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Remark on Herz-Schur Multipliers on Free Groups

Marek Bożejko

Institute of Mathematics, Wroclaw Uniwersity, Plac Grunwaldzki 2/4, Wroclaw, Polen

Introduction

Let G be a discrete group, and $A(G)=A_2(G)=l^2(G)*l^2(G)$ be the Fourier algebra of G introduced and studied by Eymard [2]. Herz considered the interesting algebra $B_2(G)$ of multipliers on G and he observed that

$$\mathrm{FS}(G)\subseteq B_2(G)\subseteq M(A(G)), \tag{1}$$

where $M(A(G))=\{\varphi:\varphi\cdot A(G)\subseteq A(G)\}$ and FS(G) is the Fourier-Stieltjes algebra of G defined as the linear space generated by positive definite functions on G.

If G is an amenable group, then all the spaces in (1) are equal.

It was shown by Figà-Talamanca and Picardello [8] that for a noncommutative free group F

$$\mathrm{FS}(F)\subsetneqq M(A(F)) \tag{2}$$

and Leinert [6] proved that for a free group F

$$\mathrm{FS}(F)\subsetneqq B_2(F). \tag{3}$$

In this note we show that

$$B_2(F)\subsetneqq M(A(F)) \tag{4}$$

and even more, namely for $1<p<\infty$

$$B_p(F)\subsetneqq M(A_p(F)). \tag{5}$$

2. Preliminaries

Let $1<p<\infty$, and X be an arbitrary set. The function u on X belongs to $l_p(X)$ if

$$|u|_p=\Big(\sum_{x\in X}|u(x)|^p\Big)^{1/p}<\infty.$$

The Banach algebra of bounded linear operators on $l_p(X)$ will be denoted by $\mathrm{END}_p(X)$.

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Every element $k \in \mathrm{END}_p(X)$ can be considered as a mapping

$$k: X \times X \to \mathbb{C} \qquad \text{such that} \qquad \|k\|_p < \infty,$$

where
$$\|k\|_p = \inf\left\{C : \left|\sum_{x,y \in X} k(x,y)\, u(y)\, v(x)\right| \leqq C\,|u|_p\,|v|_p\right\}.$$

The algebra of Schur multipliers $V_p(X)$ is defined as follows: $\varphi \in V_p(X)$ if $\varphi: X \times X \to \mathbb{C}$ and

$$\|\varphi \cdot k\|_p \leqq |||\varphi|||_p \cdot \|k\|_p$$

for every $k \in \mathrm{END}_p(X)$ with finite support in $X \times X$. $\varphi \cdot k$ is the pointwise multiplication.

Now let G be a discrete group, and $M\varphi(g,h) = \varphi(gh^{-1})$ for a function $\varphi: G \to \mathbb{C}$.

The algebra of Herz multipliers $B_p(G)$ is obtained as the space of functions φ on G such that $M\varphi \in V_p(G)$ and

$$\|\varphi\|_{B_p} = |||M\varphi|||_p.$$

Let $P: l_p \hat{\otimes} l_{p'}(G) \to c_0(G)$, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$, be defined on simple tensors in the following way:

$$P(f \otimes g) = \check{g} * f, \qquad \text{for} \qquad f \in l_p(G),\ g \in l_{p'}(G).$$

Let the Herz algebra $A_p(G) = P(l_p \hat{\otimes} l_{p'}(G))$ have the quotient norm.

It was proved by Herz [4] for an arbitrary group G, that $A_p(G)$ is a Banach algebra and

$$B_p(G) \subseteq M(A_p(G)) = \{u : uA_p(G) \subset A_p(G)\}.$$

If the group G is amenable, then

$$B_p(G) = M(A_p(G)). \qquad \text{(see [3])},$$

We show now that for a noncommutative free group F

$$B_p(F) \subsetneq M(A_p(F)).$$

3. Leinert and Strong Leinert Sets

We recall that the set $E \subseteq G$ is called *Leinert set* if there exists $C > 0$ such that for every function $f \in l_2(E)$

$$\|f\|_{VN} \leqq C\,|f|_2, \tag{6}$$

where
$$\|f\|_{VN} = \|Mf\|_2 = \sup\{|f * g|_2 : |g|_2 = 1\}$$

or equivalent by a duality

$$\chi_E \cdot A(G) = l_2(E), \tag{7}$$

where χ_E is the characteristic function of E.

By the classical argument, (7) is equivalent to the following statement: There exists $C>0$ such that

$$|\chi_E \cdot f|_2 \leqq C \|f\|_{A(G)} \quad \text{for} \quad \operatorname{supp} f \subset E. \tag{8}$$

Now we introduce the following

Definition. Let $1<p<\infty$. A set $E \subseteq G$ is called *p-Leinert set* ($E \in L(p)$) if

$$l_\infty(E) \subset M(A_p(G)).$$

A set $E \subset G$ is called *strong p-Leinert set* ($E \in L^*(p)$) if

$$l_\infty(E) \subset B_p(G).$$

The existence of an infinite Leinert set and a strong p-Leinert set in free noncommutative groups was proved by M. Leinert in his papers [5, 6].

Note that the existence of an infinite Leinert set in the group G implies that G is not amenable.

Proposition 1. (a) *A set $E \subset G$ is a Leinert set if and only if E is a 2-Leinert set.*

(b) *For $1 \subset p \leqq 2$, $L(2) \subset L(p)$.*

For the proof we use the following generalization of the Khinčin inequality given by M. Picardello [7].

Lemma (Picardello). *If* supp$f$ *is finite in a discrete group G, then there exists a function $r=r(f)$, $|r(x)|=1$ such that*

$$|f|_2 \leqq 2\sqrt{3}\, \|f \cdot r\|_{A(G)}. \tag{9}$$

Proof of Proposition 1. (a) If the set E is a Leinert set, then $\chi_E \cdot A(G) = l_2(E)$. Hence by (8) $l_\infty(E) \cdot A(G) = l_2(E)$, but $l_2(G) \subseteq A(G)$ so $l_\infty(E) \subset M(A(G))$ and this means that E is a 2-Leinert set.

Conversely, let E be a 2-Leinert set and take $f \in A(G)$ with finite support in G then by the Lemma we obtain:

$$|f \cdot \chi_E|_2 \leqq 2\sqrt{3}\, \|f \chi_E \cdot r\|_{A(G)} \leqq 2\sqrt{3}\, \|\chi_E \cdot r\|_{M(A)} \|f\|_{A(G)}. \tag{10}$$

But from the assumption

$$\|\chi_E \cdot r\|_{M(A)} \leqq D \|\chi_E \cdot r\|_\infty = D \quad \text{for fixed} \quad D = D(E),$$

follows now

$$|f \cdot \chi_E|_2 \leqq 2\sqrt{3}\, D \|f\|_{A(G)}.$$

So E is a Leinert set.

(b) If E is a Leinert set, then for $1<p \leqq 2$ by the interpolation theorem, we have the following inequality

$$|f * g|_p \leqq C_p |f|_p |g|_p$$

for every $f \in l_p(E)$ and $g \in l_p(G)$. Moreover, using a simple duality argument we get for $\varphi \in l_\infty(E)$ $a \in l_p(G)$, $b \in l_{p'}(G)$

$$\|\varphi(a*b)\|_{A_p} \leqq C_p |\varphi|_\infty |a|_p |b|_{p'},$$

hence $\|\varphi\|_{M(A_p)} \leqq C_p |\varphi|_\infty$, therefore $E \in L(p)$.

4. Construction

Let F be a noncommutative free group with free generators $\{x_1, x_2, \ldots\} = E$. Let $E^k = \{a_1 a_2 \ldots a_k : a_i \in E\}$.

In our construction we apply the following nice theorem of Haagerup [9]:

Theorem (Haagerup). *E^k is a Leinert set, $(k = 1, 2, \ldots)$*

The main results of this note are the following:

Theorem 1. *If $E = \{x_1, x_2, \ldots\}$ is the set of free generators in a free group F and $1 < p < \infty$, then E^2 is not a strong p-Leinert set.*

Theorem 2. *If F is a noncommutative free group and $1 < p < \infty$, then $B_p(F) \nsubseteq M(A_p(F))$.*

Proof of Theorem 1. It is enough to construct a sequence of functions φ_n such that:

(a) $\operatorname{supp} \varphi_n \subseteq E^2$

(b) $\|\varphi_n\|_{B_p} \geqq n^{1-1/p} |\varphi_n|_\infty$.

For this we produce a special unitary matrix as follows: Let $\mathbb{Z}_n = \{g_1, g_2, \ldots, g_n\}$ be the finite cyclic abelian group of order n, let $\hat{\mathbb{Z}}_n = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be the dual group and $a_{ij} = \gamma_i(g_j)$, $1 \leqq i, j \leqq n$.

Denote the matrix $(a_{ij})_{i,j \leqq n}$ by $A = A_n$, and $B = n^{-1/2} A$.

One can verify the following facts:

(x) B is the unitary matrix on $\mathbb{C}^n$

(xx) $\|A\|_2 = \sqrt{n}$

(xxx) $\||A|\|_2 = n$, where $|A| = (|a_{ij}|)$

(xxxx) For $1 < p \leqq 2$

$$\|B\|_p \leqq n^{1/p - 1/2}$$

(xv) $\|A\|_p = \|\bar{A}\|_p \leqq n^{1/p}$, where $\bar{A} = (\bar{a}_{ij})$

(xvv) $\||A\||_p \geqq \dfrac{\|A\bar{A}\|_p}{\|A\|_{p'}} \geqq n^{1-1/p}$

Now we can obtain our "bad" multipliers on a free group F with free generators $x_1, x_2, \ldots$ in the following manner:

For fixed $n = 1, 2, 3, \ldots$, let φ_n be a function, supported by the set $E_n = \{x_i x_j, 1 \leqq i, j \leqq n\} \subset E^2$, which is defined in the following way:

$$\varphi_n(x_i x_j) = a_{ij}$$

We show that $|||M\varphi_n|||_p \geqq n^{1-1/p}$, $(1<p\leqq 2)$. Let consider the matrix A_n as the matrix of the operator on $l_2(F)$ with respect to the orthonormal basis $\{\delta_x\}_{x\in G}$

$$\langle A_n(\delta_x), \delta_y\rangle = \begin{cases} a_{ij} & x=x_i, y=x_j, i,j\leqq n \\ 0 & \text{otherwise} \end{cases}$$

then $(M\varphi_n)\cdot A_n = |A_n|$ and by (xvv) we have

$$|||M\varphi_n|||_p = \|\varphi_n\|_{B_p} \geqq n^{1-1/p}.$$

Hence we get (b) for $1<p\leqq 2$. Since $\check{A}_p(G)=A_{p'}(G)$, $\check{B}_p(G)=B_{p'}(G)$ and $M(A_p)=M(A_{p'}(G))$.

The Theorem 1 is also true for $p>2$.

The Theorem 2 follows directly if we apply the theorem of Haagerup, the theorem 1 and the Proposition 1.

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