Math. Ann. 255, 159–198 (1981)

Mathematische Annalen

Equivalence of Glancing Hypersurfaces. II

R. B. Melrose
Department of Mathematics, M.I.T., Cambridge, MA 02139, USA

1. Introduction

In [5], hereafter referred to as EQ, the geometry of intersecting hypersurfaces F, G in a symplectic manifold was analysed, near points at which F and G intersect normally but at which the Hamilton foliation of F (resp. G) is simply tangent to G (resp. F). In this second part the more degenerate situation, in which the Hamilton foliation of F is still simply tangent to G but that of G is tangent to F to second order, is considered. It is no longer true that dimension is the only symplectic invariant of such a system; we show the existence of a countable family of (functional) obstructions to equivalence, all of which can be seen in the formal power series analysis on the submanifold of maximal degeneracy.

Whereas in EQ symplectic manifolds were treated initially and then reconsidered with the addition of homogeneity, contact manifolds, the case of principal interest in the application to differential boundary problems, will be considered directly here.

In a contact manifold (E, M^+), where M^+ is the (oriented) contact line subbundle of T^*E, the space $C^\infty(M^*)$ of smooth sections of the dual bundle to M is equipped with the structure of a Lie algebra, through the Lagrange bracket $[,]$. Any hypersurface F can be defined, near any $p \in F$, as the zero set of a section $f \in C^\infty(M^*)$, with the bundle-valued 1-form df non-vanishing at p. The conditions we wish to consider on the intersection of F and G can be written, in terms of any $f, g \in C^\infty(M^*)$ defining them near p as

(1.1) N_p^*F, N_p^*G, M_p are linearly independent,

(1.2) $f(p) = g(p) = [f, g](p) = 0$,

(1.3) $[f,[f,g]](p) \neq 0$,

(1.4) $[g,[g,f]](p) = 0, \quad [g,[g,[g,f]]](p) \neq 0$.

The definition of equivalence will be somewhat weaker than that used in EQ, since we shall treat F as the boundary of a region of interest. Thus, let e be an orientation of F at p and write G^e for the part of G on the positive side of F, near p.

0025-5831/81/0255/0159/$08.00

(1.5) *Definition.* Two systems $(F_i, G_i, E_i, M_i^+, p_i, e_i)$ $i=1,2$, consisting of contact manifolds (E_i, M_i^+) containing hypersurfaces F_i, G_i intersecting at p_i, at which point F_i has orientation e_i, in such a way that (1.1)–(1.4) are satisfied are said to be *equivalent* if there is a germ of contact transformation

$$\phi: E_1, p_1 \to E_2, p_2, \quad \phi^* M_2^+ = M_1^+$$

such that

(1.6) $\phi(F_1) = F_2, \quad \phi(G_1^{e_1}) = G_2^{e_2}.$

The full equivalence property used in EQ, requiring $\phi(G_1) = G_2$, will now be referred to as *two-sided equivalence.* Consider the problem of EQ, where (1.4) is replaced by $[g,[g,f]](p) \neq 0$, with respect to the weaker one-sided definition of equivalence. Two such systems are equivalent if, and only if, $[g,[g,f]]$ has the same sign in each, where $f = \bar{f}\alpha^*$, $\alpha^* \in C^\infty(M^*, p)$ has $\alpha^*(M_p^+) > 0$ and $d_p \bar{f} \in e$, and the dimensions of the underlying manifolds are the same. The difficulty of the proof of the one-sided equivalence depends on the sign; for one choice (>0, "diffraction") the problem is formal, involving no convergence arguments beyond Borel's theorem whereas for the other choice (<0, "gliding") the arguments needed are no simpler than those for the proof of two-sided equivalence, as in EQ.

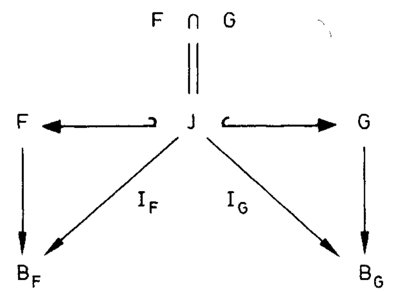
In the present case there is a somewhat subtler orientation invariant, discussed below. However, the alternative definition of one-sided equivalence, in which G has an orientation and in place of (1.6) one requires

$$\phi(F_1^{e_1}) = F_2^{e_2}, \quad \phi(G_1) = G_2$$

does exhibit a marked change with change of orientation. Indeed, two systems are not, then, equivalent unless $[f,[f,g]]$ has the same sign when $g = \bar{g}\alpha^*$, $\alpha^* > 0$, $d_p \bar{g} \in e$ and then the problem is formal if this is positive, whereas when this is negative the equivalence of formally equivalent systems involves convergence arguments beyond those presented below.

As in EQ, the main part of the analysis concerns the intersections maps and associated relations. According to (1.1), F and G are both non-radial (see Guillemin and Schaeffer [3] for a discussion of radial points on hypersurfaces), their normals being outside M_p, so they have well-defined, local, Hamilton foliations V_F and V_G with leaves the "bicharacteristic curves". Since any non-radial hypersurface, F, projects naturally onto the quotient contact manifold $F/V_F = B_F$ the inclusion maps $J \hookrightarrow F$, $J \hookrightarrow G$, where $J = F \cap G$, define *intersection maps* I_F, I_G

(1.7)



with the contact property

(1.8) $I_F^*(M_F^+) = A_J^+, \quad I_G^*(M_G^+) = A_J^+.$

Here M_F^+, M_G^+ are the induced contact bundles on B_F, B_G, and A_J^+ is the pull-back of M^+ to J; A_J^+ does not quite define a contact structure on J but is minimally degenerate and defines what we shall call a *folded contact structure* (see Sect. 2). Indeed, as a consequence of (1.3) I_F has a simple fold (singularity of type $S_{1,0}$) at p. Clearly if two intersection systems are equivalent the corresponding maps I_F are equivalent in the following sense.

(1.9) *Definition.* If (J_i, A_i^+, p_i) and (B_i, M_i^+, q_i) are, respectively, a folded contact manifold and a contact manifold, for $i=1,2$, the C^∞ germs

$$I_i : J_i, p_i \to B_i, q_i \qquad I_i^* M_i^+ = A_i^+$$

are said to be equivalent under *contact transformation* if there are diffeomorphisms $\phi : B_1, q_1 \to B_2, q_2$, $\psi : J_1, p_1 \to J_2, p_2$ such that

$$\phi^*(M_2^+) = M_1^+, \qquad \psi^*(A_2^+) = A_1^+$$

and the following square commutes

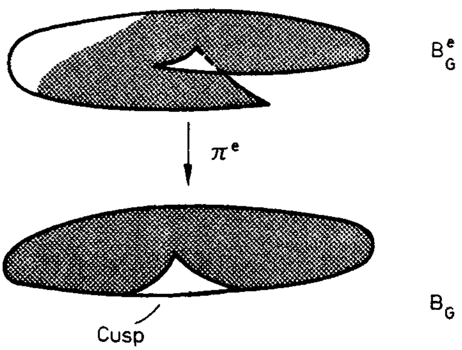
$$\begin{array}{ccc} J_1, p_1 & \stackrel{\psi}{\longleftrightarrow} & J_2, p_2 \\ {\scriptstyle I_1}\downarrow & & \downarrow{\scriptstyle I_2} \\ B_1, q_1 & \underset{\phi}{\longleftrightarrow} & B_2, q_2 . \end{array}$$

This definition leads to the following refinement of a classical theorem of Whitney.

(1.10) **Theorem.** *Between manifolds of a fixed dimension, $2n+1$, all germs of maps exhibiting $S_{1,0}$ singularity are equivalent under contact transformation, as are all germs exhibiting $S_{1,1,0}$ singularity.*

The point of the second part of this theorem is that, under the assumptions (1.1)–(1.4) I_G has a Whitney cusp at p (a singularity of type $S_{1,1,0}$, see for example Golubitsky and Guillemin [2]), as is proved in Sect. 3.

Now, when F has an orientation e, let $B_G^e = \mathring{G}^e / \sim_e$ be defined by the relation, $p_1 \sim_e p_2$ if and only if p_1 and p_2 lie on a segment $[p_1, p_2] \subset \mathring{G}^e$ of G-bicharacteristic, in the interior $\mathring{G}^e$ of G^e. Clearly, there is a projection $\pi^e : B_G^e \to B_G$, under which B_G^e covers B_G simply outside (and on) the cusp in B_G, defined by I_G, and doubly inside, and a map $I_G^e : J \backslash K^- \to B_G^e$ defined on the open dense submanifold of J where V_G is not tangent to F from the outside such that $\pi^e \circ J_G^e = I_G | J \backslash K^-$. Here, $K^- \subset J$ is defined by $[g, f] = g = f = 0$, $[g, [g, f]] \leqq 0$.



Thus, when F is oriented, in place of (1.7), we have the following system of *oriented-intersection maps*

$$(1.11)\quad B_F \xleftarrow{I_F} J \hookleftarrow J\backslash K^- \xrightarrow{I_G^e} B_G^e \xrightarrow{\pi^e} B_G.$$

which are all contact maps, inducing local isomorphisms of the various line bundles M_F, A_J, $M_G^e = (\pi^e)^* M_G$.

(1.12) *Definition.* Two systems of oriented-intersection maps, as in (1.11), are equivalent under contact transformation if there exist germs of smooth contact diffeomorphism ϕ_F, ψ, ϕ_G^e giving a commutative diagram

$$(1.13)\quad \begin{array}{ccccccccc}
B_F & \xleftarrow{I_F} & J & \hookleftarrow & J/K^- & \xrightarrow{I_G^e} & B_G^e & \xrightarrow{\pi^e} & B_G \\
\varphi_F \updownarrow & & \updownarrow \psi & & \updownarrow \psi^e & & \updownarrow \varphi_G^e & & \\
B'_F & \xleftarrow[I'_F]{} & J' & \hookleftarrow & J'/K^{-\prime} & \xrightarrow[I_G^{e\prime}]{} & B_G^{e\prime} & \xrightarrow[\pi^{e\prime}]{} & B'_G
\end{array}$$

such that $\pi^{e'} \circ \phi_G^e \circ I_G^e$ and $\pi^e \circ \phi_G^e \circ I_G^{e\prime}$ extend smoothly to J and J' respectively.

(1.14) **Theorem.** *Two systems of intersecting hypersurfaces are equivalent in the sense of Definition 1.5 if, and only if, the associated systems of oriented-intersection maps are equivalent in the sense of Definition 1.12.*

Apart from the need for joint conjugation of I_F and I_G there is also an orientation invariant between Theorem 1.10 and the hypotheses of Theorem 1.14. Consider the obvious definition of joint equivalence: the existence of a diagram of contact maps, with vertical diffeomorphisms

$$(1.15)\quad \begin{array}{ccccc}
B_F & \xleftarrow{I_F} & J & \xrightarrow{I_G} & B_G \\
\phi_F \updownarrow & & \updownarrow \psi & & \updownarrow \phi_G \\
B'_F & \xleftarrow[I'_F]{} & J' & \xrightarrow[I'_G]{} & B'_G.
\end{array}$$

Since the orientation e of F fixes an orientation of V_G, namely the positive direction along bicharacteristics on G leads (eventually, locally) into G^e, and at L, where I_G has $S_{1,1,0}$ singularity, V_G is tangent to K and non-zero, e defines an orientation e_L of L in K. Thus, in addition to (1.15) we must have

$$(1.16)\quad \psi^*(e'_L) = e_L.$$

Equivalence in the sense of Definition 1.12 is weaker than (1.15), with (1.16) since then ϕ_G lifts to ϕ_G^e.

In outline, the paper proceeds as follows. In Sect. 2, folded contact structures are discussed, leading to the proof of the first part of Theorem 1.10, the second part is proved in Sect. 3. Sections 4 and 5 contain the treatment of the joint conjugation problem (1.13) [and (1.15)] at the level of formal power series, including the display of the obstructions to such conjugation. Sections 6 and 7 give factorization theorems which show that there are no obstructions to the replacement of the formal equivalence by C^∞ equivalence in the sense of (1.13). Theorem 1.14 is proved in Sect. 8. Section 9 contains a complete set of examples. For the convenience of the reader some, essentially classical, material on the Lagrange bracket and extension theorems for contact manifolds is presented in the appendix.

To a certain extent, the results of this paper, as they affect boundary value problems, are negative. The appearance of large spaces of moduli, distinguishing between apparently similar situations in "geometric optics" indicates that the construction of parametrices for boundary value problems with bicharacteristic tangency to higher order than first (see [1, Sect. 4]) is a formidable task. It should be noted however that certain results (as illustrated by Morawetz et al. [7]) can be expected to follows from more qualitative aspects of the canonical geometry. In particular using the results of Sect. 9, the (C^∞) propagation of singularities for boundary value problems with (non-degenerate) bicharacteristic inflection will be analysed in [6].

2. Folded Contact Structure

Much of the analysis in subsequent sections is concerned with minimally degenerate contact structures, which we term folded contact structures, their reflective involutions and Lagrange algebras.

(2.1) *Definition.* A *folded contact structure* on a $(2n+1)$-manifold J is an oriented line subbundle $A^+ \subset T^*J$ for any local, non-vanishing section, α, of which the $(2n+1)$-form $\alpha \wedge (d\alpha)^n$ vanishes simply on a hypersurface $\iota_K : K \hookrightarrow J$ and, if $\alpha_K = \iota_K^* \alpha$,

$$(2.2) \quad \alpha_K \wedge (d\alpha_K)^{n-1} \neq 0 .$$

Martinet [4] showed that closed 2-forms with the property analogous to this ("$\Sigma_{1,0}$ singularity") can be brought to a fixed form by change of coordinates. We shall show that folded contact structure can similarly be brought to normal form. As our basic example we take $\mathbb{R}^{2n+1}$ with coordinates (y, η, x, ξ, z) $y, \eta \in \mathbb{R}^{n-1}$, $x, \xi, z \in \mathbb{R}$, and let $\bar{A} \subset T^*\mathbb{R}^{2n+1}$ be spanned by

$$(2.3) \quad \bar{\alpha} = \sum_{j=1}^{n-1} \eta_j dy_j + x\xi dx + dz > 0 .$$

(2.4) **Theorem.** *If (J, A^+, p) is a germ of folded contact structure at $p \in J$ then there is a germ of diffeomorphism $\phi: J, p \to \mathbb{R}^{2n+1}, 0$ such that $\phi^* \bar{A}^+ = A^+$.*

Proof. Suppose $\alpha \in C^\infty(A, p)$, $\alpha > 0$ then $\alpha \wedge (d\alpha)^n = f\gamma$ where γ is a non-vanishing volume form and $f \in C^\infty(J, p)$ has $f(p) = 0$, $d_p f \neq 0$. If $\sigma \in C^\infty(J, p)$ then

$$(2.5) \quad (d(\sigma\alpha))^n = (d\sigma \wedge \alpha + \sigma d\alpha) \wedge \ldots \wedge (d\sigma \wedge \alpha + \sigma d\alpha)$$
$$= \sigma^n (d\alpha)^n + n\sigma^{n-1} d\sigma \wedge \alpha \wedge (d\alpha)^{n-1} .$$

Pulling this equation back to K (defined by $f=0$), gives (2.5) with α replaced by $\alpha_K=\iota_K^*\alpha$. If γ_K is a volume form on K then

$$\alpha_K\wedge(d\alpha_K)^{n-1}=i_X\circ\gamma_K$$

defines a non-vanishing vector field, by assumption (2.2). So, on K, we define σ to satisfy

$$0=(d(\sigma\alpha_K))^n=\sigma^n(d\alpha_K)^n+n\sigma^{n-1}d\sigma\wedge i_X\gamma_K$$

which can be rewritten

$$X\sigma+g\sigma=0,\ \frac{1}{n}(d\alpha_K)^n=g\gamma_K.$$

This certainly has a solution with $\sigma(p)=1$. If $\tilde{\sigma}\in C^\infty(J,p)$ restricts to σ on K then $\tilde{\alpha}=\tilde{\sigma}\alpha$ satisfies $(d\tilde{\alpha}_K)^n\equiv 0$. By the version of Darboux's theorem given by Sternberg [9, Theorem 6.2] coordinates $y_1,\dots,y_{n-1},\eta_1,\dots,\eta_n,\xi,z$ can be introduced at p in K so that

$$(2.6)\quad \tilde{\alpha}_K=\sum_1^{n-1}\eta_j dy_j+dz.$$

Returning to (2.5), with α replaced by $\tilde{\alpha}$ note that $(d\tilde{\alpha}_K)^n=0$ so at K $(d\tilde{\alpha})^n=df\wedge v$. Now, $\tilde{\alpha}_K\wedge(d\tilde{\alpha}_k)^{n-1}\neq 0$ and if $\sigma\equiv 1$ on K, then $d\sigma=\sigma_1 df$ at K, $\sigma_1\in C^\infty(K,0)$ and

$$(d(\sigma\tilde{\alpha}))^n=df\wedge v+n\sigma_1 df\wedge(dy_1\wedge d\eta_1\wedge\dots\wedge dy_{n-1}\wedge d\eta_{n-1})\quad\text{at}\quad K.$$

Since $d\xi\wedge d\tilde{\alpha}_K=dz\wedge d\tilde{\alpha}_K=0$, $d\xi\wedge v=dz\wedge v=0$ the equation $(d(\sigma\tilde{\alpha}))^n=0$ can be solved for σ_1, on K, providing us with $\tilde{\alpha}\in C^\infty(A,0)$, $\tilde{\alpha}>0$ such that (2.6) holds and

$$(d\tilde{\alpha})^n=0\quad\text{at}\quad K.$$

Now, extend ξ to an element of $C^\infty(J,p)$ and define $V_{\xi\tilde{\alpha}^*}$, in $f\neq 0$, where A is a contact bundle (see the appendix):

$$(2.7)\quad 2d\tilde{\alpha}(V,\cdot)=-d\xi+\varrho\tilde{\alpha},\qquad \tilde{\alpha}(V)=\xi.$$

From (A.5), ϱf vanishes on K, so $\varrho\in C^\infty(J,p)$. Evaluating (2.7) on the vector fields ∂_{y_j}, ∂_{η_j}, ∂_ξ, ∂_z on K shows that V is not C^∞ across K, *however* fV is smooth and transversal to K, so the equations

$$(2.8)\quad V(\chi)=1,\qquad Vy_j=Vz=[\xi,\eta_j\tilde{\alpha}^*]=0$$

have unique solutions such that $\chi=0$, on K and y_j,η_j,z are extensions of these functions off that surface. Now χ vanishes to exactly second order on K so, reversing the sign of ξ if necessary, $\chi=\frac{1}{2}x^2$. From Jacobi's identity and Remark A.15 it follows that

$$\tilde{\alpha}=\sum_{j=1}^{n-1}\eta_j dy_j+\xi d(\tfrac{1}{2}x^2)+dz$$

as desired.

We shall use this result to prove a stronger one, namely the first part of Theorem 1.10.

Let $\mathrm{Con}(J, A^+, p) = \mathrm{Con}$ be the group of germs of diffeomorphism $J, p \to J, p$ leaving invariant the oriented bundle A^+. The subset $\mathrm{Ref}(J, A^+, p) = \mathrm{Ref}$, consisting of the involutions which leave K pointwise fixed, excluding the identity so that all $\mathscr{J} \in \mathrm{Ref}$ exchange the local components at p of $J \backslash K$, is of special importance. If $\mathscr{J} \in \mathrm{Ref}$ and $\phi \in \mathrm{Con}$ then $\phi \circ \mathscr{J} \circ \phi^{-1} \in \mathrm{Ref}$.

(2.9) **Proposition.** *The action of* $\mathrm{Con}(J, A^+, p)$ *on* $\mathrm{Ref}(J, A^+, p)$, *by conjugation, is transitive.*

Proof. Using Theorem 2.4 we only need show that any $\mathscr{J} \in \mathrm{Ref}(\mathbb{R}^{2n+1}, \bar{A}^+, 0)$ is conjugate by some $\phi \in \mathrm{Con}(\mathbb{R}^{2n+1}, \bar{A}^+, 0)$ to the basic involution

$$\mathscr{J}_0(y, \eta, x, \xi, z) = (y, \eta, -x, \xi, z). \tag{2.10}$$

Let $\Xi = \frac{1}{2}(\mathscr{J}^*(\xi\bar{\alpha}^*) + (\xi\bar{\alpha}^*))$ be the $\mathscr{J}$-even part of $\xi\bar{\alpha}^*$, using the assumption $\mathscr{J}^2 = \mathrm{Id}$. The Hamilton field V_Ξ (see Appendix), well-defined in $x \neq 0$, satisfies

$$2d\bar{\alpha}(V, \cdot) + d\Xi(\bar{\alpha}) = \varrho\bar{\alpha}, \qquad \bar{\alpha}(V) = \Xi(\bar{\alpha}).$$

From (A.5) it follows that $\varrho \in C^\infty(J, p)$ and hence that

$$V = V_\Xi = \frac{1}{x}(\partial_x + h\partial_\xi) + W,$$

where h and W are C^∞ across $x = 0$. Thus, we can follow the proof of Theorem 2.4 above, solving equations (2.8) to give functions X, Y_j, and section H_j, $\alpha^* \in C^\infty(A^*, p)$, (with initial conditions x, y_j, $\eta_j\bar{\alpha}^*$, α^* on K), such that

$$\alpha = \Sigma H_j(\alpha) dY_j + X\Xi(\alpha) dX + dZ > 0.$$

Now since $\mathscr{J}$ is itself a contact diffeomorphism these functions $\frac{1}{2}X^2$, Y_j, Z and sections H_j, α^* are all $\mathscr{J}$-invariant as V and the initial data are $\mathscr{J}$-invariant. Thus, $\mathscr{J}^* Y_j = Y_j$, $\mathscr{J}^* H_j = H_j$, $\mathscr{J}^* Z = Z$, $\mathscr{J}^* \alpha^* = \alpha^*$, $\mathscr{J}^* \Xi = \Xi$ and $\mathscr{J}^* X = -X$, since $\mathscr{J}$ exchanges the components of $J \backslash K$. We can therefore take

$$\phi(Y, H, X, \Xi, Z) = (y, \eta, x, \xi, z).$$

Proof of Theoem 1.10 (first part). Suppose the given map is $I: J, p \to B, q$ where B has contact bundle M^+, and I has a fold singularity at p, and $A^+ = I^* M^+$ is a folded contact bundle on J. Using Darboux's theorem on B and Theorem 2.4 on J, I is equivalent under contact transformation to a map, with $S_{1,0}$ singularity $I': \mathbb{R}^{2n+1}, \bar{A}^+, 0 \to \mathbb{R}^{2n+1}, \bar{M}^+, 0$ where $\bar{M}$ is the standard contact structure (A.11). Clearly, the involution $\mathscr{J}$ on J, defined by exchange of points identified by I', lies in $\mathrm{Ref}(J, \bar{A}^+, 0)$; using Proposition 2.9 a further change of coordinates on J, leaving $\bar{A}^+$ fixed, reduces $\mathscr{J}$ to $\mathscr{J}_0$ as in (2.10) and shows that I is equivalent under contact transformation to

$$I_0: \mathbb{R}^{2n+1} \to (y, \eta, x, \xi, z) \to (y, \eta, \tfrac{1}{2}x^2, \xi, z).$$

Next let us note how the Lagrange isomorphism and bracket carry over to folded contact structures. The space, $\mathscr{CV}(A^+, p)$, of germs at p of contact vector fields, $V \in \mathscr{CV}(A, p)$ satisfies

$$\mathscr{L}_V \alpha = \varrho_{V,\alpha} \alpha \quad \forall \alpha \in C^\infty(A^+, p),$$

is again mapped into $C^\infty(A^*,p)$ by the evaluation map $V\mapsto\alpha(V)$. This map is certainly injective but, because of the degeneracy of A, it is not surjective; its range is the Lagrange algebra, $\mathscr{L}a(A,p)\subset C^\infty(A^*,p)$.

(2.11) **Lemma.**

$$\mathscr{L}a(A,p)=\{f\in C^\infty(A^*,p);\quad \forall\alpha\in C^\infty(A,p)\quad d(f(\alpha))\wedge\alpha\wedge(d\alpha)^{n-1}-nf(\alpha)(d\alpha)^n=0\quad\text{on}\quad K\}.$$

Proof. Observe that the given condition on f holds for all α if it holds for one α, $\alpha(p)\neq 0$, since if $\alpha=\varrho\alpha$

$$\begin{aligned}&d(f(\alpha)\varrho)\wedge\varrho\alpha\wedge(d(\varrho\alpha))^{n-1}\\&=\varrho^{n+1}df(\alpha)\wedge\alpha\wedge(d\alpha)^{n-1}+nf(\alpha)\varrho^n d\varrho\wedge\alpha\wedge(d\alpha)^{n-1}\\&=n[\varrho^n(d\alpha)^n+\varrho^{n-1}d\varrho\wedge\alpha\wedge(d\alpha)^{n-1}]f(\alpha')\\&=f(\alpha')(d\alpha')^n.\end{aligned}$$

Moreover, from Theorem 2.4 we can introduce local coordinates y, η, x, ξ, z and take $\alpha=\bar{\alpha}$. From (A.13) we deduce that the contact vector field V_f, defined on $J\backslash K$ by any $f\bar{\alpha}^*\in C^\infty(\bar{A}^*,p)$, $f\in C^\infty(J,0)$ is in these coordinates

$$\text{(2.12)}\quad \begin{aligned}V_f=&\sum_{j=1}^{n-1}\left(\frac{\partial f}{\partial\eta_j}\partial_{y_j}-\left(\frac{\partial f}{\partial y_j}-\eta_j\frac{\partial f}{\partial z}\right)\partial_{\eta_j}\right)\\&+\frac{1}{x}\left(\frac{\partial f}{\partial\xi}\partial_x-\left(\frac{\partial f}{\partial x}-\xi\frac{\partial f}{\partial z}\right)\partial_\xi\right)\\&+\left(f-\sum\eta_j\frac{\partial f}{\partial\eta_j}-\xi\frac{\partial f}{\partial\xi}\right)\partial_z.\end{aligned}$$

Thus, V_f is smooth precisely when $\dfrac{\partial f}{\partial\xi}=\dfrac{\partial f}{\partial x}=0$ on K $(x=0)$, and since $(d\bar{\alpha})^n=0$ on K this is the given condition, $df(\bar{\alpha})\wedge\bar{\alpha}\wedge(d\bar{\alpha})^{n-1}=0$, proving the lemma.

We need a further refinement of this result. Note, from (2.12) that in coordinates in which $A=\bar{A}$ if $f\bar{\alpha}^*\in\mathscr{L}a$, $V_f=0$ at 0 exactly when

$$\text{(2.13)}\quad \frac{\partial f}{\partial y_j}=\frac{\partial f}{\partial\eta_j}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial\xi^2}=0\quad\text{at}\quad 0$$

which condition is certainly independent of the isomorphism to $(\mathbb{R}^{2n+1},\bar{A},0)$ so defines the subalgebra $\mathscr{L}a'(J,A,p)\subset\mathscr{L}a(J,A,p)$ consisting of the elements with Hamilton vector fields V_f exponentiating locally:

$$\mathscr{L}a'(J,A,p)\ni f\mapsto\exp(V_f)\in\operatorname{Con}(J,A,p).$$

3. Cusp Singularities

We continue the analysis of folded contact structure by proving the second part of Theorem 1.10, that any contact map $I:J,B\to P,p$, with $S_{1,1,0}$ singularity can be brought to normal form by contact transformation. This result is considerably more delicate than the theory of contact folds above. As basic example we shall take the map

(3.1) $\tilde{I}:\mathbb{R}^{2n+1}\ni(y,\eta,x,\xi,z)\mapsto(y,\eta,x,x\xi+\xi^3,z)\in\mathbb{R}^{2n+1}$,

where the image has the contact structure $\bar{M}$, spanned by (A.13). The folded contact bundle $\tilde{A}^+=I^*\bar{M}^+$ is spanned by the form

(3.2) $\tilde{\alpha}=\sum_{j=1}^{n-1}\eta_j dy_j+(x\xi+\xi^3)dx+dz>0$.

Using Theorem 2.4 and Darboux's theorem on the domain and range of I, respectively, the second part of Theorem 1.10 is reduced to the following result.

(3.3) **Proposition.** *If $I:\mathbb{R}^{2n+1}$, $0\to\mathbb{R}^{2n+1}$, 0 has singularity of type $S_{1,1,0}$ and $I^*\bar{M}^+=\tilde{A}^+$ then there are maps $\psi,\phi:\mathbb{R}^{2n+1}$, $0\to\mathbb{R}^{2n+1}$, 0 with $\psi^*\tilde{A}^+=\tilde{A}^+$, $\phi^*\bar{M}^+=\bar{M}^+$ such that the diagram*

$$\begin{array}{ccc}\mathbb{R}^{2n+1},0 & \xrightarrow{\psi} & \mathbb{R}^{2n+1},0\\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle \tilde{I}}\\ \mathbb{R}^{2n+1},0 & \xrightarrow[\phi]{} & \mathbb{R}^{2n+1},0\end{array}$$

commutes.

Proof. First note that as a simple "cusp" singularity, by ignoring the contact structure I can certainly be transformed to $\tilde{I}$ in (3.1). In particular, if we denote by L the codimension 2 submanifold of $\mathbb{R}^{2n+1}$ where I is triple then $I:L\to I(L)$ is a contact diffeomorphism, where the contact bundles on L and $I(L)$ are the pull-backs of $\tilde{A}$ and $\bar{M}$. Indeed, it suffices to show that $\tilde{A}$ defines a contact structure on L. Since $\tilde{A}$ is a folded contact bundle any section $\alpha\in C^\infty(\tilde{A},p)$ pulls back to the fold K, where $x+3\xi^2=0$, as α_K satisfies (2.2), the rank of the tangent map $dI:T_pK\to T_p\mathbb{R}^{2n+1}$ is clearly $2n-1$ so $\alpha_K\wedge(d\alpha_K)^{n-1}$ does not vanish when pulled back to L, in fact $\alpha_K\wedge(d\alpha_K)^{n-1}>0$.

As remarked above, one can find $X,\Xi\in C^\infty(\mathbb{R}^{2n+1},0)$ such that the cusp, $I(K)$ is defined by

(3.4) $\left(\frac{X}{3}\right)^3+\left(\frac{\Xi}{2}\right)^2=0$,

so $I(L)$ is $X=\Xi=0$ and then

$[\Xi,X]\neq 0$,

$I(L)$ being a contact submanifold of $\mathbb{R}^{2n+1}$. Changing the sign of Ξ if necessary we can assume

(3.5) $[\Xi,X]>0$.

Suppose that we choose $\mu\in C^\infty(\bar{M},0)$, $\mu>0$, and define $\bar{X}\in C^\infty(\bar{M}^*_{2/5},0)$ $\bar{\Xi}\in C^\infty(\bar{M}^*_{3/5},0)$ by $\bar{X}(\mu)=X$, $\bar{\Xi}(\mu)=\Xi$. Then $I(K)$ is defined by $(\bar{X}/3)^3+(\bar{\Xi}/2)^2=0$ and $[\bar{\Xi},\bar{X}]\in C^\infty(\mathbb{R}^{2n+1},0)$. If μ were so chosen that

(3.6) $[\bar{\Xi},\bar{X}]=1$

then Proposition A.16 would allow the extension of any chosen contact diffeomorphism $\phi:I(L)$, $0\to\mathbb{R}^{2n+1}$,0 to a contact diffeomorphism on $\mathbb{R}^{2n+1}$, giving new coordinates Y', H', X', Ξ', Z' in the range of I with $X'=\bar{X}(\mu')$, $\Xi'=\bar{\Xi}(\mu')$, $\mu'=\sum H'_j dY_j+\Xi' dX'+dZ'>0$ spanning $\bar{M}^+$ and (3.4) still holding. Thus we could define new coordinates on the domain of I by

$$(\bar{y}_j,\bar{\eta}_j,\bar{x},\bar{z})=I^*(\bar{Y}_j,\bar{H}_j,\bar{X},\bar{Z})$$

and with $\bar{\xi}$ satisfying $\bar{x}\bar{\xi}+\bar{\xi}^3=I^*\bar{\Xi}'$. Note that $\bar{\xi}$ is then smooth because it is some non-vanishing multiple of the original coordinate ξ. Clearly these new coordinate systems in domain and range give the desired equivalence under contact transformation.

Thus to complete the proof of Proposition 3.3 we must show that (3.6) can always be arranged.

(3.7) **Proposition.** *Given* $X\in C^\infty(\bar{M}^*_{2/5},0)$, $\Xi\in C^\infty(\bar{M}^*_{3/5},0)$ *with* $X=\Xi=0$ *at* 0 *and*

$$[\Xi,X]>0 \quad \text{at} \quad 0$$

there exists $\sigma\in C^\infty(\mathbb{R}^{2n+1},0)$, $\sigma(0)>0$ *such that*

(3.8) $[\sigma^3\Xi,\sigma^2X]=1$.

This result suffices to prove (3.6) since if the initial choice $\mu\in C^\infty(\bar{M},0)$ does not ensure (3.6) then $\mu'=\sigma^{-5}\mu$ will do so.

Proof of Proposition 3.7. First note that there is a unique section $X'\in C^\infty(\bar{M}^*_{2/5},0)$ such that

$$[\Xi,X']=1, \qquad X'=0 \quad \text{on} \quad X=0.$$

So, for some $h\in C^\infty(\mathbb{R}^{2n+1},0)$, $h(0)\neq 0$, $X=hX'$ and (3.8) becomes

(3.9) $[\sigma^3\Xi,\sigma^2hX']=1$.

Using Proposition A.16 we can choose $\beta\in C^\infty(\bar{M},0)$ and functions y_j,η_j,z such that, if $x=X'(\beta)$, $\xi=\Xi(\beta)$, (A.11) holds. Then, using (A.7) we can write (3.9) as

$$(3.10)\quad h\sigma^4\Big(2x[\Xi,\sigma](\beta)-3\xi[X',\sigma](\beta)+\frac{3x\xi}{h}[\sigma,\alpha](\beta) +\frac{\sigma}{h}[\Xi,\alpha X'](\beta)\Big)=1.$$

From (A.9), (A.12), and (A.13)

$$[\Xi,\sigma]=\partial_x\sigma-\tfrac{2}{5}\xi\partial_z\sigma, \qquad [x,\sigma]=-\partial_\xi\sigma+\frac{2x}{5}\partial_z\sigma$$

so that (3.10) becomes

$$(3.11)\quad 2x\partial_x\sigma+3\xi\partial_\xi\sigma+x\xi V\sigma+R\sigma-(h\sigma^4)^{-1}=0,$$

where V is a smooth vector field (depending on h) and $R\in C^\infty(\mathbb{R}^{2n+1},0)$ is equal to 1 at $x=\xi=0$. To prove the existence (and uniqueness) of a smooth solution to (3.11) we shall first sove it formally and then use an argument of Nelson [8] to obtain a C^∞ solution.

Thus let

$$\mathfrak{g}=C^\infty(\mathbb{R}^{2n-1},0)[[x,\xi]]$$

be the ring of formal power series in x, ξ, with real-valued coefficients germs of functions of y, η, z. $\mathfrak{g}$ is the countable direct product of the spaces of homogeneous polynomials in x, ξ

(3.12) $\mathfrak{g}=\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots$

and (3.11) is meaningful as an equation for $\sigma \in \mathfrak{g}$ if R, h and the coefficients of V are replaced by their Taylor series at $x=\xi=0$. Moreover σ can then be constructed using the homogeneous gradation (3.12) of $\mathfrak{g}$. The projection of (3.11) into $\mathfrak{g}_0$ is

$$\sigma_0-(h_0\sigma_0^4)^{-1}=0,$$

where, h_0 is the projection of h into $\mathfrak{g}_0$ and hence is positive. Thus $\sigma_0=(h_0)^{-1/5}>0$ is uniquely fixed, being real. Inductively, we can assume that the projections σ_j of σ into $\mathfrak{g}_j$ have been chosen for $j<n$ in such a way that the projection of (3.11) into $\mathfrak{g}_j$, $j<n$ is valid. Then, σ_n must satisfy

(3.13) $T\sigma_n=2x\partial_x\sigma+3\xi\partial_\xi\sigma+(1+h_0\sigma_0^{-3})\sigma_n=T_n,$

where $T_n \in \mathfrak{g}_n$ is determined by the earlier choices σ_j, $j<n$. The linear map $T:\mathfrak{g}_n \to \mathfrak{g}_n$ is invertible for all $n \geqq 1$ since the functions $x^{n-s}\xi^s$ are always eigenvectors with eigenvalues

$$2n+3+(1+4h_0^{-2/5})>0.$$

Thus, (3.11) has a unique formal power series solution, let $\tilde{\sigma} \in C^\infty(\mathbb{R}^{2n+1},0)$ be some function with this series as Taylor series at $x=\xi=0$. Returning to the beginning of the proof, but replacing Ξ by $\tilde{\sigma}^3\Xi$, X by $\tilde{\sigma}^2X$ and repeating these arguments we arrive at (3.9) where now $h-1$ vanishes to all orders at $x=\xi=0$. We now use a variant of the Sternberg linearization theorem to simplify this equation. We no longer work at the germ level but, since we only need a local solution σ of (3.10) we shall take, for some $\varepsilon>0$,

(3.14) $$\begin{cases} h=1 \text{ if } |y_j|, |\eta_j|, |x|, |\xi| \text{ or } |z|>\varepsilon \text{ (for some } j) \\ h>\frac{1}{2} \text{ everywhere and } h-1 \text{ vanishes to all orders at } x=\xi=0. \end{cases}$$

Then, (3.11) becomes

(3.15) $2x\partial_x\sigma+3\xi\partial_\xi\sigma-2x\xi\partial_z\sigma+V\sigma+(1+\gamma)\sigma-\sigma^{-4}/h=0,$

where V is a vector field and γ a function, both vanishing outside a compact set and to infinite order on $x=\xi=0$. To solve (3.15) we use Nelson's proof of Sternberg's linearization theorem to eliminate the "small" V term in (3.15). First however, replace z by $z'=z+\frac{2}{5}x\xi$ so that (3.15) becomes

(3.16) $2x\partial_x\sigma+3\xi\partial_\xi\sigma+V\sigma+\gamma\sigma-\sigma^{-4}/h=0.$

The vector fields

$$Q=-3\xi\partial_\xi-2x\partial_x-V \quad Q_0=-3\xi\partial_\xi-2x\partial_x$$

both define global flows $U(t)$, $U_0(t)$ on $\mathbb{R}^{2n+1}$, given (3.14), and the wave operator

(3.17) $W(y,\eta,x,\xi,z)=\lim_{t\to\infty} U_0(-t)U(t)(y,\eta,x,\xi,z)$

is a local diffeomorphism, near 0, intertwining U and U_0 and so conjugating Q to Q_0, $W_* Q = Q_0$.

We refer to Nelson [8, pp. 42–44] for the proof of the C^∞ convergence in (3.17) and note here only the minor modifications required to that argument. The only estimate used by Nelson that our U and U_0 do not satisfy is the local exponential decay, for $t>0$,

$$\|U(t)(y,\eta,x,\xi,z)\| \leqq e^{-at}\|(y,\eta,x,\xi,z)\| \; a>0.$$

Instead we have the weaker condition of exponential decay in the x,ξ components. Putting $(y,\eta,z)=\phi$ and $U(t)(\phi,x,\xi)=(\Phi(t),X(t),\Xi(t))$,

$$|X(t)| \leqq e^{-at}|x|, \qquad |\Xi(t)| \leqq e^{-at}|\xi| \qquad a>0.$$

Exact compensation for this deficiency is provided by the fact that $Q-Q_0$ vanishes to all orders at $x=\xi=0$, as opposed to the corresponding assumption in [8] that this difference vanish to all orders at the base point. Thus, the components of $(Q-Q_0)$ can be estimated by $c_K(|x|+|\xi|)^k$ for each k and so we can conclude that W is well-defined by (3.17).

Working in the new coordinates provided by W, which we shall continue to denote by (ϕ,x,ξ) (3.16) becomes

$$3\xi\partial_\xi\sigma + 2x\partial_x\sigma + (1+\gamma)\sigma - \sigma^{-4}/h = 0,$$

where $h-1$ and γ vanish to all orders at $x=\xi=0$. Polar coordinates, $\xi=r^3\cos\theta$, $x=r^2\sin\theta$, reduce this to an ordinary differential equation in which ϕ and θ are parameters:

$$r\frac{\partial\sigma}{\partial r} + (1+\gamma)\sigma - \sigma^{-4}/h = 0$$

which has a unique smooth solution σ. Indeed, putting $\varrho=\sigma-1$ gives

$$f\frac{d\varrho}{dr} + (1+\varrho-(1+\varrho)^{-4}) + f(\varrho,r,\theta,\phi) = 0,$$

where $f(\varrho,r,\theta,\phi)$ vanishes to all orders at $r=0$ and depends smoothly on ϱ, ϕ, θ. Let $s(\varrho)$ be the unique smooth solution, near $\varrho=0$, of

$$s(\varrho) = s'(\varrho)(1+\varrho-(1+\varrho)^{-4}),$$

so that $\frac{d}{dr} rs(\varrho) = -s'(\varrho)f(\varrho,r,\phi,\theta)$. Since $s(0)=0$, $s'(0)\neq 0$ this can be rewritten as an integral equation

$$\text{(3.18)} \quad s = -\frac{1}{r}\int_0^r g(s,r,\phi,\theta)\,dr,$$

where g is a C^∞ function vanishing to all orders at $r=0$. Clearly the operator on the right of (3.18) is a contraction near the origin of the Banach space of functions $r^k\psi$, $\psi\in C^k$ (in all variables) provided $0\leqq r\leqq\varepsilon$ with ε small. Thus (3.18) has a unique C^∞ solution. We have therefore solved (3.8) for σ and so proved Proposition 3.7, and consequently Theorem 1.10.

Now, we need to verify that $I_G: J \to B_G$ has a Whitney cusp at p, where (1.1)–(1.4) hold. Using Proposition A.16 coordinates can be introduced in E, at p, so that M is spanned by the form (A.11) and G is the surface $x_1 = 0$. Since V_G is then spanned by ∂_{ξ_1}, (1.4) and the Malgrange-Weierstrass Preparation Theorem show that F is defined by

$$\xi_1^3 + a\xi_1^2 + b\xi_1 + c = 0,$$

where a, b, c are independent of ξ_1 and vanish at $x = \xi = z = 0$. Using Proposition A.16 again, ξ_1 can be replaced by $\xi_1 + \frac{a}{3}$ so one can assume $a \equiv 0$. Since (1.3) implies that db and dc are independent it follows that the intersection map $J \to B_G$, where B_G can be identified with $x_1 = \xi_1 = 0$, has a simple cusp. It is clearly a contact map.

4. Formal Power Series on L

The folded contact structure $(\mathbb{R}^{2n+1}, \tilde{A}, 0)$ defined by (3.2) arose in connection with the relation $\mathscr{R}$ defined by the map $\tilde{I}$ in (3.1),

$$(\bar{y}, \bar{\eta}, \bar{x}, \bar{\xi}, \bar{z}) \in \mathscr{R}(y, \eta, x, \xi, z) \Leftrightarrow y = \bar{y}, \eta = \bar{\eta}, z = \bar{z}, x = \bar{x}, \bar{x}\bar{\xi} + \bar{\xi}^3 = x\xi + \xi^3 .$$

We need also to consider the "positive part" of $\mathscr{R}$, $\mathscr{R}_+$, defined to be the trivial relation outside the region

$$(4.1) \quad D(\mathscr{R}_+) = \{(y, \eta, x, \xi, z) \in \mathbb{R}^{2n+1} = J; \quad \xi \leqq 0 \text{ and } -3\xi^2 \geqq x \text{ or } \xi \geqq 0 \text{ and } -\tfrac{3}{4}\xi^2 \geqq x\},$$

and inside $D(\mathscr{R}_+)$ to be $\mathscr{R}$ itself. Thus $p \in \mathscr{R}_+(p')$ if, and only if, $p = p'$ or $p \in \mathscr{R}(p')$ and $p, p' \in D(\mathscr{R}_+)$. The negative part of $\mathscr{R}$ is defined analogously, being non-trivial on

$$D(\mathscr{R}_-) = \{(y, \eta, x, \xi, z) \in \mathbb{R}^{2n+1}; \quad \xi \leqq 0 \text{ and } -\tfrac{3}{4}\xi^2 \geqq x \text{ or } \xi \geqq 0 \text{ and } -3\xi^2 \geqq x\}.$$

Let

$$\mathfrak{a} = C^\infty(L, 0)[[x, \xi]]$$

be the ring of formal power series in x, ξ with coefficients in $C^\infty(L, 0)$ where L, defined by $x = \xi = 0$, is the surface on which $\mathscr{R}$ is triple. To utilize the quasi-homogeneity of $\mathscr{R}_+$ we shall grade $\mathfrak{a}$ by assigning weight 2 to x and weight 1 to ξ

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_N \oplus \ldots,$$

where $\mathfrak{a}_N$ is spanned, as a module over $C^\infty(L, 0)$, by the polynomials $x^s \xi^{N-2s}$; *let* $\mathfrak{a}_{(N)} = \bigoplus_{k \geqq N} \mathfrak{a}_k$ be the associated filtration.

The Taylor series map

$$T_L : C^\infty(J, 0) \to \mathfrak{a}$$

is, according to Borel's theorem, surjective. Let $C^\infty(\mathscr{R}_+)\subset C^\infty(J,0)$ be the ring of germs of functions constant on the $\mathscr{R}_+$-equivalence classes

(4.2) **Lemma.** *$T_L: C^\infty(\mathscr{R}_+)\to C^\infty(L,0)[[x, x\xi+\xi^3]]=\mathfrak{a}(\mathscr{R})$ is surjective.*

Proof. Given $f\in C^\infty(\mathscr{R}_+)$ let $f_N\in\mathfrak{a}_N$ be the projection of the Taylor series of f into $\mathfrak{a}_N$. If $\bar{f}$, f_j are representations of f, f_j then, uniformly near 0

$$\left|f(y,\eta,r^2x,r\xi,z)-\sum_{j=0}^{N} f_j(y,\eta,r^2x,r\xi,z)\right|\leqq C_N|r|^{N+1}. \tag{4.3}$$

Since $\mathscr{R}_+$ is positively quasi-homogeneous, $r>0$ and $(y,\eta,x,\xi,z)\in\mathscr{R}_+(y,\eta,x,\xi',z)$ implies that $(y,\eta,r^2x,r\xi,z)\in\mathscr{R}_+(y,\eta,r^2x,r\xi',z)$ so, from (4.2) and the $\mathscr{R}_+$-invariance of f

$$\left|\sum_{j=0}^{N}(f_j(y,\eta,r^2x,r\xi,z)-f_j(y,\eta,r^2x,r\xi',z))\right|\leqq 2C_N|r|^{N+1}.$$

Thus each f_N is itself locally $\mathscr{R}_+$-invariant. The f_N are polynomials and $\mathscr{R}_+$ is algebraic so the f_N are actually $\mathscr{R}$-invariant globally in the complexified (x,ξ)-space. Cauchy's formula then easily shows that $f_N\in C^\infty(L,0)[[x,x\xi+\xi^3]]$. The surjectivity of the map, even when restricted to $\mathscr{R}$-invariant germs, is obvious from Borel's theorem so the proof is complete.

Next, consider $\phi\in\mathrm{Con}(J,\tilde{A}^+,0)$. We shall denote by $\mathrm{Con}_0\subset\mathrm{Con}$ the subgroup of the diffeomorphisms leaving L invariant. The Taylor series, on L, of such a germ determines, and is determined by, the induced action ϕ^* of ϕ on $\mathfrak{a}$. Let $\mathrm{Con}_k\subset\mathrm{Con}_0$ consist of the subgroup whose elements satisfy

$$(\phi^*-\mathrm{Id})\mathfrak{a}_{(l)}\subset\mathfrak{a}_{(l+k)}\qquad\forall l\geqq 0. \tag{4.4}$$

As ϕ^* is a ring homomorphism and $\mathfrak{a}$ is generated by its elements of degree 0, 1, 2 it suffices to check (4.4) for $l=0,1,2$. $\phi\in\mathrm{Con}_0$ satisfies (4.4) for $k=0$.

Consider the Lagrange algebra $\mathscr{L}a'$ defined at the end of Sect. 2. Denote by $\mathscr{L}a_0\subset\mathscr{L}a'$ the subalgebra whose elements f have V_f tangent to L. Using the 1-form $\tilde{\alpha}$ in (3.2) to trivialize $\tilde{A}^*$ we can apply the Taylor series map and so easily conclude

$$T_L(\mathscr{L}a_0)=\mathfrak{m}\oplus(x+3\xi^2)^2\mathfrak{a}_{(1)}\tilde{\alpha}^*,$$

where $\mathfrak{m}\subset C^\infty(L,0)$ is the principal ideal. For every $k\geqq 0$ put

$$\mathfrak{b}_{(k)}=(x+3\xi^2)^2\mathfrak{a}_{(k+1)}\tilde{\alpha}^*. \tag{4.5}$$

(4.6) **Proposition.** *To each $\phi\in\mathrm{Con}_1$ there corresponds a unique $f\in\mathfrak{b}_{(1)}$ such that*

$$(\phi^*-\exp(V_f)^*)\mathfrak{a}=\{0\}. \tag{4.7}$$

The map $AT_L:\phi\mapsto f$ maps Con_k onto $\mathfrak{b}_{(k)}$ for every $k\geqq 1$ and if $\phi_i\in\mathrm{Con}_{k_i}$, $k_i\geqq 1$ for $i=1,2$ $AT_L(\phi_1\circ\phi_2)\equiv AT_L\phi_1+AT\phi_2$ modulo $\mathfrak{b}_{(k_1+k_2+1)}$.

Proof. Given $\phi\in\mathrm{Con}_1$ the vector field V, with coefficients in $\mathfrak{a}$ is uniquely defined by induction over the filtration. Thus, defining $V^{(1)}y_j=\phi^*y_j-y_j$, $V^{(1)}\eta_j=\phi^*\eta_j-\eta_j$ etc., we find $V^{(1)}\mathfrak{a}_{(k)}\subset\mathfrak{a}_{(k+1)}$, because of (4.4). Similarly, $V^{(l)}$, defined by

$$V^{(l)}y_j=\phi^*y_j-\sum_{k=0}^{l-1}\frac{1}{k!}\left(\sum_{p=1}^{l-1}V^{(p)}\right)^k y_j$$

etc., satisfies $V^{(l)}\mathfrak{a}_{(k)}\subset\mathfrak{a}_{(k+l)}$ for all k. Clearly $V=\sum V^{(j)}$ is uniquely defined by this argument and is moreover a contact vector field, again by induction. Thus $f_\alpha=\alpha(V)$ defines f. Since, for $g\in\mathfrak{a}$, $Vg=[f,g]$ one easily concludes that $f\in\mathfrak{b}_{(k)}$ precisely when $\phi\in\mathrm{Con}_k$. The other parts of the proposition follow similarly.

Define $\mathrm{Con}_k(\mathscr{R}_+)\subset\mathrm{Con}_k$ to be the group whose elements leave the relation $\mathscr{R}_+$ fixed. In particular, if $\phi\in\mathrm{Con}_k(\mathscr{R}_+)$ then $\phi^*C^\infty(\mathscr{R}_+)\subset C^\infty(\mathscr{R}_+)$. From Lemma 4.2 it follows that

(4.8) $\phi^*\mathfrak{a}(\mathscr{R})\subset\mathfrak{a}(\mathscr{R})$.

For each $k\geqq 1$ *put*

(4.9) $\mathfrak{b}_{(k)}(\mathscr{R})=\mathfrak{b}_{(k)}\cap(\mathfrak{a}_{(5)}(\mathscr{R})\tilde{\alpha}^*)$.

(4.10) **Lemma.** *$\Lambda T_L:\mathrm{Con}_k(\mathscr{R}_+)\to\mathfrak{b}_{(k)}(\mathscr{R})$ is, for each $k\geqq 1$, surjective.*

Proof. Since $\mathfrak{a}(\mathscr{R})$ is a graded (a homogeneous) subring in $\mathfrak{a}$ we conclude from (4.7) and (4.8) that, if $f=\Lambda T_L\phi$ with $\phi\in\mathrm{Con}_k(\mathscr{R}_+)$ then

$$[f,\mathfrak{a}(\mathscr{R})]\subset\mathfrak{a}(\mathscr{R}).$$

Since $f=g\tilde{\alpha}^*$ with $g\in(x+3\xi^2)^2\mathfrak{a}_{(k+1)}$,

$$[f,x]=-\frac{1}{(x+3\xi^2)}\partial_\xi g\in\mathfrak{a}(\mathscr{R}).$$

The equation $\partial_\xi g=(x+3\xi^2)x^k(x\xi+\xi^3)^p$ has solutions $g=x^k(x\xi+\xi^3)^{p+1}/(p+1)+r(x)\in\mathfrak{a}(\mathscr{R})$ so ΛT_L maps $\mathrm{Con}_k(\mathscr{R}_+)$ into $\mathfrak{b}_{(k)}(\mathscr{R})$.

The surjectivity of the map is a consequence of the fact that each $g\in(x+3\xi^2)^2\cdot\mathfrak{a}_{(k+1)}\cap\mathfrak{a}(\mathscr{R})$ is the image under T_L of a function $\bar{g}\in C^\infty(\mathscr{R})$ which vanishes to second order on the fold $x+3\xi^2=0$. From Lemma 4.2 there certainly exists $\bar{g}_1\in C^\infty(\mathscr{R})$ with $T_L\bar{g}_1=g$. Moreover $\bar{g}_1\upharpoonright(x+3\xi^2=0)$ vanishes to infinite order at $\xi=0$, so is equal to $h(-2\xi^3)$ for some C^∞ function h. Then put

$$\bar{g}=\bar{g}_1-h(x\xi+\xi^3)\in C^\infty(\mathscr{R}).$$

Clearly $T_L\bar{g}=g$ and $\bar{g}$ vanishes on $x+3\xi^2=0$, in fact $\bar{g}$ vanishes to second order on this surface if $\bar{g}_1$ was selected to be a C^∞ function of $x, x\xi+\xi^3, y, \eta, z$, since $\partial_\xi\bar{g}=(x+3\xi^2)p=0$ there.

Together with these results on the formal power series of objects associated to $\mathscr{R}_+$ we need the corresponding results for objects associated to an arbitrary $\mathscr{J}\in\mathrm{Ref}(J,\tilde{A}^+,0)$. Since $\mathscr{J}$ is an involution leaving K, and hence L, pointwise fixed, $\frac{1}{2}(\mathscr{J}^*+\mathrm{Id})$ defines a projection from $C^\infty(J,0)$ to $C^\infty(\mathscr{J})$, the space of $\mathscr{J}$-invariant germs, and also from $\mathfrak{a}$ to $\mathfrak{a}(\mathscr{J})$, the space of $\mathscr{J}$-invariant series. For each $k\geqq 0$ put

(4.11) $c_k=C^\infty(L,0)[[\xi,(x+3\xi^2)^2]]\cap\mathfrak{a}_k$,

and let $\mathrm{Con}_k(\mathscr{J})\subset\mathrm{Con}_k$ be the group of diffeomorphisms satisfying (4.4) and cummuting with $\mathscr{J}$.

(4.12) **Proposition.** *For any $\mathscr{J}\in\mathrm{Ref}(J,\tilde{A}^+,0)$ the projection of $\mathfrak{a}_{(k)}(\mathscr{J})=\mathfrak{a}_{(k)}\cap\mathfrak{a}(\mathscr{J})$ into $\mathfrak{a}_k$ lies in c_k and the sequence*

(4.13) $0\to\mathfrak{a}_{(k+1)}(\mathscr{J})\hookrightarrow\mathfrak{a}_{(k)}(\mathscr{J})\xrightarrow{\pi_k}c_k\to 0$

is exact. For each $k \geqq 1$

$$\Delta T_L : \mathrm{Con}_k(\mathscr{J}) \to \mathfrak{a}(\mathscr{J})\tilde{\alpha}^* \cap \mathfrak{b}_{(k)} = \mathfrak{b}_{(k)}(\mathscr{J})$$

is surjective.

Proof. Since $\mathscr{J}$ leaves K pointwise fixed $\mathscr{J}^*y_i - y_j$, $\mathscr{J}^*\eta_i - \eta_j$, $\mathscr{J}^*\xi - \xi$ and $\mathscr{J}^*z - z$ all vanish at $x = -3\xi^2$ and therefore lie in $\mathfrak{a}_{(2)}$. Similarly $\mathscr{J}^*(x+3\xi^2) \equiv \mathscr{J}^*x + 3\xi^2 \bmod \mathfrak{a}_{(3)}$, must vanish at $x = -3\xi^2$. Since $\mathscr{J}^2 = \mathrm{Id}$, $\mathscr{J} \neq \mathrm{Id}$ this implies that $\mathscr{J}^*x \equiv -x - 6\xi^2 \pmod{\mathfrak{a}_{(3)}}$ and therefore leads to (4.13).

As in the proof of Lemma 4.10, to see that $\Delta T_L = f \in \mathfrak{a}(\mathscr{J})\tilde{\alpha}^*$ when $\phi \in \mathrm{Con}_k(\mathscr{J})$; recall that (4.7) implies (by an inductive argument)

$$[f, \mathfrak{a}(\mathscr{J})] \subset \mathfrak{a}(\mathscr{J}).$$

In particular $[f, \xi] = \dfrac{1}{(x+3\xi^2)} \partial_x f_{\tilde{\alpha}} - \xi \dfrac{\partial f_\alpha}{\partial z} \in \mathfrak{a}(\mathscr{J})$. Using the first part of the lemma,

$$\partial_x f_{\tilde{\alpha}} \in c_{k+3} \Rightarrow f_{\tilde{\alpha}} \in c_{k+5}.$$

Then, there exists $f^{(k)} \in \mathfrak{a}(\mathscr{J})\tilde{\alpha} \cap \mathfrak{b}_{(k)}$ with $f - f^{(k)} \in \mathfrak{b}_{(k+1)}$. Moreover, from Borel's theorem there exist $\phi_{(k)} \in \mathrm{Con}_k(\mathscr{J})$ with $\Delta T_L \phi_{(k)} = f^{(k)}$ so we proceed inductively applying the last part of Proposition 4.6 to $\phi \circ \phi_{(k)}^{-1} \in \mathrm{Con}_{k+1}(\mathscr{J})$. The surjectivity of this map is, as already noted, obvious.

5. Obstructions to Equivalence

Using the calculations of Sect. 4 we shall analyse, at the level of formal power series, equivalence in the sense of Definition 1.12. By Proposition 3.3 there are coordinates in J, B_G such that I_G takes the form (3.1) and A^+ is spanned by $\tilde{\alpha}$ in (3.2). The subset $K^- \subset J$ where V_G is tangent to J from the "outside" is then either the part $\xi \geqq 0$ of $x = -3\xi^2$ or else the part $\xi \leqq 0$; K^- is the part of J on the negative side of L with respect to the orientation of e_L, L in K induced by the orientation of F. Corresponding to these two cases the relation on $J \backslash K^-$ induced by I_G^e is either $\mathscr{R}_-$ or $\mathscr{R}_+$. We shall only treat the latter case, since the map

$$(y, \eta, x, \xi, z) \mapsto (y, -\eta, x, -\xi, -z)$$

transforms one to the other, and $\tilde{\alpha}$ to $-\tilde{\alpha}$, so the case in which I_G^e induces $\mathscr{R}_-$ can be handled by reversing the orientation of A troughout. Coordinates in which J_G^e induces $\mathscr{R}_+$ will be called *G-coordinates.* The other intersection map, I_F defines an element $\mathscr{J} \in \mathrm{Ref}(\mathbb{R}^{2n+1}, \tilde{A}, 0)$. Clearly, $\mathscr{J}$ is well-defined up to conjugation by an element of $\mathrm{Con}(\mathscr{R}_+)$; we shall denote the set of conjugacy classes by $\mathrm{Ref}_{\mathscr{R}_+}$.

In the forward direction the following result is trivial and the reverse implication will be proved in Sect. 8.

(5.1) **Theorem.** *Two systems of oriented-intersection maps are equivalent in the sense of Definition 1.12 if, and only if, they have the same orientation and define the same conjugacy class in* $\mathrm{Ref}_{\mathscr{R}_+}$ *through the introduction of G-coordinates.*

Put

$$\mathrm{Con}_\infty = \mathrm{Con}_\infty(\mathbb{R}^{2n+1}, A, 0) = \bigcap_k \mathrm{Con}_k.$$

Then we can define a weaker equivalence relation, *formal equivalence at L*, by

$$\mathscr{J} \underset{L}{\sim} \mathscr{J}' \Leftrightarrow \exists \mu \in \mathrm{Con}(\mathscr{R}_+), \quad \phi \in \mathrm{Con}_\infty \text{ such that}$$
$$\mu \mathscr{J}' \mu^{-1} = \phi^{-1} \mathscr{J} \phi .$$

From the proof of Proposition 2.9 we note that two involutions are conjugate under some $\phi \in \mathrm{Con}(\mathbb{R}^{2n+1}, A^+, 0)$ which leaves K, and hence L, pointwise invariant. Thus, the action of Con_0 on Ref, by conjugation, is transitive.

$$(5.2) \quad \mathscr{J}' = \phi^{-1} \mathscr{J} \phi \qquad \phi \in \mathrm{Con}_0 ,$$

so, $\mathscr{J}' \underset{L}{\sim} \mathscr{J}$ if, and only if, $\mathscr{J}'$ projects onto the identity element of

$$\Gamma(\mathscr{J}) = \mathrm{Con}_\infty \circ \mathrm{Con}_0(\mathscr{J}) \backslash \mathrm{Con}_0 / \mathrm{Con}_0(\mathscr{R}_+)$$

under $\mathscr{J}' \mapsto [\phi]$ since this means precisely

$$\phi = \bar{\psi} \psi \mu , \quad \psi \in \mathrm{Con}_0(\mathscr{J}), \quad \mu \in \mathrm{Con}_0(\mathscr{R}_+), \quad \bar{\psi} \in \mathrm{Con}_\infty$$

[note that $\mathrm{Con}_\infty \circ \mathrm{Con}_0(\mathscr{J})$ is a subgroup of Con_0].

Recall that L, defined by $x = \xi = 0$ in G-coordinates, is a contact manifold with local diffeomorphism group $\mathrm{Con}(L)$.

(5.3) **Lemma.** *There is an extension map*

$$u : \mathrm{Con}(L) \ni s \mapsto \phi \in \mathrm{Con}_0(\mathscr{R}_+) .$$

Proof. Transfer s, using the map $\tilde{I}_G$ of (3.1), to a contact transformation ψ on the manifold $X = \Xi = 0$ in $(\mathbb{R}^{2n+1}, M, 0)$. By Proposition 1.6, ψ extends (uniquely) to $\bar{\psi} \in \mathrm{Con}(\mathbb{R}^{2n+1}, \bar{M}, 0)$ if we demand that the sections $\bar{X} \in C^\infty(\bar{M}^*_{2/5}, 0)$, $\bar{\Xi} \in C^\infty(\bar{M}^*_{3/5}, 0)$, equal to X, Ξ at $\bar{\mu}$, be invariant. This implies that the cusp (3.4) is invariant under $\bar{\psi}$ so this map, just as in the proof of Proposition 3.3, lifts to the desired element $\phi = u(s) \in \mathrm{Con}(\mathscr{R}) \subset \mathrm{Con}(\mathscr{R}_+)$. Note also that ϕ has the useful property

$$(5.4) \quad \phi^*(x \tilde{\alpha}^*_{2/5}) = x \tilde{\alpha}^*_{2/5} , \qquad \phi^*(\xi \tilde{\alpha}^*_{1/5}) = \xi \tilde{\alpha}^*_{1/5} .$$

Using coordinates in which (2.2) spans A and $\mathscr{J}$ is the simple fold it is even easier to show the existence of an extension

$$(5.5) \quad v : \mathrm{Con}(L) \to \mathrm{Con}_0(\mathscr{J}) .$$

Given $s \in \mathrm{Con}(L)$ we shall denote by $\mathrm{Con}_0(\mathscr{R}_+, s)$ the subset of germs, in $\mathrm{Con}_0(\mathscr{R}_+)$, which restrict to s on L. Put

$$\Gamma(\mathscr{J}, s) = \mathrm{Con}_\infty \circ \mathrm{Con}_0(\mathscr{J}) \backslash \mathrm{Con}_0 / \mathrm{Con}_0(\mathscr{R}_+, s) .$$

From (5.2), (5.4), and Lemma 5.3, each $[\phi] \in \Gamma(\mathscr{J}, s)$ has a representative $\phi \in \mathrm{Con}_1$. For each $k \geqq 1$ define

$$\Gamma^k(\mathscr{J}, s) \subset \Gamma(\mathscr{J}, s)$$

to be the subset consisting of the elements having a representative in Con_k.

(5.6) **Proposition.** *For each* $k \geqq 1$, $\mathscr{J} \in \mathrm{Ref}(\mathbb{R}^{2n+1}, A, 0)$ *and* $s \in Con(L)$ *there is an exact sequence*

(5.7) $$0 \to \Gamma^{k+1}(\mathscr{J}, s) \hookrightarrow \Gamma^k(\mathscr{J}, s) \xrightarrow{O_k} Ob_k \to 0,$$

where Ob_k *is a module over* $C^\infty(L, 0)$ *of dimension*

(5.8) $$[k/12] + d(k);$$

with $d(k)$ *depending only on k modulo* 12, $d(k) = 1$ *unless* $k = 0, 1, 3, 4, 7 \pmod{12}$ *when* $d(k) = 0$.

Before proving this proposition we shall discuss the space of obstructions Ob_k. Recall from (4.13) that the spaces C_k, consisting of polynomials of quasi-degree k, are the leading parts of the spaces $\mathfrak{a}_{(k)}(\mathscr{J})$ of $\mathscr{J}$-invariants.

(5.9) **Lemma.** *For every* $k \geqq 0$

$$c_{k+4} \cap \mathfrak{a}_{k+4}(\mathscr{R}) \cap (x + 3\xi^2)^2 \mathfrak{a}_k = \{0\}.$$

Proof. Suppose $p \in c_{k+4} \cap \mathfrak{a}_{k+4}(\mathscr{R}) \cap (x + 3\xi^2)^2 \mathfrak{a}_k$, then p is constant on the $\mathscr{R}$-equivalence classes, vanishes at $x = -3\xi^2$ and is invariant under the (non-contact) involution $T(y, \eta, x, \xi, z) = (y, \eta, -x - 6\xi^2, \xi, z)$. A direct calculation shows that two surfaces $x = s_1 \xi^2$, $x = s_2 \xi^2$ are $\mathscr{R}$-related precisely when $r(s_1) = r(s_2)$,

$$r(s) = (s+1)^2 s^{-3}.$$

In particular $x = -3\xi^2$ and $x = -3/4\xi^2$ are so related and p must therefore vanish on the latter. T maps the surface $x = s\xi^2$ to $x = (-s-6)\xi^2$ so p must vanish on $x = s_1 \xi^2$, $s_1 = -\frac{21}{4}$. A sketch of the graph of r shows that for any $s < -3$ there is a unique $\bar{s}$ in the range $-3 < \bar{s} < -1$ with $r(s) = r(\bar{s})$; we shall write $\bar{s} = \varrho(s)$. Define s_j, $j \in N$ inductively by

(5.10) $$\begin{cases} s_j = -s_{j-1} - 6 & j \text{ odd} \\ s_j = \varrho(s_{j-1}) & j \text{ even} \end{cases}$$

starting with $s_1 = -21/4$. Clearly $s_j < -3$ or $-3 < s_j < -1$ as j is odd or even and p must vanish on all the surfaces $x = s_j \xi^2$. We claim that s_{2p+1} increases monotonically to -3 as $p \to \infty$, necessitating $p \equiv 0$ as stated. From (5.10) it suffices to note that

$$r(-s-6) - r(s) = \frac{2(s+3)^3((s+3)^2 - 5)}{s^3(-s-6)^3} < 0$$

when $-3 < s < -1$, to complete the proof.

For each $k \geqq 1$ define

(5.11) $$Ob_k = \mathfrak{b}_k / [(\mathfrak{b}_k \cap c_{k+s} \tilde{\alpha}^*) \oplus (\mathfrak{b}_k \cap \mathfrak{a}_{k+s}(\mathscr{R}) \tilde{\alpha}^*)].$$

The map O_k in (5.7) is then defined by projection of $\Lambda T_L \phi$ into Ob_k, where $\phi \in \mathrm{Con}_k$ is a representative of $[\phi] \in \Gamma^k(\mathscr{J}, s)$. Observe that O_k is well-defined, if $\phi' \in \mathrm{Con}_k$ is another representative then

$$\phi_1' \phi' \phi_2' = \phi_1 \phi \phi_2$$

with $\phi_2, \phi_2 \in \mathrm{Con}_0(\mathscr{R}_+, s)$, $\phi_1, \phi_1, \in \mathrm{Con}_\infty \circ \mathrm{Con}_0(\mathscr{J})$. Thus

$$\phi' = \bar{\phi}_1 \phi \bar{\phi}_2,$$

where $\bar{\phi}_1 \in \mathrm{Con}_\infty \circ \mathrm{Con}_1(\mathscr{J})$, $\bar{\phi}_2 \in \mathrm{Con}_1(\mathscr{R}_+)$, since both restrict to the identity on L. Applying Propositions 4.6 and 4.11 and Lemmas 4.10 and 5.9 one quickly concludes that $\bar{\phi}_1 \in \mathrm{Con}_k(\mathscr{J})$, $\bar{\phi}_2 \in \mathrm{Con}_k(\mathscr{R}_+)$ and therefore, again by Proposition 4.5 $\Lambda T_L \phi = \Lambda T_L \phi'$ modulo $\mathfrak{b}_{(k+1)} + \mathfrak{b}_{(k)}(\mathscr{J}) + \mathfrak{b}_{(k)}(\mathscr{R})$. So, from Proposition 4.11 O_k is well-defined.

Proof of Proposition 5.6. The surjectivity of O_k is obvious and from the discussion above if $O_k[\phi]=0$ then $[\phi]$ has a representative $\phi' \in \mathrm{Con}_{k+1}$, thus, (5.7) is exact. It remains to calculate the dimension of Ob_k as given in (5.8). From (5.11),

$$\dim Ob_k = \dim \alpha_{k+1} - (\dim C_{k+5} - 1) - (\dim \mathfrak{a}_{k+5}(\mathscr{R}) - 1)$$

since, if $p \in c_{k+5}$ or $p \in \mathfrak{a}_{k+5}(\mathscr{R})$ the condition $p \in (a+3\xi^2)^2 \mathfrak{a}_{k+1}$ is only one extra (non-trivial) linear constraint. By direct observation,

$$\dim \mathfrak{a}_k = [k/2] + 1$$

$$\dim \mathfrak{a}_k(\mathscr{R}) = \begin{cases} [k/6] + 1 & k \equiv 1 \pmod 6 \\ [k/6] & k \not\equiv 1 \pmod 6 \end{cases}$$

$$\dim c_k = [k/4] + 1$$

from which (5.8) follows easily.

Summarizing these results we have

(5.12) **Proposition.** *If $\mathscr{J}, \mathscr{J}' \in \mathrm{Ref}(\mathbb{R}^{2n+1}, A, 0)$ then $\mathscr{J}' \underset{L}{\sim} \mathscr{J}$ if and only if, there exists $s \in \mathrm{Con}(L)$ such that, for each $k \geqq 1$, the successive obstructions O_k vanish on the image of $\mathscr{J}'$ in $\Gamma_k(\mathscr{J}, s)$.*

Proof. Certainly $\mathscr{J}' \underset{L}{\sim} \mathscr{J}$ implies the vanishing of all the obstructions. Conversely, if $\mathscr{J}' \in \Gamma^k(\mathscr{J}, s)$ for every k then there exists $\mu_0 \in \mathrm{Con}_0(\mathscr{R}_+, s)$ and for each $k \geqq 1$, $\mu_k \in \mathrm{Con}(\mathscr{R}_+, \mathrm{Id})$, $\phi_k \in \mathrm{Con}_k$ such that

$$\mu_k \mu_0 \mathscr{J}' \mu_0^{-1} \mu_k^{-1} = \phi_k^{-1} \mathscr{J} \phi_k.$$

Moreover the image of $\Lambda T_L(\mu_k)$ in $\mathfrak{b}_{(1)}/\mathfrak{b}_{(l)}$ is stable for $k > l$, so using the surjectivity of the Taylor series maps in Sect. 4 one can choose $\mu \in \mathrm{Con}(\mathscr{R}_+, s)$ so that $\mu \mathscr{J}' \mu^{-1} = \phi^{-1} \mathscr{J} \phi$, $\phi \in \mathrm{Con}_\infty$, as desired.

To apply this result to the intersection maps it is convenient to transform the first non-trivial obstruction map O_k somewhat. We shall choose as "basic" involution

$$\bar{\mathscr{J}}(y, \eta, x, \xi, z) = (y, \eta, -x - 6\xi^2, \xi, z - 4\xi^3(x + 3\xi^2)).$$

$\bar{\mathscr{J}} \in \mathrm{Ref}(\mathbb{R}^{2n+1}, A, 0)$, indeed $\bar{\mathscr{J}}^* \tilde{\alpha} = \tilde{\alpha}$. Recall the property (5.4) of the extension map, u, of Lemma 5.3. Clearly the sections $(x+3\xi^2)\tilde{\alpha}^*_{2/5}$, $\xi \tilde{\alpha}^*_{1/5}$ are invariant under both $\phi = u(s)$ and $\bar{\mathscr{J}}$. Since ϕ is defined by the solution of differential equations with $\bar{\mathscr{J}}$-invariant data on L it follows that u has the useful dual extension property

$$u:\mathrm{Con}(L)\to\mathrm{Con}_0(\mathscr{R})\cap\mathrm{Con}_0(\bar{\mathscr{J}}).$$

Now, note that

$$\mathfrak{a}_7(\mathscr{R})\cap(x+3\xi^2)^2\mathfrak{a}_3=\{0\}$$
$$\mathfrak{c}_7\cap(x+3\xi^2)^2\mathfrak{a}_3=(x+3\xi^2)^2\xi^3.$$

So, given $\mathscr{J}\in\mathrm{Ref}(\mathbb{R}^{2n+1},A,0)$ we can define a germ $f\in C^\infty(\mathbb{R}^{2n-1},0)$ by

(5.13) $\quad \Lambda T_L(\phi)\equiv f(y,\eta,z)(x+3\xi^2)^2x\xi\tilde{\alpha}^*$

modulo $\mathfrak{b}_{(3)}$, where $\phi\in\mathrm{Con}_2$ is such that

(5.14) $\quad \mu\mathscr{J}\mu^{-1}=\phi^{-1}\bar{\mathscr{J}}\phi \qquad \mu\in\mathrm{Con}_1(\mathscr{R}_+).$

Given a pair of intersection maps I_F, I_G we can define the *principal invariant*

$$PI(F,G)\in C^\infty((A_L)^*_{-2/5}),$$

where A_L is the contact bundle on L, by taking the section with value f on $\tilde{\alpha}$ in any G-coordinates, where f is defined through (5.13), (5.14), with $\mathscr{J}=\mathscr{J}_F\in\mathrm{Ref}(\mathbb{R}^{2n+1},A,0)$. We must verify that, despite the freedom to choose different G-coordinates this does indeed define a section of the line bundle $(A_L)^*_{-2/5}$ over L.

Suppose two systems of G-coordinates give the two germs $f_1,f_2\in C^\infty(\mathbb{R}^{2n-1},0)$. The coordinate systems are connected by some $\mu\in\mathrm{Con}(\mathscr{R}_+,s)$, $s\in\mathrm{Con}(L)$, so the corresponding involutions $\mathscr{J}_1,\mathscr{J}_2\in\mathrm{Ref}(\mathbb{R}^{2n+1},A,0)$ are related by

$$\mu\mathscr{J}_1\mu^{-1}=\mathscr{J}_2.$$

Thus, the two Eq. (5.13) give

$$\mu\mu_1^{-1}\phi_1^{-1}\bar{\mathscr{J}}\phi_1\mu_1\mu^{-1}=\mu_2^{-1}\phi_2^{-1}\mathscr{J}\phi_2\mu_2$$

which is simply $\psi=\phi_2(\mu_2\mu\mu_1^{-1})\phi_1^{-1}\in\mathrm{Con}(\bar{\mathscr{J}})$. Put $\phi=u(s)$, then

$$\psi'=(\phi^{-1}\phi_2\phi)(\phi^{-1}\mu_2\mu\mu_1^{-1})\phi_1^{-1}\in\mathrm{Con}_1(\bar{\mathscr{J}}).$$

In fact, $\phi^{-1}\phi_2\phi\in\mathrm{Con}_2$ and $\phi^{-1}\mu_2\mu\mu_1^{-1}\in\mathrm{Con}_1(\mathscr{R}_+)$ so, from Lemma 5.9, $\phi^{-1}\mu_2\mu\mu_1^{-1}\in\mathrm{Con}_2(\mathscr{R}_+)=\mathrm{Con}_3(\mathscr{R}_+)$. So, we need to find the projection of $\Lambda T_L(\phi^{-1}\phi_2\phi)$ in $\mathfrak{b}_{(2)}$, it is clearly of the form $f_1(x+3\xi^2)^2x\xi\tilde{\alpha}^*$. Thus we need to calculate

$$(\phi^{-1}\phi_2\phi)^*(\xi\tilde{\alpha}^*_{1/5})(\tilde{\alpha})=\xi-3f_1(x\xi+\xi^3)\qquad \mathrm{mod}\,\mathfrak{a}_{(4)}.$$

Now, $\phi^*(\xi\tilde{\alpha}^*_{1/5})=\xi\tilde{\alpha}^*_{1/5}$, $\phi^*(x\tilde{\alpha}^*_{2/5})=x\tilde{\alpha}^*_{2/5}$ and

$$\phi_2^*(\xi\tilde{\alpha}^*_{1/5})=\xi\tilde{\alpha}^*_{1/5}-3f_2(x\tilde{\alpha}^*_{2/5}\xi\tilde{\alpha}^*_{1/5}+(\xi\tilde{\alpha}^*_{1/5})^3)\tilde{\alpha}^*_{-2/5}$$

modulo $(\mathfrak{a}_{(4)}\tilde{\alpha}^*_{1/5})$. Thus,

$$(\phi^{-1}\phi_2\phi^*)(\xi\tilde{\alpha}^*_{1/5})(\tilde{\alpha})$$
$$\equiv\xi-3(\phi^*f_2)(x\xi+\xi^3)(\phi^*\tilde{\alpha}^*_{-2/5})(\tilde{\alpha})\qquad(\mathrm{mod}\,\mathfrak{a}_{(4)})$$

and since, modulo $\mathfrak{a}_{(1)}$, $\phi^*\tilde{\alpha}^*(\tilde{\alpha}) = s^*\tilde{\alpha}_L^*(\tilde{\alpha}_L)$, $\phi^* f_2 = s^* f_2$,

$$f_1 = (s^* f_2)[(s^*\tilde{\alpha}_L^*)(\tilde{\alpha}_L)]^{-2/5}.$$

Thus, $f_1\tilde{\alpha}^*_{-2/5} = s^*(f_2\tilde{\alpha}^*_{-2/5})$ is invariantly defined as asserted.

6. Equivalence at K

In the folded contact manifold $(\mathbb{R}^{2n+1}, A, 0)$, where A is spanned by (2.3), let $\mathfrak{m} = C^\infty(K,0)[[t]]$ be the ring of Taylor series on K; $\mathfrak{m}$ projects onto the ring of Taylor series at L with kernel

$$\mathfrak{n} = C_L^\infty(K,0)[[t]],$$

where $f \in C_L^\infty(K,0)$ if $f \in C^\infty(K,0)$ vanishes to all orders on L. Let $\{\mathfrak{n}_k\}$ and $\{\mathfrak{n}_{(k)}\}$ be the homogeneous gradation and filtration of $\mathfrak{n}$ in these coordinates. Each $\mathscr{I} \in \mathrm{Ref}(\mathbb{R}^{2n+1}, A, 0)$ acts on both $\mathfrak{m}$ and $\mathfrak{n}$; let $\mathfrak{n}(\mathscr{I}) \subset \mathfrak{n}$ be the ring of $\mathscr{I}$-invariants.

(6.1) **Lemma.** *For each* $\mathscr{I} \in \mathrm{Ref}(\mathbb{R}^{2n+1}, A, 0)$ *and each* $k \in \mathbb{N}$ *the sequence*

$$(6.2)\quad 0 \to \mathfrak{n}_{(2k+2)}(\mathscr{I}) \hookrightarrow \mathfrak{n}_{(2k)}(\mathscr{I}) \xrightarrow{\pi_{2k}} \mathfrak{n}_{2k} \to 0$$

is exact, where $\mathfrak{n}_{(j)}(\mathscr{I}) = \mathfrak{n}(\mathscr{I}) \cap \mathfrak{n}_{(j)}$.

Proof. $\mathscr{I}^* y_j - y_j$, $\mathscr{I}^*\eta_j - \eta_j$, $\mathscr{I}^*\zeta - \zeta$ and $\mathscr{I}^* t + t$ all lie in $\mathfrak{m}_{(2)}$ whereas $\mathscr{I}^*\tau - \tau \in \mathfrak{m}_{(1)}$ so $(\mathscr{I}^* - \mathscr{I}_0^*)\mathfrak{n}_{(k)} \subset \mathfrak{n}_{(k+1)}$ for all k, where

$$\mathscr{I}_0(y, \eta, t, \tau, \zeta) = (y, \eta, -t, \tau, \zeta)$$

and the exactness of (6.2) then follows from its obvious exactness for $\mathscr{I} = \mathscr{I}_0$.

The singular relation $\mathscr{R}$, transferred from $(\mathbb{R}^{2n+1}, \tilde{A}, 0)$ to $(\mathbb{R}^{2n+1}, A, 0)$ by

$$(6.3)\quad (y, \eta, x, \xi, z) = (y, \eta, x + 3\xi^2, \xi, z - 2x\xi^3 - \tfrac{18}{5}\xi^5) = (y, \eta, t, \tau, \zeta)$$

(which satisfies $\psi^*\alpha = \tilde{\alpha}$) also defines an action on $\mathfrak{n}$ (but not $\mathfrak{m}$) through the singular involution near $t = 0$

$$R(y, \eta, t, \tau, \zeta) = (y, \eta, \bar{t}, \bar{\tau}, \bar{\zeta})$$

$$(6.4)\quad \begin{aligned} \bar{t} &= t - 3\tau^2 + 3\bar{\tau}^2 \\ \bar{\tau} &= 1/2(-\tau + (\operatorname{sgn}\tau)\sqrt{9\tau^2 - 4t}) \\ \bar{\zeta} &= \zeta + 2(\tau^3 - \bar{\tau}^3)(t - 3\tau^2) + \tfrac{18}{5}(\tau^5 - \bar{\tau}^5). \end{aligned}$$

Using the same reasoning as in Lemma 6.1 we obtain, for each $k \in \mathbb{N}$ an exact sequence

$$(6.5)\quad 0 \to \mathfrak{n}_{(2k+2)}(\mathscr{R}_+) \hookrightarrow \mathfrak{n}_{(2k)}(\mathscr{R}_+) \xrightarrow{\pi_{2k}} \mathfrak{n}_{2k} \to 0,$$

where $\mathfrak{n}_{(j)}(\mathscr{R}_+) = \mathfrak{n}_{(j)} \cap \mathfrak{n}(\mathscr{R}_+)$ and $\mathfrak{n}(\mathscr{R}_+)$ is the ring of R-invariants in $\mathfrak{n}$. We use this notation because of the following result on the Taylor series map $T_K : C_L^\infty(\mathbb{R}^{2n+1}, 0) \to \mathfrak{n}$.

(6.6) Lemma. *$T_K: C_L^\infty(\mathscr{R}_+)\to \mathfrak{n}(\mathscr{R}_+)$ is surjective where $C_L^\infty(\mathscr{R}_+) = C^\infty(\mathscr{R}_+)\cap C_L^\infty(\mathbb{R}^{2n+1},0)$ is the space of $\mathscr{R}_+$-invariant C^∞ functions vanishing to all orders at $t=\tau=0$.*

Proof. The involution R defined by (6.4) clearly corresponds to the involution defined by $\mathscr{R}_+$ near $t=0$, for $\tau>0$. On the other hand, the R-invariance of the Taylor series $T_K f$ near $t=0$ for $\tau<0$ corresponds exactly to the fact that $f\in C^\infty(\mathscr{R}_+)$ pulls back to a C^∞ function on $t=\frac{9}{4}\tau^2$ under the singular map R' defined by replacing $\operatorname{sgn}\tau$ by $-\operatorname{sgn}\tau$ in (6.4). Thus, $T_K C_L^\infty(\mathscr{R}_+)\subset \mathfrak{n}(\mathscr{R}_+)$ and the converse is similar.

Next, consider the group $\mathrm{Con}_\infty(\mathbb{R}^{2n+1}, A, 0)$ of contact diffeomorphisms leaving L fixed to infinite order. Clearly this group acts on $\mathfrak{n}$ and we denote by $\mathrm{Con}_{\infty,k}$ the subgroup of those ϕ with

$$(\phi^* - \mathrm{Id})\mathfrak{n}_{(j)} \subset \mathfrak{n}_{(j+k)} \quad \forall j.$$

The following analogue of Proposition 4.5 can be proved by the same method.

(6.7) Proposition. *To each $\phi\in \mathrm{Con}_{\infty,1}$ there corresponds a unique $f\in \mathfrak{n}_{(3)}\alpha^*$ such that*

$$(\phi^* - \exp(V_f)^*)\mathfrak{n} = \{0\}.$$

The map $AT_K: \phi\mapsto f$ maps $\mathrm{Con}_{\infty,k}$ onto $\mathfrak{n}_{(k+2)}\alpha^*$ for each $k\geqq 1$ and if $\phi_i\in \mathrm{Con}_{\infty,k_i}$ $k_i\geqq 1$, $i=1,2$, then

$$(6.8)\quad AT_K(\phi_1\circ\phi_2) \equiv AT_K(\phi_1) + AT_K(\phi_2)$$

modulo $\mathfrak{n}_{(k_1+k_2+2)}\alpha^*$.

Similarly, we have the following analogues of Lemma 4.9 and Proposition 4.10 for the stability subgroups, $\mathrm{Con}_{\infty,k}(\mathscr{R}_+)$ and $\mathrm{Con}_{\infty,k}(\mathscr{I})$, that the two sequences

$$(6.9)\quad \{\mathrm{Id}\}\to \mathrm{Con}_{\infty,k+1}(\mathscr{R}_+) \hookrightarrow \mathrm{Con}_{\infty,k}(\mathscr{R}_+) \xrightarrow{\pi\circ AT_K} \mathfrak{n}_{(k+2)}(\mathscr{R}_+)\alpha^*/\mathfrak{n}_{(k+3)}(\mathscr{R}_+)\alpha^* \to 0$$

$$(6.10)\quad \{\mathrm{Id}\}\to \mathrm{Con}_{\infty,k+1}(\mathscr{I}) \hookrightarrow \mathrm{Con}_{\infty,k}(\mathscr{I}) \xrightarrow{\pi\circ AT_K} \mathfrak{n}_{(k+2)}(\mathscr{I})\alpha^*/\mathfrak{n}_{(k+3)}(\mathscr{I})\alpha^* \to 0$$

are exact. In particular, because of (6.2) and (6.5)

$$\mathrm{Con}_{\infty,2k+1}(\mathscr{R}_+) = \mathrm{Con}_{\infty,2k+2}(\mathscr{R}_+),\ \mathrm{Con}_{\infty,2k+1}(\mathscr{I}) = \mathrm{Con}_{\infty,2k+2}(\mathscr{I}).$$

Now, the main result of this section concerns the factorization of Con_∞. Put

$$\mathrm{Con}_{\infty,\infty} = \bigcap_k \mathrm{Con}_{\infty,k}.$$

(6.11) Proposition. *For each $\mathscr{I}\in \mathrm{Ref}(\mathbb{R}^{2n+1}, A, 0)$*

$$\mathrm{Con}_\infty = \mathrm{Con}_\infty(\mathscr{I})\circ \mathrm{Con}_{\infty,\infty}\circ \mathrm{Con}_\infty(\mathscr{R}_+).$$

Before proceeding to the proof we shall discuss the linearized analogue of this splitting.

(6.12) **Lemma.** *For each* $k\in\mathbb{N}$, $\mathscr{J}\in\mathrm{Ref}(\mathbb{R}^{2n+1},A,0)$

$$\mathfrak{n}_{(2k)}=\mathfrak{n}_{(2k)}(\mathscr{R}_+)+\mathfrak{n}_{(2k)}(\mathscr{J}).$$

Proof. We shall show that $\mathfrak{n}_{(2k)}\subset\mathfrak{n}_{(2k)}(\mathscr{R}_+)+\mathfrak{n}_{(2k)}(\mathscr{J})+\mathfrak{n}_{(2k+2)}$ since then the desired result follows by induction. Given $f\in\mathfrak{n}_{(2k)}$ we wish to find $p\in\mathfrak{n}_{(2k)}(\mathscr{J})$, $q\in\mathfrak{n}_{(2k)}(\mathscr{R}_+)$ such that $f_{2k}=p_{2k}+q_{2k}$, $f_{2k+1}=p_{2k+1}+q_{2k+1}$. From Lemma 6.1 and (6.5) the maps

$$V_{\mathscr{J}}:\mathfrak{n}_{2k}\ni p_{2k}\mapsto p_{2p+1}=\pi_{2k+1}1/2(p_{2k}+\mathscr{J}^*p_{2k})\in\mathfrak{n}_{2k+1}$$
$$V_R:\mathfrak{n}_{2k}\ni p_{2k}\mapsto q_{2k+1}=\pi_{2k+1}1/2(q_{2k}+R^*q_{2k})\in\mathfrak{n}_{2k+1}$$

are well-defined and extend trivially to $\mathfrak{n}_{(2k)}$. Thus, the desired decomposition of f is fixed by the equation

$$(6.13)\quad V_Rq_{2k}+V_{\mathscr{J}}(f_{2k}-q_{2k})=f_{2k+1}.$$

Clearly, $V_{\mathscr{J}}$ is a first-order differential operator on K with C^∞ coefficients and differential part everywhere tangent to the Hamilton foliation of K

$$V_{\mathscr{J}}=a\partial_\tau+b.$$

V_R is of a similar form but singular at $\tau=0$; directly from (6.4)

$$V_Rq_{2k}=-\frac{1}{3\tau}\frac{\partial}{\partial\tau}q_{2k}-\frac{2k}{9\tau^2}q_{2k},$$

since $\bar{\tau}=\tau-\dfrac{t}{3\tau}+O(t^2)$, $\bar{t}=-t+\dfrac{t^2}{9\tau^2}+O(t^3)$, $\bar{\zeta}=\zeta+O(t^2)$. Thus, (6.13) is

$$(6.14)\quad (1+3\tau a)\frac{\partial q_{2k}}{\partial\tau}+\left(\frac{2k}{3\tau}+3\tau b\right)q_{2k}=\tau r_{2k+1},$$

where $r_{2k+1}\in C_L^\infty(K,0)$, and we require $q_{2k}\in C_L^\infty(K,0)$. Defining

$$s(y,\eta,\tau,z)=\int_0^\tau[(2k+q\tau^2b)/(1+3\tau^2a)-2k]/3\tau.$$

(6.14) becomes

$$3\tau\frac{\partial}{\partial z}e^sq_{2k}+2ke^sq_{2k}=v'\in C_L^\infty(L,0)$$

which has the unique solution

$$q_{2k}=e^{-s}\frac{\tau^{-2k/3}}{3}\int_0^\tau e^{2k/3-1}r'\in C_L^\infty(K,0),$$

proving the lemma.

Proof of Proposition 6.11. First we show that

$$(6.15)\quad \mathrm{Con}_\infty=\mathrm{Con}_{\infty,1}\circ\mathrm{Con}_\infty(\mathscr{R}_+)=\mathrm{Con}_{\infty,1}\circ\mathrm{Con}_\infty(\mathscr{J}).$$

This involves the extension of $\phi_K=\phi 1 K$, where $\phi\in\mathrm{Con}_\infty$ so ϕ_K has the form

$$\phi_K^* y_j=y_j,\ \phi_K^*\eta_j=\eta_j,\ \phi_K^*\zeta=\zeta,\ \phi_K^*\tau=(1+g)\tau$$

with $g\in C_L^\infty(K,0)$, to an element of $\mathrm{Con}_\infty(\mathscr{R}_+)$; the construction for $\mathscr{J}$ being similar, but simpler. Using ψ in (6.3) to transfer the problem to $(\mathbb{R}^{2n+1},\tilde{A},0)$ in which $\mathscr{R}_+$ is simple, we can project ϕ_K using $\tilde{I}_G$ onto a transformation of the cusp (3.4), leaving L fixed to all orders and preserving $\tilde{A}$. The condition that the cusp be fixed gives initial data for X on it and the choice of a smooth extension of $\phi_K^*\Xi$ allows one to apply the method of Proposition A.16 to construct a contact transformation which lifts to the desired element of $\mathrm{Con}_\infty(\mathscr{R}_+)$. We leave the reader to check that regularity of this extension follows from the fact that ϕ_K leaves L fixed to all orders.

Thus, by (6.15) given $\phi\in\mathrm{Con}_\infty$ we can choose $\phi_1\in\mathrm{Con}_\infty(\mathscr{R}_+)$ so that $\phi\phi_1\in\mathrm{Con}_{\infty,1}$. Next then we must show that given $\phi\in\mathrm{Con}_{\infty,1}$ there exist $\psi_1\in\mathrm{Con}_\infty(\mathscr{R}_+)$, $\phi_2\in\mathrm{Con}_\infty(\mathscr{J})$ such that

(6.16) $\quad\psi_2^{-1}\phi\psi_1\in\mathrm{Con}_{\infty,2}$.

Clearly, $\phi_1 1K=\psi_2 1K$ and then $\psi_2^{-1}\phi\psi_1\in\mathrm{Con}_{\infty,1}$, so we need to calculate $\Lambda T_K(\psi_2^{-1}\phi\psi_1)$ modulo $\mathfrak{n}_{(4)}\alpha^*$ in terms of $\psi_1 1K$. Since $\psi_1\in\mathrm{Con}_\infty$,

$$\psi_1^*\tau=f+\beta t+O(t^2),$$

where $f=(1+g)\tau$, $g\in C_L^\infty(K,0)$, $\beta\in C^\infty(K,0)$. From the invariance of y_j, η_j, ζ on K we note that

$$\psi_1^*\alpha^*=(1+O(t))\alpha^*$$

so the Lagrange bracket $[1/2t^2,\tau]=\alpha_{-1}^*$ shows that

$$\psi_1^*t=(f_\tau')^{-1/2}t+O(t^2).$$

Suppose that $\tau+\gamma t+O(t^2)$, $\gamma\in C^\infty(K,0)$, is the $\mathscr{J}$-even part of τ, then

$$\psi_1^*(\tau+\gamma t)=f+\beta t+\gamma(y,\eta,f,\zeta)(f_\tau')^{-1/2}t+O(t^2)$$

must be $\mathscr{J}$-invariant, modulo $O(t^2)$, so

$$f(y,\eta,\tau+2\gamma t,\zeta)-2\beta t-2\gamma(y,\eta,f,\zeta)(f_\tau')^{-1/2}t \\ -f=O(t^2).$$

This determines the first-order term β in terms of γ and f:

$$\beta=\gamma f_\tau'-\gamma(f)(f_\tau')^{-1/2}.$$

Carrying out the same calculation for $\psi_2\in\mathrm{Con}_\infty(\mathscr{R}_+)$

$$\psi_2^*\tau=f+\varrho t+O(t^2)$$

using the invariance of $\tau-\dfrac{t}{6\tau}$ modulo $O(t^2)$ we find

$$\varrho=-\frac{f_\tau'}{6\tau}+\frac{(f_\tau')^{-1/2}}{6f}.$$

Returning to (6.16) we must choose f so that, modulo $O(t^2)$,

$$\begin{aligned}\psi_1^*\phi^*(\psi_2^{-1})^*(f+\beta t) &\equiv \psi_1^*\phi^*\tau\\ &\equiv \psi_1^*(\tau+cf) \equiv f+\varrho t + c(y,\eta,f,\zeta)(f'_\tau)^{-1/2}t\\ &\equiv f+\beta t,\end{aligned}$$

where $AT_K(\phi)=c\frac{\tau^3}{3}\alpha^*$ mod $\mathfrak{n}_{(4)}\alpha^*$. Thus, $\phi_2^{-1}\phi\psi_1\in \mathrm{Con}_{\infty,2}$ becomes the non-linear analogue of (6.14):

$$-\frac{f'_\tau}{6\tau}+\frac{(f'_\tau)^{-1/2}}{6f}+c(f)(f'_\tau)^{-1/2}-\gamma f'_\tau+\gamma(f)(f'_\tau)^{-1/2}=0.$$

Rewritten as an integral equation, including the condition $f=0$ at $\tau=0$, this becomes

$$(6.17)\quad f(y,\eta,\tau,\zeta)=\int_0^\tau\left[\frac{1+6f\gamma(f)+6fc(f)}{f/2+6f\gamma}\right]^{2/3}\lambda\tau$$

which is easily shown to have a unique solution, of the form $f=(1+g)\tau$, $g\in C_L^\infty(K,0)$, by contraction arguments.

Thus, we conclude that the first step, the factorization (6.16) can be carried out. It remains to verify by induction that $\phi\in \mathrm{Con}_{\infty,2k}$ can be factorized as

$$\phi=\psi_1\phi'\psi_2^{-1}$$

$\psi_1\in \mathrm{Con}_{\infty,2k}(\mathscr{J})$, $\psi_2+\mathrm{Con}_{\infty,2k}(\mathscr{R}_+)$, $\phi'\in \mathrm{Con}_{\infty,2k+2}$. Applying Proposition 6.7 this reduces to the equation

$$AT_K\phi\equiv AT_K\psi_1-AT_K\psi_2$$

modulo $\mathfrak{n}_{(2k+4)}\alpha^*$ and Lemma 6.12 shows that such a decomposition is always possible (and is unique). Clearly the inductive construction converges in the sense of formal power series so, using Lemma 6.5 and the corresponding result for $\mathscr{J}$ we can sum these series to give the desired factorization with "error" in $\mathrm{Con}_{\infty,\infty}$.

7. One-Sided Factorization on J

Recall that $\mathrm{Con}_{\infty,\infty}(\mathbb{R}^{2n+1},A,0)$ is the group of contact diffeomorphisms leaving the fold K of A fixed to infinite order, and $\mathrm{Con}_{\infty,\infty}(\mathscr{J})$, $\mathrm{Con}_{\infty,\infty}(\mathscr{R}_+)$ are the stability subgroups of $\mathscr{J}\in \mathrm{Ref}(\mathbb{R}^{2n+1},A,0)$ and $\mathscr{R}_+$.

(7.1) **Proposition.** *For each* $\mathscr{J}\in \mathrm{Ref}(\mathbb{R}^{2n+1},A,0)$

$$\mathrm{Con}_{\infty,\infty}=\mathrm{Con}_{\infty,\infty}(\mathscr{J})\circ \mathrm{Con}_{\infty,\infty}(\mathscr{R}_+).$$

The remainder of this section is devoted to the proof of this proposition. First we note a simple extension result serving to simplify the analysis.

(7.2) **Lemma.** *Given* $\phi\in \mathrm{Con}_{\infty,\infty}$ *there exists* $\psi\in \mathrm{Con}_{\infty,\infty}(\mathscr{J})$ *such that, for each* $\varepsilon>0$, $\psi^{-1}\phi$ *extends to a global contact diffeomorphism of* $(\mathbb{R}^{2n+1},A,0)$ *which is the* *identity outside the region*

$$G_\varepsilon = \{|y|<\varepsilon, |y|<\varepsilon, |\tau|<\varepsilon, |\zeta|<\varepsilon, t \geqq 0\}.$$

Proof. First we choose ψ. Let $\bar{\phi}$ be some representative of g, put

$$\bar{\psi} = \begin{cases} \bar{\phi} & \text{in} \quad t \leqq 0 \\ \mathscr{J}\bar{\phi}\mathscr{J} & \text{in} \quad t \geqq 0, \end{cases}$$

clearly $\psi \in \mathrm{Con}_{\infty,\infty}(\mathscr{J})$ and $\chi = \psi^{-1}\phi$ has the representative $\bar{\chi} = \bar{\psi}^{-1}\bar{\phi}$ which is the identity in $t \leqq 0$. If $\bar{\chi}(y, \eta, t, \tau, \zeta) = (\bar{y}, \bar{\eta}, \bar{t}, \bar{\tau}, \bar{\zeta})$, so $\bar{\tau} = \tau$ in $t \leqq 0$, choose a C^∞ function s which has the same germ as $\bar{\tau}$ at 0, with $\dfrac{\partial s}{\partial \tau} \neq 0$ everywhere and $s = \tau$ outside G_ε. The differential operator Λ_s (see the appendix), which is such that $t\Lambda_s$ is smooth and transversal to K, allows one to extend the initial data $(y, \eta, t, \zeta, \alpha)$ on K, using the Lagrange bracket conditions, to a global contact transformation on $\mathbb{R}^{2n+1}$ with all the desired properties.

(7.3) *Remark.* The construction extends to show that any germ $\phi \in \mathrm{Con}$ which leaves K pointwise fixed has a global representative $\bar{\phi}: \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ which is a contact transformation and reduces to the identity outside $J_\varepsilon = \{|y|<\varepsilon, |\eta|<\varepsilon, |\tau|<\varepsilon, |\zeta|<\varepsilon\}$, where ε is preassigned. We shall use this observation to note that any $\mathscr{J} \in \mathrm{Ref}$ has a globally defined representative equal, outside J_ε, to the simple reflection $\mathscr{J}_0$ defined in Sect. 6.

Combining Lemma 7.2 and Remark 7.3 the factorization problem of Proposition 7.1 is replaced by a global problem with ϕ and $\mathscr{J}$ simple outside J_ε. We wish to construct ψ_1, ψ_2 in $\mathrm{Con}_{\infty,\infty}(\mathscr{J})$, $\mathrm{Con}_{\infty,\infty}(\mathscr{R}_+)$ respectively, such that

$$(7.4) \quad \phi = \psi_1\psi_2, \quad \psi_1 \equiv \mathrm{Id}, \psi_2 \equiv \mathrm{Id} \quad \text{on} \quad K,$$

where we write $\phi \equiv \phi'$ on S for the equivalence relation of equality of Taylor series on the hypersurface S. Note that $\mathscr{R}_+$ is defined, in the coordinates $(y, \eta, t, \tau, \zeta)$ by the singular map

$$R_+ : \Omega = (y, \eta, t, \tau, \zeta) \to (y, \eta, t_+, \tau_+, \zeta_+)$$

with $\Omega = \{(y, \eta, t, \tau, \zeta); t \leqq 0\}$ and

$$t_+ = t - 3\tau^2 + 3\tau_+^2, \quad \tau_+ = 1/2(-t + \sqrt{9\tau^2 - 4t})$$
$$\zeta_+ = \zeta + 2(\tau^3 - \tau_+^3)(t - 3\tau^2) + \tfrac{18}{5}(\tau^5 - \tau_+^5).$$

Now, R_+ maps the region $\tau \leqq 0$ of $K = \{t = 0\}$ onto the hypersurface S_0 on which $t = 9\tau^2/4$, $\tau \geqq 0$ so

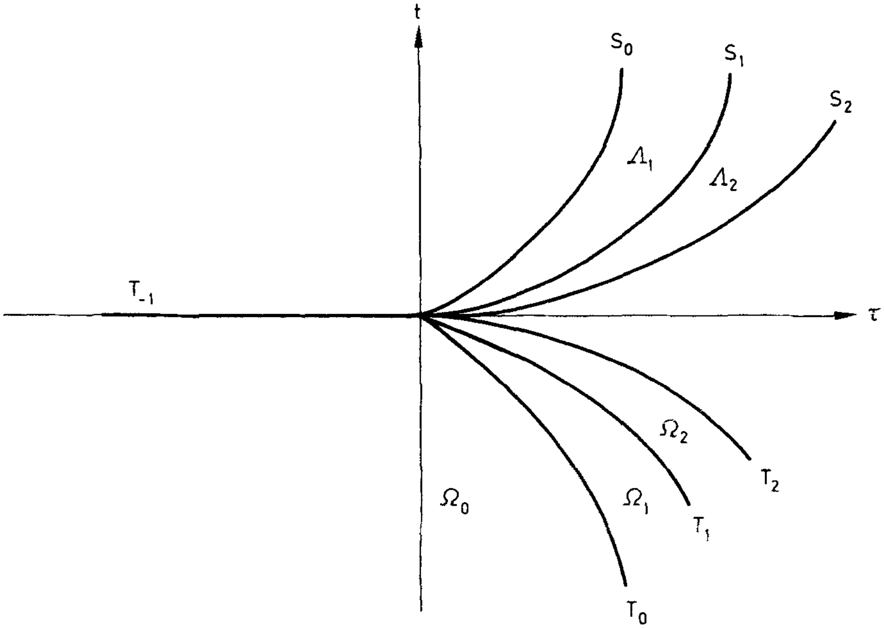
$$(7.5) \quad \psi_2 \equiv \mathrm{Id} \quad \text{on} \quad S_0, \psi_1 \equiv \phi \quad \text{on} \quad S_0.$$

By assumption $\psi_1\mathscr{J} = \mathscr{J}\psi_1$ so from (7.5)

$$(7.6) \quad \psi_1 \equiv \mathscr{J}\phi\mathscr{J} \quad \text{on} \quad T_0 = \mathscr{J}(S_0), \psi_2 = \mathscr{J}\mathscr{J}' \quad \text{on} \quad T_0' = \mathscr{J}'(S_0),$$

where we have written $\mathscr{J}'$ for the involution $\psi^{-1}\mathscr{J}\psi$. Clearly, T_0 is a smooth hypersurface in $t \leqq 0$, tangent to $t = \tau = 0$ and of the form $t = -9\tau^2/4$ for large τ. We shall denote by Ω_0 the closed region in Ω lying below T_0. The R_+ image, Λ_1 of Ω_0

lies in $D_+ = \left\{(y,\eta,t,\tau,\zeta);\, 0 \leqq t \leqq \frac{9\tau^2}{4},\, \tau \geqq 0\right\}$, and is bounded above by S_0 and below by a hypersurface S_1, it is not immediately clear that $S_1 = R_+(T_0)$ is C^∞ at its



boundary $t=\tau=0$, but it is certainly smooth elsewhere and is shown to be C^∞, below. Inductively, we define the closed regions $\Omega_j \subset \Omega$, $\Lambda_j \subset D_+$ and their boundaries T_j, S_j by

(7.7) $\Lambda_j = R_+(\Omega_{j-1}),\ R_j = \mathscr{J}(\Lambda_j),\ S_j = R(T_{j-1}),\ T_j = \mathscr{J}(S_j).$

Similarly we define Λ'_j, Ω'_j, S'_j, T'_j by replacing the involution $\mathscr{J}$ by $\mathscr{J}'$, and initially $T'_{-1} = T_{-1}$, $S'_0 = S_0$. Now, we choose ψ_2 in Ω'_0

(7.8) $\psi_2 = \mathscr{J}\mathscr{J}' : \Omega'_0 \to \Omega_0\,,$

certainly satisfying the compatibility condition (7.6), on T'_0. This choice fixes ψ_1 in $\phi(\Omega'_0)$ through (7.4), moreover the demand of $\mathscr{R}_+$-invariance fixes ψ_2 in Λ'_1 as $R_+\psi_2 R_+^{-1}$, so defining ψ_1 in $\phi(\Lambda'_1)$ through (7.4) and hence fixing ψ_1 in $\phi(\Omega'_1)$ through the requirement of $\mathscr{J}$-invariance. Thus, our initial choice (7.8) of ψ_2 fixes ψ_2 on each Ω'_j, Λ'_j through the following commutative diagram.

$$\begin{array}{ccccc} \Lambda'_{j+1} & \xleftarrow{R_+} & \Omega'_j & \xrightarrow{\mathscr{J}'} & \Lambda'_j \\ \downarrow{\scriptstyle\psi_2} & & \downarrow{\scriptstyle\psi_2} & & \downarrow{\scriptstyle\psi_2} \\ \Lambda_{j+1} & \xleftarrow[R_+]{} & \Omega_j & \xrightarrow[\mathscr{J}]{} & \Lambda_j\,. \end{array} \tag{7.9}$$

Note that all the horizontal maps are homomorphisms which are diffeomorphisms on the interiors of their domains.

We shall see below that, if $\varepsilon>0$ is sufficiently small and we put $D_+(\varepsilon)=\{(y,\eta,t,\tau,\zeta)\in D_+;\ \tau\leqq\varepsilon\}$

$$(7.10)\quad \bigcup_{j\geqq 1}(\Lambda'_j\cap D_+(\varepsilon))=\{(y,\eta,t,\tau,\zeta)\in D_+(\varepsilon);\ t>0\}.$$

Thus, the diagrams (7.9) serve to define ψ_2 in a dense subset of $D_+(\varepsilon)$. We must show that this definition is consistent across the common boundary S'_j of Λ'_j and Λ'_{j+1} for each j, and then show that ψ_2 extends smoothly to $D_+(\varepsilon)$.

The consistency of the definition of ψ_2 is built into (7.8) and (7.9). By construction, $\psi_2=R_+\mathscr{J}\mathscr{J}'R_+^{-1}$ in Λ'_1 so $\psi_2\equiv\mathrm{Id}$ on S_0. Now, in Λ'_2

$$(7.11)\quad \psi_2=R_+\mathscr{J}\psi_2\mathscr{J}'R_+^{-1}$$

so has Taylor series on S'_1, $\psi_2=R_+\mathscr{J}\mathscr{J}'R_+^{-1}$ agreeing with the limit from Λ'_1. In general, the regularity of ψ_2 across S'_{j+1} follows from its regularity across S'_j because of the recurrence formula (7.11) and the fact that $\mathscr{J}'R_+^{-1}(S'_{j+1})=S'_j$.

Thus, ψ_2 is smooth in $\bigcup_j \Lambda'_j$. To prove (7.10) and the regularity of ψ_2 up to $t=0$ it is convenient to introduce new, and singular, coordinates. Consider the mapping $\kappa(y,\eta,t,\tau,\zeta)=(y,\eta,s,\tau,\zeta)$, $s=t\tau^{-2}$ which is a diffeomorphism from $\{(y,\eta,t,\tau,\zeta);\ \tau>0\}$ onto the corresponding region $\tau>0$. We shall denote by tildas the transforms of objects into the new coordinates. Thus,

$$\tilde{D}_+(\varepsilon)=\{(y,\eta,s,\tau,\zeta);\ 0\leqq\tau\leqq\varepsilon,\ 0\leqq s\leqq 9/4\}=\kappa D_+(\varepsilon).$$

(7.12) **Proposition.** *With ψ_2 defined by (7.8), (7.9), $\tilde{\psi}_2=\kappa\psi_2\kappa^{-1}$ defines a C^∞ map $\tilde{D}_+(\varepsilon)\to\tilde{D}_+$ for $\varepsilon>0$ sufficiently small, with Taylor series the identity on both $s=0$ and $\tau=0$.*

The proof of this proposition is based on the following five lemmas which analyse the problem in the new coordinates.

(7.13) **Lemma.** *The map $\tilde{F}=\kappa R_+\mathscr{J}\kappa^{-1}$: $\tilde{D}_+\to\tilde{D}_+$ is C^∞, mapping $\tilde{\Lambda}_j=\kappa\Lambda_j$ into $\tilde{\Lambda}_{j+1}$ for each j. The hypersurfaces $\tilde{S}_j=\kappa S_j$, separating $\tilde{\Lambda}_j$ and $\tilde{\Lambda}_{j+1}$ are C^∞ and of the form*

$$s=\sigma_j+\tau\beta_j(y,\eta,\tau,\zeta)$$

near $\tau=0$.

Proof. We shall show that $\tilde{R}_+=\kappa R_+\kappa^{-1}$ and $\tilde{\mathscr{J}}=\kappa\mathscr{J}\kappa^{-1}$ are C^∞ on domains $\tilde{V}_r=\{(y,\eta,s,\tau,\zeta);\ \tau\geqq 0,\ -r\leqq s\leqq 0\}$ and $\tilde{D}_+$ respectively and that $\tilde{\mathscr{J}}(\tilde{D}_+)\subset\tilde{V}_r$ for large r. Both maps are certainly C^∞ away from $s=0$, since this is the only singularity of κ^{-1}.

Writing $\mathscr{J}(y,\eta,s,\tau,\zeta)=(\bar{y},\bar{\eta},\bar{s},\bar{t},\bar{\zeta})$ we know that $\bar{y}-y$, $\bar{\eta}-\eta$ and $\bar{z}-z$ are $O(t^2)$ and $\bar{\tau}=\tau+\gamma t$ so these are C^∞ functions of y,η,s,τ,ζ, when $t=s\tau^2$, and

$$(7.14)\quad \bar{\tau}=(1+s\tau\tilde{\gamma}(y,\eta,s,\tau,\zeta)).$$

Since $\bar{t}=-t(1+\delta t)$, $\tilde{\mathscr{J}}(y,\eta,s,\tau,\zeta)=(\bar{y},\bar{\eta},\bar{s},\bar{\tau},\bar{\zeta})$ with

$$(7.15)\quad \bar{s}=\bar{t}\bar{\tau}^{-2}=-s(1+\tau^2 s\tilde{\delta})(1+s\tilde{\gamma}\tau)^{-2}.$$

Thus, $\mathscr{J}$ is C^∞ and clearly $\mathscr{J}(\tilde{D}_+)\subset \tilde{V}_r$ for large r. From the form of R_+, $\tilde{R}_+(y,\eta,s,\tau,\zeta)=(y,\eta,s_+,\tau_+,\zeta_+)$ where

(7.16) $\tau_+=\tau l^{-1}(s),\ s_+=(s-3)l^2(s)+3,\ l^{-1}(s)=-1/2+1/2(9-4s)^{1/2}$

and

$$\zeta_+=\zeta+2\tau^5[(s-3)(1-l^{-3}(s))+\tfrac{9}{5}(1-l^{-5}(s))].$$

Thus, $\tilde{R}_+$ is also C^∞ so it only remains to demonstrate the form of $\tilde{S}_j$ near $\tau=0$. Thus follows from (7.14), (7.15), and (7.16) since, at $\tau=0$, $\tilde{\mathscr{J}}=\tilde{\mathscr{J}}_0$ when $\tilde{\mathscr{J}}_0$ is the reflection $s\to -s$; the constants σ_j are therefore defined by

$$\sigma_0=9/4\sigma_{j+1}=3-(3+\sigma_j)l^2(-\sigma_j).$$

For each j, $\varepsilon>0$ put $\tilde{A}_j(\varepsilon)=\tilde{A}_j\cap\tilde{D}_+(\varepsilon)$.

(7.17) **Lemma.** *Given $\delta>0$ there exist constants $\varepsilon_0>0$, C_1, C_2 such that if $\varepsilon<\varepsilon_0$, $\tilde{A}_{j+1}(\varepsilon)\subset\tilde{F}(\tilde{A}_j(\varepsilon))$ and*

$$(\tfrac{5}{9}-\delta)j+C_1\leqq\frac{1}{s}\leqq(\tfrac{5}{9}+\delta)j+C_2 \quad \text{in} \quad \tilde{A}_j(\varepsilon).$$

Proof. From (7.16), (7.17), and (7.18) and the Taylor series expansion $l^2(-s)=s-\frac{11}{9}s^2+0(s^3)$ it follows that, if $\tilde{F}(y,\eta,s,\tau,\zeta)=(y',\eta',s',\tau',\zeta')$ then

(7.18) $s'=s(1-\frac{5}{9}s+O(s\tau))y'-y,\eta'-\eta,\zeta'-\zeta=O(s^2\tau^2)$
$\tau'=\tau(1+\frac{5}{3}+O(s\tau))$.

So, if ε_0 is small, $\tau\leqq\varepsilon<\varepsilon_0$ implies that $\tau'\geqq\tau$ so certainly $\tilde{A}_{j+1}(\varepsilon)\subset\tilde{F}(\tilde{A}_j(\varepsilon))$. If $p_k=\tilde{F}^{-k}p=(y_k,\eta_k,s_k,\tau_k,\zeta_k)$ is the sequence of preimage points of $p_0=(y,\eta,s,\tau,\zeta)\in\tilde{A}_j(\varepsilon)$, $k=0,\ldots,j-1$ then, from (7.18) if $\varepsilon_0=\varepsilon_0(\delta)$ is small

$$-(\tfrac{5}{9}+\delta)\leqq\frac{1}{s_k}-\frac{1}{s_{k-1}}\leqq-(\tfrac{5}{9}-\delta)\ \forall k.$$

Summing this inequality over k, and noting that $p_{j-1}\in\tilde{A}_1(\varepsilon)$ implies that s_{j-1} satisfies a fixed estimate $0<C_1'\leqq 1/s_{j-1}\leqq C_2'$ gives the desired bound on $s_0=s$.

(7.19) **Lemma.** *There are positive constants ε_0, C_1, C_2, μ_1, μ_2 such that if $\varepsilon<\varepsilon_0$ the successive preimages $p_k=\tilde{F}^{-k}p=(y_k,\eta_k,s_k,\tau_k,\zeta_k)$ of any point $p\in\tilde{A}_j(\varepsilon)$, $k=0,\ldots,j-1$ satisfy estimates*

$$\tau_k s_k^{\mu_1}\geqq C_1\tau_0 s_0^{\mu_1},\ \tau_k s_k^{\mu_2}\leqq C_2\tau_0 s_0^{\mu_2}$$

independently of p, j, and k.

Proof. Lemma 7.17 implies that for some constants $\beta_1,\beta_2>0$ independent of k and j, $\beta_1/(j-k)\leqq s_k\leqq\beta_2/(j-k)$. From (7.18) it follows that

$$\prod_{r=2}^{k}(1+(\tfrac{1}{3}-\delta)s_r)\leqq\frac{\tau_0}{\tau_k}\leqq\prod_{r=2}^{k}(1+(\tfrac{1}{3}+\delta)s_r).$$

So, estimating the product on the left by $C_1'\left(\frac{j}{j-k}\right)^{\mu_1}$, $\mu_1=\beta_1(\frac{1}{3}-\delta)$ from below, and that on the right similarly from above it follows from Lemma 7.17 again that

$$C_1^{-1}\left(\frac{S_k}{S_0}\right)^{\mu_1} \leqq C_1'\left(\frac{j}{j-k}\right)^{\mu_1} \leqq \frac{\tau_0}{\tau_k} \leqq C_2'\left(\frac{j}{j-k}\right)^{\mu_2} \leqq C_2^{-1}\left(\frac{S_k}{S_0}\right)^{\mu_2},$$

proving the lemma.

(7.20) **Lemma.** *There exists* $\varepsilon_0>0$ *such that if* $\varepsilon<\varepsilon_0$ *and* $\tilde{F}^r p$, $\tilde{F}^r q\in\tilde{D}_+(\varepsilon)$ *for* $0\leqq r\leqq l-1$ *then, in terms of the Euclidean norm,*

$$|\tilde{F}^l p - \tilde{F}^l q| \leqq K l^{\mu} |p-q|,$$

where K, $\mu>0$ *are independent of* p, q, l, *and* ε.

Proof. From (7.18) it is clear that if ε is small, $p, q\in\tilde{D}_+(\varepsilon)$ and s take the values $s(p)$ and $s(q)$ at p, q then

$$|\tilde{F}p-\tilde{F}q| \leqq (1+C(s(p)+s(q)))|p-q|$$

with C an absolute constant. Lemma 7.17 shows that the assumption $\tilde{F}^r p\in\tilde{D}_+(\varepsilon)$ implies that $s\leqq C'/(r+1)$ with C' independent of p and r, so iterating this estimate

$$|\tilde{F}^l p-\tilde{F}^l q| \leqq |p-q| \prod_{r=1}^{l}\left(1+\frac{2CC'}{r}\right).$$

This product is bounded by Kl^{μ}, $\mu=2CC'$, proving the lemma.

The final preparatory result we need concerns the difference between $\tilde{F}$ and the map $\tilde{F}'=\kappa R_+\mathscr{J}'\kappa^{-1}$ corresponding to $\mathscr{J}'=g^{-1}\mathscr{J}g$. Lemmas 7.13, 7.17, 7.19, and 7.20 apply equally well to $\tilde{F}'$. Note that $\tilde{F}$ and $\tilde{F}'$ have the same Taylor series at both $s=0$ and $\tau=0$, so given $l>0$, $\mu>0$ there exists $C_{l,\mu}>0$ such that

$$(7.21)\quad |p-\tilde{F}\tilde{F}'^{-1}p| \leqq C_{l,\mu}(\tau s^{\mu})^l$$

for all $p=(y,\eta,s,\tau)\in\tilde{F}'(\tilde{D}_+(\varepsilon))$.

(7.22) **Lemma.** *There exists* $\varepsilon_0>0$ *and, for each* $k\in\mathbb{N}$, *constants* C_k, $J(k)$ *such that if* $\varepsilon<\varepsilon_0$ *and* $p=(y,\eta,s,\tau,\zeta)\in\tilde{\Lambda}_{j+1}(\varepsilon/2)$ *then* $\tilde{F}^l\tilde{F}'^{-l}p\in\tilde{D}_+(\varepsilon)\,\forall l\leqq j$, $j\geqq J(k)$ *and*

$$|p-\tilde{F}^j\tilde{F}'^{-j}p| \leqq C_k s^k.$$

Proof. By the triangle inequality,

$$|p-\tilde{F}^l\tilde{F}'^{-l}p| \leqq \sum_{r=0}^{l-1} |\tilde{F}^r\tilde{F}'^{-r}p-\tilde{F}^r(\tilde{F}\tilde{F}'^{-1})\tilde{F}'^{-r}p|$$

so, assuming $\tilde{F}^r\tilde{F}'^{-r}p\in\tilde{D}_+(\varepsilon)$ for $r\leqq l-1$ we deduce from Lemma 7.20 that

$$(7.23)\quad |p-\tilde{F}^l\tilde{F}'^{-l}p| \leqq Kl^{\mu}\sum_{r=0}^{l-1}|\tilde{F}'^{-r}p-(\tilde{F}\tilde{F}'^{-1})\tilde{F}'^{-r}p|.$$

From Lemma 7.19 we know that, at $\tilde{F}'^{-r}p$, τs^{μ_2} is dominated by $C_2\, s^{\mu_2}$ with s evaluated at p and C_2 some absolute constant. Using (7.21) with $\mu=\mu_2$ we conclude that

$$|\tilde{F}'^{-r}p-(\tilde{F}\tilde{F}'^{-1})\tilde{F}'^{-r}p|\leqq(C'(C\varepsilon)^l)s^{\mu_2 l}$$

and inserting this into (7.23) gives

$$|p-\tilde{F}^l\tilde{F}'^{-l}p|\leqq C'l^{C+1}s^{\mu_2 l}.$$

Since $l\leqq j$ and $j\leqq C''/s$ in $\tilde{A}_{j+1}(c)$ we obtain the desired estimate, for each k, by choosing μ, $l\geqq k+c+1$,

$$|p-\tilde{F}^l\tilde{F}'^{-l}p|\leqq C_k s^k.$$

If we choose $J(k)$ so large that $C_k s^k\leqq\varepsilon/2$ for $p\in\tilde{A}_{j+1}(\varepsilon/2)$ it follows that $\tilde{F}^l\tilde{F}'^{-l}p\in\tilde{D}_+(\varepsilon)$ proving the inductive hypothesis and therefore the lemma.

Proof of Proposition 7.12. We have defined $\tilde{\psi}_2$ in $\tilde{A}'_{j+1}(\varepsilon)$ from its definition in $\tilde{A}_1(\varepsilon)$ by

$$\tilde{\psi}_2 p=\tilde{F}^j\tilde{\psi}_2(\tilde{F}')^{-j}p$$

and have already shown that this defines a map C^∞ in $\tilde{D}_+(\varepsilon)$ away from $s=0$. So it remains to examine the behavior as $s\downarrow 0$: Recall that in $\tilde{A}_1(\varepsilon)$ $\tilde{\psi}_2\equiv\mathrm{Id}$ at $\tau=0$. Thus, if $p\in\tilde{A}_{j+1}(\varepsilon)$

$$\begin{aligned}|p-\tilde{\psi}_2 p|&=|p-\tilde{F}^j\tilde{\psi}_2(\tilde{F}')^{-j}p|\\&\leqq|\tilde{F}^j(\tilde{F}'^{-j}p)-\tilde{F}^j q|+|\tilde{F}^j\tilde{F}'^{-j}p-p|,\end{aligned}$$

where $q=\tilde{F}'^{-j}p\in\tilde{A}'_1(\varepsilon)$. From Lemmas 7.20 and 7.22

$$|p-\tilde{\psi}_2 p|\leqq Kj^\mu|\tilde{F}'^{-j}p-q|+C_k s^k$$

with s evaluated at p. Since $|\tilde{\psi}_2 q-q|\leqq C_k\tau^k(q)$ and from Lemma 7.19 we know that $\tau(q)\leqq Cs^{\mu_2}(p)$ (since $q=\tilde{F}'^{-j}p$) we deduce that

$$|p-\tilde{\psi}_2 p|\leqq C'_k s^k$$

uniformly in j. Thus $\tilde{\psi}_2$ is certainly continuous as $s\downarrow 0$ in $\tilde{D}_+(\varepsilon)$, indeed is equal to the identity to all orders there.

The fact that $\tilde{\psi}_2\in C^\infty$ at $s=0$ can be proved inductively over the order of differentiability. Thus the Jacobian of $\tilde{\psi}_2$ at $p\in\tilde{A}_{j+1}(\varepsilon)$ is

$$J_{\tilde{\psi}_2}(p)=J_{\tilde{F}}(q_0)\dots J_{\tilde{F}}(q_j)J_{\tilde{\psi}_2}(p_j)J_{\tilde{F}'}^{-1}(p_{j-1})\dots J_{\tilde{F}'}^{-1}(p_0)\tag{7.24}$$

where $p_l=\tilde{F}'^{-l}p$, $q_j=\tilde{\psi}_2 p_j$, $p_r-\tilde{F}^{j-r}p_j$ $r\leqq j$. From Lemma 7.19 we note that $\tau(p_j)\leqq Cs^\mu(p)$ so that

$$J_{\tilde{\psi}_2}(p_j)\leqq(1+C_k s^k)\mathrm{Id},\tag{7.25}$$

where the inequality between matrices means that the absolute value of each entry of the matrix on the left is bounded by the corresponding entry on the right. From Lemma 7.22 it follows that $|p_l-q_l|\leqq C_k s^k$, uniformly in l and j, provided $l\leqq j$ so from Lemma 7.19 and the fact that $\tilde{F}$, $\tilde{F}'$ have the same Taylor series at $s=0$ and $\tau=0$

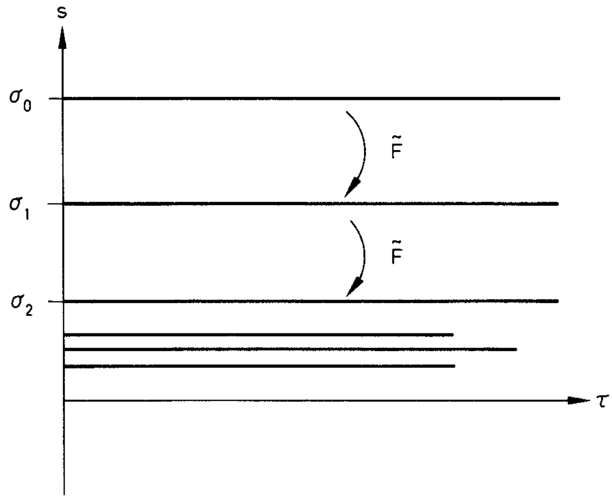
$$J_{\tilde{F}'}^{-1}(p_l)-J_{\tilde{F}}^{-1}(q_l)\leqq C_k s^k\mathrm{Id}$$

in the same sense as 7.25. Combining those estimates and estimating each conjugation in (7.24) successively we find that

$$J_{\tilde{\psi}_2}(p) - \mathrm{Id} \leqq jCC_k s^k \mathrm{Id},$$

where C is an absolute constant estimating the norms of $J_{\tilde{F}}$, $J_{\tilde{F}'}$ and their inverses. From Lemma 7.17 we note again that $j \leqq C'/s$ so this shows that the Jacobian of $\tilde{\psi}_2$ is continuous as $s \downarrow 0$. Continuity of higher derivatives follows similarly. Thus Proposition 7.12 is proved.

Proof of Proposition 7.1. Using the global extensions of $\phi \in \mathrm{Con}_{\infty,\infty}$ and $\mathscr{J} \in \mathrm{Ref}(\mathbb{R}^{2n+1}, A, 0)$ proved by Lemma 7.2 and Remark 7.3 we have, in Proposition 7.12, constructed the map $\psi_2 = K^{-1}\tilde{\psi}_2 K : D_+(\varepsilon) \to D$, for small ε which is C^∞ and satisfies $\psi_2 \equiv \mathrm{Id}$ on S_0 and $\tau = 0$. Thus $\psi_2 = R_+ \psi_2 R_+^{-1}$ defined on Ω, near 0, is C^∞ and is the restriction to $t \leqq 0$ of an element of $\mathrm{Con}_{\infty,\infty}$. Then $\psi_1 = \psi_2^{-1}$ in $t \leqq 0$ [recall $\phi = \mathrm{Id}$ in $t \leqq 0$ by (7.8)] extends to an element $\psi_1 \in \mathrm{Con}_{\infty,\infty}$, $\mathscr{J}$), uniquely, and by the construction of ψ_2, in particular (7.9), $\psi_1^{-1}\phi \in \mathrm{Con}_{\infty,\infty}(\mathscr{R}_+)$ is an extension of ψ_2, defined near 0 in $D_+ \cup \Omega$. This provides the factorization stated in the proposition.



8. Extension Off J

The principal conclusion of Sects. 6 and 7 is that Proposition 5.12 can be strengthened.

(8.1) **Proposition.** $\mathscr{J}, \mathscr{J}' \in \mathrm{Ref}(\mathbb{R}^{2n+1}, \tilde{A}, 0)$ lie in the same $\mathrm{Con}(\mathscr{R}_+)$-conjugacy class if, and only if, there exists $s \in \mathrm{Con}(L)$ such that, for each $k \geqq 1$, the successive obstructions O_k vanish on the image of $\mathscr{J}'$ in $\Gamma^k(\mathscr{J}, s)$.

Proof. The forward implication is already covered by Proposition 5.12. Conversely, again by Proposition 5.12 the vanishing of all obstructions implies the existence of $\phi \in \mathrm{Con}_\infty$, $\mu \in \mathrm{Con}(\mathscr{R}_+)$ such that

$$\mu \mathscr{J}' \mu^{-1} = \phi^{-1} \mathscr{J} \phi .$$

Combining Propositions 6.11 and 7.1 we can decompose $\phi=\psi\bar{\mu}$, $\psi\in\mathrm{Con}_\infty(\mathscr{J})$, $\bar{\mu}\in\mathrm{Con}_\infty(\mathscr{R}_+)$ and therefore

$$\bar{\mu}\mu\mathscr{J}'(\bar{\mu}\mu)^{-1}=\psi^{-1}\mathscr{J}\psi=\mathscr{J}$$

as claimed.

Proof of Theorem 5.1. As noted earlier, the forward implication is trivial. Given a system of oriented-intersection maps with contact structures $I_F^*M_F=I_G^*M_G=A$ we know, following Sects. 2 and 3 that local G-coordinates can be introduced as in Sect. 5. Thus, the commutative diagram we must construct, to prove equivalence in the sense of Definition 1.12 is (1.13) in G-coordinates.

Our hypothesis is the existence of ψ such that $\psi\in\mathrm{Con}(\mathscr{R}_+)$ and $\psi\mathscr{J}_F=\mathscr{J}_{F'}\psi$ where $\mathscr{J}_F,\mathscr{J}_{F'}\in\mathrm{Ref}(\mathbb{R}^{2n+1},\tilde{A},0)$.

By examining Taylor series on L and K one easily verifies that $\psi\in\mathrm{Con}(\mathscr{R}_+)$ projects under $I_G^e=I_G^{e'}=\tilde{I}^e$ to a smooth map, ϕ_G^e on the closure of

$$(8.2)\quad B_G^e=B_G^{e'}=\{(y,\eta,x,\xi,z);\, x+3\xi^2>0,\ \xi\leqq 0 \quad\text{or}\quad x+\tfrac{3}{4}\xi^2>0,\ \xi\geqq 0\}$$

which has C^∞ structure defined by y, η, x, $x\xi+\xi^3$, z. Indeed, $\tilde{I}^e$ identifies only those points which are $\mathscr{R}_+$-related. Similarly, we need to show that ψ projects onto $I_{F'}$, I_F to a diffeomorphism which can be extended to give ϕ_F. The regularity, near the image $\psi=0$ of the fold, follows easily from the fact that a $\mathscr{J}_{F'}$-invariant function vanishing exactly to second order on K is transformed by ψ to a $\mathscr{J}_F$-invariant function of the same type, and these project under $I_{F'}$, I_F to give smooth functions on $B_{F'}$, B_F. Again using Proposition A.16 one can easily extend ϕ_F so defined.

Thus, the next step is to prove Theorem 1.14 and so finally show that the formal equivalence $\mathscr{J}_F\underset{L}{\sim}\mathscr{J}_{F'}$, in G-coordinates, implies the equivalence of the original system of intersecting hypersurfaces.

Proof of Theorem 1.14. Starting from the diagram (1.13) we wish to extend ψ to a contact diffeomorphism, ϕ, as required by Definition 1.5. We shall apply Proposition A.16, but not directly, first we must contruct suitable functions which will become X_1 in the hypotheses of that result.

In the two manifolds E_i, $i=1$, 2, choose hypersurfaces Σ_i. Through the base point and transversal to the bicharacteristic foliation of G_i. Since this foliation is tangent to K_i at p_i the Σ_i intersect F_i, J_i, and K_i normally. Now, on Σ_2 we choose a function Q_2 which is C^∞, vanishes on $\Sigma_2\cap K_2$ but has differential non-zero at p_2 in $\Sigma_2\cap J_2$. Extend Q_2 as the solution of

$$(8.3)\quad V_{g_2}Q_2=0,$$

where $g_2\in C^\infty((M_2)^*,p_2)$ vanishes simply on G_2. Then, as V_{f_2}, where f_2 defines F, is tangent to J_2 but not K_2,

$$(8.4)\quad V_{f_2}Q_2\neq 0 \quad\text{at}\quad p_2.$$

Next, define $Q_1 \upharpoonright J_1=\psi^*(Q_2\upharpoonright J_2)$, and note that Q_1 then projects from J_1 to $B_{G_1}^{e_1}$ as a C^∞ function. In particular, Q_1 extends uniquely to $G_1^{e_1}$ as a C^∞ function constant on the bicharacteristics. We choose an arbitrary C^∞ extension of Q_1 off

$G_1^{e_1}$, so

(8.5) $V_{g_1}Q_1 = h_1, h_1 = 0$ on $G_1^{e_1}$,

where $g_1 \in C^\infty((M_1)^*, p_1)$ defines G_1. Clearly Q_1 satisfies the condition

(8.6) $V_{f_1}Q_1 \neq 0$ at p,

as Q_2 vanishes on $K_1 \cap \Sigma$, and has differential non-zero in $\Sigma_1 \cap J_1$. In particular, ψ can be extended to a diffeomorphism $\phi_0 : F_1 \to F_2$ by defining $\Phi_0^* Q_2 = Q_1$ and noting that the surfaces $Q_i = \text{const}$ in F_i are naturally isomorphic to the projected spaces B_{F_i}. Thus, the commutativity of (1.13) allows us to define ϕ_0 as ϕ_F on each such surface, thereby assuring

$$\phi_0^* M_2 = M_1, \phi_0 \upharpoonright J_1 = \psi.$$

By Darboux's Theorem each F_i is certainly isomorphic, with its contact structure, to $x_1 = 0$ in $(\mathbb{R}^{2n+1}, \bar{M}, 0)$. So, using Proposition A.16 twice, for $i = 1, 2$, with $X_1 = Q_i$ in each case, we can extend ϕ_0 to a contact diffeomorphism $\phi : E_1 \to E_2$ which certainly maps F_1 to F_2. We must check that $\phi(G_1^{e_1}) = G_2^{e_2}$. This follows from the fact that each $G_i^{e_i}$ is the Q_i-flow out of J_i because of (8.3), (8.4), (8.5), and (8.6), which build the transversality of $V(Q_i, p_i)$ to F_i and the tangency of Λ_{Q_i} to $G_i^{e_i}$ into the definition of Q_i. Thus, the proof of Theorem 1.14 is complete.

9. A Complete Set of Examples

In the manifold $\mathbb{R}^{2n+1}$ with base point 0, coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n, z)$ and contact bundle $\bar{M}$ spanned by (A.11) we consider the hypersurfaces F, defined by $x_1 = 0$, and G defined by

$$g = \xi_1^2 \pm x_1 x_n + x_1^2 h(x, \xi'', z) - \xi_n = 0,$$

where $\xi'' = (\xi_1, \dots, \xi_{n-1})$ and h is a C^∞ function. For these examples of intersecting hypersurfaces satisfying (1.1), (1.2), (1.3), and (1.4), J is the manifold $x_1 = 0 = \xi_1^2 - \xi_n$ with induced folded contact bundle spanned by $\alpha_J = \sum_{j=2}^{n-1} \xi_j dx_j + \xi_1^2 dx_n + dz$ in the coordinates $x_2, \dots, x_n, \xi_1, \dots, \xi_{n-1}, z$; K is the submanifold $x_1 = \xi_1 = \xi_n = 0$ and L is defined by $x_1 = \xi_1 = x_n = \xi_n = 0$. The quotient manifold $B_F = F/V_F$ is isomorphic to $x_1 = \xi_1 = 0$ and in the coordinates $x_2, \dots, x_n, \xi_2, \dots, \xi_n$ has contact bundle spanned by $\mu_F = \sum_2^n \xi_j dx_j + dz$. The F intersection map $I_F : J \to B_F$ is then always

(9.1) $I_F(x_2, \dots, x_n, \xi_1, \dots, \xi_{n-1}, z) = (x_2, \dots, x_n, \xi_2, \dots, \xi_{n-1}, \xi_1^2, z)$.

Similarly, we can identify $B_G = G/V_G$ with the surface $x_n = 0$ in G, in which the coordinates $x_1, \dots, x_{n-1}, \xi_1, \dots, \xi_{n-1}, z$ give the contact form $\mu_G = \sum_1^{n-1} \xi_j dx_j + dz$.

(9.2) **Theorem.** *Every system of intersecting hypersurfaces* (F, G, E, M, p, e) *satisfying* (1.1), (1.2), (1.3), *and* (1.4) *is equivalent, in the sense of Definition 1.5, to this standard system for some choice of sign and* h.

To prove Theorem 9.2 it suffices, according to Theorem 1.14, to show that every system of oriented-intersection maps (1.11) is isomorphic, in the sense of Definition 1.12, to that derived from F, G above for some h. According to Theorem 5.1 and Proposition 8.1 once the sign (orientation) is correctly chosen this is only a problem in formal power series, so our first task is to find the Taylor series of $I_G: J \to B_G$ at $x_n = \xi_1 = 0$ and to relate it to the Taylor series of h

$$h \equiv \sum_{k,p} x_1^k x_n^p h_{k,p}(x'', \xi'', z) \quad \text{on} \quad x_1 = x_n = 0.$$

For simplicity we shall assume that the orientation e of F corresponds to the choice, "+", of sign above, the other case is very similar.

Now, I_G is defined by flow along V_G from J to $x_n = 0$, so if functions X_j, Ξ_j, Z of $x_2, \ldots, x_n, \xi_1, \ldots, \xi_{n-1}, z, \varrho$ are defined by integration along the vector field $V_{g\alpha*}$:

$$(9.3) \quad \frac{dX_1}{d\varrho} = 2\Xi_1 \qquad \frac{d\Xi_1}{d\varrho} = -X_n - (X_1^2 h)'_{X_1} + \Xi_1 h'_2 X_1^2$$

$$\frac{dX_n}{d\varrho} = -1 \qquad \frac{dZ}{d\varrho} = -(2\Xi_1^2 + \Xi'' h'_{\Xi''} X_1^2 - \Xi_n)$$

$$\frac{dX''}{d\varrho} = X_1^2 h''_{\Xi''} \qquad \frac{d\Xi''}{d\varrho} = -X_1^2(h'_{X''} - \Xi'' h'_Z)$$

with $\Xi_n = \Xi_1^2 + X_1 X_n + X_1^2 h(X, \Xi'', Z)$ throughout and initial conditions on $J, X_1 = 0$, $X'' = x''$, $X_n = x_n$, $\Xi = \xi$, $Z = z$ at $\varrho = 0$ then,

$$I_G(x_2, \ldots, x_n, \xi_1, \ldots, \xi_{n-1}, z) = (X_1, X'', \Xi_1, \Xi'', Z) \upharpoonright (\varrho = x_n).$$

Indeed, $X_n = x_n - \varrho$ so $X_n = 0$ exactly when $\varrho = x_n$. To determine this map we only need to know $\Xi_1 \upharpoonright (\varrho = x_n)$ since the other functions and the factor σ, for which $I_G^* \alpha_G = \sigma \alpha_J$, are determined by their Lagrange brackets with Ξ_1 from the initial conditions at $x_n = 0$:

$$(X, \Xi, Z, \sigma) \upharpoonright (\varrho = x_n = 0) = (x, \xi, z, 1) \quad (\text{on } J).$$

Recall the quasi-homogeneous filtration of the formal power series ring $\mathfrak{a} = C^\infty(L, 0)[[x_n, \xi_1]]$ where, because of the change of coordinates from Sect. 4, ξ_1 is now assigned weight 2 and x_n weight 1; we shall denote by $\bar{X}_j$ etc., the projection of $X_j \upharpoonright (\varrho = x_n)$ into $\mathfrak{a}$. Directly from (9.3)

$$\bar{X}_1 - 2\xi_1 x_n + 2\frac{x_n^3}{3} \in \mathfrak{a}_{(4)}$$

and

$$\bar{X}'' - x'', \bar{\Xi}'' - \xi'', \bar{Z} - z, \sigma - 1 \in \mathfrak{a}_{(4)}.$$

Integrating the equation for Ξ_1 gives the Taylor series at $\varrho = 0$

$$\Xi_1 \equiv \xi_1 - x_n \varrho + \varrho^2/2 - \int_0^\varrho \sum_{k,p \geq 0} [(k+2) X_1^{k+1} X_n^p h_{k,p}(X'', \Xi'', Z) - \Xi_1 X_1^{k+2} X_n^p (h_{k,p})'_Z(X'', \Xi'', Z)] \, d\varrho.$$

Thus, the projection of $\bar{\Xi}_1$ into $\mathfrak{a}/\mathfrak{a}_{(l+1)}$ for any l depends only on the $h_{k,p}$ with $3k+p+4\leq l$. Moreover, the choice of each $h_{k,p}$ adds to $\bar{\Xi}_1$, modulo $\mathfrak{a}_{(3k+p+5)}$ a term $h_{k,p}(x'', \xi'', z)T_{k,p}(x_n, \xi_1)$

$$(9.4)\quad T_{k,p}(x_n, \xi_1) = \int_0^{x_n} (k+2)(2\xi_1\varrho - x_n\varrho^2 + \varrho^3/3)^{k+1}(x_n-\varrho)^p d\varrho \in \mathfrak{a}_{3k+p+4}.$$

Clearly,

$$(9.5)\quad T_{k,p}(x_n, \xi_1) = \sum_{s\leq k+1} t_{k,p,s} x_n^{3k+p+4-2s}\xi_1^s,$$

where

$$(9.6)\quad t_{k,p,k+1} = (k+1)2^{k+1}\int_0^1 \varrho^{k+1}(1-\varrho)^p d\varrho \neq 0.$$

Note that the projection of $\bar{X}_1$ and the $\bar{X}''$, $\bar{\Xi}''$ into $\mathfrak{a}/\mathfrak{a}_{(l+1)}$ depends only on the $h_{k,p}$ with $3k+p+4<l$.

Now, given an arbitrary oriented-intersection system (1.11) we know that we can introduce coordinates so that $J' = \mathbb{R}^{2n+1}$ with contact bundle spanned by α_J, $B_{G'} = B_G$, $B_{F'} = B_F$ with their contact structure and I_G, as the "standard" G map corresponding to $h\equiv 0$ above. Then $\mathscr{J}_{F'} \in \mathrm{Ref}(\mathbb{R}^{2n+1}, A_J, 0)$ is some contact involution and we need to exhibit a formal contact diffeomorphism $\exp(V_q)$, where $q \in \mathfrak{a}_{(5)}^{\mathrm{reg}}\alpha^*$, such that

$$(9.7)\quad \exp(V_q)^*\mathfrak{a}(\mathscr{R}_0) = \mathfrak{a}(\mathscr{R}_h),$$

$$(9.8)\quad \exp(V_q)^*\mathfrak{a}(\mathscr{J}_{F'}) = \mathfrak{a}(\mathscr{J}).$$

Here, we have written $\mathfrak{a}(\mathscr{R}_h)$ for the pull-back to J under I_G (defined by h) of the ring of formal power series at $x_1 = \xi_1 = 0$ on B_G, then $\mathfrak{a}(\mathscr{R}_h)$ is the ring of $\mathscr{R}_h$-invariants. Similarly, $\mathfrak{a}(\mathscr{J}_{F'})$ is the ring of $\mathscr{J}_{F'}$-invariants where $\mathscr{J} = \mathscr{J}_F$ is the involution defined by I_F in (9.1).

Now, (9.7) and (9.8) can be analysed in terms of the quasi-homogeneous filtration of $\mathfrak{a}$. First, as noted above (9.7) is equivalent to the condition

$$(9.9)\quad \bar{\Xi}_1 - \exp(V_q)^*\left(\xi_1 - \frac{x_n^2}{2}\right) \in \mathfrak{a}_{(3)}(\mathscr{R}_h) = \mathfrak{a}(\mathscr{R}_h)\cap\mathfrak{a}_{(3)}.$$

Similarly, if S is the $\mathscr{J}_{F'}$-even part of x_n then (9.8) is equivalent to

$$(9.10)\quad \exp(V_q)^*(S) \in \mathfrak{a}(\mathscr{J}).$$

Thus, we shall show that for arbitrary S with $S - x_n \in \mathfrak{a}_{(2)}$, h and q can be chosen so that (9.9) and (9.10) hold.

Since $\mathfrak{a}_{(l)}(\mathscr{R}_h) = \mathfrak{a}(\mathscr{R}_h)\cap\mathfrak{a}_{(l)}$ projects into $\mathfrak{a}_l$ onto the subspace, $\mathfrak{k}$, spanned, over $C^\infty(L, 0)$, by the polynomials

$$(9.11)\quad (\xi_1 x_n - x_n^3/3)^j(\xi_1 - x_n^2/2)^k \qquad 3j+2k = l,$$

which is independent of h, we can analyse (9.9) inductively as follows: If $q = (q_5 + q_6 + \ldots)\alpha^*$, $q_j \in \mathfrak{a}_j$ and (9.9) is known to hold modulo $\mathfrak{a}_{(l)}$ then it holds modulo $\mathfrak{a}_{(l+1)}$ provided

$$\left[q_{l+3}, \xi_1 - \frac{x_n^2}{3}\right] + V_l - P_l \in \mathfrak{a}_l,$$

where $V_l, P_l \in \mathfrak{a}_l$ are, respectively, the remainder term determined by the $q_j, j<l+3$ and $\bar{\Xi}_1$ modulo $\mathfrak{a}_{(l)}$, and the term in $\bar{\Xi}_1$ of homogeneity l. From (9.4) and the preceding remarks this can be written

$$\text{(9.12)}\quad \left[q_{l+3}\alpha^*, \xi_1 - \frac{x_n^2}{2}\right] + V_l' - \sum_{3k+p+4=l} h_{k,p} T_{k,p} \in \mathfrak{a}_l,$$

where V_l' is determined by the q_j, $j<l+3$ and the $h_{k,p}$ $3k+p+4<l$. The second equation, (9.10) is even simpler to interpret in terms of the filtration. If it holds modulo $\mathfrak{a}_{(l-1)}$ then, the condition that it holds modulo $\mathfrak{a}_{(l)}$ is simply

$$q_{l+3} - W_l \in \mathfrak{a}_{l-1}(\mathscr{I})$$

that is, it prescribes the ξ_1-odd part of q_{l+3} in terms of the earlier choices of $q_j, j<l+3$, which determine W_l. Thus, in solving (9.12) we have the freedom to choose the ξ_1-even part of q_{l+3}, subject to $q_{l+3} \in \mathfrak{a}_{l+3}^{\text{reg}}$, as well as the $h_{k,p}$. The problem is therefore reduced to linear algebra.

The subspace, $\mathfrak{d}_l \supset \mathfrak{a}_l$, spanned by the $\xi_1^r x_n^{l-2r}$ with $r \leqq \frac{l-1}{3}$ (integral) together with $\mathfrak{k}_l$ spanned by the polynomials (9.11), spans $\mathfrak{a}_l$. From (9.5) and (9.6) we conclude that the $T_{k,p}$, with $3k+p+4=l$ are independent in $\mathfrak{d}_l$ and, together with the element x_n^l, span it. Finally then, we need to use the freedom to choose the even part of q_{l+3} (that is we cannot choose $q \equiv 0$ because of the x_n^l term). Since

$$\left[\xi_1^2 x_n^{l-1}\alpha^*, \xi_1 - \frac{x_n^2}{2}\right] = (n-1)\xi_1 x_n^{l-2} + 2x_n^l$$

and $\xi_1^2 x_n^{l-1}$ is independent of $T_{0,l-3}$ we conclude that (9.12) does indeed always have a solution $q_{l+3} \in \xi_1^2 x_n^{l-1} C^\infty(L,0) \subset \mathfrak{a}_{l+3}^{\text{reg}}$, for an appropriate choice of h. A similar argument shows that the first step, solution of (9.10) and (9.12) modulo $\mathfrak{a}_{(2)}$ and $\mathfrak{a}_{(3)}$ (the second of these being trivial) respectively, is possible.

This completes the proof of Theorem 9.2.

(9.13) *Note.* The arguments above apply just as well to show that any intersecting pair satisfy (1.1)–(1.4) can be transformed to the normal form in $(\mathbb{R}^{2n+1}, \bar{M}, 0)$ in which F is $x_1 = 0$ and G is defined by

$$\xi_1^2 \pm x_1\left(x_n - \sum_{j=2}^{n-1}(x_j^2 + \xi_j^2) + z^2\right) + x_1^2 h(x, \xi'', z) - \xi_n = 0.$$

Appendix. Contact Transformations

Recall that a contact manifold (P, M) is a $(2n+1)$-manifold, P, with a distinguished, oriented, line subbundle $M^+ \supset T^*P$ any local, non-vanishing, section μ of which defines a volume form

$$\text{(A.1)}\quad \mu \wedge (d\mu)^n \neq 0.$$

Since all our considerations are local, all structure will be considered at the germ level at a base point $p \in P$. Thus, $C^\infty(M, p)$ is the space of germs at p of C^∞ sections of M. A vector field, $V \in C^\infty(TP, p)$, is a *contact vector field* if for every $\mu \in C^\infty(M, p)$ there exists

$\varrho_{V,\mu}\in C^\infty(P,p)$ such that

(A.2) $\mathscr{L}_V\mu=\varrho_{V,\mu}\mu$.

We shall write $\mathscr{CV}(M,p)$ for the Lie algebra of such fields.

Interior multiplication defines the *Lagrange isomorphism*

(A.3) $\mathscr{CV}(M,p)\ni V\mapsto i_V\circ\mu=\mu(V)\in C^\infty(M^*,p)$

onto the space of sections at p of the dual bundle to M. For $f\in C^\infty(M^*,p)$ the vector field, V_f, giving the inverse to (A.3) is determined by (A.2), which can be written

(A.4) $2d\mu(V,\cdot)=-df(\mu)+\varrho_{f,\mu}\mu$,

together with

$$\mu(V)=f(\mu).$$

Note in particular that

(A.5) $\varrho_{f,\mu}(\mu\wedge(d\mu)^n)=df(\mu)\wedge(d\mu)^n$.

The Lie algebra structure on $C^\infty(M^*,p)$ carried from $\mathscr{CV}(M,p)$ by (A.3) is the Lagrange bracket, if $f,g\in C^\infty(M^*,p)$ and $\mu\in C^\infty(M,p)$

(A.6) $[f,g](\mu)=V_fg(\mu)-\varrho_{f,\mu}g(\mu)$.

If $s\in\mathbb{R}$ let M_s^* be the line bundle with fibre at p' the space of functions $L_{p'}\backslash\{0\}\to\mathbb{R}$, homogeneous of degree s. Thus, $M_1^*=M^*$, $M_0^*=P\times\mathbb{R}$ and for each s, t there is a product isomorphism

$$M_s^*\otimes M_t^*\simeq M_{s+t}^*.$$

Given $\mu\in C^\infty(M,p)$ with $\mu(p)\neq 0$ one can define sections $\mu_s^*\in C^\infty(M_s^*,p)$, for each s, by requiring $\mu_s^*(\mu)=1$. The Lagrange bracket (A.5) extends uniquely to

$$[\ ,\ ]:C^\infty(M_s^*,p)\times C^\infty(M_t^*,p)\to C^\infty(M_{s+t-1}^*,p)$$

if we require that $[\mu_s^*,\mu_t^*]=0$ and that $[\mu_s^*,f]$ be smooth in s. It then satisfies

$$[f,g]=-[g,f]$$

(A.7) $[f,gh]=[f,g]h+[f,h]g$,

(A.8) $[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0$.

Condition (A.7) shows that every $f\in C^\infty(M_s^*,p)$ defines a firstorder differential operator

$$\Lambda_f:C^\infty(M_s^*,p)\to C^\infty(M_{s+t-1}^*,p)$$

for each t, (A.8) is then the Jacobi identity for these operators:

$$[\Lambda_f,\Lambda_g]=\Lambda_{[f,g]}.$$

In particular, if $f\in C^\infty(M^*,p)$, V_f is the operator Λ_f on $C^\infty(M_0^*,p)$; whenever $\mu\in C^\infty(M,p)$ with $\mu(p)\neq 0$ we shall put $V_{f,\mu}=V_{f,\mu_{-s}^*}$. With this notation and $f\in C^\infty(M_r^*,p)$, $g\in C^\infty(M_s^*,p)$

(A.9) $[f,g](\mu)=V_{f,\mu}g(\mu)-s\varrho_{f,\mu}g(\mu)+(r-1)\varrho_{g,\mu}f(\mu)$

where $\varrho_{f,\mu}$ is defined by (A.5) or

(A.10) $[\mu^*, f](\mu) = \varrho_{f,\mu}$.

Note that $V_{f,\mu}=0$ at p precisely when $f(\mu)=0$ at p and $d_p f(\mu)\in M_p$. If $f\in C^\infty(M_s^*, p)$ vanishes at p but $d_p f(\mu)\notin M_p$ for some $\mu\in C^\infty(M,p)$ then the line in T_pP spanned by $V_{f,\mu}$ is independent of μ, we shall denote it by $V(f,p)$. It is of course just the restriction to p of the Hamilton foliation of $f=0$.

The version of Darboux's theorem given, as Proposition 3.11, in EQ can easily be restated in terms of Lagrange brackets. We shall give a somewhat more general result along these lines and sketch a proof more directly in terms of the contact structure. The basic manifold we use is $\mathbb{R}^{2n+1}$ with coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n, z)$ and contact bundle $\bar{M}$ spanned by

(A.11) $$\beta = \sum_{j=1}^{n} \xi_j dx_j + dz > 0.$$

This choice of section trivializes all the bundles $\bar{M}_s^*$. If $f\in C^\infty(\bar{M}_s^*, 0)$ then

(A.12) $$\varrho_{f,\beta} = \frac{\partial}{\partial z} f_\beta(x,\xi,z),$$

where $f_\beta = f(\beta)$, and

(A.13) $$V_{f,\beta} = \sum_{j=1}^{n}\left(\frac{\partial f_\beta}{\partial \xi_j}\partial_{x_j} - \left(\frac{\partial f_\beta}{\partial x_j} - \xi_j\frac{\partial f_\beta}{\partial z}\right)\partial_{\xi_j}\right) + \left(f_\beta - \sum_{j=1}^{n}\xi_j\frac{\partial f}{\partial \xi_j}\right)\partial_z.$$

Thus, if $\xi_i^* = \xi_i\beta^*$ then, for $i,j=1,\ldots,n$

(A.14) $$\begin{aligned}&[x_i, x_j] = [\xi_i^*, \xi_j^*] = [z, \xi_j^*] = [\beta^*, x_i] = [\beta^*, \xi_i^*] = 0\\ &[x_i, \beta^*] = x_i,\ [\xi_j^*, x_i] = \delta_{ij},\ [\beta^*, z] = 1.\end{aligned}$$

(A.15) *Remark.* If (Q, N^+) is a second contact manifold and $\phi: P, p \to Q, q$ is a contact transformation, $\phi^* N^+ = M^+$, then the induced maps $\phi^*: C^\infty(N_s^*, p) \to C^\infty(M_s^*, p)$ preserve the Lagrange brackets. Conversely, if $x_i, z\in C^\infty(P,p)$, $\xi_i^*, \mu^*\in C^\infty(M^*, p)$ for $i=1,\ldots,n$ satisfy (A.14), with μ replacing β, and $x_i = z = \xi_i^* = 0$ at p, $\beta^* \neq 0$ at p then $\phi: p' \mapsto (x, \xi^*(\mu), z)(p')$ is a contact transformation to $(\mathbb{R}^{2n+1}, \bar{M}^+)$ such that

$$\phi^*\beta = \mu.$$

(A.16) **Proposition.** *Let* (P, M^+, p) *be a germ of contact structure on the* $(2n+1)$-*manifold* P *and let* A, $B\subset\{1,\ldots,n\}$ *be index sets. Given sections* $X_i\in C^\infty(M_{s_i}^*, p)$, $\Xi_j\in C^\infty(M_{t_j}^*, p)$ *for* $i\in A$, $j\in B$, *with* $X_i(p) = \Xi_j(p) = 0$ *and* $s_i + t_i = 1$ *if* $i\in A\cap B$, *together with a submanifold* $\iota_K: K, p \hookrightarrow P, p$ *such that*

(A.17) T_pK, $V(X_i, p)$, $V(\Xi_j, p)$ *are independent and*

$$M_p \cap N_p^* K = \emptyset$$

and an immersion $\phi: K, p \hookrightarrow \mathbb{R}^{2n+1}, 0$ *then, in order that there exist a diffeomorphism*

$$\bar{\phi}: P, p \to \mathbb{R}^{2n+1}, 0$$

extending ϕ *and such that*

(A.18) $\mu = \bar{\phi}^*\beta \in C^\infty(M,p)$, $X_i = \bar{\phi}^*(x_i\beta_{s_i}^*)$, $\Xi_j = \bar{\phi}^*(\xi_j\beta_{t_j}^*)$

it is necessary and sufficient that

(A.19) $[X_i, X_{i'}] = 0 = [\Xi_j, \Xi_{j'}],\ [\Xi_j, X_i] = \delta_{ij} \forall i, i' \in A, j, j' \in B$

and that $\phi^*\beta \in C^\infty(\iota_K^* M, p)$ *be such that* (A.18) *holds on* K.

(A.20) *Note.* It is, of course, possible to remove the restriction $s_i + t_i = 1$ but then the useful property, that (A.19) is independent of the putative section μ, is lost.

Proof of Proposition A.16. We follow the proof of Darboux's theorem, Proposition 2.1 in EQ, closely. Choose a manifold $K' \supset K$, of maximal dimension $2n+1-|A|-|B|$, such that (A.17) still holds. On K' choose $\bar{\mu}$, a section of M, subject to $\iota_K^* \mu = \phi^* \beta$ (on K). Then the section $\mu \in C^\infty(M, p)$ is fixed by the first-order equations

(A.21) $[X_i, \mu^*] = \Lambda_{x_i} \mu^* = 0,\ [\Xi_j, \mu^*] = \Lambda_{\Xi_j} \mu^* = 0 \qquad i \in A, j \in B$

and the initial condition $\mu^* = \bar{\mu}^*$ on K'. The consistency of (A.21) follows from Jacobi's identity. Then we can replace the X_i, Ξ_j by $\bar{X}_i = X_i \mu^*_{-s_i} \in C^\infty(P, p)$ and $\bar{\Xi}_j = \Xi_j \mu^*_{1-t_j} \in C^\infty(M^*, p)$. We leave the reader to verify that A and B can be increased, by the choice of extra sections $\bar{X}_i$, $\bar{\Xi}_j$ so that the meaningful equations in (A.14) hold, and (A.17) is still valid, until either $A = B = \{1, \ldots, n\}$ or $\dim K = 2n+1-|A|-|B|$. In the latter case (and in the former if $\dim K = 1$), for each point p' near p there are unique constants a_i, b_j such that

$$\pi(p') = \prod_{i \in A} \exp(a_i V_{\bar{X}_i, \mu}) \prod_{j \in B} \exp(b_j V_{\Xi_j})(p') \in K$$

so we put

$$\bar{\phi}(p') = \prod_{j \in B} \exp(-b_j V_{\xi_j^*}) \prod_{i \in A} (-a_i V_{x_i^*}) \circ \phi \circ \pi(p').$$

It is easily verified that ϕ is a diffeomorphism, having the desired properties. If $K = \{p\}$ then $Z \in C^\infty(P, p)$ is uniquely defined by

$$[\bar{X}_i, Z] = [\bar{\Xi}_j, Z] = 0,\ [\mu^*, Z] = 0 \qquad Z(p) = 0$$

and, once again from Jacobi's identity, all the Eqs. (A.14) holds so there is a suitable $\bar{\phi}$, following Remark A.15.

References

1. Andersson, K.G., Melrose, R.B.: Propagation of singularities along gliding rays. Invent. Math. (to appear)
2. Golubitsky, M., Guillemin, V.W.: Stable mappings and their singularities. Berlin, Heidelberg, New York: Springer 1973
3. Guillemin, V.W., Schaeffer, D.G.: On a certain class of Fuchsian partial differential equations. Duke Math. J. **44**, 157–199 (1977)
4. Martinet, J.: Sur les singularites des formes differentielles. Ann. Inst. Fourier (Grenoble) **20**, 95–177 (1970)
5. Melrose, R.B.: Equivalence of glancing hypersurfaces. Invent. Math. **37**, 165–191 (1976)
6. Melrose, R.B.: Transformation of boundary value problems (in preparation)
7. Morawetz, C.S., Ralston, J.V., Strauss, W.A.: Decay of solutions to the wave equation outside nontrapping obstacles. Comm. Pure Appl. Math. (to appear)
8. Nelson, F.: Topics in dynamics. Princeton: Princeton University Press 1970
9. Sternberg, S.: Lectures on differential geometry. Englewood Cliffs, New York: Prentice-Hall 1964

Received October 1, 1979