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The existence of elliptic fibre space structures on Calabi–Yau threefolds

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Introduction

If X is a Calabi–Yau threefold with an elliptic fibre space structure $\phi : X \to S$ over a surface S, there exists on X a nef integral divisor D on X with $D^3 = 0$, $D^2 \not\equiv 0$ and $D \cdot c_2 \geqq 0$. As a partial converse, given a divisor on a Calabi–Yau threefold X with $D^3 = 0$, $D^2 \not\equiv 0$ and $D \cdot c_2 > 0$, there exists an elliptic fibre space structure on X determined by D ([17], (3.2)′).

This paper considers the case when a Calabi–Yau threefold X contains a nef integral divisor D with $D^2 \not\equiv 0$ and $D \cdot c_2 = 0$. Assuming X is not the étale quotient of a torus, we know that D is not ample. If however $D^3 > 0$, then ϕ_{nD} for suitable large $n > 0$ defines a birational morphism $\phi : X \to \bar{X}$ to a Calabi–Yau model $\bar{X}$ with $c_2(\bar{X}) = 0$, and this in turn implies that $\bar{X}$ is the quotient of a torus by a finite group acting freely in codimension 2 [16]. We therefore investigate in this paper the case $D^3 = 0$; by Riemann–Roch we then have that $\chi(\mathcal{O}_X(nD)) = 0$ for all n, and so new ideas are needed to prove effectivity of the divisor nD. The main theorem of this paper proves this effectivity except when the Euler characteristic of X takes a specified value.

Theorem. *Suppose that X is a Calabi–Yau threefold and D a rational nef divisor on X with $D^3 = 0$, $D \cdot c_2 = 0$ and $D^2 \not\equiv 0$. Let $E_1, \dots, E_r$ denote the (necessarily finitely many) surfaces E on X with $D|_E \equiv 0$. Except possibly for the case when the Euler characteristic $e(X) = 2r$ and each E_i is a rational surface, some positive multiple of D will determine an elliptic fibre space structure on X.*

If we can show that some positive multiple of D is effective, then the argument from (3.2)′ of [17] shows that $\phi_{nD} : X \to S$ is an elliptic fibre space over a surface S for an appropriate choice on n (cf. also [14, 13]). Moreover, the condition that $D \cdot c_2 = 0$ says that this fibre space is very special; such special fibre spaces have been studied by Oguiso [14, 15]. In particular, it follows that X is a smooth model of a finite quotient of either an abelian threefold or a product $E \times Y$ for E an elliptic curve and Y a K3 surface. This may be compared with the result proved in [16] for the case when $D \cdot c_2 = 0$ for a nef class D with $D^3 > 0$. As in the case from [16], we should in principle be able to classify the quotients which arise in this way, and Oguiso has already made a start here [15]. The author is informed by Mark Gross that for an elliptic Calabi–Yau threefold X of the above type, a direct calculation on a suitable minimal model shows that $e(X) \geqq 0$; this would also follow from the conjectured formula of Vafa for the Euler number of an orbifold resolution with trivial canonical bundle [1], known to be true when the group is abelian. As a Corollary of our Theorem, we deduce therefore that no Calabi–Yau threefold with $e(X) < 0$ can contain a rational nef divisor D with $D \cdot c_2 = 0$ and $D^2 \not\equiv 0$.

The proof of the Theorem proceeds by three stages. First we show that by flopping curves C with $D \cdot C = 0$, we may assume that the surfaces E_i are simultaneously contractable by a birational morphism $\phi : X \to \bar{X}$. Letting $\tilde{\Omega}^q_{\bar{X}}$ denote the sheaf on $\bar{X}$ of q-forms regular in codimension 1, we show that, unless all the E_i are rational and $e(X) = 2r$, there are necessarily global sections of $H^0(\tilde{\Omega}^1_{\bar{X}}(nD))$ or $H^0(\tilde{\Omega}^2_{\bar{X}}(nD))$ for n sufficiently large. Stability arguments are then employed in Sect. 3 to show that in these cases, some multiple of D must be effective.

1 Contraction of the surfaces E_i

We wish now to contract the surfaces E_i simultaneously to curves or points, but in order to do this we may have to flop to a different model. Since $D^2 \cdot H > 0$ for H a general hyperplane of X, it follows that $D \cdot C = 0$ for only finitely many curves C on H, and hence there are only finitely many surfaces E with $D|_E \equiv 0$. The notation is the same as in the statement of the main Theorem.

Proposition 1.1. (a) *If X' is a birationally equivalent Calabi–Yau threefold and D' the divisor on X' corresponding to D on X, then some positive multiple of D is effective on X if and only if the same statement holds for D' on X'.*
(b) *Changing X birationally by means of a finite number of flops in curves C with $D \cdot C = 0$, we may reduce the Theorem down to the case when there exists a birational morphism $g : X \to \bar{X}$ to a* **Q***-factorial Calabi–Yau model $\bar{X}$ with g contracting all the surfaces E_i down to points or curves. We may also assume that D determines a Cartier divisor on $\bar{X}$.*
(c) *If $g : X \to \bar{X}$ is a birational morphism to a Calabi–Yau model $\bar{X}$ on which D is Cartier and which contracts all the surfaces E_i on X for which $D|_{E_i} \equiv 0$, then there exists a very ample linear system $|L|$ on $\bar{X}$ such that the restriction $D|_L$ of the Cartier divisor D to the general element of the linear system is ample.*

Proof. (a) is obvious once we have observed that X and X' are isomorphic in codimension one, and in fact must be related by means of a finite sequence of flops [6].

(b) For any surface E with $D|_E \equiv 0$, the Hodge Index Theorem implies that

$$(D^2 \cdot H)(E^2 \cdot H) \leqq 0$$

with equality if and only if $E|_H \equiv 0$, i.e. $E = 0$. Thus $E_i^2 \cdot H < 0$ for all i.

We now invoke the theory of directed flops as developed in [6], and elucidated further in [8, 9]; the procedure following should be viewed in the context of the general Log Minimal Model Programme as described in [5, 7]. Fixing H, the surface E gives rise to a contraction on X (determined by $E + \lambda H$ for some $\lambda \in \mathbf{Q}$; cf. [17, 18]). If this morphism does not contract all of E, it is a small contraction of some curves $Z \subset E$ with $E \cdot Z < 0$. We therefore make the corresponding E-flop on X. Continuing in this way, we will eventually terminate and reach a stage where the corresponding surface E' on X' can be contracted. We never reach the stage of E' being nef on X' since the divisor D on X' (corresponding to D on X) continues to have the numerical property that $D|_{E'} \equiv 0$, and hence as argued above that $(E')^2 \cdot H' < 0$ for H' ample on X'. We therefore contract E', obtaining a Calabi–Yau model X_1 with only $\mathbf{Q}$-factorial canonical singularities. We continue this process with the other E_i, eventually finding a model X_r with all the surfaces $E_1, \ldots, E_r$ contracted to smaller dimensions (we have of course made a number of choices in achieving this).

We now set $\bar{X} = X_r$ and let X' be any crepant resolution of the singularities of $\bar{X}$. By the theory of [6], X' will be related to our original X by a finite sequence of flops. Since the singularities and Betti numbers are unchanged under flops [8], we deduce that X' is again a smooth Calabi–Yau threefold with $e(X') = e(X)$; moreover the birational properties of the E_i are unchanged. Furthermore, since we have only flopped curves C for which $D \cdot C = 0$, the relevant numerical properties of D are unchanged, i.e. D represents a rational nef divisor on X' with $D^3 = 0$, $D \cdot c_2 = 0$, $D^2 \not\equiv 0$ and $D|_{E_i'} \equiv 0$ for the corresponding surfaces E_i' on X'. In the light of (a) therefore, for the purposes of proving the main Theorem, we may replace X by X'; i.e. we now assume $X = X'$ and that there exists a simultaneous contraction $g : X \to \bar{X}$ of the E_i. Replacing D by a multiple if necessary, we may consider D (by abuse of notation) as a Cartier divisor on $\bar{X}$.

(c) Let $|L|$ denote a very ample linear system on $\bar{X}$ and consider the countable number of families of curves C on $\bar{X}$ with $D \cdot C = 0$ (parametrized by various Hilbert schemes). Suppose first there is a covering family; the parameter space for this family will have to be of dimension two and the family unique, since otherwise it is possible to find a big divisor M on X with $D^2 \cdot M = 0$. By taking a multiple of L if necessary and choosing L general, we can assume that L contains no curves from this family.

The other families either cover surfaces in $\bar{X}$ or are isolated curves. If a surface $F \subset \bar{X}$ only contains a 1-dimensional family of curves C with $D \cdot C = 0$, then by choosing L general, we may assume that L contains no curve of the family. If however F contains a family of dimension > 1, then we can consider the corresponding surface E on X. Taking a resolution of singularities $f : Y \to X$ of the embedded surface E, we obtain a corresponding family of curves on the proper transform E' of E with $D \cdot C = 0$ for all curves C in the family (where D is also used to denote f^*D on Y). Since this family has dimension > 1, we can take Δ a sum of such curves on E', a divisor which is nef with $\Delta^2 > 0$. Since however $D \cdot \Delta = 0$, and D is nef, the Hodge Index Theorem implies that $D^2 \cdot E' = 0$ and $D|_{E'} \equiv 0$. Thus $D|_E \equiv 0$ and so E is one of the surfaces contracted by g, a contradiction. It follows therefore that by taking L general in its linear system (i.e. in the complement of countably many proper subvarieties of $|L|$), we may assume that L contains no curves C with $D \cdot C = 0$, and hence that $D|_L$ is ample.

2 Calculations for $H^0(\tilde{\Omega}^q_{\bar{X}}(nD))$

In this section, X will denote a Calabi–Yau threefold containing a divisor D and surfaces $E_1, \ldots, E_r$ with the properties as given in the statement of our main Theorem, and $g : X \to \bar{X}$ a morphism to a Calabi–Yau model $\bar{X}$ which contracts all the E_i and for which D is $\mathbf{Q}$-Cartier on $\bar{X}$; we shall furthermore assume without loss of generality that D is in fact Cartier on $\bar{X}$. By (1.1)(iii), we can choose a very ample linear system $|L|$ on $\bar{X}$ for which the restriction $D|_L$ is ample for L general in its linear system. Mainly we shall be interested in the case when $\bar{X}$ is $\mathbf{Q}$-factorial, but at one stage in the proof of (2.3)(iii) we shall need the slightly more general case.

We shall resolve the exceptional locus of $g : X \to \bar{X}$ into a divisor with smooth normal crossings $\mathscr{E}$ on a smooth threefold $\tilde{X}$. Thus we have $f : \tilde{X} \to \bar{X}$ with $\mathscr{E} = f^{-1}(\operatorname{Sing} \bar{X})$. To fix notation, we suppose that the components of $\mathscr{E}$ are smooth surfaces $F_1, \ldots, F_N$, where F_i is a resolution of E_i for $i \leqq r$. We consider the sheaf $\Omega^1_{\tilde{X}}(\log \mathscr{E})$. Since L is ample on $\bar{X}$, it follows from Corollary 6.7 of [2] that $H^1(\Omega^1_{\tilde{X}}(\log \mathscr{E})(-f^*L)) = 0$. The Leray spectral sequence then implies that $H^1(f_*\Omega^1_{\tilde{X}}(\log \mathscr{E})(-L)) = 0$. We may however assume that L has been chosen such that $H^0(\tilde{\Omega}^1_{\bar{X}}(-L)) = 0$, where as before $\tilde{\Omega}^1_{\bar{X}}$ denotes the reflexive sheaf of 1-forms regular in codimension one. We have an exact sequence of sheaves

$$0 \to f_*\Omega^1_{\tilde{X}}(\log \mathscr{E}) \to \tilde{\Omega}^1_{\bar{X}} \to Q \to 0$$

where the sheaf Q is concentrated in dimension zero (since the 1-dimensional singular locus of $\bar{X}$ consists of quotient singularities). Thus tensoring the above exact sequence by $\mathcal{O}_{\bar{X}}(-L)$ and considering the corresponding long exact sequence on cohomology, we deduce that $Q = 0$, i.e. that $f_*\Omega^1_{\tilde{X}}(\log \mathscr{E}) = \tilde{\Omega}^1_{\bar{X}}$.

Proposition 2.1. *With notation as above,* $H^1(\Omega^1_{\tilde{X}}(\log \mathscr{E})(-nD)) = 0$ *for all* n *sufficiently large.*

Proof. Since L is ample on $\bar{X}$, we saw above that $H^1(\Omega^1_{\tilde{X}}(\log \mathscr{E}(-f^*L)) = 0$. Let S be a general element of $|f^*L|$ and $\bar{S} \in |L|$ the corresponding element of $|L|$. The surface $\bar{S}$ has rational double point singularities, and S is a desingularization. We have an exact sequence from page 13 of [2]

$$0 \to \Omega^1_{\tilde{X}}(\log \mathscr{E}) \to \Omega^1_{\tilde{X}}(\log(\mathscr{E} + S)) \to \mathcal{O}_S \to 0\,.$$

Tensoring by $\mathcal{O}(-nD)$, we deduce using the Kawamata-Viehweg form of Kodaira vanishing that

$$H^1(\Omega^1_{\tilde{X}}(\log \mathscr{E})(-nD)) \cong H^1(\Omega^1_{\tilde{X}}(\log(\mathscr{E} + S))(-nD))\,.$$

From the second exact sequence on page 13 of [2], we have

$$\begin{aligned} 0 \to\ & \Omega^1_{\tilde{X}}(\log(\mathscr{E} + S))(-S - nD) \to \Omega^1_{\tilde{X}}(\log(\mathscr{E} + S))(-nD) \\ \to\ & \Omega^1_S(\log(\mathscr{E}|_S))(-nD) \to 0\,. \end{aligned}$$

We know that $H^1(\Omega^1_{\tilde{X}}(\log(\mathscr{E} + S))(-S - nD)) = 0$ by (6.7) from [2], so the Proposition follows if we show that $h^1(\Omega^1_S(\log(\mathscr{E}|_S))(-nD)) = 0$. Let M be the number of curves C_i in $\mathscr{E}|_S$, i.e. the number of exceptional curves of $S \to \bar{S}$. Thus we have an exact sequence

$$0 \to \Omega^1_S(-nD) \to \Omega^1_S(\log(\mathscr{E}|_S))(-nD) \to \oplus\mathcal{O}_{C_i} \to 0$$

and hence an exact sequence of spaces

$$0 \to \mathbf{C}^M \to H^1(\Omega^1_S(-nD)) \to H^1(\Omega^1_S(\log(\mathscr{E}|_S))(-nD)) \to H^1(\oplus\mathcal{O}_{C_i}) = 0\,.$$

Now D is by assumption ample on $\bar{S}$, and so

$$H^1(\Omega^1_S(-nD)) \cong R^1 f_*\Omega^1_S \cong \mathbf{C}^M,$$

this latter isomorphism holding for any rational surface singularity. The result therefore follows.

Lemma 2.2. *The sheaf $R^1 f_*\Omega^1_{\tilde{X}}(\log \mathscr{E})$ has at most zero dimensional support.*

Proof. On $\tilde{X}$, we have

$$0 \to \Omega^1_{\tilde{X}} \to \Omega^1_{\tilde{X}}(\log \mathscr{E}) \to \oplus\mathcal{O}_{F_i} \to 0$$

where the sum is take over all components F_i of $\mathscr{E}$. Thus

$$\begin{aligned} f_*\Omega^1_{\tilde{X}} \hookrightarrow f_*\Omega^1_{\tilde{X}}(\log \mathscr{E}) \to \oplus f_*\mathcal{O}_{F_i} \to R^1 f_*\Omega^1_{\tilde{X}} \\ \to R^1 f_*\Omega^1_{\tilde{X}}(\log \mathscr{E}) \to \oplus R^1 f_*\mathcal{O}_{F_i}\,. \end{aligned}$$

Restricting to $\bar{S}$, and letting $Z_i = F_i|_S$ (where a non-zero Z_i is either a single $\mathbf{P}^1$, or two disjoint such curves corresponding to a line pair in the corresponding fibre of E_i), we obtain:

$$\mathbf{C}^M \cong \oplus H^0(\mathcal{O}_{Z_i}) \to R^1 f_* \Omega^1_{\tilde{X}}|_{\bar{S}} \to R^1 f_* \Omega^1_{\tilde{X}}(\log \mathcal{E})|_{\bar{S}} \to \oplus H^1(\mathcal{O}_{Z_i}) = 0 .$$

But $H^0(R^1 f_* \Omega^1_{\tilde{X}}|_{\bar{S}}) \cong H^1(\Omega^1_{\tilde{X}}|_S \otimes \mathcal{O}(nL))$ for n sufficiently large. From the exact sequence

$$0 \to \mathcal{O}_S((n-1)L) \to \Omega^1_{\tilde{X}}|_S \otimes \mathcal{O}(nL) \to \Omega^1_S \otimes \mathcal{O}(nL) \to 0 ,$$

the above group is the same as $H^1(\Omega^1_S(nL)) \cong R^1 f_* \Omega^1_S \cong \mathbf{C}^M$. Thus from the previous displayed exact sequence, it follows that $R^1 f_* \Omega^1_{\tilde{X}}(\log \mathcal{E})|_{\bar{S}} = 0$, and hence the Lemma is proved.

Observe now that $\chi(\tilde{\Omega}^1_{\bar{X}}(nD)) = \chi(\tilde{\Omega}^1_{\bar{X}})$ for any $n \in \mathbf{Z}$ (by Riemann-Roch and the Leray spectral sequence), and that $h^3(\tilde{\Omega}^1_{\bar{X}}(nD)) = h^0(\tilde{\Omega}^2_{\bar{X}}(-nD))$ for any $n \in \mathbf{Z}$ (cf. [4], page 131).

Theorem 2.3. *With the notation as above, assume that $h^0(\tilde{\Omega}^q_{\bar{X}}(nD)) = 0$ for infinitely many $n > 0$, for both $q = 1,2$. Then*

(i) $\chi(\tilde{\Omega}^1_{\bar{X}}) = 0$

(ii) $R^1 f_* \Omega^1_{\tilde{X}}(\log \mathcal{E}) = 0$

(iii) $E_1, \ldots, E_r$ *are rational surfaces and* $\chi(\Omega^1_{\tilde{X}}(\log \mathcal{E})) = 0$

(iv) *The Euler characteristic* $e(X) = 2r$.

Proof. (i) We have seen that $H^1(\Omega^1_{\tilde{X}}(\log \mathcal{E})(-nD)) = 0$ for n sufficiently large, and hence by Leray that $H^1(\tilde{\Omega}^1_{\bar{X}}(-nD)) = 0$. Since, by assumption, $h^3(\tilde{\Omega}^1_{\bar{X}}(-nD)) = h^0(\tilde{\Omega}^2_{\bar{X}}(nD)) = 0$ for some large n, it follows that $\chi(\tilde{\Omega}^1_{\bar{X}}) = h^2(\tilde{\Omega}^1_{\bar{X}}(-nD)) \geqq 0$. A standard argument however shows that $H^2(\tilde{\Omega}^1_{\bar{X}}(nD)) = 0$ for n sufficiently large (since $D|_{\bar{S}}$ is ample), and hence $\chi(\tilde{\Omega}^1_{\bar{X}}) = \chi(\tilde{\Omega}^1_{\bar{X}}(nD)) = -h^1(\tilde{\Omega}^1_{\bar{X}}(nD)) \leqq 0$ for some large n. Thus $\chi(\tilde{\Omega}^1_{\bar{X}}) = 0$ as claimed.

(ii) The fact that $h^2(\tilde{\Omega}^1_{\bar{X}}(-nD)) = 0$ and $h^1(\Omega^1_{\tilde{X}}(\log \mathcal{E})(-nD)) = 0$ for some large n, implies by Leray that $h^0(R^1 f_* \Omega^1_{\tilde{X}}(\log \mathcal{E})(-nD)) = 0$. Since the sheaf has support in dimension zero (2.2), this gives $R^1 f_* \Omega^1_{\tilde{X}}(\log \mathcal{E})(-nD) = 0$.

(iii) We observe that the tangent space to the deformation space at $\bar{X}$ is $\mathrm{Ext}^1(\Omega^1_{\bar{X}}, \mathcal{O}_{\bar{X}})$, which by Serre duality is dual to $H^2(\Omega^1_{\bar{X}})$. But $\Omega^1_{\bar{X}}$ is reflexive except at the finitely many dissident points; i.e. the cokernel of $\Omega^1_{\bar{X}} \hookrightarrow \tilde{\Omega}^1_{\bar{X}}$ has support only at these points. Thus $H^2(\Omega^1_{\bar{X}}) \cong H^2(\tilde{\Omega}^1_{\bar{X}})$. Since $R^1 f_* \Omega^1_{\tilde{X}}(\log \mathcal{E}) = 0$, it follows from the Leray spectral sequence that $h^2(\tilde{\Omega}^1_{\bar{X}}) \leqq h^2(\Omega^1_{\tilde{X}}(\log \mathcal{E}))$.

However, from the exact sequence

$$0 \to \Omega^1_{\tilde{X}} \to \Omega^1_{\tilde{X}}(\log \mathcal{E}) \to \oplus \mathcal{O}_{F_i} \to 0$$

we deduce that

$$0 \to \oplus H^0(\mathcal{O}_{F_i}) \to H^1(\Omega^1_{\tilde{X}}) \to H^1(\Omega^1_{\tilde{X}}(\log \mathcal{E}))$$

is exact. Here $H^1(\Omega^1_{\tilde{X}}) \cong H^2(\tilde{X}, \mathbf{C}) \cong \mathrm{Pic}_{\mathbf{C}}(\tilde{X}) = \mathrm{Pic}\,(\tilde{X}) \otimes \mathbf{C}$ and $H^1(\Omega^1_{\tilde{X}}(\log \mathcal{E})) \cong H^2(\tilde{X} \setminus \mathcal{E}, \mathbf{C})$, the latter since $H^0(\Omega^2_{\tilde{X}}(\log \mathcal{E})) = 0 = H^2(\mathcal{O}_{\tilde{X}})$.

Moreover, the natural map $H^2(\tilde{X} \setminus \mathscr{E}, \mathbf{C}) \to H^2(\tilde{X} \setminus \mathscr{E}, \mathcal{O}_{\tilde{X}})$ factors through $H^2(\Omega^0_{\tilde{X}}(\log \mathscr{E})) = H^2(\mathcal{O}_{\tilde{X}}) = 0$, and hence is itself zero (cf. [3, p. 146]). Using the fact that the singularities of $\bar{X}$ are Cohen-Macaulay and rational, it also follows that $H^1(\tilde{X} \setminus \mathscr{E}, \mathcal{O}_{\tilde{X}}) = H^1(\bar{X} \setminus \operatorname{Sing}(\bar{X}), \mathcal{O}_{\bar{X}}) = H^1(\bar{X}, \mathcal{O}_{\bar{X}}) = 0$. This then gives a natural isomorphism $\mathrm{Pic}_{\mathbf{C}}(\tilde{X} \setminus \mathscr{E}) \cong H^2(\tilde{X} \setminus \mathscr{E}, \mathbf{C})$. Since the map $\mathrm{Pic}_{\mathbf{C}}(\tilde{X}) \to \mathrm{Pic}_{\mathbf{C}}(\tilde{X} \setminus \mathscr{E})$ is surjective, the previously displayed exact sequence is also exact on the right. This then yields an exact sequence of vector spaces

$$0 \to \oplus H^1(\mathcal{O}_{F_i}) \to H^2(\Omega^1_{\tilde{X}}) \to H^2(\Omega^1_{\tilde{X}}(\log \mathscr{E})) \to 0\,,$$

exact on the right since $H^2(\mathcal{O}_{F_i}) = 0$ for all i.

As $\tilde{X}$ is obtained from X by blowing up points or smooth curves, we know that $H^2(\Omega^1_{\tilde{X}}) \cong H^2(\Omega^1_X) \oplus \bigoplus_{j>r} H^1(\mathcal{O}_{F_j})$; from this and the above displayed sequence it follows that $h^2(\Omega^1_{\tilde{X}}(\log \mathscr{E})) \leqq h^2(\Omega^1_X)$, with equality if and only if the surfaces $F_1, \ldots, F_r$ are rational (recall that these are desingularizations of $E_1, \ldots, E_r$). Thus $h^2(\tilde{\Omega}^1_{\bar{X}}) \leqq h^2(\Omega^1_X)$, where the two numbers can be interpreted as the dimensions of tangent spaces to the deformation spaces.

However, any small deformation of X will blow down to give a small deformation of $\bar{X}$, and hence induces a map on tangent spaces (to the spaces of deformations) $H^2(\Omega^1_X)^\vee \to H^2(\tilde{\Omega}^1_{\bar{X}})^\vee$. The Kuranishi space of X is smooth (since by the theorem of Bogomolov–Tian–Todorov, the deformations of a Calabi–Yau threefold are unobstructed), and the map of (germs of) deformation spaces $\mathrm{Def}(X) \to \mathrm{Def}(\bar{X})$ has only finite fibres (the crepant resolution is determined canonically up to a finite number of choices and so there cannot be a positive dimensional family of deformations within the fibres of this map). Thus, if X is general in moduli, we will know that the corresponding map on tangent spaces $H^2(\Omega^1_X)^\vee \to H^2(\tilde{\Omega}^1_{\bar{X}})^\vee$ is injective (i.e. we are at a point for which the map $\mathrm{Def}(X) \to \mathrm{Def}(\bar{X})$ is unramified), and hence the dimensions of the two tangent spaces are equal, i.e. $h^2(\tilde{\Omega}^1_{\bar{X}}) = h^2(\Omega^1_X)$. In the case when X is rigid, this equality is trivial as both dimensions are zero.

To deal with the case when X is not necessarily general in moduli, we consider a general 1-parameter small deformation $\pi : \mathscr{X} \to \Delta$ of X. The contraction $g : X \to \bar{X}$ is determined by some nef and big divisor on X, and we may assume that this defines a contraction of the whole family (vanishing theorems ensuring that the cohomology does not jump in the family). We therefore obtain a small deformation of $\bar{X}$, say $\bar{\pi} : \bar{\mathscr{X}} \to \Delta$. Letting X_1 be a general fibre of the family $\pi : \mathscr{X} \to \Delta$, we obtain a map $g_1 : X_1 \to \bar{X}_1$ which satisfies the conditions stated at the beginning of the Section (we are only contracting curves C with $D \cdot C = 0$); here $\bar{X}_1$ is not a priori $\mathbf{Q}$-factorial even if $\bar{X}$ is taken to be a $\mathbf{Q}$-factorial model. By considering the reflexive sheaf $\tilde{\Omega}^1_{\bar{\mathscr{X}}/\Delta}$, which is flat over Δ, and applying standard semicontinuity theorems, we may assume that $h^0(\tilde{\Omega}^q_{\bar{X}_1}(nD)) = 0$ for infinitely many $n > 0$ (for both $q = 1$ and 2). Our previous argument may then be applied to the contraction $g_1 : X_1 \to \bar{X}_1$, and since X_1 has been chosen to be general in moduli, we deduce that $h^2(\tilde{\Omega}^1_{\bar{X}_1}) = h^2(\Omega^1_{X_1})$.

Semicontinuity however also implies that $h^2(\tilde{\Omega}^1_{\tilde{X}}) \geqq h^2(\tilde{\Omega}^1_{\tilde{X}_1})$; since $h^2(\Omega^1_X) = h^2(\Omega^1_{X_1})$ and $h^2(\tilde{\Omega}^1_{\tilde{X}}) \leqq h^2(\Omega^1_X)$, we deduce the equality $h^2(\tilde{\Omega}^1_{\tilde{X}}) = h^2(\Omega^1_X)$ on X; this we saw above could only happen if $F_1, \ldots, F_r$ were all rational and $h^2(\Omega^1_X) = h^2(\tilde{X}, \Omega^1_{\tilde{X}}(\log \mathscr{E})) = h^2(\tilde{\Omega}^1_{\tilde{X}})$. The Leray spectral sequence then implies (in the light of (ii)) that $R^2 f_* \Omega^1_{\tilde{X}}(\log \mathscr{E}) = 0$ and that $\chi(\Omega^1_{\tilde{X}}(\log \mathscr{E})) = \chi(\tilde{\Omega}^1_{\tilde{X}}) = 0$.

(iv) We have seen in the argument of (iii) that $h^1(\Omega^1_{\tilde{X}}(\log \mathscr{E})) = h^1(\Omega^1_{\tilde{X}}) - N = h^1(\Omega^1_X) - r$ and that $h^2(\Omega^1_{\tilde{X}}(\log \mathscr{E})) = h^2(\Omega^1_X)$, given our initial assumptions. Since $h^i(\Omega^1_{\tilde{X}}(\log \mathscr{E})) = 0$ for $i = 0$ or 3, we deduce that $0 = \chi(\Omega^1_{\tilde{X}}(\log \mathscr{E})) = \chi(\Omega^1_X) + r$, and hence $e(X) = 2r$ as claimed.

The main conclusion from this section is therefore (with notation as before) that, unless the surfaces $E_1, \ldots, E_r$ are rational surfaces and $e(X) = 2r$, either $H^0(\tilde{\Omega}^1_{\tilde{X}}(nD))$ or $H^0(\tilde{\Omega}^2_{\tilde{X}}(nD))$ has sections for all n sufficiently large. In the next section, we show by means of stability considerations that this implies that mD is effective for some $m > 0$.

3 Proof of the main theorem

In this section, we follow previous notation, but with $\tilde{X}$ now assumed to be the **Q**-factorial Calabi–Yau model of X constructed in Proposition 1.1. We saw in (2.3) that unless $e(X) = 2r$ and each E_i is rational, either $h^0(\tilde{\Omega}^1_{\tilde{X}}(nD))$ or $h^0(\tilde{\Omega}^2_{\tilde{X}}(nD))$ is positive for all n sufficiently large. We show that in either case mD is effective for some $m > 0$. We assume for simplicity that we are in the former case; the proof for the other case is entirely analogous. We may also assume (replacing D by a multiple) that D is Cartier.

We assume that for some $n_0, H^0(\tilde{\Omega}^1_{\tilde{X}}(nD)) \neq 0$ for $n \geqq n_0$, and suppose first that all the sections obtained (for different n) are dependent when considered as rational forms at the generic point of $\tilde{X}$. Thus we have the inclusion of a rank 1 saturated divisorial sheaf $\mathscr{L} \hookrightarrow \tilde{\Omega}^1_{\tilde{X}}(n_0 D)$ (corresponding to a **Q**-Cartier Weil divisor L) such that $L + rD \geqq 0$ for all $r \geqq 0$. We observe that semistability of the sheaf $\tilde{\Omega}^1_{\tilde{X}}$ (see [11]) implies that $D^2 \cdot L = 0$. Now set $L = M + F$ where F is fixed in $|L + rD|$ for all $r \geqq 0$ and is maximal with respect to this property (i.e. it is the h.c.f. of the fixed parts of the linear systems). Note that $D^2 \cdot M = D^2 \cdot F = 0$, and that $M + rD \geqq 0$ for all $r \geqq 0$. If $M = 0$, we have the required conclusion; therefore we assume $M \neq 0$. Thus for any component M_i of M, there exists an integer $r_i \geqq 0$ such that M_i is not a fixed component of $|M + r_i D|$.

The Hodge Index Theorem (with respect to the hypersurface $mD + H$ for $m \gg 0$) shows that

$$\{M^2 \cdot (mD + H)\}\{D^2 \cdot H\} \leqq \{M \cdot D \cdot (mD + H)\}^2$$

where now the righthand side is bounded, and therefore $D \cdot M^2 \leqq 0$. Given a component M_i of M, we know that $|M+r_iD|$ does not have M_i as a fixed component, and hence $D \cdot (M+r_iD) \cdot M_i \geqq 0$. But $D^2 \cdot M_i = 0$ and so $D \cdot M \cdot M_i \geqq 0$ for all i. Therefore $D \cdot M^2 = 0 = D \cdot M \cdot M_i$ for all i.

Suppose now that C is a curve with $M \cdot C < 0$; then $C \subset M_i$ for some component M_i of M. Since M_i is not a fixed component of $|M+r_iD|$, we know however that $(M+r_iD) \cdot C \geqq 0$ unless C is in the intersection $(M+r_iD) \cap M_i$, where divisors here are chosen to be general in their linear systems. In this way, we produce an integer $r > 0$ with the property that $M + rD$ is negative on only finitely many curves C, and these all have $D \cdot C = 0$.

We now make flops directed by the effective divisor $M + rD$ on $\bar{X}$ (cf. [6, 8, 9]); as shown above, we may always assume that only curves C with $D \cdot C = 0$ are flopped. Eventually, we reach a model on which $M + rD$ is also nef (we cannot reach a model for which some component M_j' is contracted, since then the corresponding surface M_j on $\bar{X}$ is necessarily fixed in $|M + rD|$ for $r \geqq 0$).

With $g : X' \to \bar{X}'$ a resolution of singularities for the flopped model $\bar{X}'$, we have divisors M' and D' on $\bar{X}'$ with $g^*(M' + rD')$ nef and effective on X' for $r \gg 0$. Thus some multiple of it will move without fixed points by [14]. The Calabi–Yau threefold X' can be obtained from X by means of flops in curves C with $D \cdot C = 0$. The fact that $D^2 \cdot (M + rD) = 0$ on $\bar{X}$ implies that $(D')^2 \cdot g^*(M' + rD') = 0$ on X' and hence that the morphism ϕ corresponding to an appropriate multiple of $g^*(M' + rD')$ cannot be birational. Moreover, if the image is one-dimensional, then $(M' + rD') \cdot (M' + rD') \equiv 0$ for all $r \gg 0$, and hence that $(D')^2 \cdot H' = 0$ for H' ample on X'. Hence $D^2 \cdot H'' = 0$ for some big divisor H'' on X, which is a contradiction.

Thus $\phi : X' \to S$ with $g^*(M' + rD') = \phi^* \Delta_r$ for some Δ_r ample on the surface S. Moreover, S has only quotient singularities, and so is in particular $\mathbf{Q}$-factorial (cf. (0.4) of [12] and (3.1) of [14]). Also $g^*(M' + (r+1)D') \geqq 0$ with

$$(M' + rD') \cdot (M' + rD') \cdot (M' + (r+1)D') = 0 .$$

This implies that $g^*(M' + (r+1)D')$ is of the form $\phi^* \Delta$ for some $\mathbf{Q}$-Cartier divisor Δ on S. Thus $D' = \phi^* \Delta_0$ for some nef $\mathbf{Q}$-Cartier divisor Δ_0 on S with $\Delta_0^2 > 0$ (since $D^2 \not\equiv 0$ implies that $(D')^2 \not\equiv 0$). It follows therefore (from Riemann-Roch on a smooth model of S) that some positive integer multiple of Δ_0 on S is effective, and hence that $mD' \geqq 0$ on X' for some $m > 0$; i.e. $mD \geqq 0$ on X as claimed.

We may reduce consideration therefore to the case when there is no integer n_0 for which the sections of $H^0(\tilde{\Omega}^1_{\bar{X}}(nD))$ are dependent for $n \geqq n_0$, when considered as rational forms at the generic point of $\bar{X}$. Thus we obtain non-zero sections of $H^0(\tilde{\Omega}^2_{\bar{X}}(nD))$ for infinitely many n. If these are not all dependent at the generic point, then we know that there are sections $\sigma \in H^0(\tilde{\Omega}^1_{\bar{X}}(n_1D))$ and $\omega \in H^0(\tilde{\Omega}^2_{\bar{X}}(n_2D))$ with $\sigma \wedge \omega \neq 0$ as a 3-form at the generic point of $\bar{X}$.

Thus

$$\sigma \wedge \omega \in H^0(\tilde{\Omega}^3_{\bar{X}}((n_1 + n_2)D)) \cong H^0(\mathcal{O}_{\bar{X}}((n_1 + n_2)D))$$

is a non-zero section, and hence $(n_1 + n_2)D \geqq 0$.

Finally we deal with the case of an infinite sequence $\omega_i \in H^0(\tilde{\Omega}^2_{\bar{X}}(n_i D))$ of dependent sections; specifically that there exist generically independent $\sigma_1 \in H^0(\tilde{\Omega}^1_{\bar{X}}(m_1 D)), \sigma_2 \in H^0(\tilde{\Omega}^1_{\bar{X}}(m_2 D))$, with $\sigma_1 \wedge \omega_i = \sigma_2 \wedge \omega_i = 0$ for all i. We now proceed in an analogous way to our previous argument (which works also in the case of only an infinite sequence of dependent sections in $H^0(\tilde{\Omega}^1_{\bar{X}}(n_i D))$), obtaining an inclusion of a rank 1 saturated divisorial sheaf $\mathscr{L} \hookrightarrow \tilde{\Omega}^2_{\bar{X}}(n_0 D)$, and a corresponding $\mathbf{Q}$-Cartier Weil divisor $L \geqq 0$ with $L + nD \geqq 0$ for $n = 0, n_1, n_2, \ldots$. Writing $L = M + F$ as before, with F fixed in $|L + n_i D|$ for all $i \geqq 0$ and maximal with respect to this property, we follow the earlier argument. If $M = 0$, we obtain $n_i D \geqq 0$ for $i \geqq 0$. If not, we deduce that for some $r \geqq 0$, the divisor $M + rD$ is negative on only finitely many curves C, and these all have $D \cdot C = 0$. With D assumed Cartier, the previous argument (modified in obvious ways) yields the fact that $mD \geqq 0$ for some $m > 0$.

We have therefore demonstrated the following result.

Proposition 3.1. *If $g : X \to \bar{X}$ as in* (1.1)(b) *and there are non-zero sections of $H^0(\tilde{\Omega}^1_{\bar{X}}(nD))$ for all n sufficiently large, then $mD \geqq 0$ for some $m > 0$. A similar result holds if there are non-zero sections of $H^0(\tilde{\Omega}^2_{\bar{X}}(nD))$ for all n sufficiently large.*

This together with (1.1) and (2.3) provides a proof of our main Theorem as stated in the Introduction.

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References

1. Dixon, L., Harvey, J., Vafa, C., Witten, E.: Strings on orbifolds. Nucl. Phys B **274**, 285–314 (1986)
2. Esnault, H., Viehweg, E.: Lectures on Vanishing Theorems. DMV Seminar, Band 20. Basel Boston Stuttgart: Birkhäuser 1992
3. Flenner, H.: Divisorenklassengruppen quasihomogener Singularitäten. J. Reine Angew. Math. **328**, 128–160 (1981)
4. Hartshorne, R.: Stable reflexive sheaves. Math. Ann. **254**, 121–176 (1980)
5. Kawamata, Y.: The cone of curves of algebraic varieties. Ann. Math. **119**, 603–633 (1984)
6. Kawamata, Y.: Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. Ann. Math. **127**, 93–163 (1988)
7. Kawamata, Y., Matsuda, K., Matsuki, K.: Introduction to the minimal model problem. In: Oda, T. (ed.) Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math. Vol. 10, pp. 283–360. Amsterdam: North Holland 1987

8. Kollár, J.: Flops. Nagoya Math. J. **113**, 14–36 (1989)
9. Kollár, J.: Flips, Flops, Minimal Models, etc. Proc. Conf., Cambridge/MA (USA) 1990, Surv. in Differ. Geom., Suppl. J. Differ. Geom. **1**, 113–199 (1991)
10. Miyaoka, Y.: Deformations of a morphism along a foliation. In: Bloch, S. (ed.) Algebraic Geometry, Bowdoin 1985, Proc. Symp. Pure Math. Vol. 46, pp. 245–268. A.M.S., Providence, 1987
11. Miyaoka, Y.: The Chern classes and Kodaira dimension of a minimal variety. In: Oda, T. (ed.) Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math. Vol. 10, pp. 449–476. Amsterdam: North Holland 1987
12. Nakayama, N.: On Weierstrass models. In: Hijikata, H. et al. (eds.) Algebraic Geometry and Commutative Algebra in Honor of M. Nagata, pp. 405–431. New York London: Academic Press 1987
13. Peternell, T.: Calabi–Yau manifolds and a conjecture of Kobayashi. Math. Z. **207**, 305–318 (1991)
14. Oguiso, K.: On algebraic fiber space structures on a Calabi–Yau 3-fold. Intern. J. Math. **4**, 439–465 (1993)
15. Oguiso, K.: On certain rigid fibered Calabi–Yau threefolds. MPI preprint, Bonn 1993
16. Shepherd-Barron, N.I., Wilson, P.M.H.: Singular threefolds with numerically trivial first and second Chern class. J. Alg. Geom. **3**, 265–281 (1994)
17. Wilson, P.M.H.: Calabi–Yau manifolds with large Picard number. Invent. Math. **98**, 139–155 (1989)
18. Wilson, P.M.H.: The Kähler cone on Calabi–Yau threefolds. Invent. Math. **107**, 561–583 (1992). Erratum: Invent. Math. **114**, 231–233 (1993)