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# On Eigenspaces of the Hecke Algebra with Respect to a Good Maximal Compact Subgroup of a $p$-Adic Reductive Group

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## Introduction

Let $\mathbb{G}$ be a connected reductive algebraic group defined over a non-archimedean local field $k$. Denote by $K$ a good maximal compact subgroup of $G=\mathbb{G}(k)$. Then the commutative algebra, what we call Hecke algebra, $H(G,K)$ acts on $C(G/K)$, the space of $\mathbb{C}$-valued functions on $G/K$, via right convolutions. Let $P_{K,G}(\omega)$ be the $\omega$-isotypic subspace of $C(G/K)$ for an algebra homomorphism $\omega:H(G,K)\to\mathbb{C}$. Under left translations by $G$, $P_{K,G}(\omega)$ is a $G$-module. This is a $p$-adic analogue of an eigenspace of invariant differential operators on a symmetric space. The aim of this paper is to show some of the properties of the $G$-module $P_{K,G}(\omega)$:

(1) We prove that $P_{K,G}(\omega)^\infty$, the subspace of smooth vectors in $P_{K,G}(\omega)$, is admissible. See Sect. 2.
(2) In Sect. 3, we define the Poisson integral which is an intertwining operator between $E'$, the dual space of an unramified principal series representation $E$, and $P_{K,G}(\omega)$. There necessary and sufficient conditions for the bijectivity of the Poisson integral are given in terms of the cyclicity of a $K$-fixed vector in $E$ plus something more. (See also Addendum.)

Here (2) is a $p$-adic analogue of the Helgason's conjecture for real groups proved in [3]. For the proof of the above results, we essentially use Borel-Matsumoto theory on representations with vectors fixed under an Iwahori subgroup (see [1, 5]; note that the arguements in [5] can be easily generalized to the case of reductive groups).

## 1. Difference Equations Invariant Under Finite Reflection Groups

Let $V$ be a finite dimensional vector space over $\mathbb{R}$. We denote by $W$ a finite subgroup of $\mathrm{GL}(V)$. Choose a $W$-invariant lattice $L$ of $V$. By $C(L)$, $\mathbb{C}[L]$ and $\mathbb{C}[L]^W$, we denote the algebra of $\mathbb{C}$-valued functions on $L$, the group algebra of $L$ over $\mathbb{C}$, and the subalgebra of $\mathbb{C}[L]$ which consists of $W$-invariants, respectively. For $v\in L$, we define a difference operator $T_v$ on $C(L)$ by $(T_vf)(x):=f(x+v)$ $(x\in L;f\in C(L))$. By linearity, this map $v\mapsto T_v$ extends to an algebra homomorphism

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of $\mathbb{C}[L]$ into the algebra of difference operators with constant coefficients $\phi \mapsto T_\phi (\phi \in \mathbb{C}[L])$. Set $X(L) = \mathrm{Hom}(L, \mathbb{C}^\times)$. Then $X(L)$ is identified with the set of algebra homomorphisms of $\mathbb{C}[L]$ to $\mathbb{C}$ and $W$ acts on $X(L)$. Consider the following system of difference equations $\Sigma_s$ for every $s \in X(L)$.

$$\Sigma_s: \qquad \sum_{w \in W} T_{w \cdot v} f = (\Sigma_{w \in W} s(w \cdot v)) f \qquad ({}^\forall v \in L).$$

Note that the above system is equivalent to

$$\Sigma'_s: \qquad T_\phi f = s(\phi) f \qquad ({}^\forall \phi \in \mathbb{C}[L]^W).$$

We denote the space of the solutions of $\Sigma_s$ by $H_s(W)$ or simply by $H_s$.

Now we are ready to state the next proposition, which is an analogue of Steinberg's theorem [7].

**Proposition 1.1.** *For all* $s \in X(L)$, $\dim H_s = |W/W_s| \dim H_1(W_s)$ *and* $\infty > \dim H_s \geqq |W|$. *The equality* $\dim H_s = |W|$ *holds if and only if* $W_s \subset \mathrm{GL}(V)$ *is a reflection group.*

Here $W_s$ is the stabilizer of $s$ in $W$ and $1 \in X(L)$ denotes the trivial character. The proof of this proposition goes along somewhat same lines as Steinberg's.

*Proof.* Let us call $f \in C(L)$ a polynomial if $f$ can be written as $f = g|_L$ for some $g \in S(V^*)$ (the algebra of polynomial functions on $V$). It is obvious that the above $g \in S(V^*)$ is uniquely determined. Now let $f$ be an element of $H_s$. Then the same argument as in [7] shows that $f$ can be written in the form,

$$f = \sum_{w \in W/W_s} (w \cdot s) \cdot f_{w \cdot s}$$

where $f_{w \cdot s}$ is a polynomial for every $w \in W/W_s$. Moreover, the linear independence of characters over polynomials (see [5; (3.4.3)]) shows that $f_{w \cdot s}$ belongs to $H_1(W_{w \cdot s})$ for each $w \in W/W_s$. This implies $\dim H_s = |W/W_s| \dim H_1(W_s)$. Hence it suffices to consider the case $s = 1$. But we have already seen that all the elements of $H_1$ are polynomials. Therefore the subspace of $S(V^*)$ which corresponds to $H_1$ is the space of harmonic polynomials on $V$ and Steinberg's theorem assures our proposition.

*Remark 1.2.* For a vector space $E$ and its subset $S$, we denote by $\langle S \rangle$ the subspace of $E$ generated by $S$. Then we can regard $H_s$ as the dual space of $\mathbb{C}[L]/\langle f\phi - s^{-1}(\phi) f \ (f \in \mathbb{C}[L];\ \phi \in \mathbb{C}[L]^W) \rangle$ since $C(L) \simeq \mathbb{C}[L]' = \mathrm{Hom}(\mathbb{C}[L], \mathbb{C})$ canonically.

## 2. Eigenspaces of the Hecke Algebra with Respect to a Good Maximal Compact Subgroup

From now on, let

$k$ = a non-archimedean local field

$\mathbb{G}$ = a connected reductive algebraic group defined over $k$[1]

$\mathbb{P}$ = a minimal $k$-parabolic subgroup of $\mathbb{G}$

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1 Algebraic groups over $k$ will usually be denoted by Special Roman, and the groups of their $k$-rational points by ordinary types, e.g. $\mathbb{G}$ and $G$

$\mathbb{A}$ = a maximal $k$-split torus of $\mathbb{G}$ in $\mathbb{P}$

$\mathbb{M}$ = the centralizer of $\mathbb{A}$

$\mathbb{N}$ = the unipotent radical of $\mathbb{P}$

$\mathbb{N}^-$ = the opposite of $\mathbb{N}$

$\Phi$ = $k$-roots of $\mathbb{G}$ with respect to $\mathbb{A}$

$\Sigma$ = the affine root system of $G$

${}^v\Sigma$ = the reduced root system associated with $\Sigma$

$W$ = the Weyl group of $(\mathbb{G}, \mathbb{A})$.

*2.1.* Let $K$ be a good maximal compact subgroup of $G$ with Iwasawa decomposition $G=KP$. We denote by $C_c(K\backslash G/K)$ the space of compactly supported $K$-biinvariant $\mathbb{C}$-valued functions on $G$.[2] We shall fix $dg$ the Haar measure of $G$ with $\mathrm{vol}(K)=1$. Then $C_c(K\backslash G/K)$ is an algebra under convolution products, which we call the Hecke algebra of $G$ with respect to $K$ to be denoted $H(G,K)$. It is known that $H(G,K)$ is semisimple and commutative.

Put $X_{nr}(M)=\{\lambda\in\mathrm{Hom}(M,\mathbb{C}^\times)|\lambda \text{ is trivial on } M\cap K\}$ (the group of unramified characters on $M$). We may choose representatives of $W$ in $K$. The group $W$ stabilizes $M\cap K$ and hence acts on $X_{nr}(M)$ canonically. For $\lambda\in X_{nr}(M)$, we denote by $E_\lambda$ the space of the unramified principal series representation associated with $\lambda$:

$$E_\lambda=\{f:G\to\mathbb{C}|\ \text{(i) } f \text{ is locally constant;}\ \text{(ii) } f(gmn)=(\lambda\delta^{1/2})(m)f(g) \text{ for } g\in G,\ m\in M \text{ and } n\in N\}.$$

Here $\delta$ is the modulus character of $P$. The space $E_\lambda$ is an admissible $G$-module under left translations by $G$. Let $1_\lambda$ be the $K$-fixed vector of $E_\lambda$ with $1_\lambda(e)=1$. Then there exists an algebra homomorphism $\omega_\lambda: H(G,K)\to\mathbb{C}$ such that $\phi * 1_\lambda=\omega_\lambda(\phi)1_\lambda$ for all $\phi\in H(G,K)$, where $*$ denotes the convolution product. The following fact is well known:

(i) $\omega_\lambda=\omega_{\lambda'}$ iff $\lambda'=w\cdot\lambda$ for some $w\in W$.

(ii) $\omega_\lambda^\sim=\omega_{\lambda^{-1}}$ where $\omega_\lambda^\sim$ is the contragredient of $\omega_\lambda$.

(iii) Every algebra homomorphism of $H(G,K)$ into $\mathbb{C}$ is equal to $\omega_\lambda$ for some $\lambda\in X_{nr}(M)$.

See [4, 5] or [6] for details.

We may view $C(G/K)$ and $C_c(G/K)$ as $H(G,K)$-modules under right convolutions. Let $\mathbb{C}_\lambda$ be the 1-dimensional $H(G,K)$-module corresponding to $\omega_\lambda(\lambda\in X_{nr}(M))$.

*Definition 2.2.* (cf. [1; (2.3)].)

$$I_{K,G}(\omega_\lambda):=C_c(G/K)\otimes_{H(G,K)}\mathbb{C}_\lambda.$$

$$P_{K,G}(\omega_\lambda):=\{f\in C(G/K)|f*\phi=\omega_\lambda(\phi)f\ (\phi\in H(G,K))\}.$$

2 For a totally disconnected space $X$, $C(X)$ and $C_c(X)$ denote the space of $\mathbb{C}$-valued functions on $X$, and the space of compactly supported ones, respectively

The space $P_{K,G}(\omega_\lambda)$ is the eigenspace of $H(G,K)$ under $\omega_\lambda$. Both of $I_{K,G}(\omega_\lambda)$ and $P_{K,G}(\omega_\lambda)$ are $G$-modules via left translations by $G$. In particular $I_{K,G}(\omega_\lambda)$ is smooth. We can see easily that $P_{K,G}(\omega_\lambda)$ is naturally isomorphic to $I_{K,G}(\omega_{\lambda^{-1}})'$, the dual space of $I_{K,G}(\omega_{\lambda^{-1}})$, and

$$I_{K,G}(\omega_\lambda) \simeq C_c(G/K)/\langle f*\phi - \omega_\lambda(\phi)f \quad (f \in C_c(G/K); \phi \in H(G,K))\rangle .$$

See [1; Sect. 2] for other properties of $I_{K,G}(\omega_\lambda)$ and $P_{K,G}(\omega_\lambda)$.

*2.3.* Now we shall investigate $P_{K,G}(\omega_\lambda)$ and $I_{K,G}(\omega_\lambda)$ together. Let $B$ be an Iwahori subgroup of $G$ contained in $K$ with an Iwahori factorization $B = N_0^-(M \cap K)N_0$ with respect to $P$. Here $N_0 \subset N$ and $N_0^- \subset N$. Define a linear map $\alpha: C_c(B\backslash G/K) \to C_c(M/M\cap K)$ by

$$\alpha(f)(m) = \delta^{-1/2}(m)\int_N f(mn)dn \qquad (m \in M;\ f \in C_c(B\backslash G/K))$$

where $dn$ denotes the Haar measure of $N$ with $\int_{N\cap K} dn = 1$. Set $T := M/M\cap K$. Then $T$ is a free $\mathbb{Z}$-module of rank equal to $rk_k G$. Note that $C_c(M/M\cap K)$ is isomorphic to $\mathbb{C}[T]$ in obvious manner. Hence the map $\alpha$ induces $\beta: C_c(B\backslash G/K) \to \mathbb{C}[T]$ by $\beta(f) = \Sigma_{t\in T}\alpha(f)(\dot{t})t$ for $f \in C_c(B\backslash G/K)$. Here $\dot{t}$ denotes a representative element of $t$ in $M$. We call *$\beta$ the Satake map*. Set $\beta_0 := \beta|_{H(G,K)}$. It is well known that the map $\beta_0$ is what is called the Satake isomorphism which satisfies $\beta_0: H(G,K) \tilde{\to} \mathbb{C}[T]^W$ (algebra isomorphism) and $\omega_\lambda(\phi) = \lambda^{-1}(\beta_0(\phi))(\lambda \in X_{nr}(M) = X(T);\ \phi \in H(G,K))$. But as in the case of $\beta_0$, we can check similar results for $\beta$.

**Lemma 2.4.** *Let $\lambda \in X_{nr}(M)$, $f \in C_c(B\backslash G/K)$ and $\phi \in H(G,K)$. Then we have*

(i) $f*1_\lambda(e) = \lambda^{-1}(\beta(f))$;
(ii) $\beta(f*\phi) = \beta(f)\beta(\phi)$.

*Proof.* It can be checked straightforwardly thanks to Iwasawa decomposition $G = KMN$ and the expression $dg = \delta^{-1}(m)dkdmdn$ where $dk$ (resp. $dm$) denotes the Haar measure of $K$ (resp. $M$) with $\int_K dk = 1$ (resp. $\int_{M\cap K} dm = 1$). cf. [4; (3.3)].

**Proposition 2.5.** *The map $\beta: C_c(B\backslash G/K) \to \mathbb{C}[T]$ is an isomorphism as a linear map.*

*Proof.* For $t \in T$, set $e_t$ the characteristic function of the double coset $B\dot{t}K$. Owing to the decomposition $G = \bigcup_{t\in T} B\dot{t}K$ (disjoint union), it suffices to show that $\{\beta(e_t) | t \in T\}$ is a basis of $\mathbb{C}[T]$. By (2.4),

$$\begin{aligned}\lambda^{-1}(\beta(e_t)) &= e_t * 1_\lambda(e)\\ &= \int_{K\dot{t}^{-1}B} 1_\lambda(g)dg\\ &= \sum_{t'\in T}(\lambda\delta^{1/2})(\dot{t}')\operatorname{vol}(K\dot{t}^{-1}B\cap K\dot{t}'N)\\ &= \sum_{t'\in T}(\lambda^{-1}\delta^{-1/2})(\dot{t}')\operatorname{vol}(K\dot{t}^{-1}B\cap K\dot{t}'^{-1}N),\end{aligned}$$

which implies

$$\beta(e_t) = \sum_{t'\in T}\delta^{-1/2}(\dot{t}')\operatorname{vol}(K\dot{t}^{-1}B\cap K\dot{t}'^{-1}N)t' .$$

As in [5; (3.2.1)], let $T^{++}$ and $T^{+}$ be the subsemigroup of $T$ associated with 'dominant coweights' and 'positive coroots' of ${}^v\Sigma$, respectively. Here the positive roots of ${}^v\Sigma$ or $\Phi$ are determined by the choice of $\mathbb{P}$. We denote $t \leqq t'$ if $t^{-1}t' \in T^+$. The relation $\leqq$ is a partial order on $T$. Let $t_d$ be the unique element of $T^{++}$, $W$-conjugate to $t$ and $f_t$ the characteristic function of $K\dot{t}K = K\dot{t}_dK$. As $K\dot{t}^{-1}B \subset K\dot{t}^{-1}K$, we have $\operatorname{supp}\beta(e_t) \subset \operatorname{supp}\beta(f_t)$ since

$$\beta(f_t) = \sum_{t' \in T} \delta^{-1/2}(t') \operatorname{vol}(K\dot{t}_d^{-1}K \cap K\dot{t}'^{-1}N)t'.$$

Note that $\beta(f_t) \in \mathbb{C}[T]^W$. Thus [2; (4.4.4)(i)] shows that $w(t') \leqq t_d$ for all $w \in W$, provided $K\dot{t}^{-1}B \cap K\dot{t}'^{-1}N \neq \emptyset$. On the other hand, we have

**Lemma 2.6.** *If* $K\dot{t}^{-1}B \cap K\dot{t}'^{-1}N \neq \emptyset$, *then* $t \leqq t'$.

Assume this for a moment. Then we have finally

$$\operatorname{supp}\beta(e_t) \subset \{w(t) \mid w \in W, t \leqq w(t) \leqq t_d\} \cup \{t' \mid t < t', w(t') < t_d \quad \text{for all} \quad w \in W\}.$$

Hence our assertion clearly follows since $\operatorname{supp}\beta(e_t)$ obviously contains $t$.

*Proof of Lemma 2.6.* Note that $K\dot{t}^{-1}B = K\dot{t}^{-1}N_0^- N_0$. Therefore

$$K\dot{t}^{-1}B \cap K\dot{t}'^{-1}N \neq \emptyset \Leftrightarrow \dot{t}^{-1}N_0^-\dot{t} \cap K\dot{t}'^{-1}N \neq \emptyset.$$

But the calculation of c-functions (see e.g. [5; (5.5.9)]) shows that the case of the right hand side occurs only if $t't^{-1} \in (T^+)^{-1}$.

Now we can prove the following which is analogous to [1; (4.4)].

**Theorem 2.7.** *The $G$-modules $I_{K,G}(\omega_\lambda)$ and $P_{K,G}(\omega_\lambda)^\infty$ (the subspace of the smooth vectors in $P_{K,G}(\omega_\lambda)$) are admissible.*

*Proof.* As $I_{K,G}(\omega_\lambda)^B = C_c(B\backslash G/K) \otimes_{H(G,K)} \mathbb{C}_\lambda$ by definition, the Satake map $\beta$ induces an isomorphism

$$I_{K,G}(\omega_\lambda)^B \xrightarrow{\sim} \mathbb{C}[T] / \langle f\phi - \lambda^{-1}(\phi)f \ (f \in \mathbb{C}[T]; \phi \in \mathbb{C}[T]^W) \rangle$$

by (2.4) and (2.5). Taking the dual of the both sides, we have

$$P_{K,G}(\omega_{\lambda^{-1}})^B \xleftarrow{\sim} \{f \in C(T) \mid T_\phi f = \lambda(\phi) f \ (\phi \in \mathbb{C}[T]^W)\}.$$

Therefore (1.1) shows that $\dim P_{K,G}(\omega_{\lambda^{-1}})^B = \dim I_{K,G}(\omega_\lambda)^B < \infty$ and this implies that $I_{K,G}(\omega_\lambda)$ is admissible since $I_{K,G}(\omega_\lambda)$ is generated by $I_{K,G}(\omega_\lambda)^B$ (see [5;(5.5.6)]). Recall that $P_{K,G}(\omega_{\lambda^{-1}})^\infty$ is the contragredient $G$-module of $I_{K,G}(\omega_\lambda)$. Hence this module is also admissible.

**Corollary 2.8.** *For every* $\lambda \in X_{nr}(M)$,

$$\begin{aligned} \dim I_{K,G}(\omega_\lambda)^B &= \dim P_{K,G}(\omega_\lambda)^B \\ &= |W/W_\lambda| \dim H_1(W_\lambda) \\ &\geqq |W|, \end{aligned}$$

*and the equality holds iff $W_\lambda$ is a reflection group in* $\mathrm{GL}(T \otimes \mathbb{R})$.

## 3. The Poisson Integrals

*3.1.* Define a $G$-homomorphism $\mathscr{F}_\lambda : C_c(G/K) \to E_{\lambda^{-1}}$ by $\mathscr{F}_\lambda(f) = f * 1_{\lambda^{-1}}$ for $f \in C_c(G/K)$. Then we can see easily that

$$\operatorname{Ker}\mathscr{F}_\lambda \supset \{f*\phi - \omega_{\lambda^{-1}}(\phi)f \ (f \in C_cG/K);\ \phi \in H(G,K))\}.$$

Hence $\mathscr{F}_\lambda$ induces a $G$-homomorphism $\mathscr{R}_\lambda : I_{K,G}(\omega_{\lambda^{-1}}) \to E_{\lambda^{-1}}$. If we take the dual of $\mathscr{R}_\lambda$, we obtain a $G$-homomorphism $\mathscr{P}_\lambda : E'_{\lambda^{-1}} \to P_{K,G}(\omega_\lambda)$, which we call *the Poisson integral* (of $E'_{\lambda^{-1}}$). Note that $(E'_{\lambda^{-1}})^\infty = E_\lambda$. It is easily seen that the restriction of $\mathscr{P}_\lambda$ to $E_\lambda$, $\mathscr{P}^\infty_\lambda : E_\lambda \to P_{K,G}(\omega_\lambda)^\infty$ is given by

$$(\mathscr{P}^\infty_\lambda f)(x) = \int_K f(xk)dk \qquad (f \in E_\lambda),$$

and this is the reason we call $\mathscr{P}_\lambda$ the Poisson 'integral'.

Clearly the bijectivity of $\mathscr{P}_\lambda$ is equivalent to the bijectivity of $\mathscr{R}_\lambda$ ($\Leftrightarrow$ the bijectivity of $\mathscr{P}^\infty_\lambda$). But by the definition of $\mathscr{R}_\lambda$, $\mathscr{R}_\lambda$ is surjective iff $1_{\lambda^{-1}} \in E_{\lambda^{-1}}$ is a cyclic vector. Assume $1_{\lambda^{-1}}$ is cyclic. Then $\mathscr{R}_\lambda$ is injective iff $\dim E^B_{\lambda^{-1}} = \dim I_{K,G}(\omega_{\lambda^{-1}})^B$ since $\operatorname{Ker}\mathscr{R}_\lambda \neq 0 \Leftrightarrow \operatorname{Ker}\mathscr{R}^B_\lambda \neq 0$ by [1, 5]. Thus the equality $\dim E^B_{\lambda^{-1}} = |W|$ and (2.8) show

**Theorem 3.2.** *The Poisson integral* $\mathscr{P}_\lambda$ *is bijective if and only if the following conditions* (i) *and* (ii) *are satisfied:*

(i) $1_{\lambda^{-1}}$ *is a cyclic vector of* $E_{\lambda^{-1}}$.

(ii) $W_\lambda$ *is a reflection group in* $\mathrm{GL}(T \otimes \mathbb{R})$.

*Remark 3.3.* It is obvious that our theorem has a close resemblance to [3; 5. Theorem]. There, the cyclicity of $1_{\lambda^{-1}}$ is expressed in terms of the **c**-function and $W_\lambda$ is always a reflection group. In our case, for the condition (i), we can see easily that $1_{\lambda^{-1}}$ *is cyclic iff* $\mathbf{c}(\lambda^{-1}) \neq 0$ *when* $\lambda \in X_{nr}(M)$ *is regular* (i.e. $W_\lambda = \{e\}$). Here **c** is the **c**-function. But the general criterion for the cyclicity of $1_{\lambda^{-1}}$ seems unknown. As for the condition (ii), it can be proved that *the condition* (i) *implies the condition* (ii) *for all* $\lambda \in X_{nr}(M)$ *under the assumption* $q_\alpha = q'_\alpha$ *for all* $\alpha \in {}^v\Sigma$ (see [5] for the notation). In particular, the condition (ii) is superfluous (at least) for split groups.

## Addendum

As for (3.3), we can prove the following:

**Proposition A.1.** *If* $1_\lambda$ *is a cyclic vector of* $E_\lambda$, *then* $W_\lambda$ $(= W_{\lambda^{-1}})$ *is a reflection subgroup of* $\mathrm{GL}(T \otimes \mathbb{R})$.

Here we shall give a proof of (A.1) by using results of [5].

Let $H(G,B)$ be the Hecke algebra of $G$ with respect to $B$. Let

$$A(w,\lambda) \in \operatorname{Hom}_{H(G,B)}(E^B_\lambda, E^B_{w\cdot\lambda}) \quad (w \in W)$$

be the intertwining operator defined in [5; (4.3.4)]. For $\alpha \in {}^v\Sigma$, we denote by $\mathbf{d}_\alpha$ (resp. $\mathbf{e}_\alpha$) the denominator (resp. numerator) of the **c**-function $\mathbf{c}_\alpha$[5; (4.3.1)]; i.e. $\mathbf{e}_\alpha$ and $\mathbf{d}_\alpha$ are relatively prime, and $\mathbf{c}_\alpha(\lambda) = \mathbf{e}_\alpha(\lambda)/\mathbf{d}_\alpha(\lambda)$ for $\lambda \in X_{nr}(M)$. Then, for $w \in W$, $A^0(w,\lambda) := (\Pi_{\alpha>0, w(\alpha)<0}\mathbf{d}_\alpha(\lambda))A(w,\lambda)$ gives a normalization of $A(w,\lambda)$, hence $\lambda \mapsto A^0(w,\lambda)$ is a (matrix-valued) everywhere-defined rational function on $X_{nr}(M)$. Put $W_{(\lambda)} = \langle w_\alpha | \mathbf{d}_\alpha(\lambda) = 0\rangle$. Here $w_\alpha$ denotes the reflection of $W$ associated with $\alpha \in {}^v\Sigma$. Therefore $W_{(\lambda)}$ is a normal subgroup of $W_\lambda$ generated by reflections. Let $W^\lambda$ be the representative set of $W_\lambda/W_{(\lambda)}$ such that $l(w) \leqq l(w')$ $(w \in W^\lambda;\, w' \in wW_{(\lambda)})$. Then (A.1) is a consequence of

**Proposition A.2.** *The elements of*

$$\{A^0(w,\lambda) | w \in W^\lambda\} \subset \mathrm{End}_{H(G,B)} E^B_\lambda$$

*are linearly independent.*

In fact, the cyclicity of $1_\lambda$ implies that $\dim \mathrm{End}_{H(G,B)} E^B_\lambda = 1$. So we have $W_\lambda = W_{(\lambda)}$ by (A.2).

But in view of the following claim (its proof is easy and omitted)

$$\Pi_{\alpha>0,\, w(\alpha)<0} \mathbf{d}_\alpha(\lambda) \neq 0 \qquad for \qquad w \in W^\lambda, \tag{A.3}$$

the linearly independence of $\{A(w,\lambda) | w \in W^\lambda\}$ is readily seen from

$$(A^0(w,\lambda) f_B)(\dot{w}') = \begin{cases} \Pi_{\alpha>0,\, w(\alpha)<0} \mathbf{d}_\alpha(\lambda) & (w^{-1} = w') \\ 0 & (l(w') \geqq l(w);\, w^{-1} \neq \dot{w}') \end{cases} \tag{A.4}$$

*where $f_B$ is the unique element of $E^B$ such that*

$$f_B(k) = \begin{cases} 1 & (k \in B) \\ 0 & (k \in K \backslash B), \end{cases}$$

*and $\dot{w}'$ is a representative element of $w' \in W$ in $K$.*
For the proof of (A.4), see [5; (4.3.4) (ii)].

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**Note added in proof.** We can now give the necessary and sufficient conditions for the cyclicity of $1_\lambda$ in $E_\lambda$. To be more precise, we can prove:

$$1_\lambda \text{ is cyclic} \Leftrightarrow \text{(i) } \mathbf{d}_\alpha(\lambda) \neq 0 \quad \text{for} \quad \alpha \in {}^v\Sigma^+; \quad \text{and}$$
$$\text{(ii) } W_\lambda = W_{(\lambda)}.$$

Details will appear elsewhere.