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Branched $\mathbb{CP}^1$-structures on surfaces with prescribed real holonomy

Ser Peow Tan

Department of Mathematics, University of Singapore, 10 Kent Ridge Crescent, Singapore 0511, Singapore

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1 Introduction

Let F_g be a closed oriented surface of genus $g \geqq 2$ with fundamental group π. Let $\rho \in \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))$ be a representation of π into $\mathrm{PSL}(2, \mathbf{R})$. If $e(\rho) \in \mathbb{Z}$ is the Euler class of the representation ρ, we have the Milnor-Wood inequality:

$$|e(\rho)| \leqq |\chi(F_g)| = 2g - 2 .$$

In [8] (see also [5]) Goldman showed that the following are equivalent:
(i) $|e(\rho)| = 2g - 2$
(ii) ρ is an isomorphism of π into a discrete subgroup of $\mathrm{PSL}(2, \mathbf{R})$.
(iii) ρ is the holonomy representation of a (possible orientation-reversing) hyperbolic structure on F_g.

Consider now the class of $\mathbf{CP}^1$-structures instead of hyperbolic structures on F_g. The topological invariant of the representation is now the Stiefel-Whitney class which in the case when the representation is in $\mathrm{PSL}(2, \mathbf{R})$ is just the congruence class of $e(\rho)$ modulo 2. In [2], Gallo et al. gave necessary and sufficient conditions for a representation $\rho \in \mathrm{Hom}\ (\pi, \mathrm{PSL}(2, \mathbf{R}))$ to occur as the holonomy representation of a $\mathbf{CP}^1$-structure:

Theorem [Gallo-Goldman-Porter] $\rho \in \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))$ *occurs as the holonomy representation of a* $\mathbf{CP}^1$*-structure on* F_g *if and only if*
(1) $e(\rho) \equiv 0 \pmod 2$ *and*
(2) $\rho(\pi)$ *is not an elementary subgroup of* $\mathrm{PSL}(2, \mathbf{R})$.

Condition (1) in the theorem above is equivalent to saying that ρ admits a lift to $\mathrm{SL}(2, \mathbf{R})$. Gallo [3] and Kapovich [12] have generalised the theorem to the case when $\rho \in \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{C}))$ is non-elementary and has Stiefel-Whitney class 0.

From the above theorem, we see that if $e(\rho) \equiv 1 \pmod 2$ or equivalently, ρ does not admit a lift to $SL(2, \mathbf{R})$, then ρ cannot occur as the holonomy representation of a $\mathbf{CP}^1$-structure. The main aim of this paper is to show that such representations nonetheless occur as the holonomy representations of branched $\mathbf{CP}^1$-structures with one branch point of degree 2. A branched $\mathbf{CP}^1$-structure is one where the coordinate patches are modelled on $\mathbf{CP}^1$ or branched covers of $\mathbf{CP}^1$ with transition functions in $PSL(2, \mathbf{C})$. The degree of a branch point is defined to be the degree of the branched cover of the coordinate chart at that point. Alternatively, a branched $\mathbf{CP}^1$-structure can be thought of as a cone $\mathbf{CP}^1$-structure where the cone points have cone angles which are multiples of 2π. A branch point of degree n is then a cone point with cone angle $2n\pi$. We have the following theorem:

Theorem 1 *Let F_g be a closed oriented surface of genus g with fundamental group π. Suppose $\rho \in \mathrm{Hom}(\pi, PSL(2, \mathbf{R}))$ and $e(\rho) \equiv 1 \pmod 2$. Then ρ occurs as the holonomy representation of a branched $\mathbf{CP}^1$-structure on F_g with one branch point of degree 2.*

Remark 1 A representation ρ whose image is an elementary subgroup of $PSL(2, \mathbf{R})$ can be deformed to the trivial representation and hence must have Euler class zero (see [8]). The condition $e(\rho) \equiv 1 \pmod 2$ therefore implies that ρ is non-elementary.

Remark 2 Theorem 1, together with the theorem of Gallo, Goldman and Porter shows that every non-elementary representation of π into $PSL(2, \mathbf{R})$ occurs as the holonomy representation of either a regular $\mathbf{CP}^1$-structure or a branched $\mathbf{CP}^1$-structure with one branch point of degree two on F_g, thus answering the question of minimising the number (and degree) of the branch points.

Remark 3 It seems likely that using Gallo's methods [3], Theorem 1 should generalise to cover all representations $\rho \in \mathrm{Hom}(\pi, PSL(2, \mathbf{C}))$ with Stiefel-Whitney class 1.

Another very natural question which arises is whether representations ρ of π into $PSL(2, \mathbf{R})$ with intermediate Euler classes necessarily occur as the holonomy of branched hyperbolic structures on F_g. Regarding the Euler class as a volume form, one sees that the holonomy representation of a branched hyperbolic structure on F_g with n branch points $\{x_1, x_2, \ldots, x_n\}$ of degrees $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ has Euler class $2 - 2g + \sum_{i=1}^{n} (\alpha_i - 1)$. We have the following question:

Question. Let $\rho \in \mathrm{Hom}(\pi, PSL(2, \mathbf{R}))$ such that $e(\rho) = 2 - 2g + k$, $0 < k < 2g - 1$. Does ρ occur as the holonomy representation of a branched hyperbolic structure with k branch points of degree 2 each.

When $k = 1$, Goldman and Neumann have shown that the answer is yes (unpublished). In general, however, the answer is no, in Sect. 7 we give an example of a representation ρ which cannot occur as the holonomy of a branched hyperbolic structure. If we allow grafts of $\mathbf{CP}^1$-tori (which are generalisations of 'bending', see [6, 11, 13] for example) or equivalently, fold singularities along curves, we do get a partially affirmative answer as follows:

Associated to a branched $\mathbf{CP}^1$-structure on F_g with real holonomy and a fixed developing map is the decomposition of F_g into the three following types:

$$\begin{aligned} F_g^+ &= \mathrm{dev}^{-1}(\mathbf{H}^+)\,, \\ F_g^R &= \mathrm{dev}^{-1}(\mathbf{R}\cup\infty)\,, \\ F_g^- &= \mathrm{dev}^{-1}(\mathbf{H}^-)\,, \end{aligned}$$

where $\mathbf{H}^+$ and $\mathbf{H}^-$ are the upper and lower half planes and $\mathbf{H}^+\cup\mathbf{H}^-\cup\mathbf{R}\cup\infty = \mathbf{CP}^1$. Each component of F_g^+ and F_g^- inherits a complete (possibly branched) hyperbolic metric from the hyperbolic metrics of $\mathbf{H}^+$ and $\mathbf{H}^-$. Suppose all the components of F_g^- are complete hyperbolic annuli. Then all the topological information is encoded in the F_g^+ components and we can associate to such structures corresponding singular hyperbolic structures on F_g with cone and fold singularities where the folding occurs along curves homotopic to the components of F_g^R and the components of F_g with negative signed area are all regular hyperbolic annuli. We have the following theorem:

Theorem 2 *Let $\rho \in \mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbf{R}))$ and $e(\rho) = \chi(F_g)+k$, where $0 \leqq k < 2g-2$. Then there exists a branched $\mathbf{CP}^1$-structure on F_g with holonomy ρ such that there are k branch points $x_1,\ldots,x_k$ of degree 2 each, $\{x_1,\ldots,x_k\} \subset F_g^+$, and F_g^- has at most k components, each of which is a complete hyperbolic annulus. Equivalently, there exists a singular hyperbolic structure on F_g with holonomy ρ and with k cone singularities of cone angles 4π each and at most $2k$ fold components where the folding occurs along invariant curves of some hyperbolic transformations. The components of F_g with negative signed area are annuli with a regular (orientation reversing) hyperbolic structure bounded by invariant curves of $\rho(\gamma)$ where γ is the non-trivial curve in the annulus.*

Remark. In [8] Goldman showed that the connected components of $\mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbf{R}))$ are $e^{-1}(k)$, $2-2g \leqq k \leqq 2g-2$ and in [10], using gauge theory techniques, Hitchin showed that the components $e^{-1}(\chi(F_g)+k)$ of $\mathrm{Hom}(\pi,\mathrm{PSL}(2,\mathbf{R}))/\mathrm{PSL}(2,\mathbf{R})$ are homotopic to Σ_k (the k-th symmetric product of F_g) for $0 \leqq k < 2g-2$. It seems possible that Theorem 2 can be used to try to recover Hitchin's theorem using a more elementary geometric approach.

This paper is organised as follows: In Sect. 2, the basic definitions and geometric notions associated with branched $\mathbf{CP}^1$-structures as well as some examples are given. Section 3 is technical and mostly adapted from [2] and [8]. The main technical results of the section, Lemmas 2 and 3 states that given a non-elementary representation ρ of the fundamental group π of F_g into $\mathrm{PSL}(2,\mathbf{R})$, there exists a decomposition of F_g into pairs of pants such that ρ is hyperbolic on the boundary curves of the pairs of pants and the relative Euler class of the pairs of pants have certain nice properties. In Sect. 4, we show

how to construct regular or branched hyperbolic structures on pairs of pants with prescribed holonomy ρ when ρ is hyperbolic on the boundary. In Sect. 5, we show how to glue together the structures constructed in Sect. 4 to obtain the required branched $\mathbf{CP}^1$-structures on F_g with the prescribed holonomy. Combined with the results of Sect. 3 and Sect. 4, this gives Theorems 1 and 2. In Sect. 6, we construct local deformations of the branched structures that leave the holonomy representation invariant. From a simple dimension count, we see that locally, these are all the deformations that can occur. Finally, in Sect. 7, we give an example of a representation of the fundamental group of a genus 3 surface that cannot occur as the representation of a branched hyperbolic structure. However, we show that by deforming the representation slightly, we can make the deformed representation the holonomy of a branched hyperbolic structure.

2 Branched $\mathbf{CP}^1$-structures

Let F_g be a closed oriented surface as above, π its fundamental group and $\tilde{F}_g$ its universal covering space.

Definition. *A (marked)* **branched $\mathbf{CP}^1$-structure** *on F_g is a covering of F_g by open sets $\{U_\alpha\}_{\alpha\in A}$ and maps $\psi_\alpha : U_\alpha \hookrightarrow \mathbf{CP}^1$ such that*
(1) ψ_α is either a homeomorphism from U_α onto its image or ψ_α is a branched covering map from U_α onto its image, and
(2) for all pairs $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, if V is a connected component of $U_\alpha \cap U_\beta$, then $\psi_\alpha \circ \psi_\beta^{-1}|_{\psi_\beta(V)}$ is the restriction of some $g \in \mathrm{PSL}(2, \mathbf{C})$.

Two marked branched $\mathbf{CP}^1$-structures on F_g are equivalent if there exists a map from F_g to itself which takes one structure to the other and is isotopic to the identity. The space of equivalence classes of structures is the deformation space. Alternatively, we can think of branched $\mathbf{CP}^1$-structures as cone $\mathbf{CP}^1$-structures with cone angles which are multiples of 2π (cf. [18, 19]).

Associated to each branched $\mathbf{CP}^1$-structure is the pair (dev, ρ) where

$$\mathrm{dev} : \tilde{F}_g \mapsto \mathbf{CP}^1$$

the developing map is a branched projective immersion and

$$\rho : \pi_1(F_g) \mapsto \mathrm{PSL}(2, \mathbf{C})$$

the holonomy representation is a homomorphism that is equivariant with respect to dev. dev is defined up to composition with elements of $\mathrm{PSL}(2, \mathbf{C})$ and ρ up to conjugation by elements of $\mathrm{PSL}(2, \mathbf{C})$.

Note that even in the case of branched constant curvature metric structures (spherical, euclidean or hyperbolic), the image of the holonomy representation ρ may not be a discrete subgroup of the group of isometries of the model space and we therefore do not generally get a tessellation of the model space.

Definition. *Let $x \in F_g$ and $\tilde{x} \in \tilde{F}_g$ be a lift of x. x is* **regular** *if the developing map is a projective immersion on some neighbourhood of $\tilde{x}$ into* $\mathbf{CP}^1$, *otherwise, x is* **singular**. *If x is singular, the developing map about a neighbourhood of $\tilde{x}$ is a branched covering map of degree n with $\tilde{x}$ mapped to the branch point. The integer n is independent of the lift of x chosen and x is called a* **branch point of degree** *n.*

The singular points are isolated and since F_g is compact, there are only a finite number of them. Troyanov [19, 20] has worked with branched metric structures in a different context where he defines the order of the branch point to be $n-1$.

We now define a **fundamental membrane** for a branched $\mathbf{CP}^1$-structure on F_g following Hejhal [9]. This is the generalisation of a fundamental domain for a metric structure on F_g.

Let $\Gamma = \{\gamma_i | 1 \leqq i \leqq s\}$ be a set of simple curves on F_g such that:

(a) $F_g - \Gamma$ is simply connected. $F_g - \Gamma$ is topologically a polygon whose sides occurs in pairs, each pair of sides corresponding to a curve $\gamma_i \in \Gamma$.

(b) The curves γ_i do not intersect except perhaps at the endpoints.

(c) The set of endpoints of the curves γ_i in Γ contain the set of singular points of F_g.

Such a set of curves Γ can always be found, for example we may take a standard set of curves $\{\alpha_i, \beta_i | 1 \leqq i \leqq g\}$ based at a regular point x_0 dissecting F_g into a $4g$-gon and add curves δ_j going from x_0 to each singular point x_j. If there are n singular points, the construction gives a $(4g+2n)$-gon.

We denote the polygon $F_g - \Gamma$ by $\mathscr{F}$. A fundamental membrane for the $\mathbf{CP}^1$-structure on F_g is just the developing image of a connected component of the lift of $\mathscr{F}$ to the universal cover. Note that the fundamental membrane lies on $\mathbf{CP}^1$ and may well be multi-sheeted. However there is no ramification over the points of $\mathscr{F}$ since all points on $\mathscr{F}$ are regular. Furthermore, we may make the sides of $\mathscr{F}$ piece-wise circular arcs. The transformations that map the pairs of sides of $\mathscr{F}$ that arise from the same curve γ_i to each other lie in PSL(2, $\mathbf{C}$). Theorems 1 and 2 are proven by constructing the required fundamental membranes.

Another interpretation of branched $\mathbf{CP}^1$-structures is as cone $\mathbf{CP}^1$-structures with cone singularities with cone angles of the form $2n\pi, n \geqq 2$. In this sense, cone-Euclidean (or spherical or hyperbolic) structures with cone angles of the form $2n\pi$ are examples of branched $\mathbf{CP}^1$-structures. We give some examples below:

Example 1 Take two closed surfaces S_1 and S_2 admitting metric structures of the same type (spherical, euclidean or hyperbolic). Make isometric slits l_1 and l_2 respectively on the two surfaces and glue S_1 to S_2 along the slits. Since l_1 and l_2 are isometric, the resulting surface has only two singular points at the end of the slits, these are branch points of degree 2 each. See Fig. 1.

Example 2 Take the regular Euclidean octagon with area one and identify the sides in the standard manner to obtain a genus two surface. Since the identifications are by means of Euclidean isometries, the resulting surface has

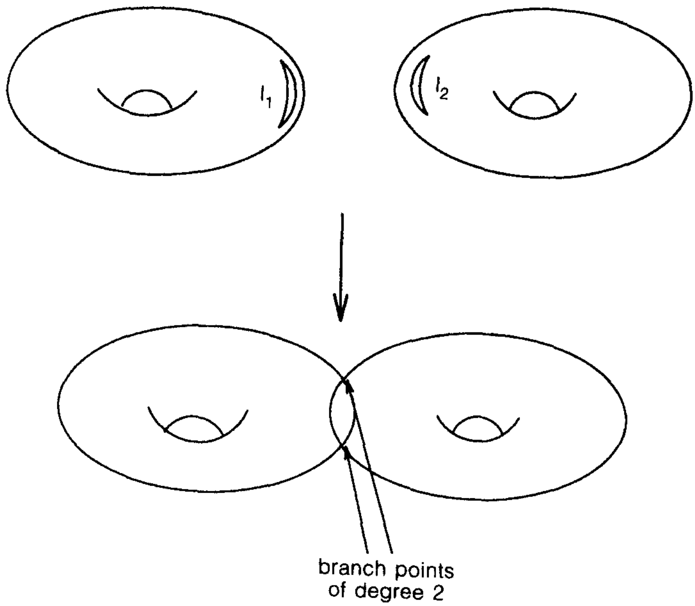

Fig. 1. Glueing two Euclidean tori along isometric slits

a Euclidean structure except at the vertex of the octagon which is a cone point with cone angle 6π (i.e. a branch point of degree 3). The octagon is then a fundamental membrane for the cone-Euclidean structure, see Fig. 2a.

Example 3 Take the regular hyperbolic octagon with interior angles $\pi/2$ and identify the sides as in Example 2 above. This gives a fundamental membrane for a branched hyperbolic structure on a genus 2 surface with one branch point of degree 2; see Fig. 2b. Note that in general, to create a branched hyperbolic structure on F_2 with one branch point of degree 2, we only require that the sides to be identified have equal hyperbolic lengths and that the sum of the interior angles is 4π. A simple parameter count (see [16]) shows that the space of such octagons has real dimension 8.

Example 4 Take any branched cover of S^2 (say, some Riemann surface) and pull-back the S^2-structure to obtain a branched spherical structure on the Riemann surface. Note that in this case, the holonomy representation may be trivial.

3 The relative Euler class and decomposition of F_g into pairs of pants

This section is mostly based on results in [2] and [8] and provides the technical tools for the proofs of our theorems. The reader is referred to [2, 5, 8] for a more detailed discussion.

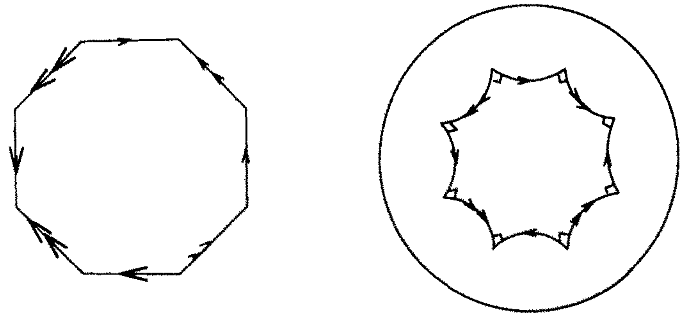

Fig. 2. Fundamental membranes for branched Euclidean and branched hyperbolic structures on a genus two surface

Let $\rho \in \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))$ where π is the fundamental group of F_g as above. Associated to ρ is a hyperbolic plane bundle E_ρ over F_g.

Definition. *The* **Euler class** *of ρ, denoted by $e(\rho)$ is defined to be the Euler class of the underlying oriented disk bundle.*

The Milnor-Wood inequality states that $e(\rho)$ lies in the finite range of values

$$2 - 2g \leqq e(\rho) \leqq 2g - 2 .$$

Considering the Euler class as a volume form and applying the Gauss-Bonnet theorem, we see that if ρ is the holonomy representation of a branched hyperbolic structure on F_g with singularities $x_1, \ldots, x_n$ of degrees $\alpha_1, \ldots, \alpha_n$ respectively, then

$$e(\rho) = 2 - 2g + \sum_{i=1}^{n} (\alpha_i - 1) .$$

By modifying Example 3 in the previous section, for every F_g with fixed $g > 1$, we can produce examples of representations with Euler class from $(2-2g)$ to -1. Conjugating with the map $z \mapsto \bar{z}$, we also obtain representations with Euler class from 1 to $(2g - 2)$. To obtain a representation with Euler class 0, we can use the trivial representation. Thus all integer values k between the bounds of the Milnor-Wood inequality are attained. In fact the topological components of $\mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))$ are $e^{-1}(k)$ (see [8]).

Definition. *A* **pair of pants** *is a topological surface with three boundary components homeomorphic to the sphere S^2 with three disks removed. By abuse of notation, we will also call the interior of the surface a pair of pants.*

Definition. *A* **maximal cut** $\mathscr{V}$ *of F_g is defined to be a collection of disjoint, non-trivial simple closed curves $\{\gamma_i\}$ in F_g such that*

$$F_g - \mathscr{V} = \bigcup_{i=1}^{2g-2} S_i$$

where all the S_i's are pair-of-pants.

Our method for constructing the required branch structures consists of finding a nice maximal cut $\mathscr{V}$ and then constructing regular or branched hyperbolic structures on each of the *pair-of-pants* components of $F_g - \mathscr{V}$ with either ideal or totally geodesic boundaries and then glueing them together. To do this, we first define a relative Euler class for surfaces with boundaries:

Definition. *Let S be a surface with boundary ∂S and fundamental group π and let ρ be a representation of π into* $\mathrm{PSL}(2,\mathbf{R})$. *Let E_ρ be the oriented hyperbolic plane bundle over the surface S associated with ρ and let σ be an ideal section over ∂S, i.e. a section of ∂S into $\mathbf{R}\cup\infty \cong \mathbf{RP}^1$. The relative Euler class of ρ with respect to σ which lies in $H^2(S,\partial S) \cong \mathbb{Z}$ is the obstruction for extending $\sigma : \partial S \to \mathbf{RP}^1$ to an ideal section $S \to \mathbf{RP}^1$. We denote it by $e(\rho, S; \sigma)$.*

The following lemma was proved in [8]:

Lemma 1 *Let E_ρ be the hyperbolic plane bundle over F_g associated to the representation $\rho, \mathscr{W}$ a collection of disjoint simple closed curves in F_g and σ a special ideal section over $\mathscr{W}$. Then*

$$e(\rho) = \sum_{i=1}^{k} e(\rho|_{S_i}, S_i; \sigma)$$

where $S_i, 1 \leqq i \leqq k$ are the components of $F_g - \mathscr{W}$ (with boundary components attached) and $\rho|_{S_i}$ is the restriction of ρ to the components S_i.

Note that the above lemma holds irrespective of the special ideal sections chosen but by defining special ideal sections for various cases when $\rho|_\gamma$ is the identity, elliptic, parabolic or hyperbolic, Goldman was able to show that $e(\rho, S; \sigma) = 0$ or ± 1 when S is a pair of pants and σ is a special ideal section. This gives an independent proof of the Milnor-Wood inequality.

We will be mostly interested in the case where the holonomy is hyperbolic on the boundary components of the S_i so we will just define the special ideal section in this case, the interested reader is referred to [5] or [8] for the other cases.

Definition. *Let S be a surface with boundary and let γ be a component of ∂S such that $\theta = \rho(\gamma)$ is hyperbolic. Let $t \to \exp(t(\log\theta))$ be the one parameter subgroup of* $\mathrm{PSL}(2,\mathbf{R})$ *containing θ. This is just the one parameter subgroup of* $\mathrm{PSL}(2,\mathbf{R})$ *consisting of hyperbolic elements with the same invariant axis as θ. We identify γ with $\mathbf{R}/\mathbb{Z}$ so that $\tilde{\gamma}$ is identified with $\mathbf{R}$ and we also identify $\partial\mathbf{H}^+ = \mathbf{R}\cup\infty$ with $\mathbf{RP}^1$. The developing map $\mathrm{dev}(t) = \exp(-t(\log\theta))y$ where $y \in \mathbf{RP}^1$ is not a fixed point of θ gives rise to a special ideal section of the $\mathbf{RP}^1$-bundle over γ. We choose this as the special ideal section of γ.*

We now return to the surface F_g. π and ρ will be as defined at the beginning of the paper. The following two propositions were proven in [2]:

Proposition 1 *If $\rho \in \mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbf{R}))$ is non-elementary, then there exists a maximal cut $\mathscr{V}$ of F_g such that ρ is hyperbolic on the components of $\mathscr{V}$.*

We call this a hyperbolic maximal cut.

Proposition 2 *Let $\mathscr{V}$ be a hyperbolic maximal cut of F_g and σ the special ideal section over $\mathscr{V}$ as defined above. Let S_i and S_j be components of $F_g - \mathscr{V}$ sharing a common boundary curve γ and satisfying one of the three conditions below:*
(a) $e(\rho|_{S_i}, S_i; \sigma) = 0$, $e(\rho|_{S_j}, S_j; \sigma) = \pm 1$, *or*
(b) $e(\rho|S_i, S_i; \sigma) = e(\rho|_{S_j}, S_j; \sigma) = 0$, *or*
(c) $e(\rho|_{S_i}, S_i; \sigma)$ *and* $e(\rho|_{S_j}, S_j; \sigma)$ *are non-zero and have opposite signs.*
Then we can find a new hyperbolic maximal cut $\mathscr{V}'$ where all the components of $\mathscr{V}'$ are the same as those of $\mathscr{V}$ except that γ is replaced by γ' and such that the components S_i' and S_j' separated by γ' satisfy the corresponding three conditions below:
(a') $e(\rho|_{S_i'}, S_i'; \sigma) = \pm 1$, $e(\rho|_{S_j'}, S_j'; \sigma) = 0$, *or*
(b') $e(\rho|_{S_i'}, S_i'; \sigma)$ *and* $e(\rho|_{S_j'}, S_j'; \sigma)$ *are non-zero and have opposite signs, or*
(c') $e(\rho|_{S_i'}, S_i'; \sigma) = e(\rho|_{S_j'}, S_j'; \sigma) = 0$, *respectively.*

We are now ready to state our two main technical lemmas concerning the decomposition of F_g into pairs of pants:

Lemma 2 *If $e(\rho) \equiv 1 \pmod 2$ then there exists a hyperbolic maximal cut $\mathscr{V}$ of F_g such that*

$$e(\rho|_{S_i}, S_i; \sigma) = \begin{cases} 0 & \text{if } i = 1 \\ \pm 1 & \text{if } 1 < i \leqq 2g-2 \end{cases}$$

where $S_i, 1 \leqq i \leqq 2g-2$ are the components of $F_g - \mathscr{V}$.

Lemma 3 *If $e(\rho) = 2 - 2g + k$, $0 \leqq k < 2g-2$ then there exists a hyperbolic maximal cut $\mathscr{V}$ of F_g such that*

$$e(\rho|_{S_i}, S_i; \sigma) = \begin{cases} 0 & \text{if } 1 \leqq i \leqq k \\ -1 & \text{if } k < i \leqq 2g-2 \end{cases}$$

where S_i, $1 \leqq i \leqq 2g-2$ are the components of $F_g - \mathscr{V}$.

Proof of Lemmas 2 and 3 The Euler class of the representation ρ in both lemmas is non-zero so ρ is non-elementary (see [8]). By Proposition 1, there exists a hyperbolic maximal cut $\mathscr{V}$ of F_g. By repeated applications of Proposition 2 (using conditions (a) and (b)) we are left with one component of $F_g - \mathscr{V}$ with relative Euler class zero. Renaming if necessary, we get Lemma 1. Lemma 2 is proved similarly by repeated applications of conditions (a) and (c) of Proposition 2.

4 Branched hyperbolic structures on a pair of pants

In this section we show how, given a representation of the fundamental group of a pair-of-pants S to $\mathrm{PSL}(2,\mathbf{R})$ which is hyperbolic on the boundary curves and has relative Euler class 0 with respect to the special ideal section, we can construct a complete branched hyperbolic structure on S with one branch point of degree two such that the holonomy representation of the branched structure is the original representation and the developing map restricts to the special ideal section on the boundary curves. We first state a result of Goldman [8] and give a brief sketch of the proof:

Lemma 4 *Let S be a pair-of-pants with fundamental group π and let $\rho \in \mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbf{R}))$ such that ρ is hyperbolic on the components of ∂S. If $e(\rho, S; \sigma) = -1$ (resp. $+1$) where σ is the special ideal section over ∂S as defined in Sect. 3, then there exists a complete orientation-preserving (resp. reversing) hyperbolic structure on S with holonomy ρ and with ∂S developing to the ideal sections defined by σ. Moreover, the boundary components can be retracted to geodesic sections, i.e. special interior sections in the $\mathbf{H}^+$ bundle over the boundary components which are geodesic with respect to the hyperbolic metric on $\mathbf{H}^+$.*

Proof of Lemma 4 (sketch) Let A, B and C be the oriented components of ∂S and let l_A, l_B and l_C be the directed axes of the hyperbolic transformations $\rho(A)$, $\rho(B)$ and $\rho(C)$ respectively in the upper half plane $\mathbf{H}^+$. If $e(\rho, S; \sigma) = -1$ (resp. $+1$), then the l_i's ($i = A, B$ or C) are pairwise disjoint, furthermore, no one of them seperates the other two in $\mathbf{H}^+$ and the directed axes has an anti-clockwise (resp. clockwise) direction. For each pair of l_i and l_j, there is a unique geodesic m_{ij} in $\mathbf{H}^+$ perpendicular to l_i and l_j; together with the axes l_i's, they form a hyperbolic right angled hexagon. Take two copies of this hexagon and glue them along the m_{ij}'s, this gives a hyperbolic structure on a pair of pants with geodesic boundaries and with holonomy ρ. The structure is orientation-preserving (resp. reversing) when $(e, \rho; \sigma) = -1$ (resp. $+1$). The complete structure can be obtained by glueing hyperbolic annuli which extend to infinity at each of the boundary components. See Fig. 3 where $\mathbf{H}^+$ is represented by the disc D^2.

We now state and prove the analogous result for the case when the relative Euler class is zero:

Lemma 5 *Suppose S and ρ are as in Lemma 4 above except that the relative Euler class $e(\rho, S; \sigma) = 0$. Then S admits a complete branched hyperbolic structure (with one branch point of degree 2) with holonomy representation ρ and with ∂S developing to the ideal sections defined by σ. Furthermore, we can choose the structure to be orientation-preserving or reversing and two of the boundary components can be retracted to special interior sections which are geodesic with respect to the hyperbolic metric on $\mathbf{H}^+$.*

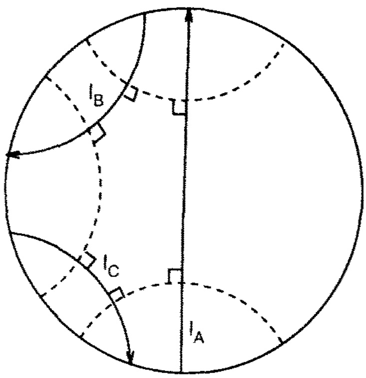

Fig. 3. Right angled hexagon associated to an Euler class -1 representation on a *pair-of-pants*

Before proceeding with the proof of Lemma 5, we give three examples of branched Euclidean structures on a pair-of-pants S with certain prescribed holonomy representations which will give an idea of the proof of Lemma 5.

Example 5 Let S be a *pair-of-pants* with fundamental group π and A, B and C be the oriented boundary curves. Suppose that ρ is a representation of π into $\mathrm{Isom}(E^2)$ such that ρ takes A, B and C into parallel Euclidean translations. Without loss of generality, we may assume that the translation length of $\rho(A)$ is greater than the translation length of $\rho(B)$ and $\rho(C)$. Consider the infinite Euclidean cylinder $E^2/\langle\rho(A)\rangle$. Make two semi-infinite slits l_1 and l_2 on the cylinder starting from p_1 and p_2 such that $\rho(B)l_1 = l_2$. Identify l_1^+ to l_2^- and l_1^- to l_2^+ via $\rho(B)$. This gives a complete branched Euclidean structure with one branch point of degree 2 (where p_1 is identified to p_2) on $S - \partial S$ with holonomy ρ. Clearly, by cutting off the infinite ends, we can get a structure on S with totally geodesic boundaries and in fact, we can make the area of S as small as we like. See Fig. 4.

Example 6 Using the notation above, if ρ takes A, B and C to non-parallel translations, we can still obtain a branched Euclidean structure with holonomy ρ as follows: Fix a point x in E^2. The points x, $\rho(A)(x)$ and $\rho(B)\rho(A)(x)$ form the vertices of a triangle where the sides of the triangle give the translation length and direction of $\rho(A)$, $\rho(B)$ and $\rho(C)$ respectively, (note that CBA is homotopic to the trivial curve in S). Call the sides l_A, l_B and l_C respectively and without loss of generality, assume that the angles between l_A and l_B and l_A and l_C are acute. Take a fundamental domain for the infinite Euclidean cylinder $E^2/\langle\rho(A)\rangle$. Again we can find infinite slits l_1 and l_2 lying on the same fundamental domain such that $\rho(B)(l_1) = l_2$. Identifying the slits l_1 and l_2 as before, we obtain a complete branched Euclidean structure on S with holonomy ρ. Again, we can retract the boundary curves to totally geodesic curves but in this case, the area of the branched structure with totally geodesic

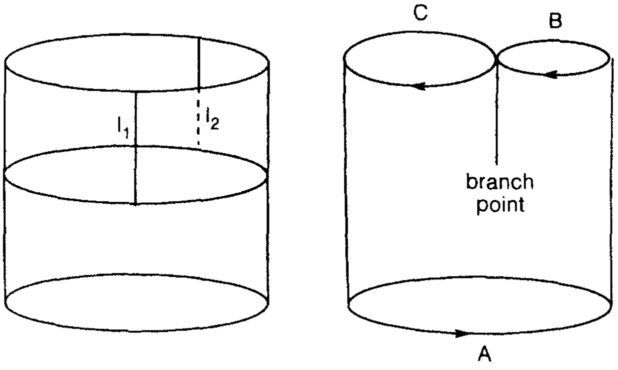

Fig. 4. Branched Euclidean structure on a *pair-of-pants*

boundary has minimum area Δ where Δ is the area of the triangle with vertices x, $\rho(A)(x)$ and $\rho(C)^{-1}(x)$. Figure 5 gives the fundamental membrane of the structure.

Example 7 Again, using the notation above, if $\rho(A)$ is a translation and $\rho(B)$ and $\rho(C)$ are rotations we can still obtain a branched Euclidean structure on S with holonomy ρ. The method is similar to Example 6 above, we again make slits l_1 and l_2 such that $\rho(B)l_1 = l_2$ (note that now $\rho(B)$ is a rotation). Making the usual identifications we obtain a branched Euclidean structure on S with holonomy ρ. Note that in this case, we cannot get a complete branched Euclidean structure. However, we can make the boundary curve A geodesic while the boundary curves for B and C can be made arbitrarily small invariant curves of $\rho(B)$ and $\rho(C)$ respectively. See Fig. 6.

Proof of Lemma 5 There are three possibilities if ρ satisfies the conditions of the lemma, namely,

Case 1 $\rho(A)$, $\rho(B)$ and $\rho(C)$ have the same invariant axis, or

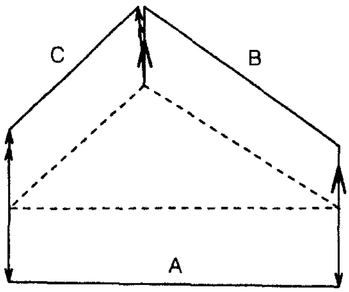

Fig. 5. Fundamental domain for branched Euclidean structure on a *p.o.p* with holonomy representation into non-parallel translations on the boundary curves

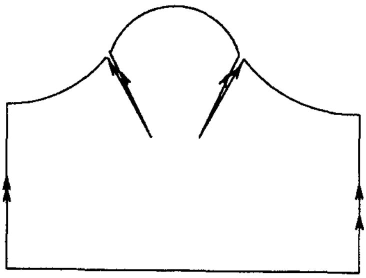

Fig. 6. Fundamental domain for branched Euclidean structure on a *pair-of-pants* with holonomy representation into translation and rotations on the boundary curves

Case 2 The invariant axes of $\rho(A)$, $\rho(B)$ and $\rho(C)$ intersect pairwise, or

Case 3 The invariant axes of $\rho(A)$, $\rho(B)$ and $\rho(C)$ do not intersect and one of them seperates the other two in $\mathbf{H}^+$.

Case 1 This case is very similar to Example 5 above. Without loss of generality, suppose that the translation length of $\rho(A)$ is greater than that for $\rho(B)$ and $\rho(C)$. Take the complete hyperbolic cylinder $\mathbf{H}^2/\langle\rho(A)\rangle$ and make semi-infinite geodesic slits l_1 and l_2 starting from the invariant axis such that $\rho(B)l_1 = l_2$. Identify l_1^+ to l_2^- and l_1^- to l_2^+ via $\rho(B)$ to obtain a complete branched hyperbolic structure on S with holonomy ρ and ideal boundary. By shifting the end point of the slits (which gives the branch point) beyond the invariant axis, we can retract the two boundary components B and C to geodesic boundaries. In this case, A can also be retracted to a special interior section that is arbitrarily close but not equal to the geodesic boundary. See Fig. 7.

Case 2 This is similar to Example 6 above. The invariant axes l_A, l_B and l_C of $\rho(A)$, $\rho(B)$ and $\rho(C)$ respectively form a hyperbolic triangle since they intersect pairwise. Assume, without loss of generality that the angle between

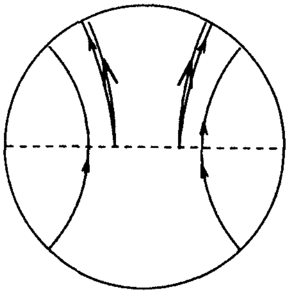

Fig. 7. Fundamental domain for complete branched hyperbolic structure on a *pair-of-pants* with hyperbolic holonomy on boundary curves all having the same invariant axis

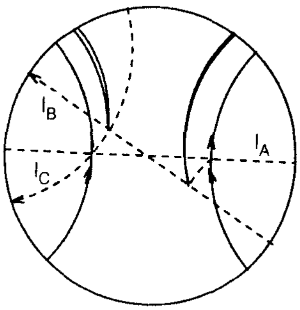

Fig. 8. Fundamental domain for complete branched hyperbolic structure on a *pair-of-pants* with hyperbolic holonomy on boundary curves having intersecting invariant axes

l_A and l_B and that between l_A and l_C are both acute. Note that the lengths of the sides of the triangle are half the translation lengths of the respective transformations. Using the Poincare disc model for hyperbolic space, we may for convenience assume that l_A and l_B are straight lines through the origin. Take the fundamental domain for the complete hyperbolic annulus $\mathbf{H}^2/\langle\rho(A)\rangle$ and we can make semi-infinite slits l_1 and l_2 lying on the same fundamental domain such that $\rho(B)l_1 = l_2$. Identify the slits in the usual way to obtain a branched complete hyperbolic structure on S with holonomy ρ. Shifting the branch point if necessary, it is possible to retract the two boundary components B and C to geodesic boundaries. In this case, the third boundary A cannot be retracted to be arbitrarily close to the geodesic boundary, see Fig. 8.

Case 3 Without loss of generality, assume that the invariant axis l_A for $\rho(A)$ separates the invariant axes l_B and l_C of $\rho(B)$ and $\rho(C)$ respectively. Take a fundamental domain for the infinite hyperbolic cylinder $\mathbf{H}^2/\langle\rho(A)\rangle$ and once again, we can find semi-infinite slits l_1 and l_2 lying on the same fundamental domain such that $\rho(B)l_1 = l_2$. Identify the sides of the slits as in Case 2 above to obtain a complete branched hyperbolic structure on S with holonomy ρ. Again, by shifting the branch point if necessary,we can retract the boundaries B and C to geodesic boundaries. However, A cannot then be retracted arbitrarily close to the geodesic boundary, see Fig. 9.

Note that in all three cases, depending on the direction (upwards or downwards) of the slits chosen, we either get an orientation-preserving or an orientation-reversing structure. This completes the proof of the lemma. □

We remark that following Example 7 above, we can also form branched hyperbolic structures when some of the boundary components have parabolic or elliptic holonomy, see [15] for details.

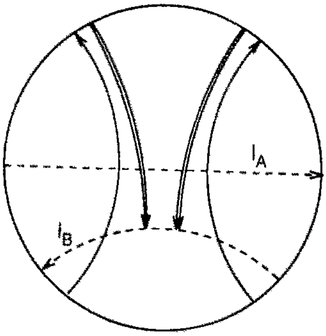

Fig. 9. Fundamental domain for complete branched hyperbolic structure on a *pair-of-pants* with hyperbolic holonomy on boundary curves having non-intersecting invariant axes

5 The glueing process

In this section, we describe how to glue the structures on the various pairs of pants components together to obtain the structure on the whole surface F_g with the prescribed holonomy.

Let $\rho \in \mathrm{PSL}(2, \mathbf{R})$, $\mathscr{V}$ a hyperbolic maximal cut of F_g such that both ρ and $\mathscr{V}$ satisfy the conditions of Lemma 2 or Lemma 3. We have seen how to construct orientation-preserving (resp. reversing) complete hyperbolic structures on the components of $F_g - \mathscr{V}$ with relative Euler class -1 (resp. $+1$) and complete orientation-preserving branched hyperbolic structures on the components with relative Euler class 0 such that the holonomy representation on each component agrees with the restriction of ρ on the component (Lemmas 4 and 5). By composing the developing maps of the orientation-reversing structures with the map $z \mapsto \bar{z}$, we get orientation-preserving developing maps into $\mathbf{H}^-$. We shall always think of orientation-reversing hyperbolic structures then as orientation preserving structures modelled on $\mathbf{H}^-$.

Let γ be a curve bounding two (not necessarily distinct) *pair-of-pants* components of $F_g - \mathscr{V}$. We have the following possibilities:

If the relative Euler class of the two components are both equal to $+1$ or -1, then we can retract the complete hyperbolic structure of these two components along γ to the totally geodesic curves and glue them along these totally geodesic curves, giving a complete hyperbolic structure on the union of the two components along γ.

If the relative Euler class of the two components are non-zero and of opposite signs, then we just glue the two structures along the ideal boundary corresponding to γ.

If one of the components has relative Euler class 0 and the other 1, we glue along an ideal boundary.

If the relative Euler classes are 0 and -1 respectively, we glue along the retract to the totally geodesic curve if possible, (recall that we can retract two of the boundary curves to totally geodesic curves for the Euler class zero component), otherwise, we add on a complete hyperbolic cylinder developing onto $\mathbf{H}^-$ and glue the ideal boundaries of the two components corresponding to γ to the ideal boundaries of the hyperbolic cylinder.

If ρ and $\mathscr{V}$ satisfies the conditions of Lemma 2, proceeding inductively, we can construct a fundamental membrane for a branched $\mathbf{CP}^1$-structure on F_g with one branch point of degree two and holonomy representation ρ, thus proving Theorem 1. To see Theorem 2, we use ρ and $\mathscr{V}$ satisfying the conditions of Lemma 3. In this case, for each of the branched structures on the components with relative Euler class zero, we can retract any two of the boundaries to totally geodesic curves. It follows that we need to add at most k complete hyperbolic cylinders developing to $\mathbf{H}^-$.

6 The fibre of the holonomy map

Suppose $\rho \in \mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbf{R}))$ with $e(\rho) = 2 - 2g + k$, $1 \leqq k < 2g - 1$. By Theorem 2, F_g has a branched $\mathbf{CP}^1$-structure with holonomy ρ with branch points $\{x_1, \ldots, x_k\}$ all of degree 2. In this section we show that there is a k-complex dimensional family of branched $\mathbf{CP}^1$-structures on F_g with k branched points all of degree 2 having the same holonomy representation. We also show that the generic branch point has degree two.

We start by showing that for each branch point of degree two, there is a complex one-dimensional family of deformations which arise from "moving" the branch point around.

Let x be a branch point of degree two for some fixed branched $\mathbf{CP}^1$-structure on F_g. Choose a small neighbourhood U of x such that U is contractible and U is mapped by the developing map onto a geometric disc D in $\mathbf{CP}^1$ and $U = \mathrm{dev}^{-1}(D)$, locally. Remove U from F_g and attach a new 'disc' as follows:

Choose any point y in D distinct from $\mathrm{dev}(x)$ and join x to y by a line l lying completely in D. l lifts to two distinct lines $\tilde{l}_1$ and $\tilde{l}_2$ in U both ending in x. Slit U along these two lines and reglue, matching $\tilde{l}_1^+$ to $\tilde{l}_2^-$ and $\tilde{l}_1^-$ to $\tilde{l}_2^+$. The two lifts of y are now identified and becomes a branch point, x is now split to two regular points. It is easy to see that the new disc U_y depends only on y and not on the choice of l and also that the boundary of U_y is isomorphic to that of U so that we can attach U_y back to $F_g - U$, resulting in a different branched $\mathbf{CP}^1$ structure on F_g with the same holonomy. Since the structures are parametrised by y, we obtain a one complex parameter family of branched structures for each branch point all sitting over the same holonomy representation, see Fig. 10.

The above construction can be applied to branch points with degree $d > 2$, in this case, there are more than two lifts of l to choose from. If we choose the lifts to differ by an angle of 2π at x, then we create two branch points with degrees 2 and $d - 1$. Continuing this inductively, we can reduce all branch points to degree 2 which is the generic case.

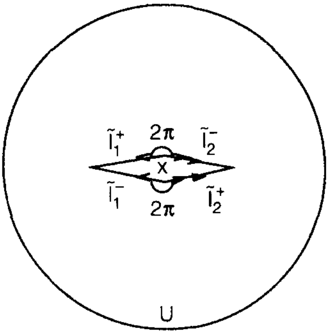

Fig. 10. Moving the position of the branch point by slitting and reglueing

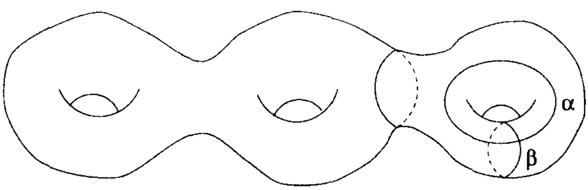

Fig. 11. Genus three surface M obtained by attaching a handle to a genus two surface

7 An example

Example 8 Let M be a genus 3 surface, obtained by attaching a handle to a genus 2 surface (Fig. 11) and let $\pi = \pi_1(M)$. We define a representation $\rho \in \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))$ as follows:

(i) ρ is a discrete, faithful representation on the original genus two surface.
(ii) ρ is trivial on the attached handle.

$\rho(\pi)$ is thus a discrete subgroup Γ of $\mathrm{PSL}(2, \mathbf{R})$ and $\mathbf{H}/\Gamma$ is a genus two surface.

Proposition 3 *M does not admit a branched hyperbolic structure with holonomy representation* ρ.

Proof. Suppose not, i.e. suppose that M has a branched hyperbolic structure with holonomy ρ.

Then $\mathrm{dev} : \tilde{M} \mapsto \mathbf{H}$ passes down to a branched map

$$\overline{\mathrm{dev}} : M \mapsto \mathbf{H}/\Gamma .$$

The induced map on the fundamental group is just that of the pinching map where the handle is pinched to a point, hence the above map is homotopic to a pinch map of degree 1, implying that dev is a homeomorphism from M to $\mathbf{H}/\Gamma$, a contradiction. $\square$

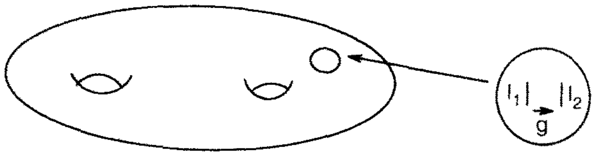

Fig. 12. Attaching a branched hyperbolic structure on a handle to $\mathbf{H}/\Gamma - \varepsilon$-disc

Although M does not admit a branched hyperbolic structure with holonomy ρ in the above example, we shall see that by perturbing the representation slightly, we will be able to construct a branched hyperbolic structure with holonomy close to that above.

Let α and β be curves on the attached handle as in Fig. 11. Note that $\rho(\alpha) = \rho(\beta) = \mathrm{Id}$. Construct a hyperbolic structure on the original genus 2 surface using $\mathbf{H}/\Gamma$ and remove an ε-disc about a base point x to obtain a hyperbolic surface with boundary, the holonomy about the boundary curve is trivial. We will attach a branched structure on a handle to the boundary as follows: Start with a hyperbolic ε-disc D_ε and make two geodesic slits l_1 and l_2 lying in D_ε having the same length. There exists some $g \in \mathrm{PSL}(2, \mathbf{R})$ mapping l_1 to l_2. Identify l_1^+ to l_2^- and l_1^- to l_2^+ via the hyperbolic isometry g. Topologically, we obtain a torus with a disc removed, i.e., a handle, geometrically we obtain a branched hyperbolic structure on the handle with two branch points of degree two each and with the boundary isometric to an ε-circle. Attaching this to the original structure, we obtain a branched hyperbolic structure on M whose holonomy representation ρ' is the same as ρ on a standard set of generators for π except that $\rho(\alpha) = g$. Clearly, g can be chosen to be arbitrarily close to the identity by making l_1 and l_2 close, so ρ' can be made arbitrarily close to ρ. Note also that g can approach the identity from the elliptic, parabolic or hyperbolic directions. By a slight modification, we can also construct branched hyperbolic structures with holonomy ρ' such that $\rho' = \rho$ on a standard set of generators except that $\rho'(\alpha) = \rho'(\beta) = g$ for some g close to the identity. See Fig. 12.

In view of the above example, we conclude by posing the following question.

Question. For $2 - 2g \leqq k \leqq -1$, consider the component $e^{-1}(k)$ of $\mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))$. Is the subset of $e^{-1}(k)$ consisting of all the representations that occur as the holonomy of branched hyperbolic structures on the surface F_g dense in $e^{-1}(k)$?

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References

1. D.B.A. Epstein, A. Marden. Convex Hulls in Hyperbolic Spaces. A Theorem of Sullivan and Measured Pleated surfaces. Lond. Math. Soc. Lect. Notes Ser. **111**, 113–254 (1987)
2. D.M. Gallo, W.M. Goldman, R.M. Porter. Projective structures with monodromy in PSL(2, **R**). Preprint
3. D.M. Gallo. Prescribed holonomy for projective structures on compact surfaces. Bull. AMS **1**, 31–34 (1989)
4. W.M. Goldman. Affine manifolds and projective geometry on surfaces, Senior Thesis, Princeton University (1977)
5. W.M. Goldman. Discontinuous groups and the Euler class. Doctoral dissertation, University of California, Berkeley (1980)
6. W.M. Goldman. Projective structures with Fuchsian holonomy. J. Differ Geom. **25**, 297–326 (1987)
7. W.M. Goldman. Geometric structures on manifolds and varieties of representations. Contemp. Math. **74**, 169–198 (1988)
8. W.M. Goldman. Topological components of spaces of representations. Invent. Math. **93**, 557–609 (1988)
9. D. Hejhal. Monodromy groups and linearly polymorphic functions. Acta Math. **135**, 1–55 (1975)
10. N.J. Hitchin. The self-duality equations on Riemann surfaces. Proc. Lond. Math. Soc. **55**, 59–126 (1987)
11. D. Johnson, J.J. Millson. Deformation spaces associated to compact hyperbolic manifolds. In: Discrete Groups in Geometry and analysis. Papers in Honor of G D. Mostow on His Sixtieth Birthday R. Howe (ed.), Prog. Math. **67**, Birkhauser, Boston-Basel-Stuttgart, 48–106 (1987)
12. M. Kapovich. Complex Projective structures with prescribed monodromy. Preprint (1992)
13. C. Kourouniotis. Deformations of hyperbolic structures on manifolds of several dimensions. Math. Proc. Camb. Phil. Soc. **98**, 247–261 (1985)
14. J.J. Millson. Deformations of Representations of finitely Generated Groups. Contemp. Math. **74**, 237–253 (1988)
15. S.P. Tan. Representations of surface groups into PSL(2, **R**) and geometric structures. Ph.D. dissertation, UCLA, (1988)
16. S.P. Tan. Singular pleated surfaces and $\mathbf{CP}^1$-structures. Preprint (1992)
17. W.P. Thurston. The Geometry and Topology of three-manifolds. Princeton University Mathematics Department notes (1979)
18. W.P. Thurston. Shapes of Polyhedra. Research Report GCG 7, Geometry Supercomputer Project
19. M. Troyanov. Metrics of constant curvature on a sphere with two conical singularities. Differential Geometry, Peniscola 1988. Lect. Notes Math. Springer **1410**, 296–306 (1988)
20. M. Troyanov. Les Surfaces euclideannes a singularities coniques. Enseign. Math. **32**, 79–94 (1986)
21. J. Wood. Bundles with totally disconnected structure groups. Comment. Math. Helv. **51**, 257–273 (1971)