u\rangle < 2d(u).$$

Under the above assumptions (indeed under much weaker ones) it is known that, if a is positive somewhere and condition $(P-S)$ holds, then f has a stationary point $v \neq 0$, i.e. $f'(v)=0$, which can be found using minimax arguments [5]. But, in view of our applications, we are interested in finding stationary points veryfying additional properties. To this purpose, we are going to expose a variational principle which will enable us to find stationary points as minima. Set

$$h(u)=\langle f'(u), u\rangle$$

and

$$M=\{u\in E, u\neq 0 : h(u)=0\}.$$

If $f'(u)=0$ for some $u\in E$, $u\neq 0$, then $u\in M$ and is a critical point of $f_{/M}$, the restriction of f to M. Conversely we have:

Proposition 1.1. *Suppose $f, h\in C^1(E,\mathbb{R})$ and*

$$(1.1)\quad \langle h'(u), u\rangle \neq 0 \quad \forall u\in M.$$

Then $f'(u)=0$, $u\neq 0$, iff u is a critical point of $f_{/M}$.

Proof. First of all, let us note that (1.1) implies that M is a C^1 manifold of codimension 1 in E. Let $\bar{u}\in M$ be such that $f'_{/M}(\bar{u})=0$. This means that $\bar{u}\neq 0$ and there exists $\lambda\in\mathbb{R}$ such that $f'(\bar{u})=\lambda h'(\bar{u})$. Then it follows

$$(1.2)\quad \langle f'(\bar{u}),\bar{u}\rangle=\lambda\langle h'(\bar{u}),\bar{u}\rangle.$$

Since $\langle f'(\bar{u}),\bar{u}\rangle=h(\bar{u})=0$, while $\langle h'(\bar{u}),\bar{u}\rangle\neq 0$ by assumption, (1.2) implies $\lambda=0$, and thus $f'(\bar{u})=0$.

In view of our existence result (Theorem 1.5 below) we first state some preliminary lemmas.

Lemma 1.2. *Assume (I_1) and (I_4). Then $h\in C^1(E,\mathbb{R})$ and $\langle h'(u), u\rangle<0$ $\forall u\in M$.*

Proof. It results

$$h(u)=\langle f'(u), u\rangle=-a(u,u)+d(u).$$

Hence $h\in C^1(E,\mathbb{R})$ and

$$(1.3)\quad \langle h'(u), u\rangle=-2a(u,u)+\langle d'(u), u\rangle.$$

If $u\in M$, then $a(u,u)=d(u)$ and hence, substituting in (1.3) and using (I_4) we get

$$(1.4)\quad \langle h'(u), u\rangle=\langle d'(u), u\rangle-2d(u)<0 \quad \forall u\in M.$$

Lemma 1.3. *Assume (I_1), (I_2), and (I_3). Then:*

(i) *there exists $\varrho>0$ such that $\forall u\in E$, $0<\|u\|<\varrho$, it results $h(u)>0$. In particular, $\|u\|\geqq\varrho$ $\forall u\in M$.*

(ii) *$\forall u\in M$ it results $f(u)\geqq c\|u\|^\alpha$, $c>0$ constant.*

Proof. Using (I_1) and (I_2) we obtain

$$(1.5)\quad h(u)=-a(u,u)+\langle b'(u), u\rangle\geqq c_1\|u\|^\alpha-c_3\|u\|^2.$$

Here and throughout in the following, we denote by $c_1, c_2, \dots$ (possibly different) constants. Since $\alpha < 2$, the right hand side in (1.5) is positive for $0 < \|u\| < \varrho$, with ϱ small enough, and this proves (i).

For all $u \in M$ it results $a(u,u) = \langle b'(u), u\rangle$ and hence f has the form

$$f(u) = b(u) - \tfrac{1}{2}\langle b'(u), u\rangle .$$

Using (I_3) it follows

$$f(u) \geqq (\theta - \tfrac{1}{2})\langle b'(u), u\rangle \quad \forall u \in M ,$$

and by (I_2) $f(u) \geqq c_4 \|u\|^\alpha$, $\forall u \in M$.

Lemma 1.4. *Assume (I_1) and (I_4). If $u \in E$ then for all $s > 0$ it results:*

$$(1.6) \quad \frac{d}{ds}\left[\frac{h(su)}{s^2}\right] < 0 ;$$

$$(1.7) \quad \frac{d}{ds}[f(su)] = \frac{1}{s} h(su) .$$

Proof. Set $\tilde{h}(s) = s^{-2} h(su)$. From the expression of h it follows:

$$\tilde{h}(s) = -a(u,u) + s^{-2} d(su) .$$

Differentiating and setting $su = v$, we have:

$$\tilde{h}'(s) = s^{-3}\,[\langle d'(v), v\rangle - 2d(v)] .$$

Thus $\tilde{h}'(s) < 0$ $\forall s > 0$ by (I_4).

As for (1.7), it is enough to calculate the derivative of

$$f(su) = -\tfrac{1}{2} s^2 a(u,u) + b(su) .$$

We are now in position to state our main abstract result:

Theorem 1.5. *Assume $(I_1$–$I_4)$. Moreover suppose:*

(1.8) *there exists $\bar{u} \in E$ such that $\bar{a} = a(\bar{u}, \bar{u}) > 0$;*

and the pair (f, M) satisfies:

$(P-S)$ *every sequence $u_n \in M$ such that $f(u_n)$ is bounded and $f'/M(u_n) \to 0$, has a converging subsequence.*

Then $f_{/M}$ assumes its minimum in a point $v \in M$. Such v is a stationary point of f on E, i.e. $f'(v) = 0$, and has the following property:

$$(*) \quad a(v,v) = \max\{a(u,u) : u \in E, u \neq 0, b(su) \leqq b(sv)\ \forall s > 0,$$
$$\langle b'(u), u\rangle \leqq \langle b'(v), v\rangle\} .$$

Proof. First of all, let us show that $M \neq \emptyset$. In fact one has

$$h(s\bar{u}) = -s^2 \bar{a} + \langle b'(s\bar{u}), s\bar{u}\rangle .$$

By (I_2) it follows

$$h(s\bar{u}) \leqq c_2 s^\alpha \|\bar{u}\|^\alpha - s^2 \bar{a} .$$

Since $\bar{a}>0$ and $\alpha<2$, then $h(s\bar{u})<0$ for $s>0$ large enough. On the other hand, we know by (i) of Lemma 1.3, that $h(s\bar{u})>0$ for s small: hence there exists $\bar{s}>0$ such that $h(\bar{s},\bar{u})=0$, i.e. $\bar{s}\bar{u}\in M$. Let us remark that such $\bar{s}$ is unique in view of (1.6): this will be useful later in the proof. Next, by Lemma 1.2, we have that (1.1) holds and, in particular (see the proof of Proposition 1.1), M is a C^1 manifold in E. Moreover, combining (i) and (ii) of Lemma 1.3, we deduce that f is bounded from below on M. Using $(P-S)$ it is well known (see for example Rabinowitz [15]) that $f_{/M}$ takes its minimum in a point $v\in M$. By Proposition 1.1 such a v is a stationary point of f on E. It remains to prove that property $(*)$ holds. By contradiction, let $u\in E$, $u\neq 0$, be such that

(1.9) $\quad b(su)\leqq b(sv)\quad \forall s>0 \quad$ and $\quad \langle b'(u),u\rangle\leqq\langle b'(v),v\rangle$;

(1.10) $\quad a(u,u)>a(v,v)$.

Using (1.9) and (1.10) and the fact that $v\in M$, we get

$$0=h(v)>-a(u,u)+\langle b'(u),u\rangle=h(u).$$

This implies that there exists a unique $\bar{s}$, $0<\bar{s}<1$, such that $\bar{s}u\in M$: in fact, it is sufficient to repeat the arguments exposed above to prove $M\neq\emptyset$. Using again (1.9) and (1.10) we have

(1.11) $\quad f(\bar{s}u)=-\tfrac{1}{2}\bar{s}^2a(u,u)+b(\bar{s}u)<-\tfrac{1}{2}\bar{s}^2a(v,v)+b(\bar{s}v)=f(\bar{s}v)$.

On the other hand, by (1.6) and (1.7) and because $0<\bar{s}<1$, it results $f(v)>f(\bar{s}v)$. Lastly, using (1.11), we obtain $f(v)>f(\bar{s}u)$, a contradiction, because $\bar{s}u\in M$ and $f(v)=\min\{f(u):u\in M\}$.

2. Periodic Solutions for a Class of Convex Hamiltonians

Let $p=(p_1,\dots,p_n)$, $q=(q_1,\dots,q_n)\in\mathbb{R}^n$ and $H\in C^2(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R})$. We will deal with the Hamiltonian system

$$(H)\quad \dot{p}_k=-\frac{\partial H}{\partial q_k},\quad \dot{q}_k=\frac{\partial H}{\partial p_k},\quad k=1,\dots,n$$

which will be also written in the compact form

$$\dot{z}=JH'(z)\quad\text{or}\quad -J\dot{z}=H'(z),$$

where $z=(p,q)\in\mathbb{R}^{2n}$, $J=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}$ with I identity in $\mathbb{R}^n$, and H' indicates the gradient of H.

We look for periodic solutions $z(t)$ of (H) having a given number $T>0$ as the *minimal* period. Denote by $(\cdot,\cdot)$ the Euclidean scalar product in $\mathbb{R}^{2n}$ and by $|\cdot|$ the corresponding norm.

On the Hamiltonian H we assume, first of all:

(H_1) $\quad H\in C^2(\mathbb{R}^{2n},\mathbb{R})$ and there exist c_1, c_2, $c_3>0$ and $\beta>2$ such that

$$c_1|z|^\beta\leqq H(z)\leqq c_2|z|^\beta\quad\forall z\in\mathbb{R}^{2n}$$

$$(H''(z)\zeta,\zeta)\geqq c_3|z|^{\beta-2}\quad\forall z\in\mathbb{R}^{2n}\quad\text{and}\quad\forall\zeta\in\mathbb{R}^{2n};\quad |\zeta|=1,$$

(H_2) $\quad \beta H(z)\leqq(H'(z),z)\quad\forall z\in\mathbb{R}^{2n}$.

Remark 2.1. (i) From the hypotheses above it follows that H is strictly convex and super-quadratic at zero and infinity.

(ii) Condition (II_2) and the first inequality of (II_1) are related in the sense that the former implies $H(z) \geqq c_1|z|^\beta$ as $z \to \infty$ and $H(z) \leqq c_2|z|^\beta$ as $z \to 0$.

Before stating our results, we have to introduce the Legendre transform G of H, defined on $\mathbb{R}^{2n}$ by

$$G(u) = \sup\{(u,z) - H(z) : z \in \mathbb{R}^{2n}\}.$$

From $(II_1 - II_2)$ several properties of G can be deduced [11]:

$(G1)$ G is everywhere defined, finite, $G \in C^2(\mathbb{R}^{2n} \backslash \{0\}, \mathbb{R})$, is strictly convex and $G(0) = G'(0) = 0$.

Recall also that

$$(2.1) \quad G'(u) = z \quad \text{iff} \quad H'(z) = u.$$

$(G2)$ Let $\alpha = \beta(\beta - 1)^{-1}$. Then

$$c_4|u|^\alpha \leqq G(u) \leqq c_5|u|^\alpha \quad \forall u \in \mathbb{R}^{2n}.$$

$(G3)$ $G(u) \geqq \theta(G'(u), u), \quad \theta = \dfrac{1}{\alpha}, \quad \forall u \in \mathbb{R}^{2n}.$

Furthermore, from the properties of the Legendre transform and $(G2)$ we get

$(G4)$ $(G'(u), u) \geqq c_4|u|^\alpha \quad \forall u \in \mathbb{R}^{2n}$,

$(G5)$ $c_6|u|^{\alpha-1} \leqq |G'(u)| \leqq c_7|u|^{\alpha-1} \quad \forall u \in \mathbb{R}^{2n}$.

Lastly, from (II_1) and $(G5)$ it follows

$(G6)$ $\|G''(u)\| \leqq c_8|u|^{\alpha-2} \quad \forall u \in \mathbb{R}^{2n}$.

We are now ready to state:

Theorem 2.2. *Assume* $(II_1 - II_2)$, *and*

(II_3) *there exists* $\mu \in]0,1[$ *such that* $\forall u \in \mathbb{R}^{2n}$, $u \neq 0$, *it results*

$$(G''(u)u, u) \leqq \mu(G'(u), u).$$

Then for all $T > 0$, (H) *has a periodic solution having* T *as minimal period.*

We remark that, in terms of the original Hamiltonian, the assumption (II_3) can be stated: $(K(z)u, u) \leqq \mu(z, u) \forall u$, where $u = H'(z)$ and $K(z) = (H''(z))^{-1}$. Examples of Hamiltonians verifying (II_3) will be given at the end of this section, as well as in Sects. 4 and 5.

The proof of the theorem is carried out by substituting the system (H) with a dual one, and studying this latter by means of the variational arguments of Sect. 1.

First we introduce some notations and the functional framework.

Let A be the operator defined in

$$\operatorname{dom} A = \{z \in W^{1,\alpha}(0, 2\pi; \mathbb{R}^{2n}) : z(0) = z(2\pi)\}$$

by

$$(2.2) \quad Az = -J\dot{z}.$$

Remark that $\operatorname{Ker} A = \mathbb{R}^{2n}$. Set

$$E = \left\{ u \in L^\alpha(0, 2\pi; \mathbb{R}^{2n}) : \int_0^{2\pi} u = 0 \right\}$$

and denote by $\|\cdot\|$ the norm in $L^\alpha(0, 2\pi; \mathbb{R}^{2n})$. For all $u \in E$ there exists $z \in \operatorname{dom} A$ such that $Az = u$; such a z is unique, provided $\int z = 0$ (here and in the following all the integrals are between 0 and 2π). Setting $Lu = z$, it turns out that L, as an operator from E to $L^\beta(0, 2\pi; \mathbb{R}^{2n})$ is compact. Let $\sigma = \dfrac{2\pi}{T}$ and consider for $u \in E$:

$$(2.3) \quad f_\sigma(u) = -\frac{1}{2\sigma} \int (u, Lu) + \int G(u).$$

From $(G2)$ and $(G5)$ it follows that $f_\sigma \in C^1(E, \mathbb{R})$; if $f'_\sigma(v) = 0$ for some v in E, then there is $\xi \in \mathbb{R}^{2n}$ such that

$$-\frac{1}{\sigma} Lv + G'(v) = \xi.$$

Setting $z = \sigma^{-1} Lv + \xi$, then $\sigma Az = v$ and $G'(v) = z$. From (2.1) we get $\sigma Az = H'(z)$. Hence, in view of (2.2), we conclude that $z(\sigma t)$ defines, by continuation, a T-periodic solution of (H). This is the so called dual action principle [8, 11, 6]. Clearly, T is the minimal period of z iff 2π is the minimal period of v. In the remainder of the section, since σ plays no rôle in the arguments, we will take $\sigma = 1$.

In order to use the results of Sect. 1, we set, for $u \in E$:

$$a(u, u) = \int (u, Lu), \qquad b(u) = \int G(u)$$

and hence (2.3) becomes

$$f(u) = -\tfrac{1}{2} a(u, u) + b(u).$$

We will use the notations introduced in Sect. 1. First we state:

Lemma 2.3. *f satisfies $(I_1$–$I_4)$.*

Proof. It is clear that (I_1) holds. Due to $(G2)$ and $(G5)$, $b \in C^1(E, \mathbb{R})$. It results:

$$\langle b'(u), u \rangle = d(u) = \int (G'(u), u)$$

and hence, from $(G4$–$G5)$, we deduce that (I_2) holds. (I_3) follows directly from $(G3)$. As for (I_4), we note that the mapping $u \to (G'(u), u)$ is $C^1(\mathbb{R}^{2n}, \mathbb{R})$; in view of $(G5$–$G6)$ we get $d \in C^1$. Lastly from (II_3) it follows:

$$(2.4) \quad \begin{aligned} \langle d'(u), u \rangle &= \int (G''(u)u, u) + \int (G'(u), u) \\ &\leqq (1 + \mu) \int (G'(u), u) < 2d(u). \end{aligned}$$

In (2.4) we mean that $G''(u)u = 0$ for $u = 0$.

Since f verifies $(I_1$–$I_4)$, we know (cf. Lemma 1.2) that $\langle h'(u), u \rangle < 0$ $\forall u \in M$. Actually, a bit sharper result can be proven:

$$(2.5) \quad \exists c > 0 : \forall u \in M \quad -\langle h'(u), u \rangle \geqq c.$$

In fact, from (2.4) and (1.4) we have

$$-\langle h'(u), u\rangle = 2d(u) - \langle d'(u), u\rangle \geqq (1-\mu)d(u).$$

Then, from (I_2), it follows:

$$-\langle h'(u), u\rangle \geqq (1-\mu)c_4\|u\|^\alpha.$$

But, $u\in M$ implies [cf. (i) of Lemma 1.3] that $\|u\| \geqq \varrho > 0$ and thus (2.5) holds.
Next we prove:

Lemma 2.4. *The pair (f, M) satisfies $(P-S)$.*

Proof. Let $u_n \in M$ be such that

(2.6) $f(u_n)$ is bounded;

(2.7) $f_{/M}'(u_n) = f'(u_n) + a_n h'(u_n) \equiv w_n \to 0 \quad (a_n \in R)$.

From (2.6) and (ii) of Lemma 1.3, it follows that $\|u_n\| \leqq c_1$. Multiplying (2.7) by u_n, and taking into account that $u_n \in M$, i.e. $\langle f'(u_n), u_n\rangle = 0$, we find

(2.8) $a_n\langle h'(u_n), u_n\rangle = \langle w_n, u_n\rangle$.

In (2.8) $\langle w_n, u_n\rangle$ tends to zero, because $w_n \to 0$ and $\|u_n\| \leqq c_1$. By (2.5), $\langle h'(u_n), u_n\rangle$ is bounded away from zero and hence $a_n \to 0$. From (2.7) and since h' is bounded on bounded sets, we infer that $f'(u_n) \to 0$. The argument is now as in [11]. We sketch it here for completeness. So, it results that $f'(u_n) = -Lu_n + b'(u_n) + \xi_n$ (with $\xi_n \in \mathbb{R}^{2n}$) tends to zero. Since ξ_n is bounded and L is compact, we deduce (passing to a subsequence) that $b'(u_n) = z_n$ is convergent. On the other side $G'(u_n) = z_n$ is equivalent to $H'(z_n) = u_n$; since H' is continuous from L^β to L^α, we conclude that u_n has a converging subsequence.

We are now in position to prove Theorem 2.2.

Proof of Theorem 2.2. We apply Theorem 1.5 to f. We have already shown in Lemmas above that the assumptions $(I_1$–$I_4)$ hold, as well as that (f, M) satisfies $(P-S)$. Moreover, choosing $\bar{u} = (\bar{\xi}\sin t, \bar{\xi}\cos t)$ with $\bar{\xi} \in \mathbb{R}^n$, $\bar{\xi} \neq 0$, it results $L\bar{u} = \bar{u}$ and thus $a(\bar{u}, \bar{u}) = 2\pi|\bar{\xi}|^2 > 0$, so that (2.8) holds, too. Let us denote by v the critical point of f given by Theorem 1.5. We complete the proof showing how property $(*)$ implies that 2π is the minimal period of v. In fact, if v has period $\frac{2\pi}{m}$ for some integer $m > 1$, we take $v^*(t) = v\left(\frac{t}{m}\right)$. It is immediate to verify that $v^* \in E$ and that

$$b(sv^*) = b(sv) \quad \forall s > 0, \quad \text{and} \quad \langle b'(v^*), v^*\rangle = \langle b'(v), v\rangle.$$

On the other hand, by a direct computation it results:

$$a(v^*, v^*) = ma(v, v).$$

Lastly, recalling that $a(v, v) > 0$ (indeed this is true for all $u \in M$ because for those u it is $a(u, u) = \langle b'(u), u\rangle$) we conclude that $a(v^*, v^*) > a(v, v)$, in contradiction with $(*)$.

Remark 2.5. It is clear that the solution found above is nonconstant.

Remark 2.6. From the proof of Theorem 2.2 we see readly that in the right hand side of $(G2)$ and $(G5)$ we could take:

$$G(u) \leqq c_1 |u|^\alpha + c_2$$
$$|G'(u)| \leqq c_3 |u|^{\alpha-1} + c_4 .$$

Analogously, in $(G6)$ we could assume

$$|G''(u)u| \leqq c_5 |u|^{\alpha-1} + c_6 .$$

As an example of Hamiltonian system to which Theorem 2.2 can be applied, we may consider a $H(z)$ which is strictly convex and homogeneous of degree $\beta > 2$, namely such that $H(sz) = s^\beta H(z)\,\forall s > 0$. In this case it is easy to verify that G is homogeneous of degree $\alpha = \beta(\beta-1)^{-1}$, and hence $(II_1–II_3)$ hold. However, in such a case, the existence of a periodic solution with minimal period $T = 2\pi$ (say), can be obtained more directly. In fact, let us consider the level surface

$$S = \{u \in E : \textstyle\int G(u) = 1\} .$$

It is easy to see that the action $a(u, u)$ has a positive maximum $\bar{v}$ on S. This $\bar{v}$ has 2π as minimal period and satisfies

$$L\bar{v} = \lambda G'(\bar{v}) - \xi$$

for some $\lambda > 0$ and $\xi \in R^{2n}$. Since G is α-homogeneous, setting $w = \lambda^{\frac{1}{\alpha-1}} \bar{v}$, we get

$$Lw = G'(w) - \tilde{\xi}, \qquad \tilde{\xi} = \lambda^{\frac{1}{\alpha-2}} \xi .$$

Then $z = Lw + \tilde{\xi}$ is a solution of (H), with minimal period 2π.

3. Energy Estimates

Given $T = 2\pi\sigma^{-1}$, we indicate with v_σ the solution of (H) which has T as minimal period. Our purpose, now, is to investigate the dependence of v_σ on σ. Apart from the possible interest in itself, this will be useful in perturbations (see Sect. 5).

First we state:

Lemma 3.1. *Assume $(II_1–II_3)$. Then there exist $c', c'' > 0$ such that*

$$c' \sigma^{\frac{\alpha}{2-\alpha}} \geqq f_\sigma(v_\sigma) \geqq c'' \sigma^{\frac{\alpha}{2-\alpha}} .$$

Proof. We set $f_\sigma^{(i)}(u) = -\dfrac{1}{2\sigma} \int (u, Lu) + c_i \int |u|^\alpha$ $(i = 1, 2)$, where c_1 and c_2 are the constants which appear in $(G2)$. In correspondence of $f_\sigma^{(i)}$ we will consider h_i and M_i. From $(G2)$ it follows

$$(3.1) \quad f_\sigma^{(1)}(u) \leqq f_\sigma(u) \leqq f_\sigma^{(2)}(u) .$$

Set $m'(\sigma) = \min\{f_\sigma^{(1)}(u) : u \in M_1\}$ and $m''(\sigma) = \min\{f_\sigma^{(2)}(u) : u \in M_2\}$, and let us denote by λ' (resp. λ'') the value of λ such that $\lambda v_\sigma \in M_1$ (resp. M_2). From Lemma 1.4 and (3.1) it follows

$$f_\sigma^{(1)}(\lambda' v_\sigma) \leqq f_\sigma(\lambda' v_\sigma) \leqq f_\sigma(v_\sigma) \leqq f_\sigma^{(2)}(v_\sigma) \leqq f_\sigma^{(2)}(\lambda'' v_\sigma) .$$

Hence

(3.2) $m'(\sigma) \leqq f_\sigma(v_\sigma) \leqq m''(\sigma)$.

To evaluate m', let $B^+ = \{w \in B : \|w\| = 1, \int (w, Lw) > 0\}$.

We have that $w \in M_1$ iff $w \in B^+$ and

$$\frac{\lambda^2}{\sigma} \int (w, Lw) = \lambda^\alpha \alpha c_1 \int |w|^\alpha .$$

Hence we deduce

(3.3) $\lambda^{2-\alpha} = \sigma \alpha c_1 \dfrac{\int |w|^\alpha}{\int (w, Lw)}$.

Since on M_1 the functional $f_\sigma^{(1)}$ has the form

$$f_\sigma^{(1)}(u) = \left(1 - \frac{\alpha}{2}\right) \int |u|^\alpha$$

taking $u = \lambda w$, $w \in B^+$, we get

$$f_\sigma^{(1)}(\lambda w) = \left(1 - \frac{\alpha}{2}\right) \lambda^\alpha \int |w|^\alpha ,$$

where λ is given by (3.3). Then from the continuity of $\int (w, Lw)$ we infer that $\exists c'' > 0$ such that

(3.4) $m'(\sigma) \geqq c'' \sigma^{\frac{\alpha}{2-\alpha}}$.

Next, we bound from above m''. Fixed $\bar{w} \in B^+$, the same calculations show that $\bar{\lambda}\bar{w} \in M_2$ iff

$$\bar{\lambda}^{2-\alpha} = \sigma \alpha c_2 \frac{\int |\bar{w}|^\alpha}{\int (\bar{w}, L\bar{w})} .$$

Hence $\exists c' > 0$ such that

$$f_\sigma^{(2)}(\bar{\lambda}\bar{w}) = \left(1 - \frac{\alpha}{2}\right) \bar{\lambda}^\alpha \int |\bar{w}|^\alpha = c' \sigma^{\frac{\alpha}{2-\alpha}} .$$

We then conclude:

(3.5) $m''(\sigma) \leqq c' \sigma^{\frac{\alpha}{2-\alpha}}$

From (3.2), (3.4), and (3.5) the lemma follows.

Set $z_\sigma(t) = G'(v_\sigma(t))$. The behaviour of the energy $C_\sigma = H(z_\sigma(t))$ is given in the following

Proposition 3.2. *Assume (II_1–II_3). Then $C_\sigma \to 0$ (resp. ∞) as $\sigma \to 0$ (resp. ∞). As consequence $\|z_\sigma\|_{L^\infty} \to 0$ (resp. ∞) as $\sigma \to 0$ (resp. ∞).*

Proof. Using Lemma 3.1 we have that $f_\sigma(v_\sigma) \to 0$ for $\sigma \to 0$. From (ii) of Lemma 1.3 we deduce that $\|v_\sigma\| \to 0$ for $\sigma \to 0$. Hence $C_\sigma = H(G'(v_\sigma(t))) \to 0$ and, from (II_1) $\|z_\sigma\|_{L^\infty} \to 0$ as well.

Analogously, from $f_\sigma(v_\sigma) \leqq \int G(v_\sigma) \leqq c\|v_\sigma\|^\alpha$ and Lemma 3.1, it follows that $\|v_\sigma\|$, C_σ, and $\|z_\sigma\|_{L^\infty}$ diverge as $\sigma \to \infty$.

Remark 3.3. Similar arguments as in Lemma 3.1 and Proposition 3.2 yield energy bounds for solutions of given period of a family of convex Hamiltonians. More precisely, let H_n, H satisfy the assumptions of Theorem 2.2 (or weaker ones as in Remark 2.6) with $\inf \beta_n > 2$. Let us fix $\sigma = 1$. Then, if $(G_n'(u), u) < (G'(u), u)$ $\forall u$, it results $\sup_n C_n < +\infty$, where $C_n \equiv H_n(z_n(t))$, and z_n is the solution given by Theorem 2.2. In fact, as in (3.1), we have, with obvious notations $f_n(v_n) \leqq f(v)$ and hence

$$2\pi C_n\left(\frac{\beta_n}{2} - 1\right) = \left(\frac{\beta_n}{2} - 1\right)\int H_n(z_n(t) \leqq \frac{1}{2}\int (H'(z_n), z_n) - \int H_n(z_n)$$

$$= -\frac{1}{2}\int (G_n'(v_n), v_n) + \int G_n(v_n) = f_n(v_n) \leqq f(v).$$

4. Hamiltonians with Positive-Definite Quadratic Form

This section is devoted to study the existence of periodic solutions of Hamiltonian systems when $H''(0)$ is no longer zero. More precisely, instead of considering the general equation (H), we will deal with the second order system:

$$(4.1) \quad -\ddot{x} = Qx + U'(x), \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where Q and U satisfy to:

(III) Q is a positive definite, symmetric matrix with eigenvalues ω_j^2, $0 < \omega_1^2 \leqq \ldots \leqq \omega_n^2$; $U: \mathbb{R}^n \to \mathbb{R}$ verifies the same conditions of H in $(II_1$–$II_3)$, with obvious modifications.

In particular, as seen in Sect. 2, convex β-homogeneous U $(\beta > 2)$ are allowed. Let us note that now, unlike than in our Remark after Theorem 2.2, it does not seem easy to find solutions of (4.1) with prescribed minimal period in the same straightforward manner.

It is known [16, Remark 2.56] that, the minimal period of the solutions of (4.1) cannot be arbitrarly large.

Our result is:

Theorem 4.1. *Assume* (III) *and let* $\sigma = \dfrac{2\pi}{T} > \omega_n$. *Then* (4.1) *has a (nonconstant) solution* v_σ *of minimal period* T. *Moreover, as* $T \uparrow \dfrac{2\pi}{\omega_n}$ *then* $\|v_\sigma\|_{L^\infty} \to 0$, *while if* $T \downarrow 0$ *then* $\|v_\sigma\|_{L^\infty} \to \infty$.

Proof. As for the existence, we indicate the differences with respect to the proof of Theorem 2.2. Set $E = L^\alpha(0, 2\pi; \mathbb{R}^n)$ and let A_σ be the densely defined operator given by $A_\sigma u = -\sigma^2 \ddot{u} - Qu$. Since $\sigma > \omega_n$, then A_σ is invertible with compact inverse L_σ. For $u \in E$ we set

$$f_\sigma(u) = -\tfrac{1}{2}\int (u, L_\sigma u) + \int G(u),$$

where G is the Legendre transform of U. In order to apply Theorem 1.5, we need only to find $\hat{u}\in E$ such that $\int(\hat{u}, L_\sigma \hat{u})>0$: the remainder is the same as in Theorem 2.2 [let us remark that the form $a(\cdot,\cdot)$ was not assumed to be definite]. Let ξ_n be the unit eigenvector of Q associated to ω_n^2. Taking

$$\hat{u}=A_\sigma(\xi_n \sin t)=(\sigma^2-\omega_n^2)\xi_n \sin t,$$

we obtain $\int(\hat{u}, L_\sigma \hat{u})=\pi(\sigma^2-\omega_n^2)>0$.

Next, to take advantage of property (*), we state:

Lemma 4.2. *Let $\sigma>\omega_n$ and $u\in E$, $u=\sum_{k\in\mathbb{Z}} u_k e^{ikt}$, $u_{-k}=\bar{u}_k\in\mathbb{C}^n$. Then*

$$\int(u, L_\sigma u)=-2\pi(u_0, Q^{-1}u_0)+\sum_{k\neq 0}\frac{1}{(\sigma k)^2}|u_k|^2$$
$$+\sum_{k\neq 0}\frac{1}{(\sigma k)^4}(u_k, Qu_k)+\sum_{k\neq 0}\frac{1}{(\sigma k)^6}(u_k, Q^2u_k)+\ldots$$

Proof. The Fourier-coefficients of $z=L_\sigma u$ satisfy

$$((\sigma k)^2 I-Q)z_k=u_k, \qquad k\in\mathbb{Z}.$$

Since, for $k\neq 0$, $\sigma k>\omega_n$, and ω_n^2 is the largest eigenvalue of Q, we have, for those k:

$$((\sigma k)^2 I-Q)^{-1}=\frac{1}{(\sigma k)^2}\left(I+\frac{1}{(\sigma k)^2}Q+\frac{1}{(\sigma k)^4}Q^2+\ldots\right).$$

Taking the scalar product $(u,z)=(u, L_\sigma u)$ and integrating, the lemma follows.

Proof of Theorem 4.1 Completed. Let $v=v_\sigma$ be the minimum of f_σ on M_σ given by Theorem 1.5. If v is $\frac{2\pi}{m}$-periodic for some $m>1$ integer, we set $v=\sum_{k\in\mathbb{Z}} v_{mk}e^{imkt}$ and $v^*=\sum_{k\in\mathbb{Z}} v_{mk}e^{ikt}$. Using Lemma 4.2 and taking into account that Q is symmetric and positive definite, we readily conclude that $\int(v, L_\sigma v)<\int(v^*, L_\sigma v^*)$, in contradiction with property (*).

As for the asymptotic behaviour, we will use the same arguments of Lemma 3.1 and Proposition 3.2 with the following modifications. Since the quadratic part of f_σ is now $\int(u, L_\sigma u)$, we need to evaluate m' and m'' in a fairly different way.

Let $B_\sigma^+=\{w\in E: \|w\|=1, \int(w, L_\sigma w)>0\}$. Note that $\hat{u}\in B_\sigma^+$ for all $\sigma>w_n$. As in Lemma 3.1, we have

(4.2) $\quad f_\sigma^{(1)}(\lambda w)>c_1\lambda^\alpha,$

where λ satisfies

(4.3) $\quad \lambda^{2-\alpha}=\dfrac{c_2}{\int(w, L_\sigma w)}$ for some constant c_2.

From Lemma 4.2, we obtain for all w, $\|w\|=1$

(4.4) $\quad \int(w, L_\sigma w)=-2\pi(w_0, Q^{-1}w_0)+\varrho(\sigma)$

where $\varrho(\sigma)=O\left(\frac{1}{\sigma^2}\right)$ uniformly with respect to w in $\{w\in E:\|w\|=1\}$. In (4.4) the right-hand side tends to $-2\pi\,(w_0, Q^{-1}w_0)<0$, as $\sigma\to\infty$. On the other hand we have $\int(w, L_\sigma w)>0$ for every w in B_σ^+. Hence we can conclude that $\sup\{\int(w, L_\sigma w): w\in B_\sigma^+\}\to 0$ as $\sigma\to\infty$. Then, from (4.2) and (4.3) it follows that $m'(\sigma)\to\infty$ as $\sigma\to\infty$. This estimate substitute the (3.4) of Lemma 3.1 and allows to show that $\|z_\sigma\|_{L^\infty}\to\infty$ as $T\to 0$.

Lastly, if we take $\hat{w}=\xi_n \sin t$, it results

$$(4.5)\quad \int(\hat{w}, L_\sigma \hat{w})=\frac{\pi}{\sigma^2-\omega_n^2}.$$

For $\sigma\to\omega_n$ the right-hand side in (4.5) tends to infinity, and this implies $m''(\sigma)\to 0$, in substitution of (3.5). Then $\|z_\sigma\|_{L^\infty}\to 0$ as $\sigma\to\omega_n$, and the proof is complete.

Remark 4.3. The proof of Theorem 4.1 could be done, alternatively, transforming, by a change of coordinates, the system 4.1 into another one, where the quadratic part is in diagonal form.

We do not know if Theorem 4.1 holds under the weaker assumption $\sigma>\omega_1$, $k\sigma\neq\omega_j, j=1,\ldots,n$ (non resonant case). Let us remark that in the example of [16] it is $\omega_1=\omega_2=1$ $(n=2)$ and the minimal period must be smaller than 2π.

Moreover, we expect that results as in Theorem 4.1 can be obtained for more general Hamiltonian systems, but this would require additional arguments involving, for example, better estimates than in Lemma.

5. Perturbations and Oscillations of Large Periods and Small Amplitude

In this section we consider the case in which the Hamiltonian H is a perturbation of a homogeneous one: this will provide a wider class of systems to which apply the results of the previous sections. More precisely we consider:

$$(5.1)\quad H(z)=\bar{H}(z)+R(z),$$

where

(IV_1) $\bar{H}(z)\in C^2(\mathbb{R}^{2n},\mathbb{R})$ is strictly convex and homogeneous of degree $\beta>2$;

(IV_2) $R\in C^2(\mathbb{R}^{2n},\mathbb{R})$, $R(0)=R'(0)=0$ and $\|R''(z)\|=o(|z|^{\beta-2})$, as $z\to 0$.

To take advantage of the asymptotic condition (IV_2), we will look for solutions of (H) of small amplitude (namely near zero). Actually, we will find them as solutions of large minimal period.

Theorem 5.1. *Assume that $H(z)=\bar{H}(z)+R(z)$ and that $(IV_1$–$IV_2)$ hold. Then there exists $T_0>0$ such that for all $T>T_0$, (H) has a solution having T as minimal period. The amplitude of such a solution tends to zero for $T\to\infty$.*

The proof of the theorem is carried out in some steps. The first one consists in modifying the Hamiltonian. Let $\chi\in C^\infty(\mathbb{R}^+,\mathbb{R}^+)$ be such that

$$\chi(s)=1 \quad\text{for}\quad 0\leqq s\leqq\tfrac{1}{2},\qquad \chi(s)=0\quad\text{for}\quad s\geqq 1,\qquad \chi'(s)\leqq 0,\qquad s>0.$$

Put $R_\delta(z)=R(\chi_\delta(|z|^2)z)$, where $\chi_\delta(s)=\left(\frac{s}{\delta}\right)$, and $H_\delta(z)=\bar{H}(z)+R_\delta(z)$.

Lemma 5.2. *If δ is sufficiently small then H_δ is convex and satisfies (II_1–II_3).*

Proof. The arguments consist in standard (but long) calculations. For brevity, we indicate only the outline of the proof of (II_3).

Suppose that δ is so small that H_δ verifies (II_1–II_2). The following notations will be used: G_δ is the Legendre transform of H_δ as well as $\bar{G}$ that of $\bar{H}$; B is $\bar{H}''^{-1}$ and B_δ is $H_\delta''^{-1}$. From (IV_2) it follows that, given $\varepsilon>0$, then

(5.2) $|R_\delta''(z)v| < \varepsilon|z|^{\beta-2}|v| \quad \forall z, v \in \mathbb{R}^{2n}$

provided δ is small. Set $S_\delta = B \circ R_\delta''$. Since $\bar{H}''$ is $(\beta-2)$-homogeneous, then $\|B(z)\| \leqq c|z|^{2-\beta}$. Thus, using (5.2), it follows $\|S_\delta(z)\| < \varepsilon$ for δ small. Hence $B_\delta = (I+S_\delta)^{-1} \circ B = (I+\tilde{S}_\delta) \circ B$, where $\tilde{S}_\delta = -S_\delta + S_\delta^2 - \dots$ and $\|\tilde{S}_\delta\| \leqq \varepsilon$ (δ small). Let $\alpha - 1 < \mu < 1$ and take $0 < \varepsilon < \alpha(\mu - \alpha + 1)$. In correspondence of such ε let δ be such that all the above inequalities hold. We evaluate:

$$(G_\delta''(u)u, u) = (B_\delta(z)u, u)$$
$$(G_\delta'(u), u) = (z, H_\delta'(z)),$$

where $z = G_\delta'(u)$ and thus $u = H_\delta'(z)$. It results

(5.3) $(B_\delta(z)u, u) = (B(z)u, u) + (\tilde{S}_\delta \circ B(z)u, u)$.

Since H_δ satisfies (II_1), it follows that $|u| \leqq c|z|^{\beta-1}$, for δ small. Hence, this fact jointly with $\|B(z)\| \leqq c|z|^{2-\beta}$ implies

$$|(\tilde{S}_\delta B(z)u, u)| \leqq \varepsilon \|B(z)\| |u|^2 \leqq \varepsilon |z|^\beta .$$

As for the first addendum in the right hand side of (5.3), setting $w_0 = \bar{H}'(z)$ and $w_\delta = R_\delta'(z)$, it results $u = w_0 + w_\delta$ and

(5.4) $(B(z)u, u) = (B(z)w_0, w_0) + (B(z)w_0, x_\delta) + (B(z)w_\delta, u)$.

In (5.4) the second and third addendum are bounded, for δ small, by $\varepsilon|z|^\beta$. Moreover, we have

$$(z, H_\delta'(z)) = (z, w_0) + (z, w_\delta) \leqq (\bar{G}'(w_0), w_0) + \varepsilon|z|^\beta .$$

Combining these inequalities, we have, for δ small enough:

$$\begin{aligned}(G_\delta''(u)u, u) - (G_\delta'(u), u) &\leqq (B(z)w_0, w_0) - (w_0, \bar{G}'(w_0)) + \varepsilon|z|^\beta \\ &= (\bar{G}''(w_0)w_0, w_0) - \mu(w_0, \bar{G}'(w_0)) + \varepsilon|z|^\beta \\ &= (\alpha(\alpha-1) - \mu\alpha)\bar{G}(w_0) + \varepsilon|\bar{G}'(w_0)|^\beta \leqq (\alpha(\alpha-1) - \mu\alpha + \varepsilon)c|w_0|^\alpha < 0 .\end{aligned}$$

Proof of Theorem 5.1. Let δ be such that Lemma 5.2 holds. By Theorem 2.2 we know that for all $T>0$ the system

$$\dot{z} = JH_\delta'(z)$$

has a solution z_σ with $T = \dfrac{2\pi}{\sigma}$ as minimal period. By Proposition 3.2 we infer that there exists $T_0 > 0$ such that for $T > T_0 \|z_\sigma\|_\infty \leqq \delta$. Since for $|z| \leqq \delta$, $H(z)$ coincides with $H_\delta(z)$, we conclude that the found solution is actually a solution of (H).

Remark. If H is subquadratic, the existence of a T-periodic solution (with T minimal period), T large enough, has been proved by Brezis and Coron [20] by means of perturbation arguments, but in the framework of the paper by Clarke and Ekeland [8].

An interwinning of our techniques and those of Clarke and Ekeland [9] yields to prove a result concerning the existence of subharmonics like Theorem 2.34 of Rabinowitz [17]. We will not carry over the details.

6. A Further Existence Result

In this section we expose another existence theorem where we can eliminate the growth restrictions in (II_1). For brevity, we will omit below some details, which either are similar to the arguments already discussed or are essentially known results of convex analysis (see for example Rockafeller [18]). We assume:

(III_1) $H \in C^2(\mathbb{R}^{2n};\mathbb{R})$, H is strictly convex, $H(z) \geqq 0$,

(III_2) $\beta H(z) \leqq (H'(z), z) \quad \forall z \in \mathbb{R}^{2n}$,

(III_3) $|H'(z)| \leqq c\|H''(z)\| \cdot |z| \quad \forall z \in \mathbb{R}^{2n}$,

(III_4) $(H''(z)\zeta, \zeta) \geqq c\|H''(z)\| \forall \zeta, z \in \mathbb{R}^{2n}, \quad |\zeta| = 1$,

(III_5) $\exists \mu]0, 1[: (G''(u)u, u) \leqq \mu(G'(u), u) \quad \forall u \in \mathbb{R}^{2n} \backslash \{0\}$.

In $(III_1$–$III_5)$ we have used the same notations as above.

We remark (see Ekeland [11]) that (III_2) is equivalent to assume that

$$(6.1) \quad H(sz) \geqq s^\beta H(z) \quad \forall z, \forall s \geqq 1$$

as well as (6.1) jointly with

$$(6.2) \quad |H'(sz)| \geqq s^{\beta-1}|H'(z)| \quad \forall s \geqq 1, \forall z$$

implies (III_3). However, we do not know if (III_3) implies (6.1).

We have introduced assumptions $(III_3$–$III_4)$ to prove:

Lemma 6.1. *Assume $(III_1$–$III_4)$. Then*

$$|G''(u)u| \leqq c_1 + c_2|u|^{\alpha-1} \quad \forall u \in \mathbb{R}^{2n}.$$

Proof. Set $z = G'(u)$. It results by (III_4):

$$|G''(u)u| \leqq \frac{|u|}{\min\limits_{|\zeta|=1}(H''(z)\zeta, \zeta)} \leqq c\frac{|u|}{\|H''(z)\|}.$$

Using (III_3) we get $|G''(u)u| \leqq c|z|$. (III_2) and the convexity of G imply that $|z| = |G'(u)| \leqq c|u|^{\alpha-1}$, provided $|z| \geqq 1$. If $|z| \leqq 1$ we have $|G''(u)u| \leqq c$.

Set

$$(6.3) \quad H_n(z) = \inf_{y \in \mathbb{R}^{2n}} \left\{ H(z-y) + \frac{n}{\beta}|z|^\beta \right\}.$$

It is well known that H_n, the inf-convolution of H and $\frac{n}{\beta}|z|^\beta$, is convex and its dual $G_n(u)$ coincides with $G(u)+\frac{1}{\alpha n^{\alpha-1}}|u|^\alpha$. Then, using also Lemma 6.1, it is easy to see that G_n satisfies all the assumptions $G1$–$G6$, with the modifications allowed in Remark 2.6. Furthermore $H_n' \to H'$ uniformly on bounded sets. We are ready to state:

Theorem 6.2. *Assume* $(III_1$–$III_5)$. *Then, for all* $T>0$, (H) *has a solution, having* T *as minimal period.*

Proof. Fix $T=2\pi$. Due to the preceding remarks, we can apply Theorem 2.2 to get a solution z_n of

$$(H_n) \quad -J\dot{z}_n = H_n'(z_n);$$

z_n has 2π as minimal period. We first show that z_n converges in C^1 to a solution z of (H). Indeed, by Remark 3.3 [with H_n given by (6.3) and $H \equiv H_1$] it follows that [$C_n \equiv H_n(z_n)$ and thus] $\|z_n\|_{L^\infty} \leqq c$. From (H_n) it follows that also $\|\dot{z}\|_{L^\infty} \leqq c$. Hence we can extract a subsequence $z_n \to z$ uniformly. Equation (H_n) implies that $z_n \to z$ in C^1 and $-J\cdot\dot{z}=H'(z)$.

It remains to prove that z has 2π as minimal period.

We set $v=H'(z)=\lim_n H_n'(z_n)$, and prove that v has minimal period 2π. Set

$$\hat{M}=\{u \in L^\infty : \int (u, Lu) = \int (G'(u), u)\}.$$

It is easy to see that v (belongs to $\hat{M}$ and) minimizes $f(u) = -\frac{1}{2}\int (Lu, u) + \int G(u)$ on $\hat{M}$ [note that, due to (III_2) and the convexity of G, it results $G(v) \geqq c|v|^\alpha$ for $v \in \mathbb{R}^{2n}$, $|v| \leqq 1$]. Now, if $\tilde{v}(t) = v\left(\frac{t}{k}\right)$ is 2π-periodic for some $k>1$, from $\delta\tilde{v} \in \hat{M}$ for a $\delta<1$, and $f(\delta\tilde{v}) < f(\delta v) < f(v)$ we get a contradiction, so the proof is complete.

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