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Hermitian-Einstein Connections and Stable Vector Bundles Over Compact Complex Surfaces

N. P. Buchdahl
Department of Mathematics, Tulane University, New Orleans, LA 70118, USA

1. Introduction

Let X be a complex manifold of dimension n and E be a holomorphic vector bundle on X. It is well-known [2] that to each hermitian metric on E there is a unique hermitian connection inducing the $\bar{\partial}$-operator on E; the curvature F of this connection is an anti-self-adjoint section of $\Lambda^{1,1}\otimes \operatorname{End} E$. If h_0, h_1 are metrics on E, then the resulting curvatures are related by $F_1=F_0+\bar{\partial}_0(u^{-1}\partial_0 u)$, where u is the positive self-adjoint endomorphism $u=h_0^{-1}h_1$. Conversely, a unitary bundle with smooth unitary connection having curvature of type (1,1) inherits a unique holomorphic structure by the Newlander-Nirenberg theorem.

If X has a Kähler metric and ω is the Kähler form, then the Yang-Mills equations for connections of this type reduce to $d\hat{F}=0$, where $\hat{F}:=*\dfrac{1}{(n-1)!}(F\wedge\omega^{n-1})$. In this case, the bundle and connection split up into the eigenspaces of the endomorphism $\hat{F}$, so if the connection is irreducible or if E is simple, then $\hat{F}=i\lambda 1$ for some $\lambda\in\mathbb{R}$. Such a connection, introduced by Kobayashi and by Hitchin, is called *Hermitian-Einstein* $(H-E)$. The constant λ is determined by $c_1(E):\lambda=\lambda_E=-\dfrac{2\pi}{(n-1)!V}\cdot\mu(E)$, where $V=\operatorname{Vol}(X)$ and $\mu(E):=(c_1(E)\cup\omega^{n-1})[X]/\operatorname{rank}(E)$.

The quantity $\mu(E)$ also features in the algebro-geometric notion of stability: E is *(semi-)stable* in the sense of Mumford and Takemoto if every coherent subsheaf $S\subset\mathcal{O}(E)$ with $0<\operatorname{rank}S<\operatorname{rank}E$ satisfies $\mu(S)<\mu(E)$ ($\mu(S)\leqq\mu(E)$). (The definition of μ for sheaves is given in Sect. 3 below.)

In [17], Narasimhan and Seshadri proved that an indecomposable holomorphic bundle on a Riemann surface is stable iff it admits an irreducible $H-E$ connection; (their theorem is expressed in terms of projective unitary representations of the fundamental group). This result was later reproved by Donaldson [4] by a different method. About the same time, Kobayashi [14] and Lübke [16] showed that if a bundle on an arbitrary compact Kähler manifold admits an irreducible $H-E$ connection, then it is stable. In [5], Donaldson showed that in the case when X is an algebraic surface $X \hookrightarrow \mathbb{P}^N$ and ω is cohomologous to the restriction of the Fubini-Study form, the converse is also true. Recently, Uhlenbeck and Yau [23] have proved the general n-dimensional Kähler version of this theorem.

In [11], Hitchin observed that the notion of stability can be extended to bundles on an arbitrary hermitian n-manifold X: a theorem of Gauduchon [7] states that any hermitian metric on X has a conformal rescaling (unique up to a positive constant) so that the associated Kähler form ω of the rescaled metric satisfies $\bar{\partial}\partial\omega^{n-1}=0$. If L is a holomorphic line bundle on X, the *degree* of L (with respect to ω) can then be defined by $\deg(L)=\deg(L,\omega):=\frac{i}{2\pi}\int_X f\wedge\omega^{n-1}$, where f is the curvature of any hermitian connection on L compatible with $\bar{\partial}_L$. Since any two such forms differ by a $\bar{\partial}\partial$-exact form, $\deg(L)$ is independent of the choice of connection. If $d\omega=0$, then $\deg(L,\omega)$ is the usual topological degree $c_1(L)\cdot[\omega^{n-1}]$, but in general, $\deg(-,\omega)$ is not a topological invariant (cf. Proposition 2 below). Having defined the degree of holomorphic line bundles, the definition of stability can be repeated verbatim, and the definition of Hermitian-Einstein connection remains unaltered. Hitchin suggested that there should be a relationship between $H-E$ connections and stable bundles in this general setting.

The case when X is a compact complex surface is perhaps the most interesting, for it is in this case that the differential topology of the underlying 4-manifold is intricately connected with this problem. For example, using a deep application of his results in [5], Donaldson has given a counterexample to the 5-dimensional h-cobordism conjecture [6]. The interaction between the complex and real analysis stems from the fact that $H-E$ connections on bundles with $\mu=0$ are precisely the anti-self-dual Yang-Mills connections. [It should be noted however that if $d\omega\neq 0$, an $H-E$ connection on E is a Yang-Mills connection compatible with $\bar{\partial}_E$ iff $\deg(E,\omega)=0$.]
The main result to be proved here is (cf.[5]).

Theorem 1. *Let X be a complex surface with an hermitian metric whose Kähler form is $\bar{\partial}\partial$-closed. Then an indecomposable holomorphic bundle on X is stable iff it admits an irreducible Hermitian-Einstein connection. This connection is unique.*

("Stability" and "Hermitian-Einstein" are, of course, with respect to the given $\bar{\partial}\partial$-closed Kähler form.)

The proof of Theorem 1 is by induction on the rank of the bundle, and is based on Donaldson's proof [4] of the theorem of Narasimhan and Seshadri. In brief outline this runs as follows: given the stable bundle E on the Riemann surface X, a functional $J(A)$ is constructed on the space of hermitian connections A on E compatible with $\bar{\partial}_E$, essentially equivalent to the L^2 norm of $\hat{F}(A)-i\lambda_E 1$. Choosing a minimizing sequence A_i for J and employing Uhlenbeck's weak compactness theorem [22] for connections on bundles, a limit connection A' is obtained with $J(A')\leq\inf J(A_i)$. Now A' might define a different holomorphic structure E' on the smooth underlying bundle, but in any case, by a semi-continuity of cohomology argument, Donaldson shows that there is a non-zero holomorphic $\phi: E\to E'$. If ϕ is not an isomorphism, he shows that $J(A')\geq 4\pi V^{-1/2}\nu_E(\ker\phi)$, where $\nu_E(S):=(\operatorname{rank}S)(\mu(E)-\mu(S))$ for $S\subset E$ and $V=\operatorname{Vol}(X)$. On the other hand, using the canonical filtrations of Harder and Narasimhan [12] and the inductive hypothesis, he can construct a connection A on E (compatible with $\bar{\partial}_E$) with $J(A) < 4\pi V^{-1/2} \nu_E(\ker\phi)$. This contradiction means that A' is compatible with $\bar{\partial}_E$ and minimizes J. A simple argument then shows that for A' to minimize J, necessarily $J(A')=0$, giving $\hat{F}(A')=i\lambda_E 1$. The "only if" part of the argument is more straightforward.

The main features of Donaldson's proof also appear here, the biggest strategic difference being that the Harder-Narasimhan filtrations are avoided by reversing the order of his arguments. However, the technical differences are somewhat more significant, owing to the appearance of singularities of one sort or another: torsion-free sheaves are no longer locally free, and sequences of connections only converge off finite sets of points. These difficulties are resolved generally by blowing-up and by appealing to the appropriate removability of singularities theorem of Hartogs, Serre or Uhlenbeck. Moreover, some of the techniques used by Donaldson in [5] can still be employed and indeed, these too play an essential role in the proof to be given here. The introduction and first section of [5] also contains more background material, and in particular, a clear description of the two equivalent formulations of the problem; namely, finding a certain connection on a fixed $U(r)$-bundle, or finding a certain hermitian metric on a fixed holomorphic r-bundle.

2. Hermitian Geometry

Let X be a compact complex surface and h be an hermitian metric on X. In local holomorphic coordinates z^a, the associated Kähler form is $\omega := \frac{i}{2} h_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$; (all conventions here follow those in [10]). The volume form is $dV = \frac{1}{2}\omega \wedge \omega$, and if $* : \Lambda^{p,q} \to \Lambda^{2-q,2-p}$ is the Hodge $*$-operator, then with respect to the inner product $(f,g) \mapsto *(\bar{f} \wedge *g)$, the adjoint of $\Lambda^{p,q} \ni g \mapsto g \wedge \omega \in \Lambda^{p+1,q+1}$ is denoted by $f \mapsto \Lambda f$. On (1,1)-forms $f = f_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$, $\Lambda f = -2ih^{a\bar{b}} f_{a\bar{b}}$, frequently denoted by $\hat{f}$. Note that $\Lambda\omega = 2$.

The $*$-operator on 2-forms satisfies $*^2 = 1$, and the decomposition into $\pm$eigenspaces is $\Lambda^2_+ = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \mathrm{span}(\omega)$, $\Lambda^2_- = \ker\Lambda : \Lambda^{1,1} \to \Lambda^0$.

With respect to the inner product $(f,g) \mapsto \int_X \bar{f} \wedge *g$, a straightforward calculation gives

$$\partial^* g = -*\bar{\partial}*g = i\Lambda\bar{\partial}g + i*(\bar{\partial}\omega \wedge g) , \quad g \in \Lambda^{1,0} , \tag{2.1a}$$

$$\partial^* f = -*\bar{\partial}*f = i(\Lambda\bar{\partial} - \bar{\partial}\Lambda)f - (*\bar{\partial}\omega)\Lambda f , \quad f \in \Lambda^{1,1} . \tag{2.1b}$$

Let P be the second-order real elliptic operator on functions $P := i\Lambda\bar{\partial}\partial$, (so if h is flat, $P = \frac{1}{2}\Delta$ where Δ is the usual Laplacian having *negative* symbol). Then $P^* f = *i\bar{\partial}\partial(\omega f) = i\Lambda\bar{\partial}\partial f + i*(\bar{\partial}\omega \wedge \partial f) - i*(\partial\omega \wedge \bar{\partial} f) + i(*\bar{\partial}\partial\omega)f$. That is,

$$P^* = P + i*\bar{\partial}\omega \wedge \partial - i*\partial\omega \wedge \bar{\partial} + i*\bar{\partial}\partial\omega . \tag{2.2}$$

From (2.1a) and its complex conjugate, it follows easily that

$$\Delta' = \partial^*\partial = P + i*\bar{\partial}\omega \wedge \partial , \tag{2.3a}$$

$$\Delta'' = \bar{\partial}^*\bar{\partial} = P - i\Lambda(\partial\bar{\partial} + \bar{\partial}\partial) - i*\partial\omega\Lambda\bar{\partial} , \tag{2.3b}$$

$$\Delta = \Delta' + \Delta'' = 2\Delta'' + i\Lambda(\partial\bar{\partial} + \bar{\partial}\partial) + i*d\omega \wedge d . \tag{2.3c}$$

[Of course, $\bar\partial\partial+\partial\bar\partial=0$ on functions, but (2.3) is valid for an arbitrary hermitian connection on a bundle, in which case $\partial\bar\partial+\bar\partial\partial$ is the (1,1) component of the curvature.] Adding (2.3a) and (2.3b) and using (2.2) also gives

$$\Delta=P+P^*-i\Lambda(\partial\bar\partial+\bar\partial\partial)-i*\bar\partial\partial\omega\ . \tag{2.4}$$

Now suppose that the metric h has been conformally scaled according to the theorem of Gauduchon [7] so that $\bar\partial\partial\omega=0$. Then a number of easy but important consequences follow from these equations. The first of these is the existence of $H-E$ connections on holomorphic line bundles. For if L is a line bundle with hermitian connection compatible with $\bar\partial_L$ and curvature $f\in\Lambda^{1,1}(X)$, any other such curvature form has curvature $f+\bar\partial\partial\log u$ for some positive function u. Thus the equation to be solved is $P\log u=-i\hat{f}-\lambda$ where $\int_X(i\hat{f}+\lambda)dV=0$. From (2.4), $\Delta=P+P^*$ on functions, so ker $P^*=\mathbb{R}$. By standard linear elliptic theory on compact manifolds, there exists a smooth solution u to $P\log u=-i\hat{f}-\lambda$, unique up to multiplication by a positive constant.

Next suppose that E is a holomorphic bundle with $H-E$ connection: $\hat{F}=\Lambda F=i\lambda 1$ for $\lambda=-2\pi V^{-1}\mu(E,\omega)$. If s is a global holomorphic section, then from (2.3)(c), $\|ds\|^2=\langle s,\Delta s\rangle=-\lambda\|s\|^2+\langle s,*\,id\omega\wedge ds\rangle$, ($ds$ denoting the covariant derivative of s). But $\langle s,*d\omega\wedge ds\rangle=\langle s,*\,\bar\partial\omega\wedge\partial s\rangle=\langle s,*\,[-\partial(\bar\partial\omega s)+\partial\bar\partial\omega s]\rangle=-\langle *\,s,\partial(\bar\partial\omega s)\rangle=-\langle\partial^**\,s,\bar\partial\omega s\rangle=\langle *\,\bar\partial s,\bar\partial\omega s\rangle=0$, so $\|ds\|^2=-\lambda\|s\|^2$. Thus, just as in the Kähler case, one has the result of Kobayashi [13]:

Proposition 1. *Let X be a compact surface with a metric whose Kähler form is $\bar\partial\partial$-closed. If E is a holomorphic bundle on X which admits an $H-E$ connection, then if $\mu(E)<0$ it follows that $H^0(X,\mathcal{O}(E))=0$, and if $\mu(E)=0$, every holomorphic section is covariantly constant.* □

Corollary 1. *If L is a holomorphic line bundle on the compact surface X such that $H^0(X,L)\neq0$, then $\deg(L,\omega)\geqq0$ for any positive $\bar\partial\partial$-closed $(1,1)$-form ω, with equality iff L is trivial.* □

If s is a holomorphic section of L, it follows from the Poincaré-Lelong theorem [10] that $\deg(L,\omega)=\mathrm{Vol}(s^{-1}(0),\omega)$.

Corollary 2. *Let ω be a positive $\bar\partial\partial$-closed $(1,1)$-form on the compact surface X, and let $\{e_1,\dots,e_m\}$ be an integral basis for $H^2(X,\mathbb{Z})/$torsion. Then there exists $\varepsilon=\varepsilon(\omega)>0$ such that any holomorphic line bundle L on X with $c_1(L)\equiv\sum n^\alpha e_\alpha \bmod$ torsion and $H^0(X,L)\neq0$ satisfies $\deg(L,\omega)\ \geqq\varepsilon\sum|n^\alpha|$.*

Proof. Let $e_\alpha\cdot e_\beta=q_{\alpha\beta}$ be the intersection matrix on $H^2(X,\mathbb{Z})/$torsion, $q^{\alpha\beta}$ the inverse. If f_α is a closed 2-form representing e_α, the $(1,1)$-component $\tilde{f}_\alpha$ of f_α is $\bar\partial\partial$-closed. If $\varepsilon>0$ is sufficiently small, $\omega\pm\varepsilon m\sum q^{\alpha\beta}\tilde{f}_\beta$ is positive for any $\alpha=1,\dots,m$. By Corollary 1, $0\leqq\deg(L,\omega\pm\varepsilon m\sum q^{\alpha\beta}\tilde{f}_\beta)=\deg(L,\omega)\pm\varepsilon mn^\alpha$, [for if $f\in\Lambda^{1,1}$ represents $c_1(L)$, $\int f\wedge f_\beta=\int f\wedge\tilde{f}_\beta$]. Thus $\deg(L,\omega)\geqq\varepsilon m|n^\alpha|$ for all α, and summing over α gives the desired conclusion. □

Corollary 3. *An $H-E$ connection on an indecomposable bundle is unique if one exists.*

Proof. (cf. [4]). If E is a smooth unitary bundle with two integrable unitary connections A_0, A_1 inducing isomorphic holomorphic structures E_0, E_1 then, by definition, there is a complex automorphism g of E such that $\bar\partial_1 = g \circ \bar\partial_0 \circ g^{-1}$ and $\partial_1 = g^{*-1} \circ \partial_0 \circ g^*$. After a unitary change of gauge of one of them $[g(g^*g)^{-1/2}]$, g can be assumed positive self-adjoint. If A_0, A_1 are $H-E$ connections, then the (holomorphic) isomorphism $g: E_0 \to E_1$ is covariantly constant by Proposition 1, implying $0 = \partial_0(g^*g) = \partial_0(g^2)$, and $\bar\partial_0(g^2) = 0$. Since E_0 is indecomposable, $g^2 = \text{const}\, 1$ and since g is positive self-adjoint, $g = \text{const}\, 1$. □

The next corollary is taken verbatim from [5]. For the proof (which is short), see that reference.

Corollary 4. *Suppose that the main theorem has been proved for bundles of rank less than r. Then any r-bundle which admits an Hermitian-Einstein connection is a direct sum* $\sum E_i$ *of stable bundles* E_i *with* $\mu(E_i) = \mu(E)$. *In particular, it is semi-stable. If E admits an irreducible such connection, it is stable.* □

A slightly different version of (2.3) (c) will be of use subsequently. Suppose that E is a bundle with integrable hermitian connecting having curvature F. Then (2.3c) gives $\Delta = 2\Delta'' + i\hat{F} + i^*d\omega \wedge d$ for the full covariant Laplacian on sections. So if s is a local holomorphic section, $\Delta|s|^2 = \Delta\langle s, s\rangle = 2\langle s, \Delta s\rangle - 2|ds|^2 = 2\langle s, i\hat{F}s\rangle + 2i\langle s, *d\omega \wedge ds\rangle - 2|ds|^2$. Using the same manipulations as before, together with $\bar\partial s = 0 = \bar\partial\partial\omega$, one computes $\langle s, *d\omega \wedge ds\rangle = -*\partial(|s|^2\bar\partial\omega)$. Thus $\Delta|s|^2 + 2i*\partial(|s|^2\bar\partial\omega) = 2\langle s, i\hat{F}s\rangle - 2|ds|^2$. Since $i\hat{F}$ is a real operator, taking the complex conjugate of this last equation and adding gives

$$\Delta|s|^2 + i*\partial(|s|^2\bar\partial\omega) - i*\bar\partial(|s|^2\partial\omega) = 2\langle s, i\hat{F}s\rangle - 2|ds|^2\,, \quad (s \text{ holomorphic})\,, \tag{2.5}$$

which is the unintegrated version of the equation used for Proposition 1. Note that since $\bar\partial\partial\omega = 0$, the operator on the left of (2.5) satisfies the maximum principle, by Theorem 3.1 of [8].

The last application of (2.1)–(2.4) is the result mentioned in the introduction on the topological invariance of $\deg(-, \omega)$.

Proposition 2. *If* ω *is a positive* $\bar\partial\partial$*-closed* (1,1)*-form on the compact surface* X, *then* $\deg(L, \omega) = \frac{i}{2\pi}\int_X F_L \wedge \omega$ *depends only on the image of* $c_1(L)$ *in* $H^2(X, \mathbb{R})$ *iff* $b_1(X)$ *is even.*

Remark. $b_1(X)$ even is equivalent to the existence of a Kähler metric on X by results of Kodaira, Siu.

Proof of Proposition. Suppose $b_1(X)$ is even. Under the map $H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*)$ induced by $0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$, a representative $\bar\partial$-closed (0,1)-form g is mapped to $\frac{i}{2\pi}\int_X (\partial g - \bar\partial\bar g) \wedge \omega$ by $\deg(-, \omega)$, and of course, this map annihilates the image of $H^1(X, \mathbb{Z})$ in $H^1(X, \mathcal{O})$. Since b_1 is even, $H^1(X, \mathcal{O})$ has real dimension b_1 [3], and since $H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O})$ is always injective, $\deg: H^1(X, \mathcal{O}) \to \mathbb{R}$ must be zero, otherwise the kernel would contain a lattice of rank greater than its dimension. Thus $\deg(L,\omega)$ depends only on $c_1(L)\in H^2(X,\mathbb{Z})$ in this case. Since $\int(\partial g-\bar\partial\bar g)\wedge\omega=0$ for all $\bar\partial$-closed (0,1)-forms g, replacing g by ig shows that $\int\partial g\wedge\omega=0$ for all sich g, and similarly $\int\bar\partial h\wedge\omega=0$ for all ∂-closed (1,0)-forms h. Thus if f_0,f_1 are (1,1)-forms such that $f_0-f_1=dh$ for some $h\in A^1$, then $\bar\partial h_{0,1}=0=\partial h_{1,0}$ giving $\int(f_0-f_1)\wedge\omega=\int(\partial h_{0,1}+\bar\partial h_{1,0})\wedge\omega=0$. Thus $\deg(L,\omega)$ depends only on the image of $c_1(L)$ in $H^2(X,\mathbb{R})$.

Now suppose that $\int(\partial g-\bar\partial\bar g)\wedge\omega=0$ for all ∂-closed (0,1)-forms g. Then as above, $\int\partial g\wedge\omega=0$ for all $\bar\partial$-closed $g\in A^{1,0}$. Given such g, the equation $Pu=i\Lambda\partial g$ has a solution u since $\int\Lambda\partial g dV=\int\partial g\wedge\omega=0$, and moreover u is unique up to the addition of a constant. But this is just $\Lambda\partial\tilde g=0$, where $\tilde g:=g+\bar\partial u$. From (3.1) (b) it now follows that $\langle\partial\tilde g,\partial\tilde g\rangle=\langle\tilde g,\partial^*\partial\tilde g\rangle=\langle\tilde g,[(\Lambda\bar\partial-\bar\partial\Lambda)+*\bar\partial\omega\Lambda]\partial\tilde g\rangle=0$, so g gives the unique $\bar\partial$-closed (1,0)-form $g':=\bar{\tilde g}$. Conversely, every holomorphic 1-form on a compact surface is closed [3], so that the map $H^1(X,\mathcal{O})\to H^0(X,\Omega^1)$ defined this way is invertible. Thus $h^{1,0}(X)=h^{0,1}(X)$ and $b_1(X)=h^{1,0}(X)+h^{0,1}(X)$ is even. □

Remark. An easy continuation of this argument shows that when $b_1(X)$ is even, any real $\bar\partial\partial$-closed (1,1)-form ω is cohomologous mod im $\partial+\bar\partial$ to a d-closed real (1,1)-form, and any two such (cohomologous) d-closed (1,1)-forms differ by a d-exact term, so ω defines a unique element of $H^2(X,\mathbb{R})$.

In order to use the inductive hypothesis to prove Theorem 1, it is necessary to find sub-bundles of a given bundle. However, in general one can expect to find at most *subsheaves* which are sub-bundles off a finite set of points. To get sub-bundles therefore, these singular points have to be blown-up, and then appropriate metrics must be constructed on the blown-up space. For details of what follows, see [10, pp. 182–187].

Let x be a point on the surface X and let $\tilde X\xrightarrow{\pi}X$ be the blow-up of X at x. Given the positive (1,1)-form ω on X, $\pi^*\omega$ is degenerate on the exceptional divisor $L=\pi^{-1}(x)$, but it can be modified as follows. If U is a sufficiently small neighbourhood of x and $\tilde U:=\pi^{-1}(U)$, then there is a holomorphic projection $\pi_2:\tilde U\to\mathbb{P}_1$. Now L is the zero set of a section $s\in\Gamma(\tilde X,\mathcal{O}(-1))$, so let h_0 be the metric on $\mathcal{O}(-1)(:=\mathcal{O}(L))$ over $\tilde X\backslash L$ such that $|s|\equiv1$, and let h_1 be the standard metric on $\mathcal{O}(-1)$ over $\mathbb{P}_1$. Let ϱ be any cut-off function with support in U such that $\varrho=1$ on a neighbourhood of x. Then $h:=(1-\varrho)h_0+\varrho\pi_2^*h_1$ is a metric on $\mathcal{O}(-1)$ and and the resulting Chern form is $\sigma:=\dfrac{i}{2\pi}\bar\partial\partial\log h\in A^{1,1}(\tilde X)$. σ is identically zero outside of $\tilde U$ and is negative definite in directions tangent to L in a neighbourhood of L. Thus, for sufficiently small ε, $\tilde\omega_\varepsilon:=\pi^*\omega-\varepsilon\sigma$ is positive.

If ω is $\bar\partial\partial$-closed, resp. d-closed, then so too is $\tilde\omega_\varepsilon$, and if ω is rational ($d\omega=0$ and $[\omega]\in H^2(X,\mathbb{Q})$), so too is $\tilde\omega$ if ε is rational. These are the metrics used for the Kodaira embedding theorem.

If ω is $\bar\partial\partial$-closed, then in a neighbourhood W of x, $\omega=\partial u+\bar\partial V$ for some $u\in A^{0,1}$, $v\in A^{1,0}$. Since $\int_{\tilde X}\pi^*\omega\wedge\sigma$ does not depend on the choice of σ, it can be supposed that supp $\sigma\subset\subset\tilde W$, from which it follows that $\int_{\tilde X}\pi^*\omega\wedge\sigma=0$. Similarly, $\deg(-,\tilde\omega_\varepsilon)$ does not depend on the choice of σ, only on ε. Note also that since L has self-intersection -1, $\int_{\tilde X}\sigma\wedge\sigma=-1$ and $\mathrm{Vol}(\tilde X,\omega_\varepsilon)=\frac12\int_{\tilde X}\tilde\omega_\varepsilon^2=\mathrm{Vol}(X)-\frac12\varepsilon^2$.

Finally, note that if f is a 1-form on X, then $\lim_{\varepsilon\to 0} |\pi^* f| \leqq \pi^*|f|$, where the norm on the left (resp. right) is with respect to $\tilde{\omega}_\varepsilon$ (resp. ω). This is easily checked by using coordinates for which $\omega(x)$ is standard; (equality holds at any point off L).

3. Desingularization of Sheaves

It is well-known that singularities on surfaces can be resolved by blowing-up [3], and the same is true for coherent analytic sheaves. This will be indicated shortly, but first a number of basic facts about sheaves will be recalled, taken directly from [18, pp. 139–160] (see also [9]).

Let B be a coherent analytic sheaf on a complex manifold X. The singularity set of B is $S(B)=\{x\in X: B_x \text{ is not a free } \mathcal{O}_x\text{-module}\}$ and is an analytic set in X of codimension $\geqq 1$. Thus B has a well-defined rank, b say. The torsion subsheaf $\tau(B)$ is defined by $\tau(B)_x=$ torsion submodule of B_x, and $\tau(B)$ is coherent. If $\tau(B)=0$, then B is *torsion-free* and $\operatorname{codim} S(B)\geqq 2$. Thus if X is compact and B is torsion-free, B has a well-defined first Chern class. An equivalent definition of torsion-free is that the canonical homomorphism $B\to B^{**}$ is injective, where $B^*:=\operatorname{Hom}(B,\mathcal{O})$. If $B=B^{**}$, then B is *reflexive* and $\operatorname{codim} S(B)\geqq 3$. In general, B is reflexive iff it is torsion-free and *normal*, where normal means that $\Gamma(U,B)\to\Gamma(U\backslash A,B)$ is injective for any analytic set A of $\operatorname{codim}\geqq 2$ in an open set $U\subset X$. Thus for arbitrary B, it follows B^* is reflexive. In general, a reflexive sheaf of rank 1 is a line bundle, so the *determinant* of a coherent analytic sheaf B of rank b is $\det B:=(\Lambda^b B)^{**}$. If $B\to C$ is a monomorphism of torsion-free sheaves of ranks $b\leqq c$, then $\Lambda^b B\to\Lambda^b C$ is also a monomorphism since the kernel is a torsion subsheaf; thus if $b=c$, $\det B\to\det C$ is also a monomorphism.

If $0\to A\to B\to C\to 0$ is an exact sequence of sheaves with B reflexive, then Lemma 1.1.16 of [18] states that A is normal if C is torsion-free. If C is not torsion-free, then the maximal normal extension $\hat{A}_B$ of A in B is given by $\hat{A}_B:=\ker[B\twoheadrightarrow C/\tau(C)]$; thus there is a monomorphism $A\to\hat{A}_B$ and in this way it generally suffices to deal with reflexive subsheaves of bundles in questions related to stability.

In the case when X is a compact surface, torsion-free sheaves are singular only at finitely many points and reflexive sheaves are locally free. If ω is a positive $\bar{\partial}\partial$-closed (1,1)-form on X, the degree of a coherent analytic sheaf B of rank b on X is $\deg(B)=\deg(B,\omega):=\deg(\det B,\omega)$, and $\mu(B)=\mu(B,\omega):=\deg(B,\omega)/b$. It follows from Corollary 1 that if $B\to C$ is a monomorphism of torsion-free sheaves of the same rank, then $\mu(B)\leqq\mu(C)$. Also, despite its possibly non-topological nature, $\deg(-,\omega)$ behaves well with respect to exact sequences $0\to A\to B\to C\to 0$ of torsion-free sheaves, for since $\det B\simeq(\det A)\otimes(\det C)$ off a finite set of points, this isomorphism extends by Hartogs' theorem to all of X, giving $\deg(B)=\deg(A)+\deg(C)$.

With these preliminaries out of the way, the desingularization of torsion-free sheaves on surfaces can now be described.

Let B be a torsion-free sheaf in a neighbourhood of $0\in\mathbb{C}^2$, singular only at 0. Then in a neighbourhood of 0, B is given by an exact sequence $0\to\mathcal{O}^m\xrightarrow{f}\mathcal{O}^n\to B\to 0$, where $f(x)$ is an $n\times m$ matrix of holomorphic functions which has rank m for $x\neq 0$.

A measure of the degree of the singularity at 0 is given by rank $f(0)$. If this is zero, a second measure is given by the smallest integer p such that $\mathfrak{m}_0^p$ is contained in the ideal $I(f)_0$ generated by the germs of the $m\times m$ subdeterminants of f, where $\mathfrak{m}_0$ is the maximal ideal of $\mathcal{O}_{\mathbb{C}^2,0}$.

By elementary row and column operations, f is equivalent to a matrix of the form $\begin{pmatrix}1 & 0\\ 0 & g\end{pmatrix}$ where 1 is the unit $k\times k$ matrix ($k=\operatorname{rank} f(0)$) and $g(0)=0$. Blowing-up the origin gives $\pi^*g=\tilde{g}s$ where $s=\operatorname{diag}(t^{a_1},\dots,t^{a_{m-k}})$, $a_i>0$, $t\in\Gamma(\mathcal{O}(-1))$ defining the exceptional divisor L, with $\tilde{g}$ nonsingular and having a non-zero entry in each column. In terms of diagrams, this is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^k\oplus\mathcal{O}^{m-k} & \xrightarrow{1\oplus\pi^*g} & \mathcal{O}^k\oplus\mathcal{O}^{n-k} & \longrightarrow & \pi^*B & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle 1\oplus s} & & \Big\| & & \Big\downarrow & & \\
0 & \longrightarrow & \mathcal{O}^k\oplus\Sigma\mathcal{O}(-a_i) & \xrightarrow{1\oplus\tilde{g}} & \mathcal{O}^k\oplus\mathcal{O}^{n-k} & \longrightarrow & \tilde{B} & \longrightarrow & 0\,.
\end{array}$$

Here $\tilde{B}$ is defined by the lower row.

Now let $\tilde{B}_1:=\tilde{B}/\tau(\tilde{B})$, $\tilde{A}:=\ker[\mathcal{O}^n\to\tilde{B}_1]$, so $\tilde{A}$ is locally free and the map $\tilde{f}:\tilde{A}\to\mathcal{O}^n$ is of rank $\geq k+\operatorname{rank}\tilde{g}$ at each point. In particular, $\tilde{f}$ has rank m off L and rank $>k$ at generic points of L. If $k=0$, then at every point $x\in L$, the smallest p such that $\mathfrak{m}_x^p\subset I(\tilde{f})_x$ is clearly less than that for $I(f)_0$. In this case, the procedure can be repeated at each of the singular points of $\tilde{B}_1$ until eventually the rank of the derived map $\tilde{f}$ is positive at every point. Thus in either case, the rank of $\tilde{f}$ can be increased by blowing-up, and after finitely many such blow-ups a diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^n & \xrightarrow{\pi^*f} & \mathcal{O}^n & \longrightarrow & \pi^*B & \longrightarrow & 0\\
 & & \Big\downarrow & & \Big\downarrow\wr & & \Big\downarrow & & \\
0 & \longrightarrow & \tilde{A} & \longrightarrow & \mathcal{O}^n & \longrightarrow & \tilde{B} & \longrightarrow & 0
\end{array}$$

is arrived at, where the lower row is an exact sequence of bundles.

It follows from the above that if $0\to A\to E\to B\to 0$ is an exact sequence of sheaves on a compact surface X with E locally free and B torsion-free, then there is a modification $\tilde{X}\xrightarrow{\pi}X$ consisting of finitely many blow-ups and vector bundles $\tilde{A},\tilde{B}$ on $\tilde{X}$ such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi^*A & \longrightarrow & \pi^*E & \longrightarrow & \pi^*B & \longrightarrow & 0\\
 & & \Big\downarrow & & \Big\downarrow\wr & & \Big\downarrow & & \\
0 & \longrightarrow & \tilde{A} & \longrightarrow & \pi^*E & \longrightarrow & \tilde{B} & \longrightarrow & 0
\end{array}\tag{3.1}$$

has exact rows, commutes, and has the lower row an exact sequence of bundles. Moreover, off the exceptional divisor, the vertical arrows are isomorphisms. This will be referred to as a *desingularization* of B.

Remarks. (a) Since A is locally free, so too is π^*A, so $\pi^*A \to \pi^*E$ is is a monomorphism of sheaves even though π is not flat. Moreover, since $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and $\pi^1_*\mathcal{O}_{\tilde{X}} = 0$ [3, Theorem I.9.1] it follows that $\pi_*\pi^*A = A$ and $\pi^1_*\pi^*A = 0$. Applying π_* to the top row of (3.1) then gives $\pi_*\pi^*B = B$ and since $\ker(\pi_*\pi^*B \to \pi_*\tilde{B})$ is a torsion sheaf and B is torsion-free, it follows $B \to \pi_*\tilde{B}$ is injective; this implies $\pi_*\tilde{A} = A$.

(b) In general, if $0 \to A' \to \pi^*E \to B' \to 0$ is exact with B' torsion-free, then π_*B' is torsion-free so $K := \ker[\pi_*B' \to \pi_*A']$ is also; this implies π_*A' is locally-free. If L is any component of the exception divisor and $A'|_L = \sum \mathcal{O}(a_i)$, then necessarily $a_i \leqq 0$ for all i because $A'|_L \to \pi^*E|_L$ is injective off a finite set and $\pi^*E|_L$ is trivial. (If all a_i vanish it is easy to show $A' = \hat{\pi}^*\hat{\pi}_*A'$, where $\hat{\pi}$ is the blowing-down map for L.)

(c) If X is compact with positive $\bar{\partial}\partial$-closed (1,1)-form ω and $\tilde{X} \xrightarrow{\pi} X$ is the blow-up of X at $x \in X$, let $\tilde{\omega}_\varepsilon = \pi^*\omega - \varepsilon\sigma$ be one of the forms constructed in Sect. 2. If $\tilde{C}$ is a line bundle on $\tilde{X}$, then by [3, Theorem I.9.1], $\tilde{C} = \pi^*C \otimes \mathcal{O}(k)$ for some $C \in \mathrm{Pic}(X)$. Since $\pi_*\mathcal{O}(k) = \mathcal{O}_X$ if $k \leqq 0$ and $\pi_*\mathcal{O}(k) = \mathfrak{m}_x^k$ for $k > 0$, $\pi_*\tilde{C} = C$ or $C \otimes \mathfrak{m}_x^k$. In either case, $\det(\pi_*\tilde{C}) = C$, so it follows that $\deg(\tilde{C}, \tilde{\omega}_\varepsilon) = \deg(C, \omega) - \varepsilon\sigma \cdot c_1(\tilde{C}) = \deg(\pi_*\tilde{C}, \omega) - \varepsilon\sigma \cdot c_1(\tilde{C})$. If now $\tilde{C}$ is an arbitrary torsion-free sheaf on $\tilde{X}$, then $\pi_*\tilde{C}$ is a torsion-free sheaf on X and the isomorphism $\det\pi_*\tilde{C} = \pi_*\det\tilde{C}$ off a finite subset extends to an isomorphism $\det\pi_*\tilde{C} = \det[\pi_*\det\tilde{C}]$ over X by Hartogs' theorem. Thus $\deg(\tilde{C}, \tilde{\omega}_\varepsilon) = \deg(\det\tilde{C}, \tilde{\omega}_\varepsilon) = \deg(\pi_*\det\tilde{C}, \omega) - \varepsilon\sigma \cdot c_1(\det\tilde{C}) = \deg(\det\pi_*\tilde{C}, \omega) - \varepsilon\sigma \cdot c_1(\det\tilde{C}) = \deg(\pi_*\tilde{C}, \omega) - \varepsilon\sigma \cdot c_1(\tilde{C})$.

(d) With $X, \tilde{X}$ is in (c), suppose that $L = \pi^{-1}(x)$ is the exceptional line and $\tilde{C}$ on $\tilde{X}$ is locally free of rank n. Suppose, moreover, that $\tilde{C}|_L = \sum \mathcal{O}(-a_i)$ for some $a_i \geqq 0$ and that $C := \pi_*\tilde{C}$ is locally free.

By the Riemann-Roch theorem, the holomorphic Euler characteristic for $\tilde{C}$ is given by $\chi(\tilde{C}) = \frac{1}{2}p_1(\tilde{C}) + \frac{1}{2}c_1(\tilde{X}) \cdot c_1(\tilde{C}) + n\chi(\mathcal{O}_{\tilde{X}})$, where $p_1 = c_1^2 - 2c_2$ and $\chi(\mathcal{O}_{\tilde{X}})$ is the birational invariant $\frac{1}{12}(c_1(\tilde{X})^2 + c_2(\tilde{X})) = \frac{1}{12}(c_1(X)^2 + c_2(X))$. Moreover, $c_1(\tilde{X}) = c_1(X) + t$, where $t = c_1(\mathcal{O}(1))$. On the other hand, using the Leray spectral sequence, $\chi(\tilde{C}) = \chi(C) - \chi(\pi^1_*\tilde{C})$. Now, $\pi^1_*\tilde{C}$ is supported at x and thus is annihilated by $\mathfrak{m}_x^{p+1}$ for sufficiently large p (by the Rückert Nullstellensatz [9]), and it follows that $\pi^1_*\tilde{C} = \pi^1_*\tilde{C}|_{L^{(p)}}$ where $L^{(p)}$ is the p-th formal neighbourhood of L in $\tilde{X}$. From the exact sequences $0 \to \mathcal{O}_L(q) \to \mathcal{O}_{L^{(q)}} \to \mathcal{O}_{L^{(q-1)}} \to 0$, it follows that if a_1, say, is the largest a_i, then $\tilde{C}(a_1)|_L$ has a non-vanishing section extending to all orders. By induction, $\tilde{C}|_{L^{(p)}}$ can be expressed in terms of extensions by line bundles $\mathcal{O}(-a_i)$, so for purposes of computing $\chi(\pi^1_*\tilde{C})$ it can be supposed that $\tilde{C} = \sum \mathcal{O}(-a_i)$. Since $\pi_*\mathcal{O}(-a_i) = \mathcal{O}_{\tilde{X}}$, the Riemann-Roch formula gives $\chi(\pi^1_*\tilde{C}) = \chi(\pi^1_*\sum \mathcal{O}(-a_i)) = \chi(\sum \mathcal{O}_X) - \chi(\sum \mathcal{O}(-a_i)) = n\chi(\mathcal{O}_X) - \sum \chi(\mathcal{O}(-a_i)) = n\chi(\mathcal{O}_X) - [\sum \frac{1}{2}a_i(1 - a_i) + \chi(\mathcal{O}_{\tilde{X}})] = \frac{1}{2}\sum a_i(a_i - 1)$. Substituting this into $\chi(\tilde{C}) = \chi(C) - \chi(\pi^1_*\tilde{C})$ and using $c_1(\tilde{C}) = c_1(C) - at$ for $a = \sum a_i$ gives $p_1(C) = p_1(\tilde{C}) + \sum a_i^2$. In particular, $p_1(C) \geqq p_1(\tilde{C})$.

(e) If E is a holomorphic bundle on the compact surface X, then the Chern classes of holomorphic subbundles $E' \subset E$ must satisfy certain restrictions. To see this, *fix* an hermitian metric on E, so E' and the quotient E'' have induced hermitian metrics. In a unitary frame, the induced connection A on E has the form

$$A=\begin{bmatrix} A' & \beta \\ -\beta^* & A'' \end{bmatrix},$$

where A', A'' are the induced connections on E', E'' and $\beta \in \Lambda^{0,1}(\mathrm{Hom}(E'', E'))$ is a $\bar{\partial}$-closed form representing the extension $0 \to E' \to E \to E'' \to 0$, (cf. e.g. [4]). (Conversely, A', A'', β gives E as smooth bundle a holomorphic structure, and any $\tilde{\beta}$ of the form $t\beta + \bar{\partial}\gamma$ for $t \in \mathbb{C}\backslash 0$ gives an isomorphic structure.) The curvature of this connection is

$$F=F(A)=\begin{bmatrix} F'-\beta \wedge \beta^* & \nabla\beta \\ -\nabla\beta^* & F''-\beta^* \wedge \beta \end{bmatrix}. \tag{3.2}$$

The characteristic class $p_1(E)=(c_1^2-2c_2)(E)$ is given by $p_1(E)=\frac{-1}{4\pi^2}\int_X \mathrm{tr} F \wedge F$, so if ω is a positive (1,1)-form on X,

$$p_1(E)=\frac{1}{4\pi^2}(\|F_+\|^2-\|F_-\|^2)=\frac{1}{4\pi^2}(\tfrac{1}{2}\|\hat{F}\|^2-\|F_-\|^2). \tag{3.3}$$

The first and second Chern forms are $c_1=\frac{i}{2\pi}\mathrm{tr} F$ and $c_2=\frac{1}{8\pi^2}[\mathrm{tr} F^2-(\mathrm{tr} F)^2]$, (where $F^2 := F \wedge F$). With $G := F' - \beta \wedge \beta^*$ and $B := \beta \wedge \beta^*$, one calculates $(c_2-c_1^2)(F')=\frac{1}{8\pi^2}[\mathrm{tr} G^2+(\mathrm{tr} G)^2+2\mathrm{tr}(G \wedge B)+2(\mathrm{tr} G \wedge \mathrm{tr} B)]-(2\pi)^{-2}\mathrm{tr}\gamma \wedge \gamma^*$, where γ is the component of $\beta \otimes \beta$ in $\Lambda^{0,2} \otimes S^2E' \otimes \Lambda^2 E''^*$ (cf. [10, pp. 416–418] for similar calculations). It follows that there are constants C_1, $C_2>0$ depending only on the sup norm of $F(A)$, and thus only on E and ω, such that $*(c_2-c_1^2)(F') \leqq C_1+C_2|\beta|^2$. Furthermore, since β is a (0,1)-form, $|\beta|^2 = -i\mathrm{tr}\Lambda\beta \wedge \beta^* = i\mathrm{tr}\hat{G} - i\mathrm{tr}\hat{F}$, so if ω is $\bar{\partial}\partial$-closed, it follows that $\int |\beta|^2 dV \leqq -2\pi \deg(E', \omega)+\text{const}$. Thus there are constants C_4, $C_5>0$ depending only on E and ω such that $(c_2-c_1^2)(E') \leqq C_4 - C_5 \deg(E', \omega)$.

Now suppose that $A \subset E$ only has torsion-free quotient. Let $\tilde{X} \xrightarrow{\pi} X$ be a desingularizing space for E/A and $\tilde{A}$ be the "desingularization" of A. For the metrics $\tilde{\omega}_\varepsilon$ on $\tilde{X}$ constructed as in Sect. 2, $|\pi^* f|$ compares uniformly with $|f|$ for a two-form f on X by choosing the scaling factors ε appropriately. By remarks (*d*), (*c*) above, $(c_2-\frac{1}{2}c_1^2)(A) \leqq (c_2-\frac{1}{2}c_1^2)(\tilde{A}) \leqq C_4 - C_5 \deg(\tilde{A}, \tilde{\omega}) + \frac{1}{2}c_1(\tilde{A})^2 \leqq C_4 - C_5 \deg(A, \omega) + \frac{1}{2}c_1(A)^2$, so the inequality

$$(c_2-c_1^2)(A) \leqq C_4 - C_5 \deg(A, \omega) \tag{3.4}$$

is valid for any $A \subset E$ with torsion-free quotient, with C_4, $C_5>0$ constants depending only on E, ω.

(f) The last observation is the following: by definition, $\deg(-, \omega)$ ignores the singularities of torsion-free sheaves. However, this is also true on the level of forms in the following sense: if Q is a torsion-free quotient of a bundle E and the latter is given an hermitian connection as above, then off $S(Q)$ the bundle Q inherits an hermitian connection and thus gives a curvature form F_Q on $X\backslash S(Q)$. The claim is that $\operatorname{tr}\hat{F}_Q$ is integrable and indeed $\frac{i}{2\pi}\int_X \operatorname{tr}\hat{F}_Q dV = \deg(Q,\omega)$, where the right-hand side is defined in the usual way. To see this, it suffices to assume that $\operatorname{rank} Q = 1$ (otherwise replace E, Q by $\Lambda^q E, \Lambda^q Q$), and then Q is the image in $\det Q$ of a holomorphic map $E \to \det Q$ which is surjective outside $S(Q)$. Locally, the singular part of F_Q is then $\bar\partial\partial \log|f|^2$, where f is a rank E-tuple of holomorphic functions whose only common zero is the singular point. Pulling back to the desingularization space $\tilde{X} \xrightarrow{\pi} X$, $\pi^*\log|f|^2 = \log|\tilde{f}|^2 + \sum a_j \log|s_j|^2$ where $\tilde{f}$ is non-vanishing, s_j is the holomorphic function defining the exceptional line L_j, and $a_j \in \mathbb{Z}$. By the Poincaré-Lelong lemma [10, p. 388], $\log|s_j|^2$ is integrable and $\pi^* F_Q = F_{\tilde{Q}} + 2\pi i \sum a_j T_{L_j}$ in the sense of currents. Since $\int_{L_j} \pi^*\omega = 0$, this gives $\int_X F_Q \wedge \omega = \int_{\tilde{X}} \pi^*(F_Q \wedge \omega) = \int_{\tilde{X}} F_{\tilde{Q}} \wedge \pi^*\omega$, and since $\tilde{Q} = (\pi^* \det Q) \otimes K$ for some line bundle K with curvature $\sum n_j \sigma_j$, it follows that $\frac{i}{2\pi}\int_X F_Q \wedge \omega = \deg(Q,\omega)$, as claimed.

In fact, since the curvature forms σ constructed in the last section lie in $L^p(X)$ for all $p<2$ (when pushed down to $X \setminus \{\text{blown-up points}\}$), the same is true of $\bar\partial\partial \log|f|^2$ and F_Q.

4. Construction of Subsheaves

Let X be a compact surface and ω be a fixed positive $\bar\partial\partial$-closed (1,1)-form on X. If B is a torsion-free sheaf on X, a subsheaf $A \subset B$ will be called *admissible* if A is coherent and $0 < \operatorname{rank} A < \operatorname{rank} B$. Then B can be one of two types; namely, B has an admissible subsheaf (type I) or, B has no admissible subsheaves (type II). All of the analysis in this section will deal exclusively with a bundle E of type I.

The following fact will be used frequently (cf. [5, p. 3]): if E is a bundle which is not stable, then there exists a stable admissible $A \subset E$ with E/A torsion-free and $\mu(A) \geqq \mu(E)$.

Lemma 1. *If E is a bundle on X, then $\{\deg((A) : A \subset E$ is admissible$\}$ is bounded above.*

Proof. If not, there exists a sequence $A_i \subset E$ with $\mu(A_i) \uparrow \infty$. Without loss of generality, E/A_i is torsion-free, and passing to a subsequence, $\operatorname{rank} A_i = a$ is constant. Then $\det A_i \to \Lambda^a E$ is injective, and $\deg(\det A_i) \uparrow \infty$. Fix a connection on $\Lambda^a E$, and on $(\det A_i)^*$ put the $H-E$ connection. Then (2.5) applied to the non-zero section of $(\det A_i)^* \otimes \Lambda^a E$ yields a contradiction for i large enough. □

If $A \subset E$ is admissible of rank a, let $\nu_E(A) := a(\mu(E) - \mu(A))$. By Lemma 1, the possible values of ν_E are bounded below, and indeed, if E is stable, then $\nu_E(A) > 0$ for all admissible A.

Lemma 2. *If E is a stable bundle on X and if there exists an admissible $A \subset E$ of rank a such that $\nu_E(A) = \inf\{\nu_E(A') : A' \subset E$ is admissible$\}$, then*

(a) *A is stable; and*

(b) *$B := E/A$ is torsion-free and stable.*

Proof. (a) If $C \subset A$ is admissible of rank c, $a(\mu(E) - \mu(A)) \leqq c(\mu(E) - \mu(C)) < a(\mu(E) - \mu(C))$ since $c < a$ and $\mu(E) > \mu(C)$.

(b) If $\hat{A}$ is the maximal normal extension of A in E, then $a(\mu(E)-\mu(A)) \leqq a(\mu(E)-\mu(\hat{A}))$, so $\mu(\hat{A}) \leqq \mu(A)$. On the other hand, $A \to \hat{A}$ is a monomorphism so $\mu(A) \leqq \mu(\hat{A})$. Thus $\mu(A)=\mu(\hat{A})$, giving $\nu_E(\hat{A})=\nu_E(A)$. By (a), $\hat{A}$ is stable, so $A \to \hat{A}$ must be an isomorphism. Thus $B=E/A$ is torsion-free.

If $C \subset B$ is admissible with torsion-free quotient, let $K := \ker(E \to B/C)$. A quick calculation gives

$$\mu(C)=\mu(E)-\frac{1}{c}(\nu_E(K)-\nu_E(A)) \leqq \mu(E)<\mu(B), \quad c=\operatorname{rank} C. \quad \square$$

The strategy of this section is to produce subsheaves $A \subset E$ with this infimum property, to desingularize these, and show that (eventually) such A can be assumed to be subbundles; this process commences with the next lemma.

Lemma 3. *Let S be a torsion-free sheaf on X and let $\{L_i\}_{i=1}^{\infty}$ be a sequence of line bundles such that $|\mu(L_i)| \leqq$ const and $\Gamma(X, L_i^* \otimes S) \neq 0$. Then there is a subsequence with $c_1(L_i)$ constant.*

Proof. By replacing S with S^{**} if necessary, it can be assumed that S is locally free. If $\operatorname{rank} S=1$, the result follows from Corollary 2. If $\operatorname{rank} S>1$, pick a non-zero homomorphism $L_1 \to S$ and let $S_1 := S/L_1$, $S_1' := S_1/\tau(S_1)$, $\hat{L}_1 := \ker S \to S_1'$. From the exact sequence $0 \to L_i^* \otimes \hat{L}_1 \to L_i^* \otimes S \to L_i^* \otimes S_1' \to 0$ it follows that the sequences $\Gamma(X, L_i^* \otimes \hat{L}_1)$ and $\Gamma(X, L_i^* \otimes S_1')$ cannot both be almost always zero, so the result follows by induction on $\operatorname{rank} S$. $\square$

The next lemma is the key lemma of this section even though its proof is trivial when (X, ω) is algebraic and straightforward when X is Kähler.

Lemma 4. *Let E be a bundle of rank r on X and suppose that the main theorem has been proved for bundles of rank less than r. Then*

(a) *If E is of type I, then there exists a stable admissible $A \subset E$ with torsion-free quotient such that $\mu(A)=\sup\{\mu(A') : A' \subset E$ is admissible$\}$.*

(b) *If, moreover, E is semi-stable, then there exists an admissible $B \subset E$ such that $\nu_E(B)=\inf\{\nu_E(B') : B' \subset E$ is admissible$\}$.*

Proof. (a) Choose a sequence of admissible $A_i \subset E$ with $\mu(A_i) \uparrow M := \sup\{\mu(A') : A' \subset E\}$, and without loss of generality, each A_i is stable and has torsion-free quotient. If $\mu(A_i)$ is eventually constant, then A_i satisfies the requirements of the lemma for large enough i, so suppose that this is not the case. By passing to a subsequence it can be supposed that $\operatorname{rank} A_i=a$ is constant and $\mu(A_i)$ is strictly increasing.

Since $\mu(\det A_i)=a\mu(A_i)$ and $\det A_i \to \Lambda^a E$ is non-zero, Lemma 3 implies that there is a subsequence with $c_1(A_i)$ constant. By Proposition 2 therefore, it must be the case that $b_1(X)$ is odd. Since each A_i is stable, it admits an $H-E$ connection by the inductive hypothesis, so by (3.3), $\{(c_1^2-2c_2)(A_i)\}$ is bounded above. On the other hand, by (3.4), $\{(c_1^2-c_2)(A_i)\}$ is bounded below, so it follows that a subsequence has $c_2(A_i)$ constant. By passing to yet another subsequence, it can be assumed that $\{A_i\}$ is topologically constant.

Now recall that $\deg : \operatorname{Pic}(X) \to \mathbb{R}$ induces $\deg : H^1(X, \mathcal{O}) \to \mathbb{R}$ and this annihilates the rank $b_1(X)$ lattice $H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathcal{O})$. Since $b_1(X)$ is odd by assumption, Proposition 2 implies that $\deg: H^1(X,\mathcal{O})\to\mathbb{R}$ is not identically zero, so $\ker(\deg)/H^1(X,\mathbb{Z})=T$, a torus, and $\mathrm{Pic}_0(X)=H^1(X,\mathcal{O})/H^1(X,\mathbb{Z})=T\times\mathbb{R}$. After picking a basis for $H^1(X,\mathcal{O})$ as $\mathbb{R}$-vector space and setting $L_i:=\det A_i$, the component of $L_1^*\otimes L_i$ in T can be assumed to converge to some element of T, and on the other hand, the component in $\mathbb{R}$ also converges since it is measured by deg and $\deg(L_i)\uparrow aM$. Thus (a subsequence of the) L_i converges to some $L_\infty\in\mathrm{Pic}(X)$ with $\mu(L_\infty)=aM$.

Now let $L\in\mathrm{Pic}_0(X)$ be a line bundle with $\mu(L)=1$, and set $\tilde{A}_i:=A_i\otimes L^{-\mu(A_i)}$, so $\mu(\tilde{A}_i)=0$, $\{\tilde{A}_i\}$ is topologically constant, and of course, $\tilde{A}_i$ is stable. By the inductive hypothesis, $\tilde{A}_i$ admits a (unique) $H-E$ connection, and this is moreover an anti-self-dual Yang-Mills connection. The curvature F_i of these connections satisfy $\|F_i\|^2_{L^2}=4\pi^2 p_1(\tilde{A}_i)=\text{constant}$, so by Uhlenbeck's weak compactness theorem [22], [19, 5], there is a finite set $S=\{x_1,\dots,x_N\}\subset X$ such that a subsequence of these connections (on the same underlying smooth bundle) converges weakly in $L^p_{1,\mathrm{loc}}(X\backslash S)$ for any p to an anti-self-dual connection over $X\backslash S$. By the removable singularities theorem [21], this connection extends across S to a smooth ASD connection on a (possibly topologically different) bundle $\hat{A}_\infty$, This ASD connection gives $\hat{A}_\infty$ a unique holomorphic structure.

Since $\det\hat{A}_i=L_i\otimes L^{-a\mu(A_i)}$ and this converges to $L_\infty\otimes L^{-aM}$, it follows that $\det\hat{A}_\infty=L_\infty\otimes L^{-aM}$ and $\mu(\hat{A}_\infty)=0$. Setting $A_\infty:=\hat{A}_\infty\otimes L^M$, it follows that $\mu(A_\infty)=M$ and $A_i\to A_\infty$ weakly in $L^p_{1,\mathrm{loc}}(X\backslash S)$ for any p (in the sense of connections).

It suffices now to produce a non-zero holomorphic map $A_\infty\to E$, for if A'_∞ is one of the stable components of A_∞ whose existence is asserted by Corollary 4, and if $A'_\infty\to E$ is non-zero, then $A'_\infty\to E$ must be a sheaf inclusion else the image I satisfies $M=\mu(A'_\infty)<\mu(I)$. Moreover, A'_∞ must be equal to its maximal normal extension $\hat{A}'_\infty$ in E (since the latter must have $\mu=M$ and is therefore semi-stable), so A'_∞ has torsion-free quotient.

The existence of a non-zero holomorphic map $A_\infty\to E$ is proved by repetition of Donaldson's argument [5, pp. 22–23], and will be an argument appearing here subsequently also.

For each j, there is a non-zero holomorphic map $s_j: A_j\to E$. Fix an hermitian connection on E compatible with $\bar\partial_E$ and, as before, A_j is equipped with its $H-E$ connection. From (2.5), $\Delta|s_j|^2+i^*\partial(|s_j|^2\bar\partial\omega)-i^*\bar\partial(|s_j|^2\partial\omega)\leqq(|\hat{F}_j|+|\hat{F}_E|)|s_j|^2 \leqq\text{const}|s_j|^2$, so by Theorem 9.20 [8] it follows that $\sup_X|s_j|^2\leqq C\|s_j\|^2_{L^8(X)}$. Choose balls B_α about the points $x_\alpha\in S$ such that A_∞, E are holomorphically trivial on them and such that $C^4\sum\mathrm{Vol}(B_\alpha)=\frac{1}{2}$, and normalize s_j so that $\|s_j\|_{L^8(X)}=1$. Since the connection connections converge weakly in $L^p_{1,\mathrm{loc}}(X\backslash S)$ for any p and $\bar\partial_j s_j=0$, it follows that $\|s_j\|_{L^8_2(K)}\leqq\text{const}(\|s_j\|_{L^8(K)}+1)\leqq\text{const}$ for $K:=X\backslash\cup B_\alpha$, (using also the C^0 bound on s_j). Thus $\{s_j\}$ has a subsequence converging weakly in $L^8_2(K)$ and strongly in $C^0(K)$ to a limit s_∞ which satisfies $\bar\partial_\infty s_\infty=0$. Since $\|s_j\|^8_{L^8(K)}\geqq\frac{1}{2}$ for all j, the limit is non-zero, and by Hartogs' theorem, it extends to X to give a non-zero holomorphic map $A_\infty\to E$. This completes the proof of (a).

The proof of (b) is essentially identical. If $B\subset E$ is not stable, then there exists stable $B'\subset B$ which has E/B' torsion-free and $\nu_E(B')\leqq\nu_E(B)$. The proof of (a) can then be repeated by choosing a minimizing sequence for ν_E and passing to a subsequence of constant rank. □

Let $\tilde{X} \xrightarrow{\pi} X$ be a modification of X consisting of N blow-ups, and let ω be a positive $\bar{\partial}\partial$-closed (1,1)-form on X. Let $\sigma_1, \ldots, \sigma_N$ be forms constructed as in Sect. 2, one for each component of the exceptional divisor and all pulled-back to $\tilde{X}$. Suppose $\alpha_1, \ldots, \alpha_N > 0$ are such that, if $\varrho := \sum \alpha_i \sigma_i$, then $\pi^*\omega - \varrho$ is positive. Then $\tilde{\omega}_\varepsilon := \pi^*\omega - \varepsilon\varrho$ is positive for any $\varepsilon \in (0,1]$ since $\pi^*\omega$ is positive semi-definite. If E is an r-bundle on X, then by Lemma 4 (a), there is for each ε a subsheaf $A(\varepsilon) \subset \pi^*E$ maximizing $\mu(A, \tilde{\omega}_\varepsilon)$ over all admissible $A \subset \pi^*E$. This can be strengthened as follows:

Lemma 5. *There exists $\varepsilon_0 > 0$ and a stable admissible $A_0 \subset \pi^*E$ such that $\mu(A_0, \tilde{\omega}_\varepsilon) = \sup\{\mu(A, \tilde{\omega}_\varepsilon) : A \subset \pi^*E$ is admissible$\}$ for all $\varepsilon \in (0, \varepsilon_0]$.*

Proof. Take $\varepsilon_1 = 1$ and choose $A_1 \subset \pi^*E$ according to Lemma 4 (a). Suppose that there exists $\varepsilon_2 < \varepsilon_1$ and $A_2 \subset \pi^*E$ with $\mu(A_2, \tilde{\omega}_{\varepsilon_2}) > \mu(A_1, \tilde{\omega}_{\varepsilon_2})$. Without loss of generality, A_2 has torsion-free quotient so by remark (b) of Sect. 3, $\varrho \cdot c_1(A_2) \leqq 0$. Moreover, using remark (c); $\mu(A_1, \tilde{\omega}_{\varepsilon_1}) = \mu(\pi_* A_1) - \varepsilon_1 \varrho \cdot c_1(A_1)/a_1 \geqq \mu(\pi_* A_2) - \varepsilon_1 \varrho \cdot c_1(A_2)/a_2 = \mu(A_2, \omega_{\varepsilon_1})$ and $\mu(A_1, \tilde{\omega}_{\varepsilon_2}) = \mu(\pi_* A) - \varepsilon_2 \varrho \cdot c_1(A_1)/a_1 < \mu(\pi_* A_2) - \varepsilon_2 \varrho \cdot c_1(A_2)/a_2 = \mu(A_2, \omega_{\varepsilon_2})$. These imply $(\varepsilon_1 - \varepsilon_2)[\varrho \cdot c_1(A_1)/a_1 - \varrho \cdot c_1(A_2)/a_2] < 0$, so $\varrho \cdot c_1(A_1)/a_1 < \varrho \cdot c_1(A_2)/a_2$. Here $a_i = \operatorname{rank} A_i$.

Now replace (ε_1, A_1) by (ε_2, A_2). This process must terminate after finitely many steps because $\varrho \cdot c_1(A_j)$ is bounded above by zero, all the α_i's are positive, and the coefficients ot the σ_i's in $c_1(A_j)$ are all non-negative integers. □

Corollary 5. *If E is ω-stable, then*

(a) *π^*E is $\tilde{\omega}_\varepsilon$-stable for all ε sufficiently small, and*

(b) *there exists $\varepsilon_0 > 0$ and admissible $B_0 \subset \pi^*E$ such that $\nu_{\pi^*E}(B_0, \tilde{\omega}_\varepsilon) = \inf\{\nu_{\pi^*E}(B, \tilde{\omega}_\varepsilon) : B \subset \pi^*E$ is admissible$\}$ for all $\varepsilon \in (0, \varepsilon_0]$.*

Proof. (a) Let $M := \sup\{\mu(A, \omega) : A \subset E$ is admissible$\}$. Since M is realized by some $A \subset E$ by Lemma 4(a) and E is stable, it follows $M < \mu(E)$. Let $A_0 \subset \pi^*E$ be the A_0 given by Lemma 5. Then $\mu(A_0, \tilde{\omega}_\varepsilon) = \mu(\pi_* A_0, \omega) - \varepsilon\varrho \cdot c_1(A_0)/a_0 \leqq M - \varepsilon\varrho \cdot c_1(A_0)/a_0 < \mu(E, \omega) = \mu(\pi^*E, \tilde{\omega}_\varepsilon)$ if ε is small enough.

(b) Take ε_1 small enough so that π^*E is $\tilde{\omega}_\varepsilon$-stable for $\varepsilon \leqq \varepsilon_1$. Choose $B_1 \subset \pi^*E$ according to Lemma 4(b) and repeat the argument of Lemma 5. □

Thus stability is preserved under pull-backs to blow-ups (in the above sense). [*Semi*-stability is not preserved!]. The following lemma shows that this is also true of the desingularization process:

Lemma 6. *With $X, \tilde{X}, \omega, \varrho$ as in Lemma 5, let B be a torsion-free sheaf of rank $\leqq r$ on X and suppose that $\tilde{B}$ on $\tilde{X}$ is a desingularization of B according to Sect. 3. Then*

(a) *If B is ω-stable, it follows that $\tilde{B}$ is $\tilde{\omega}_\varepsilon$-stable for $\varepsilon > 0$ sufficiently small;*

(b) *If B is given by an exact sequence $0 \to A \to E \to B \to 0$ with $\operatorname{rank} A \leqq r$ and $0 \to \tilde{A} \to \pi^*E \to \tilde{B} \to 0$ is the desingularization sequence, it follows that $\tilde{A}$ is $\tilde{\omega}_\varepsilon$-stable for sufficiently small $\varepsilon > 0$ if A is ω-stable.*

Proof. (a) There is nothing to prove if $\tilde{B}$ has no admissible subsheaves, so suppose that it has such subsheaves. By the Remark (a) of Sect. 3, there is an exact sequence $0 \to B \to \pi_*\tilde{B} \to Q \to 0$, where $Q :=$ quotient is supported on $S(B)$. If follows that $\det B = \det(\pi_*\tilde{B})$, so $\mu(B) = \mu(\pi_*\tilde{B})$. Now, $\pi_*\tilde{B}$ is also stable: if $A \subset \pi_*\tilde{B}$ is admissible, let I be the image of A in Q under the composition $A \hookrightarrow \pi_*\tilde{B} \twoheadrightarrow Q$. Then $A' := \ker(A \to I)$

is an admissible subsheaf of B, and since B is stable it follows $\mu(A') < \mu(B)$. But as above, $A' = A$ off a finite subset, so $\mu(A) = \mu(A') < \mu(B) = \mu(\pi_* \tilde{B})$.

By Lemma 5, there exists $A_0 \subset \tilde{B}$ such that $\mu(A_0, \tilde{\omega}_\varepsilon) = \sup\{\mu(A, \tilde{\omega}_\varepsilon) : A \subset \tilde{B}\}$ for all ε small enough. So if $a = \operatorname{rank} A_0$, $b = \operatorname{rank} B$ and $\delta := \mu(\pi_* \tilde{B}) - \mu(\pi_* A_0)$, then $\delta > 0$ and $\mu(A_0, \tilde{\omega}_\varepsilon) = \mu(\pi_* A_0, \omega) - \varepsilon \varrho \cdot c_1(A_0)/a = \mu(\pi_* \tilde{B}, \omega) - \delta - \varepsilon \varrho \cdot c_1(A_0)/a$ $= \mu(\tilde{B}, \tilde{\omega}_\varepsilon) - \delta + \varepsilon(\varrho \cdot c_1(\tilde{B})/b - \varrho \cdot c_1(A_0)/a) < \mu(\tilde{B}, \tilde{\omega}_\varepsilon)$ if ε is small enough.

(b) The same proof as (a) works (and is simpler since $\pi_* \tilde{A} = A$ is stable by hypothesis). □

The next lemma is somewhat technical and is required for the proof of the main result of this section which follows it.

Lemma 6. *Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be an element of l_2 all of whose entries α_i are positive, and let $\{a^j\}_{j=1}^{\infty}$ be a sequence in l_2 such that all entries a_i^j* in *$a^j = (a_1^j, a_2^j, \dots)$ are non-negaitve integers (so almost all a_i^j are zero for fixed j). Suppose that $A_j := \langle \alpha, a^j \rangle = \sum_{i=q}^{\infty} \alpha_i a_i^j$ is strictly increasing. Then $\{\|a^j\|_{l_2}\}$ is unbounded.*

Proof. Suppose on the contrary that $\|a^j\| \leqq B$ for all j. If, for each i, $\{a_i^j\}_{j=1}^{\infty}$ is almost always zero, choose k_0 such that $\sum_{i \geqq k_0} \alpha_i^2 < (A_2/B)^2$, and choose N so large that $a_i^j = 0$ for all $i \leqq k_0$ if $j \geqq N$. Then for $j \geqq N$, $A_2 < A_j = \sum_{i \geqq k_0} \alpha_i a_i^j \leqq \left(\sum_{i \geqq k_0} \alpha_i^2\right)^{1/2} \cdot (\sum (a_i^j)^2)^{1/2} < (A_2/B) \cdot B = A_2$, a contradiction.

So there exists k such that $\{a_k^j\}_{j=1}^{\infty}$ is not almost zero, and let k_0 be the first such k. Since $\|a^j\| \leqq B$, $\{a_{k_0}^j\}$ is bounded, there is a subsequence which has $a_{k_0}^j = a_{k_0} \neq 0$ constant, with $a_1^j, \dots, a_{k_0-1}^j = 0$ for all j.

Since $\{A_j\}$ is strictly increasing, there exists M such that $A_M > \alpha_{k_0} a_{k_0}$. If every entry after the k_0-th in the subsequence is almost always zero, choose k_1 so that $\sum_{i \geqq k_1} \alpha_i^2 < (A_M - \alpha_{k_0} a_{k_0})^2 B^{-1}$ and $N > M$ so large that $a_i^j = 0$ for all i with $k_0 < i \leqq k_1$ if $j \geqq N$. Then for $j \geqq N$, the same contradiction as above ensues, giving another entry which is not almost always zero. Repeating this argument $B^2 + 1$ times gives the desired conclusion. □

Proposition 3. *Let X be a compact surface with positive $\bar{\partial}\partial$-closed* $(1,1)$*-form ω, and suppose that the main theorem has been proved for bundles of rank less than r. If E is an ω-stable r-bundle on X which has an admissible subsheaf, then there exist*

(i) *a modification $\tilde{X} \xrightarrow{\pi} X$ consisting of N blow-ups;*

(ii) *$\alpha_1, \dots, \alpha_N > 0$ such that, if $\sigma_1, \dots, \sigma_N$ are forms constructed as in Section 2 and $\varrho := \sum \alpha_i \sigma_i$, the form $\pi^* \omega - \varrho$ is positive;*

(iii) *$\varepsilon_0 > 0$ and a* subbundle *$A \subset \pi^* E$ such that $\nu_{\pi^* E}(A, \tilde{\omega}_\varepsilon) = \inf\{\nu_{\pi^* E}(A', \tilde{\omega}_\varepsilon) : A' \subset \pi^* E$ is admissible$\}$ for all $\varepsilon \in (0, \varepsilon_0]$, where $\tilde{\omega}_\varepsilon := \pi^* \omega - \varepsilon \varrho$.*

Proof. By Lemma 4(b) there exists $A_0 \subset E$ satisfying $\nu_E(A_0) = \inf\{\nu_E(A') : A' \subset E$ admissible$\}$, and the quotient $B_0 := E/A_0$ is automatically torsion-free and stable by Lemma 2.

If B_0 is locally free, then there is nothing more to do, so suppose this is not the case. Desingularize B_0 to get $\tilde{X}_0 \xrightarrow{\pi} X$ together with $\pi^* B_0 \twoheadrightarrow \tilde{B}_0$, $\pi^* A_0 \hookrightarrow \tilde{A}_0$. Let $\{\sigma_i\}$

be any of the forms of Sect. 2 (one for each exceptional line), and choose $\alpha_i > 0$ so that $\varrho_0 := \sum \alpha_i \sigma_i$ has $\pi^*\omega - \varrho_0$ positive.

By Corollary 5(b), there exists $A_1 \subset \pi^* E$ satisfying (iii), except that it may not be a subbundle. If not, for any positive ε sufficiently small one has $\nu_E(\pi_* A_1) + \varepsilon\varrho_0 \cdot c_1(A_1) = \nu_{\pi^* E}(A_1, \tilde{\omega}_\varepsilon) \leqq \nu_{\pi^* E}(\tilde{A}_0, \tilde{\omega}_\varepsilon) = \nu_E(\pi_* \tilde{A}_0) + \varepsilon\varrho \cdot c_1(\tilde{A}_0)$. Since $\pi_* \tilde{A}_0 = A_0$ letting $\varepsilon \to 0$ gives $\nu_E(\pi_* A_1) \leqq \nu_E(A_0)$, and by definition of A_0, the reverse inequality holds also. So $\nu_E(\pi_* A_1) = \nu_E(A_0)$, giving $\varrho_0 \cdot c_1(A_1) \leqq \varrho_0 \cdot c_1(\tilde{A}_0)$. If equality holds here, then $\tilde{A}_0$ satisfies the requirements of the proposition.

Suppose then that $\varrho_0 \cdot c_1(A_1) < \varrho_0 \cdot c_1(\tilde{A}_0)$. Desingularize the torsion-free sheaf $B_1 := \pi^* E / A_1$ to get $\tilde{X}_1 \xrightarrow{\pi_1} \tilde{X}_0$, $\pi_1^* B_1 \twoheadrightarrow \tilde{B}_1$, $\pi_1^* A_1 \hookrightarrow \tilde{A}_1$. Choose more σ's and α's so that $\varrho_1 := \pi_1^* \varrho_0 + \sum \alpha_1 \sigma_i$ has $\pi^*\omega - \varrho_1$ positive, where π denotes $\tilde{X}_1 \to X$. Now choose A_2 according to Corollary 5(b) so that $\nu_{\pi^* E}(\mathrm{A}_2, \tilde{\omega}_\varepsilon) = \inf\{\nu_{\pi^* E}(A', \tilde{\omega}_\varepsilon) : A' \subset \pi^* E\}$, where $\tilde{\omega}_\varepsilon = \pi^*\omega - \varepsilon\varrho_1$. [It is important to use $\pi : \tilde{X}_1 \to X$ rather than $\pi_1 : \tilde{X}_1 \to \tilde{X}_0$ at this point.] Again one obtains $\nu_E(\pi_* A_2) \leqq \nu_E(\pi_* \tilde{A}_1)$, and since $\pi_* \tilde{A}_1 = \pi_{0*} \pi_{1*} \tilde{A}_1 = \pi_{0*} A_1$, it follows as before that $\nu_E(\pi_* A_2) = \nu_E(A_0)$, and $\varrho_1 \cdot c_1(A_2) \leqq \varrho_1 \cdot c_1(\tilde{A}_1)$. If equality holds here, then $\tilde{A}_1$ satisfies the requirements of the proposition; otherwise, repeat the process again.

If this procedure fails to terminate, then there is an infinite sequence of modifications $\ldots \to \tilde{X}_{j+1} \to \tilde{X}_j \to \ldots \to X$ with A_{j+1}, $\tilde{A}_j \subset \pi^* E$ on $\tilde{X}_{j+1}$ satisfying $\nu_E(\pi_* A_{j+1}) = \nu_E(\pi_* \tilde{A}_j) = \nu_E(A_0)$ and $\varrho_{j+1} \cdot c_1(A_{j+1}) < \varrho_{j+1} \cdot c_1(\tilde{A}_j)$, where π denotes $\tilde{X}_{j+1} \to X$. Here $\varrho_{j+1} = \pi_{j+1}^* \varrho_j + \sum \alpha_i \sigma_i$ for some $\alpha_i > 0$ and σ_i belonging to the modification $\tilde{X}_{j+1} \to \tilde{X}_j$.

Since $\tilde{A}_j$ results from the desingularization of the torsion-free sheaf $B_j = \pi^* E / A_j$ on $\tilde{X}_j$, $\varrho_{j+1} \cdot c_1(\tilde{A}_j) \leqq \varrho_j \cdot c_1(A_j)$; (indeed, this is strict). Thus $\{\varrho_{j+1} \cdot c_1(\tilde{A}_j)\}$ is a strictly decreasing sequence. By passing to a subsequence, it can be assumed that $\operatorname{rank} A_j = a$ is constant, and then the equation $\nu_E(\pi_* \tilde{A}_j) = \nu_E(A_0)$ implies $\mu(\pi_* \tilde{A}_j)$ is constant. Since $\pi_* \tilde{A}_j$ is contained in E and has torsion-free quotient, it follows from Lemma 3 that there is a subsequence with $c_1(\pi_* \tilde{A}_j)$ constant. Since $0 \to \pi_* \tilde{A}_j \to E \to \pi_* \tilde{B}_j \to 0$ is exact off a finite subset, $c_1(\pi_* \tilde{B}_j)$ is also constant. Thus if $c_1(\pi_* \tilde{A}_j) = \beta \in H^2(X, \mathbb{Z})$ and $c_1(\pi_* \tilde{B}_j) = \gamma \in H^2(X, \mathbb{Z})$, then it follows that $c_1(\tilde{A}_j) = \beta + \sum a_i^j \sigma_i$ and $c_1(\tilde{B}_j) = \gamma - \sum a_i^j \sigma_i$ for some non-negative integers a_i^j. If $\varrho_{j+1} = \sum \alpha_i \sigma_i$, then $\varrho_{j+1} \cdot c_1(\tilde{A}_j) = -\sum a_i^j \alpha_i$ is strictly decreasing with j, and since $\operatorname{Vol}(\tilde{X}_{j+1}, \pi^*\omega - \varrho_{j+1}) = \operatorname{Vol}(X) - \frac{1}{2}\sum \alpha_i^2$, the infinite sequence of α's is in l_2. By Lemma 7, $\|a^j\|^2 := \sum_i (a_i^j)^2$ is an unbounded sequence.

Now, by Lemma 2, A_j and B_j on $\tilde{X}_j$ are stable with respect to $\pi^*\omega - \varepsilon\varrho_j$ for ε sufficiently small. So by Lemma 6, $\tilde{A}_j$ and $\tilde{B}_j$ on $\tilde{X}_{j+1}$ are stable with respect to some positive $\bar{\partial}\partial$-closed (1,1)-form on $\tilde{X}_{j+1}$ (not necessarily $\pi^*\omega - \varepsilon\varrho_{j+1}$). By the inductive hypothesis, they admit $H-E$ connections and therefore satisfy Lübke's inequality [15]: with $A = A_j$, $B = B_j$, $\operatorname{rank} A = a$, $\operatorname{rank} B = b$, this states $\left(\frac{a-1}{2a} c_1^2 - c_2\right)(A) \leqq 0$ and $\left(\frac{b-1}{2b} c_1^2 - c_2\right)(B) \leqq 0$. Adding these together and substituting $c_1(A) = \beta + \sum a_i^j \sigma_i$, $c_1(B) = \gamma - \sum a_i^j \sigma_i$, $c_2(E) = c_2(A) + c_2(B) + c_1(A) \cdot c_1(B)$ gives $0 \geqq \frac{a-1}{2a} \beta \cdot \beta + \frac{b-1}{2b} \gamma \cdot \gamma + \beta \cdot \gamma - c_2(E) + \frac{r}{2ab} \|a^j\|^2$ after a short calculation with some fortuitous cancellations; ($r = a + b$ of course). Since all terms except the last on the right are independent of j in this inequality, the desired contradiction has been achieved because $\|a^j\|$ is unbounded. □

5. Proof of Theorem 1

In order to prove the main theorem, a certain functional, to be given shortly, must be minimized. The set over which this minimization is performed is the set of all integrable L_1^p connections on a fixed $U(r)$-bundle, each connection inducing the same holomorphic structure. By the Newlander-Nirenberg theorem, a *smooth* integrable connection induces a holomorphic structure, but it is not immediately clear that the same is true of general L_1^p connections. However, the following result shows that if p is large enough, this is indeed the case. The proof was suggested by the proof for the case $n=1$ in [1].

Lemma 8. *Let B_1 denote the open unit polydisc in $\mathbb{C}^n$ centred at the origin. Let A be an $r\times r$ matrix of $(0,1)$-forms with coefficients in $L_{1,\mathrm{loc}}^p(B_1)$ satisfying $\bar\partial A+A\wedge A=0$, where $p\geqq 2n$. Then locally in B_1, $A=u^{-1}\bar\partial u$ for some $u\in L_2^p$.*

Proof. Consider first the following: let $\mathscr{U}$ denote the Banach manifold of invertible $r\times r$ matrices on $\mathbb{P}_n$ with coefficients in L_2^p, $\mathscr{M}$ *denote the Banach space of* $r\times r$ matrices on $\mathbb{P}_n$ with coefficients in L^p, and $\mathscr{A}$ denote the Banach space of $r\times r$ matrices of $(0,1)$-forms on $\mathbb{P}_n$ with coefficients in L_1^p. Let $\mathscr{M}^\perp$ be the subspace of M perpendicular in L^2 to the constant matrices.

Since $p>n$, the Sobolev embedding theorem shows that the map ϕ given by

$$\mathscr{U}\times\mathscr{A}\ni(u,A)\mapsto\bar\partial^*(u^{-1}\bar\partial u+u^{-1}Au)=-i\Lambda\partial(u^{-1}\bar\partial u+u^{-1}Au)\in\mathscr{M}^\perp$$

is a smooth map of Banach manifolds $\mathscr{U}\times\mathscr{A}\to\mathscr{M}^\perp$, where the adjoint is with respect to the Fubini-Study metric on $\mathbb{P}_n$. The partial derivative of ϕ in the $\mathscr{U}$-direction at $(1,0)$ is $T\mathscr{U}\ni v\mapsto\Delta''v\in\mathscr{M}^\perp$, which is surjective with kernel the constants. By the implicit function theorem, the equation $\bar\partial^*(u^{-1}\bar\partial u+u^{-1}Au)=0$ has a solution $u\in\mathscr{U}$ for all $A\in\mathscr{A}$ sufficiently small.

Now suppose that A is simply a matrix of $(0,1)$-forms with coefficients in $L_{1,\mathrm{loc}}^p(B_1)$ satisfying $\bar\partial A+A\wedge A=0$. Pull-back $A|_{B_r}$ to B_1 by the holomorphic map $B_1\ni z\mapsto rz\in B_r$ to give $\tilde A_r\in L_1^p(B_1)$. Then $\|\tilde A_r\|_{L_1^p(B_1)}\leqq\mathrm{const}\,r^{1-2n/p}\|A\|_{L_1^p(B_r)}$. Let η be a cutoff function with support in B_1 and with $\eta=1$ on $B_{1/2}$. Then if $A_r:=\eta\tilde A_r$, $\|A_r\|_{L_1^p}\leqq\mathrm{const}\,\|\tilde A_r\|_{L_1^p(B_1)}\leqq\mathrm{const}\,r^{1-2n/p}\|A\|_{L_1^p(B_r)}$, and the last term on the right can be made arbitrarily small by shrinking r since $p\geqq 2n$ and $A\in L_1^p(B_{1/2})$.

The matrices A_r can now be regarded as defined on $\mathbb{P}_n$, so if r is small enough, there exists u such that $\bar\partial^*(u^{-1}\bar\partial u+u^{-1}A_ru)=0$. If $A_r':=u^{-1}\bar\partial u+u^{+1}A_ru$, then $\bar\partial A_r'+A_r'\wedge A_r'=u^{-1}(\bar\partial A_r+A_r\wedge A_r)u=u^{-1}[\bar\partial(\eta\tilde A_r)+(\eta\tilde A_r)\wedge(\eta\tilde A_r)]u$. Thus near 0, A_r' satisfies the (overdetermined in general) elliptic system $\bar\partial^*A_r'=0$, $\bar\partial A_r'=-A_r'\wedge A_r'$ and is therefore smooth there. By the usual Newlander-Nirenberg theorem $A_r'=v^{-1}\bar\partial v$ for some smooth v defined near 0, and if $\tilde w:=vu^{-1}\in L_2^p$ then $\tilde w^{-1}\bar\partial\tilde w=\tilde A_r$ near 0. Reverting to the original coordinates gives $A=w^{-1}\bar\partial w$ for some $w\in L_2^p$ defined near 0, and the conclusion of the lemma follows by applying this result at each point of B_1. □

Remark. With simple alterations the above proof can be sharpened to $p>n$.

The functional to be minimized can now be given – it is almost identical to Donaldson's [4], so the same notation will be used.

For hermitian $r\times r$ matrices M, the trace norm is $\nu(M):=\mathrm{tr}(M^*M)^{1/2}=\sum_{i=1}^{r}|\lambda_i|$ where $\{\lambda_i\}$ are the eigenvalues of M repeated according to multiplicity. As explained in [4], it defines a norm, and if $M=\begin{pmatrix}A & B\\ B^* & D\end{pmatrix}$ then $\nu(M)\geqq|\mathrm{tr}A|+|\mathrm{tr}D|$. If s is a section of the endomorphisms of a $U(r)$-bundle E on the compact surface X, set $N(s):=\|\nu(s)\|_{L^p(X)}$, and for a connection A on E with curvature F in $A^{1,1}(\mathrm{End}E)$, the functional is $J(A):=N(i\hat{F}+\lambda 1)$, where $\lambda=\lambda_E=\frac{1}{irV}\int_X \mathrm{tr}\hat{F}dV$. *Here p will be some fixed number greater that 4.*

The following lemma corresponds to Lemma 3 of [4].

Lemma 9. *Suppose that Theorem* 1 *has been proved for bundles of rank less than r. If E is a stable holomorphic r-bundle on X which can be expressed as an extension $0\to B\to E\to C\to 0$ with B, C stable, then there is a smooth hermitian connection A on E compatible with $\bar\partial_E$ such that $J(A)<4\pi V^{1/p-1}\nu_E(B)$.*

Proof. On B, C, fix the $H-E$ connections which exist by the inductive hypothesis, and let $\beta\in A^{0,1}(\mathrm{Hom}(C,B))$ be a $\bar\partial$-closed $(0,1)$-form representing the extension $0\to B\to E\to C\to 0$.

If Q is the operator $Q:=-i\Lambda\partial\bar\partial$, then $Q=i\Lambda\bar\partial\partial-i\Lambda(\partial\bar\partial+\bar\partial\partial)=P-i\hat{F}$ [cf. (2.2), (2.3)], so from (2.4) it follows that $Q+Q^*=P+P^*-2i\hat{F}=\Delta-i\hat{F}$. For the induced $H-E$ connection on $\mathrm{Hom}(C,B)$, $\hat{F}=i(\lambda_B-\lambda_C)1$, and since E is stable, $\lambda_B>\lambda_C$. Thus Q^* has no kernel and Q is surjective; in particular, there exists $\gamma\in\mathrm{Hom}(C,B)$ such that $\Lambda\partial(\beta+\bar\partial\gamma)=0$.

If β is thus modified so that $\Lambda\partial\beta=0$, now rescale it so that $\sup_X|\beta|=1$; ($\beta\neq 0$ since E is stable). Using $t\beta$ in place of β for $t=\bar{t}\neq 0$, (3.2) shows that the curvature of the induced connection on E has

$$i\hat{F}_E(t)+\lambda_E 1=\begin{bmatrix}(\lambda_E-\lambda_B)1-it^2\Lambda\beta\wedge\beta^* & 0\\ 0 & (\lambda_E-\lambda_C)1-it^2\Lambda\beta^*\wedge\beta\end{bmatrix}.$$

Since $\lambda_B>\lambda_E>\lambda_C$, when t is small enough all of the eigenvalues of the top term are negative and all those of the bottom are positive. For such such t, it follows that $\nu(i\hat{F}_E(t)+\lambda_E 1)=-\mathrm{tr}[(\lambda_E-\lambda_B)1-it^2\Lambda\beta\wedge\beta^*]+\mathrm{tr}[(\lambda_E-\lambda_C)1-it^2\Lambda\beta^*\wedge\beta]$ $=4\pi V^{-1}\nu_E(B)-2t^2|\beta|^2$. Since $|\beta|^2\leqq 1$, taking t sufficiently small gives $N(i\hat{F}_E(t)+\lambda_E 1)<4\pi V^{\frac{1}{p}-1}\nu_E(B)$. □

The next step is the equivalent of Lemma 1 of [4], but in the current setting, it is made considerably more complicated by the presence of singularities of one sort or another.

Suppose, as usual, that E is a stable r-bundle on the compact surface X, where stability is with respect to a fixed positive $\bar\partial\partial$-closed $(1,1)$-form ω. If E has an admissible subsheaf, pull-back E to the modification $\tilde{X}\xrightarrow{\pi}X$ given by Proposition 3 and fix one of the forms $\tilde\omega_\varepsilon$ described there. By Proposition 3 and Lemma 9, π^*E admits a smooth connection A with $J(A)<4\pi\tilde{V}^{1/p-1}m$, where $\tilde{V}=\mathrm{Vol}(\tilde{X},\tilde\omega_\varepsilon)$ and $m:=\inf\{\nu_{\pi^*E}(S,\tilde\omega_\varepsilon):S\subset\pi^*E$ is admissible$\}$. If E has no admissable subsheaves, no blowing-up is required what follows. To simplify notation, $(\tilde{X}, \pi^* E, \tilde{\omega}_\varepsilon)$ will temporarily be denoted by (X, E, ω) when E is of type I.

Now choose a sequence A_i of smooth connections on E which minimize the functional J. Since line bundles admit $H-E$ connections, it can be assumed that the induced connections on $\det E$ are all the same; namely, the $H-E$ connection.

Since $J(A_i)$ is comparable with the usual L^p norm of the self-dual component of the curvature $F(A_i)$, $\|F(A_i)\|_{L^2}$ is bounded. By the weak compactness theorem of Uhlenbeck [22], ([19, 5]), there is a finite subset $S=\{x_1, \dots, x_N\} \subset X$ and local gauge transformations such that the gauge-transformed connections converge weakly in $L^2_{1,\mathrm{loc}}(X \setminus S)$. In fact, an inspection of the proof of Corollary 23 [5] shows that the sequence can be assumed to converge weakly in $L^p_{1,\mathrm{loc}}(X \setminus S)$, for all that is required in the proof of that corollary is a uniform bound on the L^p norm of the self-dual component of the curvatures. The transition functions of the resulting "bundle" on $X \setminus S$ are then continuous, and (as in [5]), Sect. 3 of [22] applies to construct global gauge transformations from the local ones. Thus, after suitable bundle automorphisms of the underlying $U(r)$-bundle, (a subsequence of) the gauge-transformed sequence, also denoted by A_i, converges weakly in $L^p_{1,\mathrm{loc}}(X \setminus S)$ to a connection A' with $F(A') \in L^2(X)$ and $\hat{F}(A') \in L^p(X)$. By semi-continuity, $J(A') \leqq \inf J(A_i)$.

The connection A' has curvature of type (1,1), so by Lemma 8 it induces a holomorphic structure; denote this holomorphic bundle on $X \setminus S$ by E'. Since the connections on $\det E$ do not change in the sequence, $\det E' = \det E$ and $\operatorname{tr} F(A') = \operatorname{tr} F(A_0)$.

Following Donaldson [5] again, a non-zero holophorphic map $E \to E'$ will now be constructed, as in the proof of Lemma 4. Let g_j be the complex automorphism intertwining A_0 and A_j, with $\det g_j = 1$ for all j; (that is, g_j is the map which gives the isomorphism between the holomorphic structure E_0 defined by A_0 and that which is defined by A_j).

By (2.5), $\Delta |g_j|^2 + i^* \partial(|g_j|^2 \bar{\partial}\omega) - i^* \bar{\partial}(|g_j|^2 \partial\omega) \leqq 2(|\hat{F}_0| + |\hat{F}_j|)|g_j|^2$, so by Theorem 9.20 [8] there is a constant C, independent of j, such that $\sup_X |g_j|^2 \leqq C[\|g_j\|^2_{L^2(X)} + \|(|\hat{F}_0| + |\hat{F}_j|)|g_j|^2\|_{L^4(X)}]$. By Hölder's inequality, it follows that $\sup_X |g_j|^2 \leqq C\|g_j\|^2_{L^q(X)}$ for $q = 8p/(p-4)$ and some new constant C, using the uniform bound on $\|\hat{F}_j\|_{L^p}$. Since $\{A_j\}$ converges weakly in $L^p_{1,\mathrm{loc}}(X \setminus S)$ *and* $p > 4$, the A_j's are bounded in $C^0(K)$ for any compact $K \subset X \setminus S$. Repeating the argument of Lemma 4, after rescaling g_j to $\tilde{g}_j$ satisfying $\|\tilde{g}_j\|_{L^q(X)} = 1$ and choosing small balls B_α about the points $x_\alpha \in S$, a subsequence of the $\tilde{g}_j$'s can be found which converges weakly in $L^p_2(K_0)$ and strongly in $L^q(K_0)$ to a non-zero limit $\tilde{g}$ representing a holomorphic map $E_0 \to E'$, where $K_0 = X \setminus UB_\alpha$. Since ∂K_0 is pseudo-concave, $\tilde{g}$ extends to $X \setminus S$, and by diagonalization ([19]) it can be assumed that $\tilde{g}_j$ is converging weakly to $\tilde{g}$ in $L^p_{1,\mathrm{loc}}(X \setminus S)$.

Since the connections on $\det E$, $\det E'$ are the same, $\det \tilde{g}$ is a holomorphic function on $X \setminus S$, and therefore constant by Hartogs' theorem. Suppose that $\det \tilde{g} = 0$. Then $\tilde{g}$ has non-zero kernel at every point, giving a diagram on $X \setminus S$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & \tilde{g}\downarrow & & \downarrow & & \\ 0 & \longleftarrow & C & \longleftarrow & E' & \longleftarrow & I & \longleftarrow & 0 \end{array} \tag{5.1}$$

where K = kernel, Q = quotient, I = image, C = cokernel. If $\mathcal{O}(E)_x$ is generated by sections $e_1, \ldots, e_r \in \Gamma(B_\alpha, \mathcal{O}(E))$ as $\mathcal{O}_x$-module for each $x \in B_\alpha$, then the images of $e_1, \ldots, e_r$ in $\Gamma(B_\alpha \backslash \{x_\alpha\}, Q)$ generate Q_y as $\mathcal{O}_y$-module for each $y \in B_\alpha \backslash \{x_\alpha\}$. By a theorem of Serre [20], $i_* Q$ is a coherent analytic sheaf on X, where $i: X \backslash S \to X$ is inclusion. [Indeed $i_* Q$ is locally free in a neighbourhood of $x_\alpha \in S$, being torsion-free and normal there; $E \to i_* Q$ need not be surjective at x_α though.] It follows that $i_* K$ is coherent, so in particular, E has an admissible subsheaf and is therefore of type I.

Off a codimension $\geqq 1$ analytic subset T of $X \backslash S$, (5.1) is a diagram of bundles. In a unitary frame for E', the curvature $F(A')$ has the form

$$F(A') = \begin{bmatrix} F_I - \beta \wedge \beta^* & \nabla \beta \\ -\nabla \beta^* & F_C - \beta^* \wedge \beta \end{bmatrix},$$

where $\beta \in A^{0,1}(\mathrm{Hom}(C, I))$ is a $\bar\partial$-closed form representing the extension $0 \to I \to E' \to C \to 0$. Moreover, $ir\lambda_E = \mathrm{tr}\hat{F}(A_0) = \mathrm{tr}\hat{F}(A') = \mathrm{tr}\hat{F}_I + \mathrm{tr}\hat{F}_C$, and it follows from the property of v stated earlier in this section that $v(i\hat{F}(A') + \lambda_E 1) \geqq |\mathrm{tr}(i\hat{F}_I - i\Lambda\beta \wedge \beta^* + \lambda_E 1)| + |\mathrm{tr}(i\hat{F}_C - I\Lambda\beta \wedge \beta^* + \lambda_E 1)| = 2|\mathrm{tr}\, i\hat{F}_I + |\beta|^2 + q\lambda_E|$, where $q = \mathrm{rank}\, Q = \mathrm{rank}\, I$ and $|\beta|^2 = -\mathrm{tr}\, i\Lambda\beta \wedge \beta^*$. Thus $J(A') = \|v(i\hat{F}(A') + \lambda_E 1)\|_{L^p(X)} \geqq 2V^{1/p-1}\|v(i\hat{F}(A') + \lambda_E 1)\|_{L^1(X)} \geqq 2V^{1/p-1}\left|\int_X [(\mathrm{tr}\, i\hat{F}_I + q\lambda_Q + |\beta|^2) + q(\lambda_E - \lambda_Q)]dV\right|$.

If it could be shown that $\int_X (\mathrm{tr}\, i\hat{F}_I + q\lambda_Q + |\beta|^2)dV$ were non-negative, then a contradiction (to $\det\tilde{g} = 0$) would be obtained at this point. For since E is stable, $\lambda_E > \lambda_Q$, and therefore $J(A') \geqq 2V^{1/p}q(\lambda_E - \lambda_Q) = 4\pi V^{1/p-1} v_E(K)$; this contradicts Lemma 9 and $J(A') \leqq \inf J(A_i)$.

Were it not for the singularities arising from the Uhlenbeck-Sedlacek-Donaldson technique, the non-negativity of the above integral would be immediate: $\int_X (\mathrm{tr}\, i\hat{F}_I + q\lambda_Q)dV$ is the volume of the zero set of $\det Q \to \det I$.

To see that the above integral is always non-negative, note first that on $X \backslash (S \cup T)$, $\mathrm{tr} F_I = \mathrm{tr} F_Q + \bar\partial\partial \log |\tilde{g}_q|^2$, where $\tilde{g}_q$ is the induced map $\Lambda^q E \to \Lambda^q E'$. Although $\bar\partial\partial \log |\tilde{g}_q|^2$ may not be integrable, it *can* be assumed that $\bar\partial\partial |\tilde{g}_q|^2$ is in $L^1(X)$. For

$$\bar\partial\partial |g_j|^2 = \langle \partial g_j, \partial g_j \rangle + \langle g_j, F_j g_j - g_j F_0 \rangle \quad . \tag{5.2}$$

(where $\langle , \rangle$ involves only the inner product on $\mathrm{End}\, E$ and not that on $\Lambda^{1,0}$). Rescaling g_j to $\tilde{g}_j$, applying $i\Lambda$ to (5.2) and integrating gives $\|\partial \tilde{g}_j\|_{L^2} \leqq \mathrm{const}$, so $\{\partial \tilde{g}_j\}$ can be assumed to converge weakly in $L^2(X)$ and $\bar\partial\partial |\tilde{g}|^2 = \langle \partial\tilde{g}, \partial\tilde{g} \rangle + \langle \tilde{g}, F(A')\tilde{g} - \tilde{g}F_0 \rangle \in L^1(X)$. The same conclusion applies if $\tilde{g}$ is replaced by $\tilde{g}_q$.

If s is the composition $\Lambda^q E \to \Lambda^q Q \hookrightarrow \det Q$ and $\det Q$ is equipped with its $H-E$ metric, then with the induced metric on $\mathrm{Hom}(\Lambda^q E, \det Q)$ it follows $\mathrm{tr} F_Q = F_{\det Q} - \bar\partial\partial \log |s|^2$ where $\hat{F}_{\det Q} = i\lambda_{\det Q} = iq\lambda_Q$. Thus

$$\mathrm{tr}\, i\hat{F}_I + q\lambda_Q + |\beta|^2 = P \log |\tilde{g}_q|^2 - P \log |s|^2 + |\beta|^2 \ , \tag{5.3}$$

where $P = i\Lambda\bar\partial\partial$ is the operator of Sect. 2.

Although $\mathrm{tr}\, i\hat{F}_I$ and $|\beta|^2$ need not be integrable, the sum on the left of (5.3) is in $L^p(X)$ since $\hat{F}(A') \in L^p(X)$ and hermitian projection has constant norm. Thus if

$\int\limits_X (\mathrm{tr}\, i\hat{F}_I + q\lambda_Q + |\beta|^2)dV = a$, then there exists $\phi \in L_2^q(X)$ such that $P\phi = \mathrm{tr}\, i\hat{F}_I + q\lambda_Q + |\beta|^2 - a/V$. Hence

$$P\log(|\tilde{g}_q|^2 e^{-\phi}) + |\beta|^2 - a/V = P\log|s|^2 \ , \tag{5.4}$$

If $(f_1, \ldots, f_n)$ is an n-tuple of holomorphic functions which is not identically zero and $|f|^2 = \sum |f_j|^2$, then $\log|f|^2$ is plurisubharmonic; ie. $i\bar{\partial}\partial \log|f|^2 \leqq 0$. Therefore $P\log|s|^2$ is bounded above, and a smooth function ψ can be chosen so that $P\log|s|^2 + P\psi$ is negative in a neighbourhood of each of the zeroes of s. [Since $\Lambda^q Q$ is torsion-free, s has only finitely many isolated zeroes.] Thus

$$P\log(|\tilde{g}_q|^2 e^{\psi - \phi}) + |\beta|^2 - a/V = P\log(|s|^2 e^{\psi}) \ , \tag{5.5}$$

with the right hand side negative in a neighbourhood of each of the zeroes of s.

Suppose now that $a = -b^2 < 0$. By the last remark of Sect. 2, the right-hand side of (5.5) is integrable, so a smooth bump function $\eta \geqq 0$ can be found which is identically equal to 1 in neighbourhoods of the zeroes of s and is supported in the neighbourhoods where $P\log(|s|^2 e^{\psi}) < 0$ such that $\int\limits_X \eta P\log(|s|^2 e^{\psi})dV = -c^2$ with $c^2 \leqq \frac{1}{2} b^2$. Then $\int\limits_X (1-\eta) P(\log|s|^2 e^{\psi})dV = c^2$, so there exists a *smooth* function χ such that $P\chi = (1-\eta)P\log(|s|^2 e^{\psi}) - c^2/V$. Thus $P\log(|s|^2 e^{\psi}) = \eta P\log(|s|^2 e^{\psi}) + (1-\eta)P\log(|s|^2 e^{\psi}) \leqq (1-\eta)P\log(|s|^2 e^{\psi}) = P\chi + c^2/V$, so (5.5) gives

$$P\log(|\tilde{g}_q|^2 e^{\psi - \phi - \chi}) + |\beta|^2 \leqq (c^2 - b^2)/V < 0 \ , \tag{5.6}$$

But since $P\log f = f^{-1}Pf + |\partial \log f|^2$ for any positive function f, (5.6) implies $P(|\tilde{g}_q|^2 e^{\psi - \phi - \chi}) < 0$. This gives the desired contradiction, for $\bar{\partial}\partial |\tilde{g}_q|^2 \in L^1(X)$, $\phi \in L_2^q(X) \hookrightarrow C^1(X)$ and ψ, χ are smooth, so $\bar{\partial}\partial(|\tilde{g}_q| e^{\psi - \phi - \chi}) \in L^1(X)$, and a sequence of smooth function f_j such that $\bar{\partial}\partial f_j$ converges to $\bar{\partial}\partial(|\tilde{g}_q| e^{\psi - \phi - \chi})$ in $L^1(X)$ [8, Theorem 7.4] yields $0 = \lim \int i\bar{\partial}\partial f_j \wedge \omega < 0$. This means that a is, in fact, non-negative, and consequently $\det \tilde{g} \neq 0$ by the earlier argument.

Thus when E is of either type, $\tilde{g} : E \to E'$ is an isomorphism. Unfortunately, a priori this is only an isomorphism outside S and it must be shown that $\tilde{g} \in L_2^q(X)$. By emulating part of Donaldson's argument in [5], it will be be shown that S in fact is empty.

Recall that the unscaled g_j's had $\det g_j = 1$ and that $\tilde{g}_j = g_j \|g_j\|_{L^q}^{-1}$ for $q = 8p/(4-p)$. From the preceeding arguments, $\sup|g_j| \leqq \mathrm{const}\, \|g_j\|_{L^q}$ is uniformly bounded and therefore, so too is $\sup\limits_X |g_j^{-1}|$.

If $h_j := g_j^* g_j$, then $F_j = g_j(F_0 + \bar{\partial}_0(h_j^{-1}\partial_0 h_j))g_j^{-1}$, giving

$$\bar{\partial}_0 \partial_0 h_j = g_j^* F_j g_j - h_j F_0 + \bar{\partial}_0 h_j \wedge h_j^{-1} \partial_0 h_j \ .$$

Since $\{h_j\}$, $\{h_j^{-1}\}$ are uniformly bounded, $|\partial_0 h_j|^2 = -i\,\mathrm{tr}\, \Lambda \bar{\partial}_0 \mathrm{h}_j \wedge \partial_0 h_j$ compares uniformly with $-i\,\mathrm{tr}\, \Lambda \bar{\partial}_0 h_j \wedge h_j^{-1} \partial_0 h_j$, and after applying $\mathrm{tr}\, i\Lambda$ to the above equation and integrating, it follows that $\{\partial_0 h_j\}$, and hence $\{h_j^{-1}\partial_0 h_j\}$, is bounded in $L^2(X)$. By ellipticity of $\bar{\partial}_0$, $\{h_j^{-1}\partial_0 h_j\}$ is bounded in $L_1^2(X)$, implying that $\{h_j\}$ is bounded in $L_2^2(X)$. Thus a subsequence converges weakly in $L_2^2(X)$, and by compactness of the embedding $L_2^2 \hookrightarrow L^{q/2}$, strongly in $L^{q/2}(X)$.

For any $j.k$, set $g_{jk}:=g_k g_j^{-1}$ and $h_{jk}:=g_{jk}^* g_{jk}$. Then $i\Lambda\bar{\partial}_j\partial_j h_{jk}=g_{jk}^* i\hat{F}_k g_{ij} - h_{jk} i\hat{F} + i\Lambda\bar{\partial}_j h_{jk} \wedge h_{jk}^{-1}\partial_j h_{jk}$, and taking the trace gives $P\operatorname{tr} h_{jk} \leqq \operatorname{tr}[i\hat{F}_k g_k h_j^{-1} g_k^* - i\hat{F}_j g_j^{*-1} h_k g_j^{-1}] = \operatorname{tr}[i\hat{F}_k(g_k h_j^{-1} g_k^* - 1) - i\hat{F}_j(g_j^{*-1} h_k g_j^{-1} - 1)] \leqq \text{const}(|\hat{F}_k| + |\hat{F}_j|)|h_j - h_k|$, using here the uniform bounds on $\sup|h_j|$, $\sup|h_j^{-1}|$ and the fact that $\operatorname{tr}\hat{F}_k = \operatorname{tr}\hat{F}_j$. Interchanging j, k and adding gives $P\sigma(h_j, h_k) \leqq \text{const}(|\hat{F}_j| + |\hat{F}_k|)|h_j - h_k|$, where $\sigma(h_j, h_k) := \operatorname{tr} h_j^{-1} h_k + \operatorname{tr} h_k^{-1} h_j - 2r$ (cf. [5, Sect. 2]). By Theorem 9.20 of [8] and Hölder's inequality, $\sup_X \sigma(h_j, h_k) \leqq \text{const}(\|h_j - h_k\|_{L^{q/2}} + \|\sigma(h_j, h_k)\|_{L^1}) \leqq \text{const}\,\|h_j - h_k\|_{L^{q/2}}$. Since $\{h_j\}$ is converging strongly in $L^{q/2}(X)$, it follows that the sequence is uniformly Cauchy and therefore converges in $C^0(X)$. By (the proof of) Lemma 19 of [5], it follows that $\{h_j\}$ is in fact bounded in $L_2^q(X)$, and by making the unitary change of gauge $g_j \mapsto h_j^{1/2}$, a weak limit $g \in L_2^q(X)$ is obtained such that the associated connection A' minimizes the functional J.

The next task is to show that $\inf J = 0$; the argument follows closely that in [4]. Recall the operators $P = i\Lambda\bar{\partial}\partial$ and $Q = -i\Lambda\partial\bar{\partial}$. Since $P + P^* = \Delta + i\hat{F}$ and $Q + Q^* = \Delta - i\hat{F}$, $R := P + Q$ satisfies $R + R^* = 2\Delta$. Any solution $s \in L_2^q(\operatorname{End} E)$ of $Rs = 0$ is necessarily of the form $s = \text{const}\,1$; this is true even though R may not have smooth coefficients, because a sequence of smooth connections A_j' can be chosen converging strongly in L_1^p to A' and the corresponding operators R_j have the same second order term, first order terms converging in L_1^p and zeroth order terms converging in L^p. Thus $0 = \langle s, Rs\rangle = \lim\langle s, R_j s\rangle = \lim\langle d_{A_j'} s, d_{A_j'} s\rangle = \langle d_A s, d_A s\rangle$, implying $s = \text{const}\,1$.

The same type of elementary approximation argument shows that there is a unique solution $s \in L_2^q(\operatorname{End} E) \cap (\ker R)^\perp$ to $Rs = i\hat{F}(A') + \lambda_E 1$ [since $i\hat{F}(A') + \lambda_E 1$ is orthogonal to $\ker R^*$], and s is self-adjoint since $(Rs)^* = Rs^*$. If $g_t := 1 - ts$, then g_t is invertible for small t, and $F_t := F(g_t A') = F(A') - t(\bar{\partial}\partial - \partial\bar{\partial})s + O(t^2)$. Thus $i\hat{F}_t + \lambda_E 1 = (1 - t)(i\hat{F}(A') + \lambda_E 1) + O(t^2)$ implying $i\hat{F}(A') + \lambda_E 1 = 0$ else J is not minimized at $t = 0$.

In the case when E has no admissible subsheaves, it has now been shown that E admits an $H-E$ connection. In the case that E does have admissible subsheaves, it has been shown that $\pi^* E$ admits an $H-E$ connection for each of the forms $\tilde{\omega}_\varepsilon$ of Proposition 3, where $\tilde{X} \xrightarrow{\pi} X$ is the modification described in that proposition. The final task is to push these down to X.

Recall that the forms σ_i of Proposition 3 could have support in arbitrarily small neighbourhoods of the exceptional lines they represent, so $\tilde{\omega}_\varepsilon - \pi^*\omega$ can have support in an arbitrarily small neighbourhood of the exceptional divisor D. Shrinking these supports (and necessarily, the coefficients α_i at the same time) gives a sequence of forms $\{\tilde{\omega}_j\}$, say, and corresponding connections $\tilde{A}_j$ on $\pi^* E$ such that $\tilde{A}_j$ is an $H-E$ connection for $\tilde{\omega}_j$. Thus if $\{x_1, \ldots, x_M\} = \pi(D)$, then off each fixed (but arbitrarily small neighbourhood) of $\pi(D)$ the sequence $\tilde{A}_j$ can be viewed as a sequence of connections A_j on E, which for j large enough, are all $H-E$ connections for ω. The constants λ_E in this sequence are of course changing: $(\lambda_E)_j = -2\pi\mu(E)/\operatorname{Vol}(\tilde{X}, \tilde{\omega}_j)$, with $\operatorname{Vol}(X, \tilde{\omega}_j) \to \operatorname{Vol}(X)$.

Applying the argument of Uhlenbeck-Sedlacek-Donaldson once again, there exist $x_{M+1}, \ldots, x_N \in X$ such that, if $S := \{x_1, \ldots, x_N\}$, then after suitable gauge transformations the A_j converge weakly in $L_{1,\text{loc}}^p(X \backslash S)$ to an $H-E$ connection A with finite Yang-Mills action over $X \backslash S$. (The $U-S-D$ argument is still applicable even though it is being applied over $X \setminus \bigcup_\alpha B_\alpha^j$ with $B_\alpha^j \to \{x_\alpha\}$, as an inspection of [19] quickly shows.) By ellipticity, A is smooth, and since, in a neighbourhood of any point of X the connection A can be twisted by an $H-E$ connection on a trivial line bundle so that the resulting connection has $\lambda=0$, it follows from the removable singularities theorem [21] that A extends across S to an $H-E$ connection on a (possibly topologically different) bundle E'. The new holomorphic bundle E' is automatically semistable by Corollary 4. If U is any neighbourhood of S, then for sufficiently large j, $\int_{X\setminus U} \operatorname{tr}\hat{F}(A_j)dV = ir(\lambda_E)_j \operatorname{Vol}(X\setminus U)$, so $\mu(E')=\mu(E)$.

It remains therefore to construct a non-zero holomorphic map $E\to E'$ or $E'\to E$. Choose a small ball B_α about x_α and set $U := \bigcup_\alpha B_\alpha$, $\tilde{U} := \pi^{-1}(U)$. The balls B_α are chosen small enough that E has a connection A_0 (compatible with $\bar{\partial}_E$) which is smooth and moreover is *flat* in all B_α. Pull A_0 back to $\tilde{X}$ and let g_j be the endomorphism intertwining $\pi^* A_0$ with $\tilde{A}_j$. Using the Laplacian Δ_j on $\tilde{X}$ determined by $\tilde{\omega}_j$, as well as the $*$ and Λ operators for $\tilde{\omega}_j$, (2.5) gives

$$\Delta_j|g_j|^2 + i^*\partial(|g_j|^2\bar{\partial}\tilde{\omega}_j) - i^*\bar{\partial}(|g_j|^2\partial\tilde{\omega}_j) \leqq 2\langle g_j, i\hat{F}(\tilde{A}_j)g_j - g_j i\hat{F}_0\rangle\ , \qquad (5.7)$$

where $i\hat{F}(\tilde{A}_j) = 2\pi\mu(E)/\operatorname{Vol}(\tilde{X},\tilde{\omega}_j)1$. If $\mu(E)>0$, replace g_j by g_j^{-1}; otherwise leave g_j as it is. Then in $\tilde{U}$, $\hat{F}_0=0$ and the right-side of (5.7) is $\leqq 0$. Since $\bar{\partial}\partial\tilde{\omega}_j=0$, Theorem 3.1 of [8] (the maximum principle) gives $\sup_{\tilde{U}}|g_j|^2 \leqq \sup_{\partial\tilde{U}}|g_j|^2$. On the other hand, outside $\tilde{U}$ the forms $\tilde{\omega}_j$ all agree for large enough j, and in $\tilde{X}\setminus\tilde{U}$ one has the usual bound $P|g_j|^2 \leqq \text{const}\,|g_j|^2$, where P is simply determined by ω. By Theorem 9.20 of [8] it now follows that $\sup_{\tilde{X}}|g_j|^2 \leqq C\|g_j\|^2_{L^8(\tilde{X}\setminus\tilde{U}',\pi^*\omega)}$, where $U' \subset\subset U$ is slightly smaller.

Now choose $U'' \subset U'$ such that $C^4\operatorname{Vol}(U'') \leqq \frac{1}{2}$ and fix a non-singular metric $\tilde{\omega}$ on $\tilde{X}$ such that $\operatorname{supp}(\tilde{\omega}-\pi^*\omega) \subset \tilde{U}''$. Normalize g_j so that $\|g_j\|_{L^8(\tilde{X},\tilde{\omega})}=1$; [here it is assumed $\mu(E)\leqq 0$, otherwise use g_j^{-1} as above]. Then since $\operatorname{Vol}(\tilde{U}'',\tilde{\omega}) \leqq \operatorname{Vol}(U'',\omega)$, the usual calculation gives $\|g_j\|^8_{L^8(\tilde{X}\setminus\tilde{U}'',\pi^*\omega)} \geqq \frac{1}{2}$.

Now regard g_j as defined on $X\setminus S$. Then $\|g_j\|^8_{L^8(X\setminus U'')} \geqq \frac{1}{2}$, and exactly the same argument as in proof of Lemma 4 (i.e. [5, p. 23]) shows that the g_j's have a subsequence weakly convergent in $L_2^8(X\setminus U'')$ and strongly convergent in $C^0(X\setminus U'')$ to a limit g representing a non-zero holomorphic map $E\to E'$ (or $E'\to E$) over $X\setminus U''$, and by Hartogs' theorem, this extends to X. This map must be an isomorphism since E is stable, E' is semi-stable and $\mu(E)=\mu(E')$. This completes the proof of the theorem.

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